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Seminar, Summer Semester 2024

Representation theory of finite groups

INTRODUCTION[1]

Representation theory plays an important role in many areas of pure and applied mathematics. It also has interesting applications in theoretical physics and quantum chemistry. The basic idea of group representation theory is to represent the elements of a group using linear transformations on a vector space. The group operates through symmetries on this vector space.

Let G be a finite group and V be a vector space of dimension n over $\mathbb{C}$. In our context, we call a homomorphism $\rho : G \to \mathrm{GL}(V)$ *a (linear) representation of* G *of degree* n. This makes it possible to investigate properties of an abstract group using objects from linear algebra. The representation theory of finite groups is the simplest case of representation theory. Many results can be extended to representations of compact groups.

In this seminar, we will deal, mainly by using the sources [FH04, Ser77], with the following topics: *Representations of finite groups*, *Characters*, *Induced representations*, *the theorems of Artin and Brauer* and *representations defined over fields of characteristic zero which are not algebraically closed*.

PRACTICALITIES

1. Prerequisites. Students are expected to have a good understanding of the topics covered in Linear Algebra 1, Linear Algebra 2 and Algebra I.

2. Requirements of the participants.
(a) Delivery of seminar talk. The student is expected to deliver a 90-minute talk on their chosen topic. Definitions and results must be stated clearly, and where possible illustrated with concrete examples. The student will have worked through and understood all proofs; some of these proofs should be presented, though for reasons of time others may have to be omitted. Each presentation should contain at least one example.

(b) Preparation of handout. Please create a handout for the seminar participants to accompany your presentation. This document should contain the most important definitions and results from your presentation. You are also welcome to explain details of calculations and examples there for which there is no time in the lecture.

(c) Timelines for preparation of talk and handout.

(i) Start preparing your presentation in good time (approx. 4 weeks before the day of the presentation). Come two weeks before your presentation for a preliminary discussion(Appointment to be made via email with Dr. Gezmiş). Bring a draft of the handout to the preliminary meeting.
(ii) A template for the handout is available on the seminar webpage.
(iii) Please email the completed handout to Dr. Gezmiş no later than Monday morning before your presentation, so that the materials can be made available to the other seminar participants in advance.

(d) Language of talk and handout. The talks must be delivered in English. The handout should also be written in English.

[1] Vorbesprechung on Monday, 22 January 2024 at 14:00 (c.t.)

TALKS

Talk 1: Basics of linear representations

(i) **Basic definitions:** Define linear representations and the notion of isomorphism for representations. Provide examples such as representations of degree one, regular representations and permutation representations given in [Ser77, I.1.2].

(ii) **Subrepresentations:** Introduce the definition of stable subspace and its complement. Give the proof of [Ser77, Thm. 1, I.1.3] and explain the case when V is endowed with a scalar product ([Ser77, Remark, I.1.3]). If time permits, also explain the direct sum of representations.

(iii) **Irreducible representations:** Define irreducible representations and prove the result [Ser77, Thm. 2, I.1.4] on the decomposition of linear representations into irreducible representations.

(iv) **Tensor product of two representations, symmetric square and alternating square:** Explain the constructions detailed in [Ser77, I.1.5, I.1.6].

References: [Ser77, I.1].

Lecture Date: April 17th, Wednesday

Talk 2: Characters and orthogonality relations

(i) **The character of a representation:** Provide the basic properties of the character of a representation [Ser77, Prop. 1, I.2.1]. For later use, define the class function [Ser77, Remark, I.2.1]. Prove [Ser77, Prop. 2, I.2.1] and [Ser77, Prop. 3, I.2.1]. Also discuss [Ser77, Exercise 2.1, I.2.1].

(ii) **Schur's Lemma:** Detail the proof of Schur's Lemma [Ser77, Prop. 4, I.2.2]. Prove also [Ser77, Cor. 1, I.2.2] as well as its matrix form [Ser77, Cor. 2 and Cor. 3, I.2.2]. Discuss [Ser77, Remarks, I.2.2] which motivates the audience for what follows.

