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Global L-functions over function fields

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Introduction

Let C be a smooth projective curve over a finite field k with a marked point ∞ and let A be the ring of regular functions on $C \smallsetminus \{\infty\}$. Furthermore, let $\mathbb{C}_\infty$ be the completion of an algebraic closure of the function field K of C at a place above ∞. In this situation Goss attaches a global L-function to any family of Drinfeld-A-modules via an infinite Euler product on the domain $S_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p$, cf. [11]. As in the classical situation, it converges on some 'half plane' of S_∞. A similar procedure yields for any closed point v of C and any family $\underline{\mathscr{M}}$ of A-motives a global L-function $L^{(v)}(\underline{\mathscr{M}}, s)$, which converges on a half plane of a suitably defined domain S_v, [12], § 8.

The theme of the current article is to derive some consequences for such global L-functions from the theory of crystals over function fields, introduced by R. Pink and the current author in [2]. This theory encompasses A-motives as defined by Anderson, [1] — indeed any family of A-motives on a scheme X is represented by an A-crystal on X. Furthermore, for any compactifiable morphism $f\colon Y \to X$, there is a functor 'direct image with compact support' from A-crystals on Y to A-crystals on X, at least in the derived context. For such f a trace formula is given in [2] for L-functions of crystals. It is this trace formula and the relations between Goss' global L-functions and L-functions of A-crystals which we will mainly exploit.

Slightly generalizing Goss' definition, we obtain for any A-scheme X of finite type over k, any A-crystal $\underline{\mathscr{F}}$ on X and any closed point v of C a v-adic L-function $L^{(v)}(\underline{\mathscr{F}}, s)$ as an Euler product which converges on some half plane of S_v, cf. Definition 2.8. Our principal goal is to prove that any such function has a meromorphic, essentially algebraic continuation to all of S_v, in the sense of [12], §8. Along the way, we will prove some interesting

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results on special values of v-adic L-functions at negative integers $-n$. Namely they can be written as a quotient of polynomials whose degrees grow logarithmically in n. Furthermore, we will obtain a criterion for such an L-function to be entire, generalizing [22]. In particular, all these results apply to the v-adic L-function of any family ψ of Drinfeld-A-modules with everywhere good reduction.

The existence of such a meromorphic continuation was conjectured by Goss in [11], and some special cases were treated by him in [10]. For $A = k[t]$, the conjecture was completely solved by Taguchi and Wan, cf. [21]. The first general proof was sketched to us by Goss and evolved during a stay of the present author at the Ohio State University. It is based on some v-adic measure theory where at a crucial point our results on special values of L-functions at negative integers are needed.

In this article we present two proofs of the above conjecture: An algebraic one that is again based on our analysis of special values at negative integers, and an analytic one which uses the results in [21]. Our first proof is similar to that sketched to us by Goss. However it completely avoids any kind of measure theory and only uses a basic p-adic interpolation procedure of certain special values.

Our second proof, which is independent of our results on special values, takes place in the framework of [21], and is therefore analytic in nature. It also yields results for α-meromorphic v-adic φ-sheaves over an A-scheme X, cf. loc. cit., which appear not to be accessible from a purely algebraic viewpoint.

There are further conjectures by Goss, cf. [14], on v-adic L-functions attached to families of A-motives with everywhere good reduction, which can be viewed as analogues of the generalized Riemann hypothesis for number fields, and are concerned with the zeroes of v-adic L-functions. Whether the cohomological methods used here will eventually contribute to these conjectures seems unclear at the moment. A major obstacle is that the cohomological theory of crystals possesses no duality and only the first three of the usual six functors $Rf_!, f^*, \otimes, f_*, f^!, \mathrm{Hom}$ are available. Another problem is that $Rf_!$ does not preserve purity of weights. Therefore, while we can express special values of v-adic values at negative integers via an endomorphism acting on a cohomology module, we have no control over the v-adic valuations of the eigenvalues of this action.

Independently of [2], in recent work, [6, 7], M. Emerton and M. Kisin developed a theory that has features dual to the theory of crystals over function fields. It seems conceivable that one could also use their formalism to obtain the results proven here.

We outline the content of this article: Section 1 reviews some basic results of the theory of crystals over function fields including the main properties of the L-function $L(\underline{\mathscr{G}},T)$ of a general crystal $\underline{\mathscr{G}}$. The following section is dedicated to the definition of v-adic L-functions $L^{(v)}(\underline{\mathscr{F}},s)$ for an A-scheme X and a crystal $\underline{\mathscr{F}}$ on X, and it discusses various of its properties. Along the way, we recall Goss v-adic exponentiation, we introduce twisting of L-functions by characters and we rephrase Goss' definition of meromorphy and entireness. As a first application of the cohomological theory of [2], we reduce Goss' conjecture to the case where $\underline{\mathscr{F}}$ is a φ-sheaf (in the terminology of [21], Sect. 1) on the base $\operatorname{Spec} A$, cf. Corollary 2.23.

In the subsequent section, we study the L-function of Drinfeld-Hayes modules, which are the generalization of the Carlitz module to rings A other than $k[t]$. This is motivated by the work of Taguchi and Wan in [21], where they show for $A = k[t]$ and $j \in \mathbb{N}$, that $L^{(v)}(\underline{\mathscr{F}},(z,-j))$ can be expressed in terms of the L-function of the crystal $\underline{\mathscr{F}} \otimes \underline{\mathscr{C}}^{\otimes j}$, where $\underline{\mathscr{C}}$ is the crystal attached to the Carlitz module. Let h^+ denote the narrow class number of A. Then for general A, we can relate in a similar manner the value $H_{v,2jh^+}(z) := L^{(v)}(\underline{\mathscr{F}},(z,-2jh^+))$ to the L-function of the crystal $\underline{\mathscr{F}}$ twisted by the j-fold tensor power of a suitable rank one φ-sheaf $\underline{\mathscr{P}}$, cf. Theorem 3.8.

In Section 4, we give our first proof of Goss' conjecture for general A, which will be of a purely algebraic nature: Using the cohomological methods of [2], we show that for a crystal $\underline{\mathscr{F}}$ on $\operatorname{Spec} A$ and any closed point v of C, the functions $H_{v,2jh^+}(z)$ are polynomials in $A[z^{-1}]$ whose degrees grow like $O(\log j)$, cf. Corollary 4.6. Some simple estimates will then allow us to construct a continuous function on $\mathbb{Z}_p$ with values in the Fréchet space of entire functions on $\mathbb{P}^1(\mathbb{C}_v) \smallsetminus \{0\}$, which interpolates the polynomials $H_{v,2jh^+}(z) \in A[z^{-1}]$ at $-2h^+j$. Goss' conjecture is an immediate consequence, cf. Corollary 4.16.

Concerning entireness of global L-functions, we prove the following generalization of [22]: Suppose X is an affine equi-dimensional Cohen-Macaulay scheme of dimension e, with a structure morphism to $\operatorname{Spec} A$, and $\underline{\mathscr{F}}$ is a crystal which can be represented by a φ-sheaf. Then for any place v of C, the function $L^{(v)}(\underline{\mathscr{F}},s)^{(-1)^{e-1}}$ extends to an entire, essentially algebraic function on S_v, cf. Theorem 4.17. We end Section 4 by applying our methods to obtain results on Euler factors at places of bad reductions and on trivial zeros of L-functions.

For the second, the analytic proof, we construct in Section 5 a uniformly overconvergent family of v-adic rank one φ-sheaves which interpolates the tensor powers of the τ-sheaf $\underline{\mathscr{P}}$ above, cf. Theorem 5.11. This is suggested by our computations of global L-functions attached to φ-sheaves constructed from Drinfeld-Hayes modules. The approach in [21], together with some of the infrastructure developed in this article shows that for any α-meromorphic v-adic φ-sheaf over an A-scheme X, the resulting v-adic L-function converges on a half plane 'of radius $q^{-\alpha}$ around ∞_v'. For $\alpha = \infty$ one obtains a second proof of Goss' conjecture.

