Notes from Vorbesprechung (26.01.24)

$G :=$ a finite group

$V :=$ a $\mathbb{C}$-vector space of dimension n

$GL(V) :=$ group of all invertible linear transformation of $V \cong GL_n(\mathbb{C})$

Def: A (linear) representation of G in V is a homomorphism $\rho : G \to GL(V)$

$\to$ induces an action of G on V: $g.v := \rho(g)(v)$

$\to$ $\rho : G \to GL(V)$, $\rho' : G \to GL(V')$ <u>isomorphic</u> if there exists a linear isomorphism $\tau : V \to V'$ so that

$$\tau(\rho(g)(v)) = \rho'(g)(\tau(v))$$

Examples

(a) $\rho : \begin{aligned} G &\longrightarrow \mathbb{C} \\ g &\longmapsto 1 \end{aligned}$ "trivial representation"

(b) Let $|G| = n$ and V of dimension n over $\mathbb{C}$ with a basis $(e_g)_{g\in G}$

Set $\rho : G \longrightarrow GL(V)$

$$g \longmapsto \rho_g : e_h \mapsto e_{gh},\ \forall h\in G.$$

ρ is called "the regular representation of G"

[More in Talk 1 & Talk 3]

(c) $G = S_3 = \langle (12), (123) \rangle$, symmetric grp.

$= \{ Id, (12), (13), (23), (123), (132) \}$

[more in Talk 7]

$\rho : S_3 \longrightarrow GL_3(\mathbb{C}) \cong GL(V)$ with

$V \cong \mathbb{C}v_1 \oplus \mathbb{C}v_2 \oplus \mathbb{C}v_3$

$Id \mapsto Id_3$

$(12) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{array}{l} v_1 \to v_2 \\ v_2 \to v_1 \\ v_3 \mapsto v_3 \end{array}$

$$(123) \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \mapsto \begin{array}{l} v_1 \to v_2 \\ v_2 \to v_3 \\ v_3 \to v_1 \end{array}$$

(d) C_3 = cyclic group of order $3 = \langle g \rangle$.

$$\begin{aligned} \rho : C_3 &\longrightarrow \mathbb{C} \\ g &\longmapsto e^{2\pi i/3} \end{aligned}$$

[More in Talk 4]

(e) Examples for representations of $GL_2(\mathbb{F}_q)$ and $SL_2(\mathbb{F}_q)$ [in Talk 8]

Applications

(1) When G is a Galois group, "Galois representations" reveal Fermat's Last Thm.

(2) Burnside's Thm: "every finite group of order $p^a q^b$

where p and q are two distinct primes are solvalbe".

c study of molecular structure in chemistry uses represen-tation theory of finite groups

d tensor decomposition of represantations [more in Talk 1] is of interest to physicists to understand particle interactions

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→ Let $W \subseteq V$ be a subspace of V "stable under the action of G" i.e. $\rho(g)(w) \in W \; \forall w \in W$

→ The restriction $\rho^W : G \to GL(W)$ of ρ to W is called <u>a subrepresentation of W</u>

Example (i) $\{0\}$ and V are subrepresentations of ρ

(ii) Recall Example (c) above and take $W := \mathbb{C}\langle v_1 + v_2 + v_3 \rangle$

Then
$$\rho^W : G \to \mathbb{C} \cong GL(W)$$
$$g \mapsto \rho_g : w \mapsto \rho_g(w) = w$$

ρ^W is the trivial representation

Def: $\rho : G \to GL(V)$ is called <u>irreducible</u> if $V \neq 0$ and the only subrepresentation of ρ is $\{0\}$ and V.

Example $D_3 = \langle r^3 = s^2 = 1, rs = sr^{-1} \rangle$

the dihedral group

[more in Talk 5]