(iii) **The orthogonality relations for characters:** The goal is to prove the orthogonality relation for characters given in [Ser77, Thm. 3, I.2.3]. Discuss the decomposition of a given representation [Ser77, Thm. 4, I.2.3] and state [Ser77, Cor. 1 and Cor. 2, I.2.3]. Finally, prove an irreducibility criterion for characters given in [Ser77, Thm. 5, I.2.3].

Reference: [Ser77, I.2.1–I.2.3].

Lecture Date: April 24th, Wednesday

Talk 3: Decomposition of representations

(i) **Decomposition of the regular representation:** Introduce the notation for this lecture and then discuss the character of the regular representation [Ser77, Prop. 5, I.2.4] as well as [Ser77, Cor. 1 and Cor. 2, I.2.4]. Explain how the above results yield to determine the irreducible representations of a group [Ser77, Remarks, I.2.4].

(ii) **The number of irreducible representations:** Recall the definition of the class function from Talk 2 and discuss the twist of an irreducible representation with a class function [Ser77, Prop. 6, I.2.5]. Define the space H of class functions and introduce an orthonormal basis for H in terms of certain characters [Ser77, Thm. 6, I.2.5]. Moreover, explain the relation between the number of conjugacy classes of G and its number of irreducible representations [Ser77, Thm. 7, I.2.5]. Discuss an application of [Ser77, Thm. 7, I.2.5] for the group $\mathfrak{S}_3$ of all the permutations of a set of order 3 [Ser77, Example, I.2.5].

(iii) **Canonical decomposition of a representation:** Define the canonical decomposition and explain some properties of the canonical decomposition [Ser77, Thm. 8, I.2.6]. Also work out [Ser77, Example, I.2.6].

(iv) **Explicit decomposition of a representation**: Finally, sketch the proof of [Ser77, Prop. 8, I.2.7].

References: [Ser77, I.2.4–I.2.7].

Lecture Date: May 8th, Wednesday

Talk 4: Subgroups, products and algebraic integers

(i) **Abelian subgroups:** Define the notion of abelian groups and then discuss their irreducible representations [Ser77, Thm. 9, I.3.1]. Consider a finite group, D_4 for example, having an abelian subgroup. Then provide an upper bound for the degree of the irreducible representation of such a group [Ser77, Cor., I.3.1].

(ii) **Product of groups:** Define the product of two groups as well as its representations and character. Explain the difference between these objects and the tensor product of two representations detailed in Talk 1. Prove also [Ser77, Thm. 10, I.3.2].

(iii) **Examples: the cyclic groups C_n and C_∞:** Define the cyclic group C_n of order n and the group of rotations C_∞. If time permits, discuss also $\mathfrak{A}_4$ the group of even permutations of a set of order 4. [Ser77, I.5.1, I.5.2, I.5.7].

(iv) **Basic properties of algebraic integers:** Define algebraic integers and prove an equivalent definition for such elements [Ser77, II.6.4] as well as discuss [Ser77, Cor. 1 and Cor. 2, II.6.4].

References: [Ser77, I.3.1, I.3.2, I.5.1, I.5.2, I.5.7, II.6.4].

Lecture Date: May 15th, Wednesday

Talk 5: Induced representations I

(i) **The induced representations:** Recall the definition of cosets of a group and define induced representations. Discuss several examples of them [Ser77, Examples, I.3.3]. Provide a proof for [Ser77, Thm. 11, I.3.3] by using [Ser77, Lem.1, I.3.3]. Moreover, explain the notion of character of an induced representation and prove [Ser77, Thm.12, I.3.3].

(ii) **Examples:** Lastly, discuss the details of dihedral groups D_n and the group $\mathfrak{S}_4$ of all permutations of a set of order 4 [Ser77, I.5.3,I.5.8].

References: [Ser77, I.3.3, I.5.3, I.5.8].

Lecture Date: May 22nd, Wednesday

Talk 6: The group algebra

(i) **Representations and modules**: For any commutative ring K, define the group algebra $K[G]$. Construct a $K[G]$-module structure on V induced by a representation of G. Discuss the semi-simplicity of $K[G]$ [Ser77, Prop. 9, II.6.1] in the characteristic zero case as well as [Ser77, Cor., II.6.1].