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Notation

- Let p denote the characteristic of the finite field k and q its order.
- By X, Y, etc., we denote schemes of finite type over k. Their absolute Frobenius endomorphism with respect to k is denoted by σ_X, σ_Y, etc. When it seems redundant, the subscripts are often omitted. For $x \in X$ denote by k_x its residue field.
- For a field L we denote by L^{sep} and L^{alg} a separable, respectively algebraic closure.
- We fix a smooth projective curve C over k and a closed point ∞ on it.
- The ring of regular functions on $C \smallsetminus \{\infty\}$ is denoted by A, its fraction field by K, and its set of maximal ideals by $\mathrm{Max}(A)$.
- For I a non-zero ideal of A, let $\deg(I) := \dim_k A/I$ denote its degree.
- For a closed point v of C, the maximal order in the completion K_v of K at v will be A_v, its residue field k_v, and we set $d_v = [k_v : k]$ and $q_v = \mathrm{card}(k_v)$.
- If v is a finite place, $\mathfrak{p}_v$ will denote the maximal ideal of A corresponding to v, and if $\mathfrak{p} = \mathfrak{p}_v$, we also write $k_\mathfrak{p}$ for k_v and $d_\mathfrak{p}$ for d_v.
- By $|\,.\,|_v$ we denote the norm on K_v, which takes the value q_v^{-1} on any uniformizing parameter of K_v, and by $\mathrm{v}_v\colon K_v^* \twoheadrightarrow \mathbb{Z}$ the corresponding valuation.
- By $\mathbb{C}_v$ we denote the topological closure of an algebraic closure of K_v. We fix an embedding $\iota_v\colon A \to K_v \to \mathbb{C}_v$.
- For any place v of K, let $A(v)$ be the ring of regular functions on $C \smallsetminus \{\infty, v\}$.
- An A-scheme X will be a scheme over $\mathrm{Spec}\, A$. If $f : X \to \mathrm{Spec}\, A$ is the corresponding morphism of schemes, we define $X(v) \subset X$ to be $f^{-1}(\mathrm{Spec}\, A(v))$ and denote by $f(v)\colon X(v) \to \mathrm{Spec}\, A(v)$ the restriction

of f. For $\mathfrak{p} \in \operatorname{Max}(A)$, we denote by $X_{\mathfrak{p}} \to \operatorname{Spec} k_{\mathfrak{p}}$ the pullback of $X \to \operatorname{Spec} A$ along $\operatorname{Spec} k_{\mathfrak{p}} \to \operatorname{Spec} A$.
- For any ring R of characteristic p, let $R\{\tau\}$ denote the non-commutative ring of polynomials over R in the indeterminate τ subject to the non-commutation rule $\tau r = r^q \tau$.

1. Crystals and their *L*-functions

In this section, we recall various parts from the theory of crystals over function fields as developed in [2]. Except for Theorem 1.38, Corollary 1.47 and the discussion of the Frobenius twist, which is a useful tool when studying L-functions, all results are from [2].

Throughout this section, B will denote a regular noetherian ring. In the applications we will specialize B to regular rings constructed from A or k.

1.1. Basic Notions

Definition 1.1 *A coherent* τ-sheaf over B *on a scheme* X *is a pair* $\underline{\mathscr{F}} := (\mathscr{F}, \tau)$ *consisting of a coherent sheaf* $\mathscr{F}$ *on* $X \times \operatorname{Spec} B$ *and an* $\mathscr{O}_X \otimes B$*-linear homomorphism*

$$(\sigma \times \mathrm{id})^* \mathscr{F} \xrightarrow{\tau} \mathscr{F}.$$

We often simply speak of *τ-sheaves on X*. The sheaf underlying a τ-sheaf $\underline{\mathscr{F}}$ will always be denoted $\mathscr{F}$. When the need arises to indicate on which sheaf τ acts, we write $\tau = \tau_{\mathscr{F}}$.

On any affine chart $\operatorname{Spec} R \subset X$ a τ-sheaf over B corresponds to a finitely generated $R \otimes B$-module M together with a $\sigma \otimes \mathrm{id}$-linear homomorphism $\tau \colon M \to M$. We will occasionally use the notation (M, τ) and call it a τ-module. Equivalently M may be regarded as a $R\{\tau\} \otimes B$-module which is finitely generated over $R \otimes B$.

By $\mathbf{Coh}_\tau(X, B)$ we denote the category whose objects are the coherent τ-sheaves on X over B, and whose morphisms are those sheaf homomorphisms which are compatible with τ. Clearly this is an abelian B-linear category.

There will be one instance when the base field k we work over will be an issue. To indicate the ground field k in the nomenclature, we will speak of *coherent τ-sheaves on X over B relative to k.*

For a τ-sheaf $\underline{\mathscr{F}}$, we define the iterates τ^n of τ by setting inductively $\tau^0 := \mathrm{id}$ and $\tau^{n+1} := \tau \circ (\sigma \times \mathrm{id})^* \tau^n$. They are $\mathscr{O}_X \otimes B$-linear homomorphisms

$$(\sigma^n \times \mathrm{id})^* \mathscr{F} \xrightarrow{\tau^n} \mathscr{F}.$$

Definition 1.2 *A τ-sheaf $\underline{\mathscr{F}}$ is called* nilpotent *if and only if $\tau^n_{\mathscr{F}}$ vanishes for some $n > 0$.*

A homomorphism of τ-sheaves is called a nil-isomorphism *if and only if both its kernel and cokernel are nilpotent.*

It is shown in [2], Chap. 2, that the nil-isomorphisms form a saturated multiplicative system, denoted by $\mathscr{S}$, of $\mathbf{Coh}_\tau(X, B)$. One can thus make the following definition.

Definition 1.3 *The category* $\mathbf{Crys}(X, B)$ *of* B-crystals on X *is the localization of* $\mathbf{Coh}_\tau(X, B)$ *with respect* $\mathscr{S}$.

We call a τ-sheaf $\underline{\mathscr{F}}$ *(locally) free*, if its underlying sheaf $\mathscr{F}$ is (locally) free. We call a crystal *(locally) free*, if it may be represented by a (locally) free τ-sheaf. Because of the following rather trivial result, for crystals over an affine base these notions are equivalent.

Lemma 1.4 *Suppose that $X = \operatorname{Spec} R$ is affine and $\underline{\mathscr{F}}$ is a τ-sheaf on X. If $\mathscr{F}$ is locally free, then the crystal associated to $\underline{\mathscr{F}}$ can be represented by a τ-sheaf whose underlying module is free over $R \otimes B$.*

Proof: This is essentially Trick (2.2) of [21]: Suppose that $\underline{\mathscr{F}}$ is represented by the τ-module (P, τ_P), where P is projective and finitely generated over $R \otimes B$. Choose any finitely generated projective module Q over $R \otimes B$ such that $P \oplus Q$ is free, and define $\tau_Q = 0$. Then $(P \oplus Q, \tau_P \oplus \tau_Q)$ represents $\underline{\mathscr{F}}$ and has an underlying module which is free. ∎

Remark 1.5 An algebraic φ-sheaf in the terminology of [21] is a locally free τ-sheaf in our terminology.

Example 1.6 (a) An *A-motive on X of rank r* is a pair $(\underline{\mathscr{M}}, \mathrm{ch}_{\underline{\mathscr{M}}})$ where $\underline{\mathscr{M}} \in \mathbf{Coh}_\tau(X, A)$ is locally free of rank r and $\mathrm{ch}_{\underline{\mathscr{M}}} \colon X \to \operatorname{Spec} A$ is a morphism of schemes such that the following conditions hold:

(i) The sheaf $\operatorname{Coker}((\sigma \times \mathrm{id})^* \mathscr{M} \xrightarrow{\tau} \mathscr{M})$ vanishes on the complement of the graph of $\mathrm{ch}_{\underline{\mathscr{M}}}$ inside $X \times \operatorname{Spec} A$.

(ii) For every geometric point $i_{\bar{x}}\colon \bar{x} \hookrightarrow X$, such that $\bar{x}$ is the spectrum of an algebraically closed field, the τ-sheaf $i_{\bar{x}}^* \underline{\mathscr{M}}$ is a Drinfeld-Anderson A-motive of rank r in the sense of [20], Def. 5.1.

The pair $(\underline{\mathscr{M}}, \mathrm{ch}_{\underline{\mathscr{M}}})$ is also referred to as a *family of A-motives on X of rank r*, $\mathrm{ch}_{\underline{\mathscr{M}}}$ is called the *characteristic of $\underline{\mathscr{M}}$*. In the terminology of [1], condition (ii) can be expressed as follows. Choose $t \in A$ non-constant, so that $k[t] \to A$ is finite flat of rank r'. Then $i_{\bar{x}}^* \underline{\mathscr{M}}$ is a t-motive of rank rr' which admits multiplication by A.

Note that if X is reduced, then it is neither necessary to specify the map $\mathrm{ch}_{\underline{\mathscr{M}}}$, nor to require condition (i), as in this case condition (ii) implies that the projection $\mathrm{pr}_1\colon X \times \operatorname{Spec} A \to X$ induces an isomorphism between the reduced induced subscheme attached to the support of $\operatorname{Coker}((\sigma \times \mathrm{id})^* \mathscr{M} \xrightarrow{\tau} \mathscr{M})$ and X. Thus one can define $\mathrm{ch}_{\underline{\mathscr{M}}}: X \to \operatorname{Spec} A$ as the inverse of this isomorphism composed with $\mathrm{pr}_2\colon X \times \operatorname{Spec} A \to \operatorname{Spec} A$. This defines the unique $\mathrm{ch}_{\underline{\mathscr{M}}}$ which satisfies (i).

(b) For a line bundle $\mathscr{L}$ on X, denote by $\mathrm{End}_{k-\mathrm{v.sp.}}(\mathscr{L})$ the set of endomorphisms of $\mathscr{L}$ as a k-vector space scheme on X. A Drinfeld-A-module $(\psi, \mathscr{L})$ of rank r on X is then defined as follows: ψ is a ring map $A \to \mathrm{End}_{k-\mathrm{v.sp.}}(\mathscr{L})$ subject to the condition that for all points $i\colon x \hookrightarrow X$ the induced map

$$\psi_x\colon A \to \mathrm{End}_{k-\mathrm{v.sp.}}(i^*\mathscr{L}) \cong k_x\{\tau\} : a \mapsto \psi_x(a) = \sum_{i=0}^{\infty} \alpha_i(a)\tau^i$$

satisfies $\alpha_i(a) = 0$ for $i > r\deg(a)$ and $\alpha_{r\deg(a)}(a) \in k_x^*$.

