Prof. Dr. G. Böckle, C.V. Sriram

Seminar in Algebra, WS2022

Affine Algebraic groups

Tuesdays; 11 - 13 c.t. [1]

The aim of this seminar is to study affine algebraic groups. For this we will follow the book "Introduction to Affine Group Schemes" by W.C. Waterhouse [Wat].
We explain in detail the individual lectures below. No matter which lectures you give, please read the entire program carefully. References without explicit citations are from [Wat].

PRACTICALITIES

1. Prerequisites. Students are expected to have a good understanding of the topics covered in Algebra 1 and Algebra 2. Also some familiarity with the fundamental concepts of topology is welcome.
2. Requirements of the participants.
(a) Delivery of seminar talk. The student is expected to deliver a 90-minute talk on their chosen topic. Definitions and results must be stated clearly, and where possible illustrated with concrete examples. Ideally the student will have worked through and understood all proofs; some of these proofs should be presented, though for reasons of time others may have to be omitted. Each presentation should contain at least one example.
(b) Preparation of handout. Please create a handout for the seminar participants to accompany your presentation. This document should contain the most important definitions and results from your presentation. You are also welcome to explain details of calculations and examples there for which there is no time in the lecture.
(c) Timelines for preparation of talk and handout.
(i) Start preparing your presentation in good time (approx. 4 weeks in advance). Come two weeks before your presentation for a preliminary discussion. The appointment for the preliminary discussion is then on the corresponding Tuesday at 9:30 a.m. Bring a draft of the handout to the preliminary meeting.
(ii) A template for handout is available on the seminar webpage.
(iii) Please email the completed handout to Mr. Sriram no later than Monday morning before your presentation, so that the materials can be made available to the other seminar participants in advance.
(d) Language of talk and handout. The talks must be delivered in English. The handout should also be written in English.

TALKS

1. Affine Algebraic Groups

The first lecture is concerned with the definition of an affine algebraic group.[2]
Go through the entire chapter 1. An essential component of the definition of an affine group is the Yoneda lemma, the statement of which is relatively easy to understand. Nevertheless, it is advisable to explain this using an example (like $det : GL_2 \to \mathbb{G}_m$ in the book) – the speaker should also do exercise 1.1 (a) - (c) for this, in order to get a better understanding of it. The introduction of the Hopf algebras is done in a similar way: simple examples are presented ($\mathbb{G}_a$ and $\mathbb{G}_m$ from p. 9); Exercises 1.5 and 1.6 should be done for gaining a clarity on this aspect.

[1] First talk on October 25, 2022

[2] Throughout the seminar we use the more 'modern' terminology, according to which an affine algebraic group scheme in the sense of [Wat] (affine group scheme) is simply called "affine algebraic group"

General remark: The exercises do not necessarily have to be explained in the lecture (the lecture time may be too short for that). They mainly serve as a means for deeper understanding of the terminology, which has a direct impact on the lecture. The speaker can also briefly point out what "algebraic" in the definition of an affine algebraic group means - preferably verbally, without going into details, because this will be explained in detail later in the seminar (see p. 24, last paragraph before the corollary). The book [Mi] starts out very much like Waterhouse. It is nevertheless worth taking a look at it, at least at its introduction – although it might be a bit difficult to understand right at the beginning of the seminar.

References: [Wat, Ch. 1], supplementary [Mi, Ch. 1].

Lecture Date: October 25, 2022

Speaker:

2. Examples of Affine Algebraic Groups

The goal of this talk is to introduce different affine (algebraic) groups: diagonalizable/constant/finite affine groups. Particular attention should be paid to [Wat, § 2.4]. The reference [Pi, Lecture 2] can also be helpful for this.
The exercises from [Wat] to be worked out by the speaker (see *General Remarks* in lecture 1 for the details about working out exercises) are 2.1 and 2.9.

Reference: [Wat, Ch. 2], [Mi, Ch. 2], [Pi, Lecture 2].

Lecture Date: November 8, 2022

Speaker:

3. Representations

For this talk we study the entire chapter 3 in detail. The notion of affine algebraic group and its meaning is introduced (cf. the short comment from Lecture 1). One important result here is that affine algebraic groups over fields are isomorphic to closed subgroups of GL_n. Another one is the construction of all finite-dimensional representations of closed subgroups of GL_n.
The exercises from [Wat] to be worked out by the speaker are 3.1 and 3.6.

References: [Wat, Ch. 3], [Mi, Ch. 4].

Lecture Date: November 15, 2022

Speaker:

4. Algebraic Matrix Groups, Connected Components

We start with simple results about the Zariski topology in k^n, where k is a field [3]. We then define algebraic matrix group and construct the corresponding representable functors, which allow us, if the field k is not finite, to realize affine algebraic groups with dense rational points[4] as algebraic matrix groups – cf. first corollary on p. 32.
We then study first general structural statements for these algebraic matrix groups (§4.5 and §4.6). In addition, a nice summary of results possibly known from previous courses in Algebra about irreducible or connected components (§§5.1 - 5.5) is to be presented. The exercises to be worked out by the speaker are 4.3 and 4.7 from [Wat].

References: [Wat, §§4.1-5.5].