(ii) **Decomposition of $\mathbb{C}[G]$**: Discuss the details of the fact that $\mathbb{C}[G]$ is a product of matrix algebras over $\mathbb{C}$ which follows from [Ser77, Cor., II.6.1]. Then define the homomorphisms $\tilde{\rho}_i$ and $\tilde{\rho}$ and prove that $\tilde{\rho}$ is an isomorphism [Ser77, Prop. 10, II.6.2]. Discuss also Fourier inverse formula [Ser77, Prop. 11, II.6.2].

(iii) **The center of $\mathbb{C}[G]$**: Recall the necessary definitions and provide a proof for [Ser77, Prop. 12 and Prop. 13, II.6.3].

(iv) **Integrality properties of characters**: For a character χ of a representation of G, prove that $\chi(g)$ is an algebraic integer for any $g \in G$ [Ser77, Prop. 15, II.6.5] as well as discuss [Ser77, Prop. 16, Cor. 1 and Cor.2, II.6.5]. Finally, compare the degree of irreducible representations of G with the index of the center of G [Ser77, Prop. 17, II.6.5].

References:[Ser77, II.6].

Lecture Date: May 29th, Wednesday

Talk 7: The example $\mathfrak{S}_d$

(i) **Young tableau and the main theorem:** Define Young diagram and Young Tableau which may be used to determine the irreducible representations of $\mathfrak{S}_d$. Define also Young symmetrizer. State [FH04, Thm. 4.3, I.4.1] as well as its immediate consequence that every irreducible representation of $\mathfrak{S}_d$ may be defined over $\mathbb{Q}$. Discuss [FH04, Ex. 4.5, I.4.1]. Later on, sketch the proof of [FH04, Thm. 4.3, I.4.1] given in [FH04, I.4.2].

(ii) **Representations of $\mathcal{A}_d$:** Explain [FH04, Prop. 5.1, I.4.1] and then apply it to $\mathcal{A}_d$ which is a subgroup of $\mathfrak{S}_d$ of index two. Finally, state [FH04, Prop. 5.3, I.4.1] and explain the details of its proof given in [FH04, Exercise 5.4, I.4.1].

References: [FH04, I.4.1, I.4.2, I.5.1]

Lecture Date: June 5th, Wednesday

Talk 8: Examples: $\mathrm{GL}_2(\mathbb{F}_q)$ and $\mathrm{SL}_2(\mathbb{F}_q)$

(i) **For $\mathrm{GL}_2(\mathbb{F}_q)$:** Provide the conjugacy classes in $\mathrm{GL}_2(\mathbb{F}_q)$ as well as the associated irreducible representations together with the character table given in [FH04, pg.67–71].

(ii) **For $\mathrm{SL}_2(\mathbb{F}_q)$:** Discuss the conjugacy classes, construction of all irreducible representations by restriction and the character table explained in [FH04, pg.71–74].

References:[FH04, I.5.2].

Lecture Date: June 12th, Wednesday

Talk 9: Induced representations II

(i) **Induced representations**: Recall the definition of induced representations from Talk 5 and give an alternative definition of an induced representation using the group algebra. Prove a necessary and sufficient condition for a representation to be induced by another representation [Ser77, Prop. 18, II.7.1]. State [Ser77, Remarks, II.7.1] as well as [Ser77, Prop. 19, II.7.1].

(ii) **The Frobenius reciprocity formula:** Define the notion of induced class function by a function on G and prove [Ser77, Prop. 20, II.7.2]. Prove also the Frobenius reciprocity formula [Ser77, Thm. 13, II.7.2] and discuss [Ser77, Remarks and Prop. 21, II.7.2].

(iii) **Restriction to subgroups and Mackey's irreducibility criteria:** Explain the restriction of a representation of G to its subgroups and sketch a proof for [Ser77, Prop. 22, II.7.3]. Finally, provide a proof for Mackey's irreducibility criteria [Ser77, Prop. 23, II.7.4].