To define the A-motive $(\mathscr{M}(\psi), \tau)$ on X of rank r attached to $(\psi, \mathscr{L})$, we denote by $\tau' \in \mathrm{End}_{k-\mathrm{v.sp.}}(\mathbb{G}_a)$ the Frobenius on $\mathbb{G}_a$ relative to X. Define

$$\mathscr{M}(\psi) := \mathrm{Hom}_{k-\mathrm{v.sp.}}(\mathscr{L}, \mathbb{G}_a).$$

This is naturally a quasi-coherent sheaf of $\mathscr{O}_X$-modules. The action of $a \in A$ is defined as right composition with $\psi(a)$, and the action of τ as left composition with τ'. This defines an $\mathscr{O}_X \otimes A$-linear map $\tau\colon \mathscr{M}(\psi) \to (\sigma \times \mathrm{id})_* \mathscr{M}(\psi)$, i.e., it makes $(\mathscr{M}(\psi), \tau)$ into a τ-sheaf $\underline{\mathscr{M}}(\psi)$. The sheaf $\mathscr{M}(\psi)$ is in fact locally free of rank r on $X \times \operatorname{Spec} A$, cf. [5].

We define $\mathrm{ch}_\psi\colon X \to \operatorname{Spec} A$ as the scheme map corresponding to the ring map

$$A \to \mathrm{End}_X(\mathrm{Lie}(\mathscr{L})) \cong \Gamma(X, \mathscr{O}_X)$$

induced from ψ, where $\mathrm{Lie}(\mathscr{L})$ is the tangent space to $\mathscr{L}$ along the zero section. The pair $(\underline{\mathscr{M}}(\psi), \mathrm{ch}_\psi)$ is an A-motive of rank r. The verification of condition (ii) is given in [1], (0.2), (0.3), (0.4). The verification of (i) is an easy consequence.

In the special case where $A = k[t]$, $X = \operatorname{Spec} R$ and ψ is of standard form, cf. [4], §5, the above can be made more explicit. Here ψ is simply a ring homomorphism

$$\psi\colon A \to R\{\tau\} : a \mapsto \psi(a) = \sum_{i=0}^{r\deg(a)} \alpha_i(a)\tau^i,$$

where $\alpha_{r\deg(a)}(a)$ is a unit in R for all $a \in A$. Let $M(\psi)$ be the module underlying $\mathscr{M}(\psi)$. Then $M(\psi) = R\{\tau\}$ where R acts by multiplication on the left, $a \in A$ acts via multiplication on the right with the element $\psi(a)$, and τ acts by multiplication on the left. As the leading coefficient of $\psi(t)$ is a unit in R, the module $M(\psi)$ is free over $R \otimes A \cong R[t]$ with basis $\tau^0, \dots, \tau^{r-1}$. The map ch_ψ is induced from the ring map $A \to R : a \to \alpha_0(a)$.

1.2. Functors

In [2] various functors were constructed on crystals, namely pullback of τ-sheaves, tensor product, extension by zero and direct image with compact support. We will discuss them in this order. For details, we refer to Sections 3 and 5 of loc. cit. To describe these functors, we fix a morphism $f : Y \to X$ of finite type, an open immersion $j : U \to X$ and a closed complement $i\colon Z \to X$ of j. Also let $\mathscr{I}$ be the ideal sheaf of Z.

A word on notation: For $n \in \mathbb{N}$, we write $\mathscr{I}^n$ for the n-th power of $\mathscr{I}$. For line bundles $\mathscr{L}$ we usually write $\mathscr{L}^{\otimes n}$ to denote their n-th power and $\mathscr{L}^{-1}$ to denote their inverse. By $\mathrm{pr}_1 : X \times \operatorname{Spec} B \to X$, the projection onto the first factor is denoted. For a sheaf $\mathscr{F}$ on $X \times \operatorname{Spec} B$, we abbreviate by $\mathscr{I}^n\mathscr{F}$ the subsheaf $\mathrm{pr}_1^*(\mathscr{I}^n)\mathscr{F}$ of $\mathscr{F}$.

Definition 1.7 *For any τ-sheaf $\underline{\mathscr{F}}$ on X over B we let $f^*\underline{\mathscr{F}}$ denote the τ-sheaf on Y over B consisting of the sheaf $(f\times\mathrm{id})^*\mathscr{F}$ and the composite homomorphism*

$$\begin{tikzcd}
(\sigma\times\mathrm{id})^*(f\times\mathrm{id})^*\mathscr{F} \arrow[r, "\tau(f^*\mathscr{F})"] \arrow[d, equal] & (f\times\mathrm{id})^*\mathscr{F}. \\
(f\times\mathrm{id})^*(\sigma\times\mathrm{id})^*\mathscr{F} \arrow[ur, "(f\times\mathrm{id})^*\tau_{\mathscr{F}}"'] &
\end{tikzcd}$$

This defines a B-linear functor $f^* : \mathbf{Coh}_\tau(X, B) \longrightarrow \mathbf{Coh}_\tau(Y, B)$ which induces a functor $f^*\colon \mathbf{Crys}(X, B) \longrightarrow \mathbf{Crys}(Y, B)$ on crystals.

By $\mathbf{C}^b(\mathbf{Crys}(X, B))$ and $\mathbf{C}^-(\mathbf{Crys}(X, B))$ we denote the category of bounded, respectively bounded above complexes of crystals on X over A.

The corresponding derived categories are denoted $\mathbf{D}^b(\mathbf{Crys}(X,B))$ and $\mathbf{D}^-(\mathbf{Crys}(X,B))$, respectively.

Theorem 1.8 (Pullback) *On crystals, the functor f^* is exact and the left derived functors $L^i f^*$ vanish for $i > 0$. Hence there is an induced exact functor*

$$f^*\colon \mathbf{D}^b(\mathbf{Crys}(X,B)) \longrightarrow \mathbf{D}^b(\mathbf{Crys}(Y,B)).$$

For later applications, we quote the following simple lemma.

Lemma 1.9 *The natural transformation $\tau\colon \sigma_X^*\underline{\mathscr{F}} \to \underline{\mathscr{F}}$ on $\mathbf{Coh}_\tau(X,B)$ between the functors σ_X^* and* id *induces an natural isomorphism between the same functors considered on $\mathbf{Crys}(X,B)$.*

Definition 1.10 *For τ-sheaves $\underline{\mathscr{F}}$ and $\underline{\mathscr{G}}$ on X over B, we let $\underline{\mathscr{F}} \otimes \underline{\mathscr{G}}$ denote the τ-sheaf on X over B consisting of the sheaf*

$$\mathscr{F} \otimes_{\mathscr{O}_{X\times \operatorname{Spec} B}} \mathscr{G}$$

and the composite homomorphism

$$\begin{tikzcd}
(\sigma\times\mathrm{id})^*\big(\mathscr{F}\otimes\mathscr{G}\big) \arrow[r, "\tau_{\mathscr{F}\otimes\mathscr{G}}"] \arrow[d, equal] & \mathscr{F}\otimes\mathscr{G}. \\
\big((\sigma\times\mathrm{id})^*\mathscr{F}\big)\otimes\big((\sigma\times\mathrm{id})^*\mathscr{G}\big) \arrow[ur, "(\tau_{\mathscr{F}})\otimes\tau_{\mathscr{G}}"'] &
\end{tikzcd}$$

This defines a bifunctor $\mathbf{Coh}_\tau(X,B)\times\mathbf{Coh}_\tau(X,B) \longrightarrow \mathbf{Coh}_\tau(X,B)$, which is B-bilinear. Passing to crystals, it induces a B-bilinear bifunctor

$$\otimes\colon \mathbf{Crys}(X,B)\times\mathbf{Crys}(X,B) \longrightarrow \mathbf{Crys}(X,B).$$

Definition 1.11 *We let $\mathbb{1}_{X,B}$ denote the B-crystal on X consisting of the structure sheaf $\mathscr{O}_{X\times\operatorname{Spec} B}$ and the natural isomorphism*

$$(\sigma\times\mathrm{id})^*\mathscr{O}_{X\times\operatorname{Spec} B} \xrightarrow{\sim} \mathscr{O}_{X\times\operatorname{Spec} B}.$$

The crystal $\mathbb{1}_{X,B}$ is the neutral object for the tensor product in the category $\mathbf{Crys}(X,B)$.

Definition 1.12 *A B-crystal is called* of pullback type *if it can be represented by a τ-sheaf $\underline{\mathscr{F}}$ such that there exists a coherent sheaf $\mathscr{F}_0$ on X for which $\mathscr{F} = \mathrm{pr}_1^*\mathscr{F}_0$.*

By Lemma 1.4, any locally free crystal over an affine base X is of pullback type.

An important property of crystals of pullback type is given by the following theorem.