[3] We use k generally to denote a field. Please emphasize it, if k denotes a commutative ring, or use letters A, B, C, R or S in that case.
[4] i.e. $G(k)$ is dense in G.

Lecture Date: November 22, 2022

Speaker:

5. Connected Components and Separable Algebras

In this lecture we deal with the connected components of affine (algebraic) groups over fields. We motivate the problem (§6.1) and then introduce the notion of separable algebras (§6.2). The proof of the theorem from §6.3 is sketched (a bit of Galois theory is used here). The remaining sections are to be presented in detail - important points are the definition of the connected component of an affine group and the structure theorem of finite affine groups over perfect fields.
The exercises to be solved here are 6.12 and 6.13 from [Wat].

References: [Wat, Ch. 6], [Pi] can also be helpful.

Lecture Date: November 29, 2022

Speaker:

6. Groups of Multiplicative Type

We study certain commutative affine groups: those of multiplicative type. These are described by their character group (as Galois modules) – last theorem on p. 55. Then we study anisotropic and so-called 'split' tori, as well as examples of them. The lecture ends with the result that every action (via automorphisms) of affine connected groups on affine groups of multiplicative type must be trivial.
Exercise 7.4 from [Wat] is to be worked out by the speaker.

References: [Wat, Ch. 7].

Lecture Date: December 6, 2022

Speaker:

7. Unipotent Groups

Another type of affine groups is introduced in this talk: the unipotent groups. We start with the definition of unipotent elements of a (matrix) group, then prove the Kolchin fixed point theorem. There are different characterizations for unipotent affine algebraic groups (theorem on p. 64). We then study the structural differences of unipotent affine groups when the field k has a positive characteristic (§8.4 - §8.5) - Exercises 8.10 and 8.11 from [Wat] are instructive on this.
The lecture ends with the so-called Lie-Kolchin triangularization theorem (§10.2, p. 74) and its corollary - here the following result can be assumed to be true (will be proved in the following lecture): *irreducible representations of abelian affine groups are one-dimensional.*

References: [Wat, Ch. 8, §§10.1-10.2]

Lecture Date: December 13, 2022

8. Jordan Decomposition

The classical Jordan decomposition of matrices is transferred in a natural way to algebraic matrix groups. This allows abelian algebraic matrix groups to be described over perfect fields as a direct product of canonically defined unipotent and multiplicative subgroups. The lecture ends with Sections 10.3 and 10.4, and with the structure theorem for nilpotent algebraic matrix groups.
The exercises from [Wat] to be worked out are 9.2 and 10.3.

References: [Wat, Ch. 9, §§10.3-10.4].

Lecture Date: January 10, 2023

Speaker:

9. Differentials

The main goal of the lecture is a criterion for the reducibility of $k[G] \otimes \bar{k}$ for G an affine algebraic group over a field k (§11.6). This is expressed in terms of the rank of the differentials - or in terms of smoothness of the affine algebraic group G. To understand this, we study the (relative) derivations and differentials over rings (§11.1 - §11.5). However, the proof of the theorem in §11.6 is not presented until Lecture 11, with results from (pure) commutative algebra.
Exercises 11.3 and 11.11 from [Wat] should be completed for this talk.

References:: [Wat, Ch. 11].

Lecture Date: January 17, 2023

Speaker:

10. Lie Algebras

We study further infinitesimal properties of affine groups, this time using the Lie algebras of affine groups. We define them, prove elementary properties - especially in relation to smooth affine algebraic groups - and study some examples. Finally, we study representations of affine algebraic groups via representations of the corresponding Lie algebras.
The exercises to be understood from [Wat] here are 12.2 and 12.13.

References: [Wat, Ch. 12].

Lecture Date: January 24, 2023

Speaker:

11. Faithful Flatness

First recall(from Algebra 2) the definition of a flat module over a ring. Then present the first theorem in §13.1 as a definition of faithful flatness. State and prove second theorem in §13.2. State the results from §13.3, whenever it is required in the talk. State and prove(briefly) the theorem in §13.4. Also outline of strategy of the proof in §13.5.

Next present the main result of the talk, [Wat, §14.1, Theorem]. It asserts that extensions of Hopf algebras are faithfully flat. This is done by first considering the smooth case, and then the general case, where nilpotents may be present. It is worthwhile to recall that over base fields k of characteristic zero, all Hopf algebras are smooth. If time permits, one can also discuss applications of this result(§14.3).

Exercises 11.5, 13.1 and 14.2 from [Wat] should be completed.[5]

References: [Wat, Ch. 13 and 14].

lecture Date: January 31, 2023

Speaker:

[5] For 13.1(b), if $\{m_i \otimes b_i\}$ generates $M \otimes_A B$ as a B modules, then investigate the map $A^n \to M$ sending $e_i \mapsto m_i$. Similarly one solves 13.1(c)

The seminar could be concluded with two further lectures on the topic: QUOTIENTS BY (FINITE) AFFINE GROUPS. In addition to [Wat, Chs. 15 and 16], also [Pi] might be useful.

Speaker:

References

[Mi] JS Milne, Basic theory of algebraic group schemes, v2.00 (December 20, 2015).

[Pi] R Pink, Finite group schemes, Lecture Notes, 2004.

[Wat] WC Waterhouse, Introduction to affine group schemes, GTM 66, Springer, 1979.