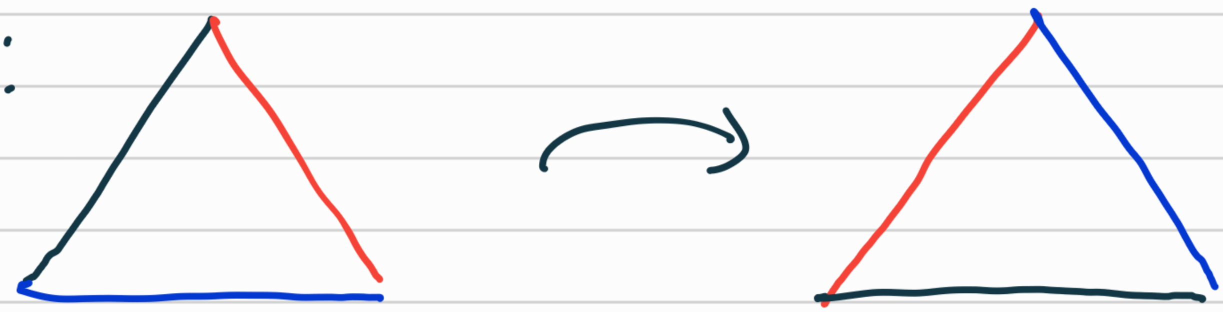
$\to$ determines "symmetries" of an equilateral triangle.

$$\rho : D_3 \longrightarrow GL_2(\mathbb{C})$$

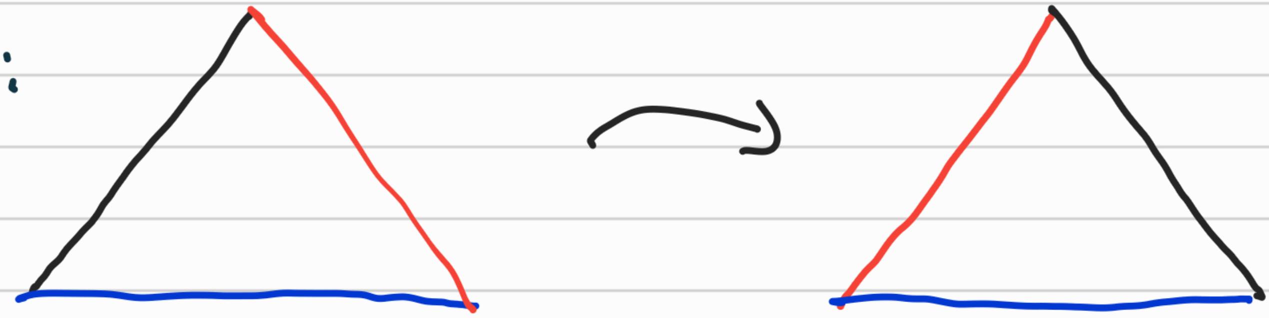
$$r \longmapsto \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}$$

$$s \longmapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

r:



s:



Then ρ is irreducible as r does not fix any line.

→ Every representation is a direct sum of irreducible representations. (in char 0) (Maschke)

Def: <u>The character of $\rho: G \to GL(V)$</u> is the map $\chi_\rho : G \to \mathbb{C}$
$g \mapsto Tr(\rho(g))$.

→ satisfy some orthogonality relations [more in Talk 2]

→ same character $\chi_\rho, \chi_{\rho'} \Leftrightarrow \rho \cong \rho'$

$\rightarrow \chi_\rho(g)$ is an algebraic integer

[More in Talk 4]

[More on in Talk 6].

Induced representations

$H \leq G$ a subgroup

Repr. of G $\xrightarrow{\text{"easy"}}$ Repr. of H

$\rho: G \to GL(V) \longrightarrow \rho^H: H \to GL(V)$

$h \mapsto \rho(h)$

Repr. of H $\xrightarrow{\text{"not so easy"}}$ Repr. of G

$\rho: H \to GL(W) \longrightarrow \mathrm{Ind}_H^G(\rho): G \to GL(V)$

"induced representation"

[More in Talk 5 & Talk 9]

Example: ⓐ If $W \cong \mathbb{C}^{|H|}$ with a basis $\{e_h\}_{h \in H}$ and $\rho: H \to GL(W)$ is the regular representation.

then $\mathrm{Ind}_H^G(\rho): G \to GL(V)$ is the regular representation where $V \cong \mathbb{C}^{|G|}$ with the basis $\{e_g\}_{g \in G}$.

6. More examples [in Talk 10]

Artin's Theorem A character of G is a "rational" linear combination of characters induced from all "cyclic" subgroups of G. [Talk 11]

Brauer's Theorem: replace above "rational" with "integral" and "cyclic" with "elementary" [Talk 12]

→ How about represantations over $\mathbb{Q}$ or $\mathbb{R}$ [Talk 13].