Theorem 1.13 *If $\underline{\mathscr{F}}$ is of pullback type, the functor $\underline{\mathscr{G}} \mapsto \underline{\mathscr{G}} \otimes \underline{\mathscr{F}}$ is exact.*

One can show that every complex in $\mathbf{C}^-(\mathbf{Crys}(X,B))$ has a resolution by a complex all of whose objects are of pullback type. Thus standard methods in homological algebra show the following:

Theorem 1.14 (Tensor product) *The functor $\otimes$ gives rise to a left derived functor*

$$\overset{L}{\otimes} \colon \mathbf{D}^-(\mathbf{Crys}(X,B)) \times \mathbf{D}^-(\mathbf{Crys}(X,B)) \longrightarrow \mathbf{D}^-(\mathbf{Crys}(X,B)).$$

Because we assumed B to be regular, the previous result can be extended to the derived category of bounded complexes. We need the following definition.

Definition 1.15 *We say that a complex $(\underline{\mathscr{F}}^\bullet)$ in $\mathbf{D}^b(\mathbf{Crys}(X,B))$ is of* bounded Tor-dimension *if there exists $n \in \mathbb{Z}$ such that for any $\underline{\mathscr{G}}$ in $\mathbf{Crys}(X,B)$, considered as a complex concentrated in degree zero, the complex $(\underline{\mathscr{F}}^\bullet) \overset{L}{\otimes} \underline{\mathscr{G}}$ is exact in degrees less or equal to n.*

The regularity of B yields the following result, which may be obtained as a consequence of Proposition 1.32 below.

Theorem 1.16 *Every complex in $\mathbf{D}^b(\mathbf{Crys}(X,B))$ is of bounded Tor-dimension. Therefore the bifunctor $\overset{L}{\otimes}$ restricts to a bifunctor*

$$\overset{L}{\otimes} \colon \mathbf{D}^b(\mathbf{Crys}(X,B)) \times \mathbf{D}^b(\mathbf{Crys}(X,B)) \longrightarrow \mathbf{D}^b(\mathbf{Crys}(X,B)).$$

Definition 1.17 *Consider a homomorphism $h : B \to B'$ of regular k-algebras. For any τ-sheaf $\underline{\mathscr{F}}$ on X over B we let $\underline{\mathscr{F}} \otimes_B B'$ denote the τ-sheaf on X over B' consisting of the sheaf $\mathscr{F} \otimes_B B' := (\mathrm{id} \times h)^* \mathscr{F}$ and the composite homomorphism*

$$\begin{tikzcd}
(\sigma \times \mathrm{id})^*(\mathrm{id} \times h)^*\mathscr{F} \arrow[r, "\tau_{(\mathscr{F}\otimes_B B')}"] \arrow[d, equal] & (\mathrm{id} \times h)^*\mathscr{F}. \\
(\mathrm{id} \times h)^*(\sigma \times \mathrm{id})^*\mathscr{F} \arrow[ur, "(\mathrm{id}\times h)^*\tau_{\mathscr{F}}"'] &
\end{tikzcd}$$

This defines a B-linear functor $\_\_ \otimes_B B' \colon \mathbf{Coh}_\tau(X,B) \longrightarrow \mathbf{Coh}_\tau(X,B')$ which induces a functor $\_\_ \otimes_B B' : \mathbf{Crys}(X,B) \longrightarrow \mathbf{Crys}(X,B')$ on crystals. To indicate h, we sometimes write $\_\_ \otimes_B^h B'$.

Because B is regular, one obtains the following theorem:

Theorem 1.18 (Change of Coefficients) *The functor $\_\_ \otimes_B B'$ is of finite* Tor*-dimension on sheaves, therefore it induces an exact functor*

$$\_\_ \otimes_B B' \colon \mathbf{D}^b(\mathbf{Crys}(X,B)) \longrightarrow \mathbf{D}^b(\mathbf{Crys}(X,B')).$$

Before continuing our discussion of functors, we will consider an important example of changing coefficients. Let $\sigma_B : B \to B$ denote the absolute Frobenius on B relative to k.

Definition 1.19 *For $\underline{\mathscr{F}} \in \mathbf{Coh}_\tau(X,B)$ we define $\underline{\mathscr{F}} \mapsto \underline{\mathscr{F}}^{(q)} := \underline{\mathscr{F}} \otimes_B^{\sigma_B} B$ and call it* the Frobenius twist of $\underline{\mathscr{F}}$. *Analogously, we define this operation for B-crystals over X.*

Remark 1.20 Suppose that $(\underline{\mathscr{M}}, \mathrm{ch}_{\underline{\mathscr{M}}})$ is an A-motive on X of rank r. We leave it as an easy exercise to check that $(\underline{\mathscr{M}}^{(q)}, \sigma_A \circ \mathrm{ch}_{\underline{\mathscr{M}}})$ and $(\sigma_X^* \underline{\mathscr{M}}^{(q)}, \mathrm{ch}_{\underline{\mathscr{M}}})$ are also A-motives on X of rank r. As $\underline{\mathscr{M}}^{(q)}$ and $\sigma_X^* \underline{\mathscr{M}}^{(q)}$ are nil-isomorphic, this shows that nil-isomorphic τ-sheaves can have different characteristics. Furthermore it shows that an A-motive and its Frobenius twist will in general have different characteristics.

Because B is regular, the map σ_B is flat and one obtains:

Proposition 1.21 *The endofunctor $\underline{\mathscr{F}} \mapsto \underline{\mathscr{F}}^{(q)}$ is exact on the categories $\mathbf{Coh}_\tau(X,B)$ and $\mathbf{Crys}(X,B)$.*

The functor $\sigma^*_{X \times \operatorname{Spec} B}$, a priori defined on sheaves of $\mathscr{O}_{X \times \operatorname{Spec} B}$-modules, extends by functoriality to a functor on the categories $\mathbf{Coh}_\tau(X,B)$ and $\mathbf{Crys}(X,B)$, which we again denote by the same symbol. This is an operation simultaneously on the base and on coefficients which is functorially isomorphic to $(\sigma_X^* \_\_) \otimes_B^{\sigma_B} B$. For a coherent sheaf $\mathscr{F}$ on a variety Z, we denote by $\mathrm{Sym}^n \mathscr{F}$ its n-th symmetric power.

Lemma 1.22 *Let Z be any scheme over k. There exists a unique natural transformation $\gamma_{\mathscr{F}}$ from the functor $\mathscr{F} \to \sigma_Z^* \mathscr{F}$ to $\mathscr{F} \to \mathrm{Sym}^q \mathscr{F}$ on coherent sheaves on Z which has the following description on $M = \Gamma(V, \mathscr{F})$ for an affine open $V = \operatorname{Spec} S \subset Z$:*

$$S^{\sigma_S} \otimes_S M \longrightarrow \mathrm{Sym}^q M : s \otimes m \mapsto s(\underbrace{m \cdot m \cdot m \cdot \ldots \cdot m}_{q}).$$

Suppose now that $Z = X \times \operatorname{Spec} B$. *By functoriality,* γ *extends to a natural transformation* $\gamma_{\underline{\mathscr{F}}}$ *between the functors* $\underline{\mathscr{F}} \mapsto \sigma^*_{X\times \operatorname{Spec} B}\underline{\mathscr{F}}$ *and* $\underline{\mathscr{F}} \mapsto \operatorname{Sym}^q \underline{\mathscr{F}}$ *on* $\mathbf{Coh}_\tau(X,B)$. *This induces a natural transformation, also denoted by* γ*, between the corresponding functors on* B*-crystals on* X.

If $\underline{\mathscr{F}}$ *is a locally free* τ*-sheaf of rank one, then* $\gamma_{\underline{\mathscr{F}}}$ *is an isomorphism between* $\sigma^*_{X\times \operatorname{Spec} B}\underline{\mathscr{F}}$ *and* $\underline{\mathscr{F}}^{\otimes q} \cong \operatorname{Sym}^q \underline{\mathscr{F}}$.

The simple if lengthy proof of the lemma is left to the reader.

Proposition 1.23 *If* $\underline{\mathscr{F}}$ *is a locally free* τ*-sheaf of rank one, then* $\underline{\mathscr{F}}^{(q)}$ *and* $\underline{\mathscr{F}}^{\otimes q}$ *are naturally isomorphic as crystals.*

Proof: By Lemma 1.9, one has the isomorphism of crystals

$$\sigma^*_X\underline{\mathscr{F}}^{(q)} \xrightarrow{\tau_{\underline{\mathscr{F}}^{(q)}}} \underline{\mathscr{F}}^{(q)}\ .$$

As remarked above, $\sigma^*_X\underline{\mathscr{F}}^{(q)}$ and $(\sigma_{X\times\operatorname{Spec} B})^*\underline{\mathscr{F}}$ are isomorphic. Finally, by the previous lemma the latter crystal is isomorphic to $\underline{\mathscr{F}}^{\otimes q}$ via the natural transformation $\gamma_{\underline{\mathscr{F}}}$. ∎

We now resume our discussion of functors.