References: [Ser77, II.7]

Lecture Date: June 19th, Wednesday

Talk 10: Examples of induced representations

(i) **Normal subgroups**: Recall the definition of a normal subgroup and then prove [Ser77, Prop. 24, II.8.1] revealing a relation between irreducible representations of G and its normal subgroups. Later on, compare [Ser77, Cor, II.8.1] with [Ser77, Cor,, I.3.1] discussed in Talk 4.

(ii) **Semidirect products by an abelian group:** Sketch a proof for [Ser77, Prop. 25, II.8.2], and explain why, in this case, we are able to recover all the irreducible representations of G. Apply it to D_n, $\mathfrak{A}_4$ and $\mathfrak{S}_4$.

(iii **Sylow's theorem**: Recall the definition of a p-group and a Sylow p-subgroup as well as solvable, supersolvable and nilpotent groups [Ser77, II.8.3]. State also Sylow's theorem [Ser77, Thm. 15, II.8.4].

(iv) **Linear representations of supersolvable groups:** Finally, prove [Ser77, Thm. 16, II.8.5] discussing the irreducible representations of a supersolvable group. If time permits, work out the details of [Ser77, Exercise 8.10].

References:: [Ser77, II.8].

Lecture Date: June 26th, Wednesday

Talk 11: Artin's theorem

(i) **The statement of Artin's theorem:** Define the ring $R(G)$ [Ser77, II.9.1] and then state Artin's theorem [Ser77, Thm. 17,II.9.2].

(ii) **First proof and an alternative proof of (i) implies (ii):** Explain the details of the proof of Artin's theorem and discuss an alternative proof for one direction of its statement [Ser77, II.9.3, II.9.4].

References:: [Ser77, II.9].

Lecture Date: July 3rd, Wednesday

Talk 12: Brauer's theorem

(i) **Induced characters arising from p-elementary subgroups:** Define the notion of p-regular elements and p-elementary subgroups [Ser77, II.10.1]. Sketch a proof for [Ser77, Thm. 18, II.10.2] whose details are explained in [Ser77, II.10.3-II.10.4].

(ii) **Brauer's theorem** Define the notion of elementary subgroups and prove Brauer's theorem [Ser77, Thm. 19, II.10.5].

(iii) **The applications of Brauer's theorem** Explain some of the applications of Brauer's theorem, such as on characterization of characters [Ser77, II.11.1] and proving a theorem of Frobenius given in [Ser77, Thm. 23, II.11.2].

References:: [Ser77, II.10, II.11.1, II.11.2].

Lecture Date: July 10th, Wednesday

Talk 13: Rationality questions

(i) **The Rings $R_K(G)$ and $\bar{R}_K(G)$:** Firstly, explain the motivation of this talk to the audience and then define the rings $R_K(G)$ and $\bar{R}_K(G)$ where K is a field of characteristic zero. Define the notion of realizable and rational representations over K. Prove a necessary and sufficient condition for a representation to be realizable over K [Ser77, Prop. 33, II.12.1]. Later on discuss the realizability over cyclotomic fields and prove [Ser77, Cor., II.12.3].

(ii) **The rank of $R_K(G)$:** Define the notion of Γ_K-conjugacy and Γ_K-classes of G. Then sketch a proof of [Ser77, Cor. 2, II.12.4], describing the rank of $R_K(G)$ which is also the number of irreducible representations of G defined over K.

(iii) **Generalization of Artin's and Brauer's theorems:** Explain a generalized version of Artin's and Brauer's theorems and sketch their proof [Ser77, II.12.5–II.12.7].

(iv) **Representations over $\mathbb{Q}$ and $\mathbb{R}$:** Finally, in the light of the results proved in [Ser77, II.12], explain the results obtained for representations of G over $\mathbb{Q}$ [Ser77, II.13.1] and $\mathbb{R}$ [Ser77, II.13.2].

References:: [Ser77, II.12, II.13].

Lecture Date: July 17th, Wednesday

References

[FH04] W. Fulton and J. Harris, *Representation theory*, Graduate Texts in Mathematics Volume 129, Springer-Verlag, New York, 2004.

[Ser77] J. P. Serre, *Linear representations of finite groups*, Translated from the second French edition by Leonard L. Scott, Springer Graduate Texts in Mathematics No:42, Springer-Verlag, New York, 1977.