Definition 1.24 *Suppose that* f *is proper. For any coherent* τ*-sheaf* $\underline{\mathscr{F}}$ *on* Y *we let* $f_*\underline{\mathscr{F}}$ *denote the* τ*-sheaf on* X *consisting of the sheaf* $(f\times \mathrm{id})_*\mathscr{F}$ *and the composite homomorphism*

$$\begin{tikzcd}
(\sigma\times\mathrm{id})^*(f\times\mathrm{id})_*\mathscr{F} \arrow[r, "\tau_{f_*\mathscr{F}}"] \arrow[d, "\text{base change}"'] & (f\times\mathrm{id})_*\mathscr{F}\,. \\
(f\times\mathrm{id})_*(\sigma\times\mathrm{id})^*\mathscr{F} \arrow[ur, "(f\times\mathrm{id})_*\tau_{\mathscr{F}}"'] &
\end{tikzcd}$$

As before, this induces a B-linear functor $f_*\colon \mathbf{Crys}(Y,B)\to\mathbf{Crys}(X,B)$, and one can show that it is right adjoint to f^*.

Theorem 1.25 (Direct image) *Suppose* f *is proper. Then there exists a natural functor*

$$Rf_*\colon \mathbf{D}^b(\mathbf{Crys}(Y,B)) \longrightarrow \mathbf{D}^b(\mathbf{Crys}(X,B)).$$

Let $\mathfrak{U}$ be a finite affine cover of Y. We use the notation $\check{C}^\bullet_{\mathfrak{U}}(\underline{\mathscr{F}^\bullet})$ for the total complex associated to the bicomplex obtained from $(\underline{\mathscr{F}^\bullet})$ by applying the usual Čech resolution with respect to $\mathfrak{U}$ to each object. Note that in general $\check{C}^\bullet_{\mathfrak{U}}(\underline{\mathscr{F}^\bullet})$ is no longer in $\mathbf{D}^b(\mathbf{Crys}(Y,B))$. It is a complex of quasi-coherent sheaves carrying a $\sigma \times \mathrm{id}$-linear operation τ.

Using $(\underline{\mathscr{F}^\bullet}) \mapsto f_*\check{C}^\bullet_{\mathfrak{U}}(\underline{\mathscr{F}^\bullet})$, in [2], Chap. 5, the functor Rf_* is constructed as a derived functor between suitably defined derived categories which are naturally isomorphic to $\mathbf{D}^b(\mathbf{Crys}(Y,B))$ and $\mathbf{D}^b(\mathbf{Crys}(X,B))$, respectively. To give the precise definitions is beyond the scope of this article. As a consequence of this, one can show the following which is of prime importance in order to compute the i-th cohomology $R^if_*(\underline{\mathscr{F}^\bullet})$ of $Rf_*(\underline{\mathscr{F}^\bullet})$.

Proposition 1.26 *There is an isomorphism*

$$R^if_*(\underline{\mathscr{F}^\bullet}) \cong (\mathbb{R}^if_*(\mathscr{F}^\bullet), \mathbb{R}^if_*\tau_{(\mathscr{F}^\bullet)}),$$

where $\mathbb{R}^if_*(\mathscr{F}^\bullet)$ is the i-th hypercohomology for quasi-coherent sheaves and where $\mathbb{R}^if_*\tau_{(\mathscr{F}^\bullet)}$ is the endomorphism on it induced from τ.

The following theorem exhibits a rigidity of crystals which does not hold for sheaves or τ-sheaves.

Theorem 1.27 *Suppose f is finite, radicial and surjective. Then the functors*

$$\mathbf{Crys}(X,B) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{Crys}(Y,B)$$

are mutually inverse equivalences of categories.

In particular, the above applies to $\sigma_X \colon X \to X$ and to the closed immersion $X_{\mathrm{red}} \to X$, where X_{red} denotes the induced reduced subscheme of a given scheme X.

The operation extension by zero, while ill-behaved on coherent sheaves, is a good operation on crystals:

Theorem 1.28 (Extension by zero) *There exists a functor*

$$j_! \colon \mathbf{Crys}(U,B) \longrightarrow \mathbf{Crys}(X,B),$$

which is uniquely characterized by the following properties:

(a) $j^*j_! \cong \mathrm{id}_{\mathbf{Crys}(U,B)}$ *and*
(b) $i^*j_! = 0$.

Note that the above theorem also gives a criterion for a τ-sheaf to represent the extension by zero of a given crystal. Passing to the derived category, one obtains:

Theorem 1.29 *There exists a functorial distinguished triangle*

$$j_!j^* \longrightarrow \mathrm{id} \longrightarrow i_*i^* \longrightarrow j_!j^*[1].$$

Important for the existence of $j_!\underline{\mathscr{F}}$ in the proof of Theorem 1.28 is the following lemma:

Lemma 1.30 *Let $\underline{\mathscr{F}}$ be in $\mathbf{Crys}(U,B)$ and suppose that $\widetilde{\mathscr{F}}$ is a coherent extension of $\mathscr{F}$ to $X \times \operatorname{Spec} B$. Then $j_!\underline{\mathscr{F}}$ can be represented by a τ-sheaf whose underlying sheaf is $\mathscr{I}^n\widetilde{\mathscr{F}}$ for any $n >> 0$.*

As any coherent sheaf on U has a coherent extension to X, the following is an immediate consequence.

Corollary 1.31 *If $\underline{\mathscr{F}}$ is of pullback type, then so is $j_!\underline{\mathscr{F}}$.*

We use the theory as developed so far to derive a result on the representability of complexes by complexes of pullbacks. We find this interesting, as being of pullback type is a property that is essentially preserved under all our functors, and as crystals of pullback type have fibers which are free crystals.

By $\mathbf{D}^b(\mathbf{Crys}(X,B))_{\mathrm{pb}} \subset \mathbf{D}^b(\mathbf{Crys}(X,B))$ we denote the triangulated subcategory generated by bounded complexes all of whose objects are of pullback type. Using the regularity of B, we will show the following:

Proposition 1.32 *The inclusion $\mathbf{D}^b(\mathbf{Crys}(X,B))_{\mathrm{pb}} \to \mathbf{D}^b(\mathbf{Crys}(X,B))$ is an equivalence of categories.*

We first need an auxiliary result. Suppose $X = \operatorname{Spec} R$ is a regular affine scheme of finite type over k. Then $R \otimes B$ is regular (combine [17], Thm. 30.2, Thm. 30.3, § 28, Lem. 1, to see that B is smooth over k). Therefore any module over $R \otimes B$ admits a finite projective resolution. The following lemma from [2] is a simple consequence.

Lemma 1.33 *Let $X = \operatorname{Spec} R$ be affine and regular. Given any complex $(\underline{\mathscr{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(X,B))$, there exists a complex $(\underline{\mathscr{G}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(X,B))$ whose objects are locally free, and hence of pullback type, and which is quasi-isomorphic to $(\underline{\mathscr{F}}^\bullet)$.*

Proof of Proposition 1.32: It suffices to prove that the inclusion is essentially surjective, and so let $(\underline{\mathscr{F}}^\bullet)$ be in $\mathbf{D}^b(\mathbf{Crys}(X,B))$. By Theorem 1.27, we may assume that X is reduced. Let $\varnothing = X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_n = X$ be an increasing sequence of closed subsets of X such that $X_i \smallsetminus X_{i-1}$ is regular and affine. By Corollary 1.31 the functor $j_!$ preserves crystals of pullback type, and it is clear that i_* has the same property. Repeatedly applying Theorem 1.29 to $X_i \smallsetminus X_{i-1} \hookrightarrow X_i \hookleftarrow X_{i-1}$, we may assume that X is regular and affine. Thus we need to show that for X regular affine the complex $(\underline{\mathscr{F}}^\bullet)$ is in the essential image of $\mathbf{D}^b(\mathbf{Crys}(X,B))_{\mathrm{pb}}$. This follows from the previous lemma. ∎

Finally we come to the definition of the functor $Rf_!$, which computes the direct image with compact support. By a result of Nagata, cf. [19], any morphism f between (separated) schemes of finite type over a field can be compactified. This means that there exists a commuting diagram

$$\begin{tikzcd} Y \arrow[r, "j"] \arrow[d, "f"'] & \bar{Y} \arrow[dl, "\bar{f}"] \\ X. & \end{tikzcd}$$

such that j is an open immersion and $\bar{f}$ is proper.

Definition 1.34 *In the above situation we define*

$$Rf_! := R\bar{f}_* \circ j_! : \mathbf{D}^b(\mathbf{Crys}(Y,B)) \to \mathbf{D}^b(\mathbf{Crys}(X,B)).$$

It can be shown that, in a suitable sense, Definition 1.34 is independent of the chosen compactification. (For this one considers the set of all compactifications as a direct filtered system and establishes various compatibilities for the transition morphism, cf. [18], VI.3.)

We write $R^if_!$ for $R^i\bar{f}_* \circ j_!$. In the special case where $f: Y \to \operatorname{Spec} k$ is the structure morphism, we also write $\mathbb{H}^i_c(Y, (\underline{\mathscr{F}}^\bullet))$ for $R^if_!(\underline{\mathscr{F}}^\bullet)$, respectively $H^i_c(Y, \underline{\mathscr{F}})$ for $R^if_!\underline{\mathscr{F}}$, where we regard $\underline{\mathscr{F}}$ as a complex concentrated in degree zero.

Concerning the effect of $Rf_!$ on pullbacks, we have the following proposition.

Proposition 1.35 *Suppose $\underline{\mathscr{F}} \in \mathbf{Crys}(Y,B)$ is of pullback type. Then the crystals $R^if_!\underline{\mathscr{F}}$ are of pullback type.*

PROOF: Choose a compactification $f = \bar{f}j$ where j is an open immersion and $\bar{f}$ is proper. By Corollary 1.31, the crystal $j_!\underline{\mathscr{F}}$ is of pullback type. Using Čech resolutions to compute the higher direct images $R^i\bar{f}_*$, cf. Proposition 1.26, the assertion follows. ∎

The following results describe compatibilities of image with compact support, pullback and tensor product:

Theorem 1.36 (Base Change) *Let $(\underline{\mathscr{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(Y,B))$ and suppose we are given the following pullback square:*

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"'] & X. \end{tikzcd}$$

Then there is a functorial isomorphism $g^*Rf_!(\underline{\mathscr{F}}^\bullet) \cong Rf'_!g'^*(\underline{\mathscr{F}}^\bullet)$.

Theorem 1.37 (Projection Formula) *There exists a natural isomorphism of functors*

$$\begin{aligned} &Rf_!(\_\_) \overset{L}{\otimes} \_\_ \overset{\cong}{\longrightarrow} Rf_!(\_\_ \overset{L}{\otimes} f^*(\_\_)) : \\ &\quad \mathbf{D}^b(\mathbf{Crys}(Y,B)) \times \mathbf{D}^b(\mathbf{Crys}(X,B)) \longrightarrow \mathbf{D}^b(\mathbf{Crys}(X,B)). \end{aligned}$$

1.3. The functor $Rf_!$ on affine Cohen-Macaulay varieties

The following result will play an essential role for our main result on entireness of L-functions.

Theorem 1.38 *Let X be an affine Cohen-Macaulay variety of dimension e with structure morphism $g_X : X \to \operatorname{Spec} k$. Suppose $\underline{\mathscr{F}} \in \mathbf{Crys}(X,B)$ is locally free. Let $j : X \to \bar{X}$ be any compactification and represent $j_!\underline{\mathscr{F}}$ by some locally free τ-sheaf $\underline{\tilde{\mathscr{F}}}$. Then $Rg_{X!}\underline{\mathscr{F}}$ is represented by the complex $H^e(\bar{X}, \underline{\tilde{\mathscr{F}}})[e] \in \mathbf{D}^b(\mathbf{Crys}(\operatorname{Spec} k, B))$, which is concentrated in degree e.*

Before giving the proof of the above theorem, we want to explain the main obstacle that has to be overcome in the proof. Suppose X has a compactification $j : X \to \bar{X}$, such that $\bar{X}$ is projective and Cohen-Macaulay and such that $\bar{X} \smallsetminus X$ is a divisor D. We may represent $\underline{\mathscr{F}}$ by a free τ-sheaf on X, cf. Lemma 1.4, and thus represent $j_!\underline{\mathscr{F}}$ by a τ-sheaf $\underline{\tilde{\mathscr{F}}}$ with underlying sheaf $\mathscr{O}_X(-nD) \otimes B$ for some $n \gg 0$, Lemma 1.30. As $\mathscr{O}_X(-D)$ is the inverse of an ample line bundle on $\bar{X}$, Serre duality for $\bar{X}$, cf. [15], Thm. III.7.6, implies that $H^i(\bar{X}, \widetilde{\mathscr{F}}) = 0$ for $i \neq e$. The theorem now follows from Proposition 1.26.

In general such a compactification may not exist - at least we do not know this. The main point of the proof given below is to use the nilpotency of $j_! \underline{\mathscr{F}}$ on $\bar{X} \smallsetminus X$ for any compactification $\bar{X}$ of X, in order to show that the singularities of $\bar{X}$ on the complement of X are irrelevant when computing the absolute cohomology of the crystal $Rg_{X!}\underline{\mathscr{F}}$.

Proof: We choose a closed immersion $i : X \to \mathbb{A}^N$ for some N and regard $\mathbb{A}^N$ as the complement of a hyperplane in $\mathbb{P}^N$. By the remark below Definition 1.34, it suffices to prove the theorem for the compactification $X \hookrightarrow \bar{X}$, where $\bar{X}$ is the closure of X in $\mathbb{P}^N$. We depict the situation in the following diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "j"] & \mathbb{A}^N \arrow[d, "j'"] \\ \bar{X} \arrow[r, "\bar{i}"] & \mathbb{P}^N \end{tikzcd}$$

Let R denote the coordinate ring of X and S that of $\mathbb{A}^N$ and let (M, τ) be a projective τ-module on $R \otimes B$ representing $\underline{\mathscr{F}}$. Lemma 1.33 in combination with Lemma 1.4 shows that one can find a resolution $(\underline{\mathscr{G}^i})^{i \leq 0}$ of $i_* \underline{\mathscr{F}}$ in $\mathbf{D}^-(\mathbf{Coh}_\tau(\mathbb{A}^N, B))$ by free τ-sheaves, so that $\Gamma(\mathbb{A}^N, \underline{\mathscr{G}^i}) \cong S^{\oplus m_i} \otimes B$ for some $m_i \in \mathbb{N}_0$.

Our next aim is to describe an extension by zero for $j' \colon \mathbb{A}^N \hookrightarrow \mathbb{P}^N$ of this resolution. For simplicity, we write $\mathscr{O}$ instead of $\mathscr{O}_{\mathbb{P}^N}$. By an inductive procedure based on Lemma 1.30 and starting at $i = 0$, one can construct a complex $(\widetilde{\underline{\mathscr{G}}^\bullet})$ of τ-sheaves which represents the complex $j'_! \underline{\mathscr{G}}^\bullet$ in such a way that the underlying sheaf in degree i is given by $\mathscr{O}(-n_i)^{\oplus m_i}$ where $0 < n_0 < n_{-1} < n_{-2} < \ldots$. Considered as a complex of crystals, it is a resolution of $j'_! i_* \underline{\mathscr{F}}$. Note that any twist of this complex by a power of $\mathscr{O}(-1)$ yields a complex of τ-sheaves with the same property.

Let $\bar{g}$ denote the structure morphism of $\mathbb{P}^N$. By [15], Thm. III.5.1, for $l > 0$ we have $R^i \bar{g}_* \mathscr{O}(-l) = 0$ if $i \neq N$. Thus $Rg_{X!}\underline{\mathscr{F}} \cong R\bar{g}_*(j'_! \underline{\mathscr{G}}^\bullet)$ is represented by $(H^N(\mathbb{P}^N, \widetilde{\underline{\mathscr{G}}^i}))_{i \leq 0}$. We claim that the latter complex has cohomology only in degree e when regarded as a complex of crystals. The claim yields the desired result, as by Proposition 1.26 we have $R^e g_{X!}\underline{\mathscr{F}} \cong H^e(\bar{X}, j_! \underline{\mathscr{F}}')$. We now prove the claim:

For each l, define the complex $(M_l^i)_{i \in \mathbb{Z}}$ of free B-modules as

$$\ldots \longrightarrow H^N(\mathbb{P}^N, \mathscr{O}(-n_{i-1} - l)^{\oplus m_{i-1}}) \longrightarrow H^N(\mathbb{P}^N, \mathscr{O}(-n_i - l)^{\oplus m_i}) \longrightarrow \ldots .$$

The natural inclusion $\mathscr{O}(-n_i - l - 1) \to \mathscr{O}(-n_i - l)$ induces a morphism of complexes $f_l : (M_{l+1}^\bullet) \to (M_l^\bullet)$ which is epimorphic on objects. The modules M_l^i carry a B-linear operation τ_l^i which is compatible with the differentials and the maps f_l. Considered as τ-modules f_l is degreewise a nil-isomorphism, so that all these complexes are isomorphic as complexes of crystals.

By Serre duality, disregarding the τ-operation, the complex $(M_l^\bullet)$ is dual to the complex $(N_l^\bullet)$ defined as:

$$\ldots \longrightarrow \mathrm{Hom}(\mathscr{O}(-n_i-l)^{\oplus m_i}, \mathscr{O}) \longrightarrow \mathrm{Hom}(\mathscr{O}(-n_{i-1}-l)^{\oplus m_{i-1}}, \mathscr{O}) \longrightarrow \ldots$$

Simply dualizing the operations $\tau_{M_l^i}$, we obtain endomorphisms compatible with the differentials. The direct limit of the $(N_l^\bullet)$ for $l \to \infty$ is the complex $\mathrm{Hom}_S(S^{\oplus m_\bullet}, S)$ whose cohomology is $\mathrm{Ext}_S^\bullet(M, S)$. The module M is finitely generated and projective over the Cohen-Macaulay ring R. In the following paragraph we will show that $\mathrm{Ext}_S^i(R, S) = 0$ for $i \neq N-e$. Hence $\mathrm{Ext}_S^i(M, S) = 0$ for $i \neq N-e$, by the projectivity of M over R. The proof of the above claim is now a consequence of Lemma 1.39 below.

To see that $\mathrm{Ext}_S^i(M, S) = 0$ for $i \neq N-e$, it suffices to show that this holds after localizing at any maximal ideal $\mathfrak{m}$ of S which lies in the image of $\mathrm{Spec}\, R \to \mathrm{Spec}\, S$. For such an $\mathfrak{m}$ we need to show that $\mathrm{Ext}_{S_\mathfrak{m}}^i(R_\mathfrak{m}, S_\mathfrak{m}) = 0$ for $i \neq N-e$. The ring $R_\mathfrak{m}$ has depth e, and thus by [17], Thm. 17.1, all Ext-modules vanish for $i < N-e$. By the theorem of Auslander-Buchsbaum, cf. [15], Prop. III.6.12A, the projective dimension of R over S is $N-e$. This implies that all Ext-modules vanish for $i > N-e$, cf. [15], Prop. III.10.A, and the proof is completed. ∎

Lemma 1.39 *Suppose for each l we are given complexes $(M_l^\bullet)$ of free, finitely generated B-modules concentrated in negative degrees:*

$$\ldots \longrightarrow M_l^{i-1} \xrightarrow{\partial^{i-1}} M_l^i \xrightarrow{\partial^i} M_l^{i+1} \longrightarrow \ldots$$

We assume that the M_l^i carry B-linear endomorphisms τ_l^i which are compatible with the differentials. Furthermore, we assume that there are maps of complexes $f_l\colon (M_{l+1}^\bullet) \to (M_l^\bullet)$ which are epimorphisms on objects, compatible with the operation of the τ_l^i, and such that τ_{l+1}^i acts nilpotently on the kernel of $M_{l+1}^i \to M_l^i$.

Define $(N_l^\bullet) := \mathrm{Hom}_B((M_l^{-\bullet}), B)$ as the dual complex of $M_l^\bullet$. Correspondingly define maps g_l dual to f_l and endomorphisms κ_l^i on N_l^i dual to τ_l^i. Clearly the maps g_l are degreewise monomorphisms, and the maps κ_l^i act nilpotently on $\mathrm{Coker}(N_l^i \to N_{l+1}^i)$.

If the cohomology of $\varinjlim(N_l^\bullet)$ is concentrated in degree e, then for all $i \neq -e$ and all l, there exists an integer $n_{i,l}$ such that $(\tau_l^i)^{n_{i,l}}(\mathrm{Ker}\,\partial_l^i) \subset \mathrm{Im}\,\partial_l^{i+1}$, i.e., the operator induced by τ_l^i on $H^i(M_l^\bullet)$ is nilpotent.

Proof: We fix $l \geq 0$ and $i \neq -e$. As direct limits commute with taking cohomology, we have $\varinjlim H^i(N_l^\bullet) = 0$. Thus for each l, there exists an $l' > l$ such that the map $H^i(N_l^\bullet) \to H^i(N_{l'}^\bullet)$ is zero. As the M_l^i are free B-modules, this morphism is dual to $H^i(M_{l'}^\bullet) \to H^i(M_l^\bullet)$, which therefore must be zero, too.

We now consider the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(M_{l'}^i \to M_l^i) \arrow[r] \arrow[d, "\partial_{l'}^i"] & M_{l'}^i \arrow[r] \arrow[d, "\partial_{l'}^i"] & M_l^i \arrow[r] \arrow[d, "\partial_l^i"] & 0 \\
0 \arrow[r] & \mathrm{Ker}(M_{l'}^{i+1} \to M_l^{i+1}) \arrow[r] & M_{l'}^{i+1} \arrow[r] & M_l^{i+1} \arrow[r] & 0.
\end{tikzcd}$$

Let x be in $\mathrm{Ker}\,\partial_l^i \subset M_l^i$. Because $M_{l'}^i \to M_l^i$ is surjective, we can find $y \in M_{l'}^i$ that maps onto x. Hence $\partial_{l'}^i(y)$ lies in the kernel of $M_{l'}^{i+1} \to M_l^{i+1}$. Because $\tau_{l'}^{i+1}$ acts nilpotently on this kernel, we can find $n \in \mathbb{N}$, which depends on i and l, such that $(\tau_{l'}^{i-1})^n$ annihilates this kernel. By the compatibility of τ with the differentials it follows that $y' := (\tau_{l'}^i)^n(y) \in M_{l'}^i$ lies in the kernel of $\partial_{l'}^i$. Using square brackets for cohomology classes, the class $[y']$ maps to $[(\tau_l^i)^n x] = (\tau_l^i)^n[x]$. By our choice of l', it follows that $(\tau_l^i)^n[x] = 0$, and hence that $(\tau_l^i)^n$ annihilates $[x]$. Since $H^i(N_l^\bullet)$ is finitely generated, it is therefore annihilated by a power of τ_l^i. ∎

1.4. L-functions of τ-sheaves

Let E be the fraction field of B. Let X^0 denote the set of closed points of X and for $x \in X^0$, let d_x denote its degree.

Definition 1.40 *We define the* local L-factor $L(x, \underline{\mathscr{F}}, T)$ *of a crystal* $\underline{\mathscr{F}}$ *at a closed point* x *of* X *via*

$$L(x, \underline{\mathscr{F}}, T)^{-1} := \det_E(\mathrm{id} - T\tau | \mathscr{F}_x \otimes_B E) \in E[T],$$

where $\underline{\mathscr{F}}_x$ *is any* τ*-sheaf representing the fiber of the crystal* $\underline{\mathscr{F}}$ *at* x *and where the determinant is taken over* E.

Remark 1.41 The above definition is independent of the choice of the $\underline{\mathscr{F}}_x$. Furthermore, one can show that $L(x, \underline{\mathscr{F}}, T)^{-1}$ is also represented by

$$\det_{k_x \otimes E}(\mathrm{id} - T^{d_x}\tau^{d_x} | \mathscr{F}_x \otimes_B E) \in k_x \otimes E[T^{d_x}],$$

where the determinant is computed over the ring $k_x \otimes E$. It follows that $L(x, \underline{\mathscr{F}}, T)^{-1} \in E[T^{d_x}]$. Based on the fact that, as a regular ring, B is normal, it is shown in [2] that $L(x, \underline{\mathscr{F}}, T)^{-1} \in B[T^{d_x}]$.

The *local L-factor* of $(\underline{\mathscr{F}^\bullet}) \in \mathbf{C}^b(\mathbf{Crys}(X,B))$ at $x \in X^0$, is defined as

$$L(x, (\underline{\mathscr{F}^\bullet}), T) := \prod_{i \in \mathbb{Z}} L(x, \underline{\mathscr{F}^i}, T)^{(-1)^i} \in 1 + T^{d_x} B[[T^{d_x}]].$$

The number of points in X^0 whose degree is below any given constant is finite. Thus for any crystal $\underline{\mathscr{F}}$ on X, the product $\prod_{x \in X^0} L(x, \underline{\mathscr{F}}, T)$ converges to an element in $1 + TB[[T]]$ and we can make the following definition.

Definition 1.42 *For* $(\underline{\mathscr{F}^\bullet}) \in \mathbf{C}^b(\mathbf{Crys}(X,B))$ *we define the* L-function of $(\underline{\mathscr{F}^\bullet})$ on X *as*

$$L(X, (\underline{\mathscr{F}^\bullet}), T) := \prod_{x \in X^0} L(x, (\underline{\mathscr{F}^\bullet}), T) \in 1 + TB[[T]].$$

Unless we want to stress the base scheme X, we often write $L((\underline{\mathscr{F}^\bullet}), T)$ for $L(X, (\underline{\mathscr{F}^\bullet}), T)$.

Proposition 1.43 *The above definition induces a function*

$$L \colon \mathbf{D}^b(\mathbf{Crys}(X,B)) \longrightarrow 1 + TB[[T]]$$

which satisfies the following properties:

(a) If $(\underline{\mathscr{F}^\bullet}) \longrightarrow (\underline{\mathscr{G}^\bullet}) \longrightarrow (\underline{\mathscr{H}^\bullet}) \longrightarrow (\underline{\mathscr{F}^\bullet})[1]$ *is a distinguished triangle, then*

$$L((\underline{\mathscr{F}^\bullet}), T) \cdot L((\underline{\mathscr{G}^\bullet}), T) \cdot L((\underline{\mathscr{H}^\bullet}), T) = 1.$$

(b) For a bounded complex $(\underline{\mathscr{F}^\bullet})$ *in* $\mathbf{Crys}(X,B)$*, denote by* $H^\bullet(\underline{\mathscr{F}^\bullet})$ *the complex consisting of the cohomology groups of* $(\underline{\mathscr{F}^\bullet})$ *with zero differentials. Then*

$$L((\underline{\mathscr{F}^\bullet}), T) = L(H^\bullet(\underline{\mathscr{F}^\bullet}), T).$$

Remark 1.44 It is also possible to define L-functions of crystals via 'projective resolutions': Assume first that $\underline{\mathscr{F}}$ is a crystal which is represented by a τ-sheaf of pullback type. Then the fiber above any closed point $x \in X$ is free, finitely generated over $k_x \otimes B$. Define

$$L'(x, \underline{\mathscr{F}}, T)^{-1} := \det_B(\mathrm{id} - T\tau | \mathscr{F}_x) \in 1 + T^{d_x} B[T^{d_x}]$$

and denote by $L'(X, \underline{\mathscr{F}}, T)$ the corresponding global L-function. It is rather trivial to note $L = L'$.

Let now $\underline{\mathscr{F}}$ be any crystal. Using Proposition 1.32, which ultimately rests on the use of projective resolutions, every crystal $\underline{\mathscr{F}}$ regarded as an element of $K'_0(\mathbf{Crys}(X,B))$ can be written as a finite sum $\sum n_i[\underline{\mathscr{G}_i}]$

with $n_i \in \mathbb{Z}$ where the $\underline{\mathscr{G}}_i$ are of pullback type and we write $[\underline{\mathscr{G}}]$ for the element in $K_0'(\mathbf{Crys}(X,B))$ represented by $\underline{\mathscr{G}}$. As an L-function should factor via $K_0'(\mathbf{Crys}(X,B))$, one defines $L'(\underline{\mathscr{F}},T) := \prod_i L'(\underline{\mathscr{G}}_i,T)^{n_i}$, and checks (on stalks) that this is independent of the chosen representatives. Using the above proposition it easily follows that $L = L'$.

The central result on L-functions in [2] is the following.

Theorem 1.45 *Let $f\colon Y \to X$ be a morphism between schemes of finite type. Then for $(\underline{\mathscr{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(Y,B))$ one has*

$$L(Y,(\underline{\mathscr{F}}^\bullet),T) = L(X,Rf_!(\underline{\mathscr{F}}^\bullet),T).$$

In the case of the structure morphism $Y \to \operatorname{Spec} k$ the above implies:

Corollary 1.46 *For $(\underline{\mathscr{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(Y,B))$, the series*

$$L(Y,(\underline{\mathscr{F}}^\bullet),T) = \prod_i L(\operatorname{Spec} k, \mathbb{H}^i_c(Y,(\underline{\mathscr{F}}^\bullet)),T)^{(-1)^i}$$

is a rational function of T.

The following is a rather general criterion for an L-function to be a polynomial (or the inverse of such), and not just a rational function. Namely, the above corollary and Theorem 1.38 yield.

Corollary 1.47 *Let X be a Cohen-Macaulay variety over k of dimension e. For any locally free crystal $\underline{\mathscr{F}}$ in $\mathbf{Crys}(X,B)$, the L-function $L(X,\underline{\mathscr{F}},T)^{(-1)^{e-1}}$ lies in $B[T]$.*

We conclude this section by studying the effect of the Frobenius twist on L-functions.

Lemma 1.48 *One has $L(\underline{\mathscr{F}},T)^q = L(\underline{\mathscr{F}}^{(q)},T^q)$ for any crystal $\underline{\mathscr{F}}$ on X.*

Proof: Clearly it suffices to prove the above pointwise for any $x \in X^0$. Thus we may assume that X is the spectrum of a finite field extension k' of k. Also we may change coefficients and assume that B is a field E. Then $\underline{\mathscr{F}}$ corresponds to a finitely generated $k' \otimes E$-module M with a $\sigma \times \mathrm{id}$-linear operation τ. Write

$$L(x,M,T)^{-1} = \det_E(1 - T\tau|M) = 1 + a_1T + \ldots a_nT^n \in E[T].$$

Because Frobenius on k' and on E commute, one has $L(x,\underline{\mathscr{F}}^{(q)},T)^{-1} = 1 + a_1^qT + \ldots a_n^qT^n \in E[T]$, and the assertion follows readily. ∎

2. Global L-functions

For the remainder of this article, we fix an A-scheme X. Let $f : X \to \operatorname{Spec} A$ denote the structure morphism to $\operatorname{Spec} A$ and $g_X : X \to \operatorname{Spec} k$ the structure morphism to $\operatorname{Spec} k$. If $X = \operatorname{Spec} A$ we usually assume that it is an A-scheme via the identity.

Given a complex $(\underline{\mathscr{F}}^\bullet)$ in $\mathbf{D}^b(\mathbf{Crys}(X, A))$ and a closed point v of C, we will define a v-adic L-function $L^{(v)}((\underline{\mathscr{F}}^\bullet), s)$. Our treatment will follow closely that of [12], Chap. 8. As a first result, we show that any such L-function can be expressed as the v-adic L-function of a complex $(\underline{\mathscr{G}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(\operatorname{Spec} A, A))$ all of whose objects are locally free crystals. We conclude this section by recasting Goss' definition of entireness and meromorphy, cf. [12], § 8, in a slightly different form.

The main example to keep in mind is that of an A-motive $(\underline{\mathscr{M}}, \mathrm{ch}_{\underline{\mathscr{M}}})$ of rank r on X. The map f will then be $\mathrm{ch}_{\underline{\mathscr{M}}}$ and the crystal we consider is the one represented by $\underline{\mathscr{M}}$.

2.1. Exponentiation of ideals

We first consider the place ∞. Let $W_\infty := \mathbb{Z}_p$ and $S_\infty := \mathbb{C}_\infty^* \times W_\infty$. An element $s \in S_\infty$ will have components (z, w). One defines an addition by $(z_1, w_1) + (z_2, w_2) = (z_1 \cdot z_2, w_1 + w_2)$. The exponentiation map will be a map

$$\{\text{fractional ideals of } A\} \times S_\infty \to \mathbb{C}_\infty^* : (I, s) \mapsto I^s,$$

which is bilinear if we use multiplication on ideals, addition on S_∞ and multiplication on $\mathbb{C}_\infty^*$.

We choose a uniformizing parameter π_∞ of A_∞. For some technical reasons, cf. Remark 2.26, we assume that there exists an $n > 0$ such that $\pi_\infty^n \in K$. To obtain π_∞, one first chooses an element $a \in A$ whose valuation $-m < 0$ at ∞ is not divisible by p. For this one may use the theorem of Riemann-Roch. Let π'_∞ be any uniformizer of A_∞. Then a^{1-q_∞} is the product of a 1-unit u of A_∞ with ${\pi'_\infty}^{m(q_\infty-1)}$. As $m(q_\infty - 1)$ is prime to p, we can write $u = {u'}^{m(q_\infty-1)}$ for some 1-unit u' of A_∞. We now take $\pi_\infty := \pi'_\infty {u'}^{-1}$.

Via this choice we identify K_∞ with $k_\infty((\pi_\infty))$. An element $a \in K^*$ is called *positive* if under the map $K^* \to K_\infty^* \cong k_\infty((\pi_\infty))^*$, the element a is mapped to an expression $\pi_\infty^n + a_{n+1}\pi_\infty^{n+1} + a_{n+2}\pi_\infty^{n+2} \ldots$, where $a_i \in k_\infty$ and $n = \mathrm{v}_\infty(a)$. Let P^+ be the set of fractional ideals of A which are principal and have a positive generator. This is a subgroup of finite index of the set J of all fractional ideals of A. The class field H^+ corresponding to P^+ is the narrow Hilbert class field of K, i.e., it is the maximal abelian extension of K which is unramified outside ∞, and such that all places of H^+ above ∞ are tamely ramified of order dividing $(q_\infty - 1)/(q-1)$ and have residue field k_∞.

For any $\mathbb{R}$-valued field F, we denote by $U_1(F)$ its 1-units. If $a \in K$ is positive, we define $\langle a \rangle := a/\pi_\infty^{\mathrm{v}_\infty(a)} \in U_1(K_\infty)$. Note that for any $I \in P^+$ there exists a unique positive generator a_I. Based on the fact that $U_1(\mathbb{C}_\infty)$ is uniquely divisible, the following is shown in [12], Prop. 8.2.4:

Proposition 2.1 *The map*

$$\langle \_\_ \rangle \colon P^+ \to U_1(K_\infty) : I \to \langle a_I \rangle$$

extends to a unique homomorphism $\langle \_\_ \rangle \colon J \to U_1(\mathbb{C}_\infty)$.

Definition 2.2 *For* $I \in J$, $s = (z, w) \in S_\infty$ *define* $I^s := z^{\deg I} \langle I \rangle^w$.

Note that this exponentiation depends on the choice of the uniformizing parameter π_∞.

To obtain an exponentiation by $\mathbb{Z}$, we make the following definition: Let π_* be a d_∞-th root of π_∞, and define

$$s_? \colon \mathbb{Z} \to S_\infty : j \mapsto s_j := (\pi_*^{-j}, j).$$

Thus for $j \in \mathbb{Z}$ and $I \in J$, we have $I^{s_j} \in \mathbb{C}_\infty$. In particular, for $I \in P^+$ the element I^{s_j} is the unique positive generator of I^j.

We now follow Goss, [12], §8, to obtain an exponentiation for the finite places of K. Define $\mathbb{V} := K(I^{s_1} : I \in J)$. This is a finite extension of K. For a fixed place $v \neq \infty$, we choose an extension $\beta = \beta_v : \mathbb{V} \to \mathbb{C}_v$ of $\iota_v : K \to \mathbb{C}_v$, and set $K_{v,\beta} := K_v(\mathbb{V}) \subset \mathbb{C}_v$ with ring of integers $A_{v,\beta}$. Unless we want to stress it explicitly, we drop the subscript v at β.

It follows that if a is positive in A, then $\beta((a)^{s_1}) = a \in K_{v,\beta}$. Based on this, one can show that for any fractional ideal I prime to $\mathfrak{p}_v$, the element $\beta(I^{s_1})$ is in $A_{v,\beta}^*$. We can write any $a \in A_{v,\beta}^*$ as $a = u_{v,0}(a)u_{v,1}(a)$ where $u_{v,1}(a)$ is a one-unit and $u_{v,0}(a)$ is a root of unity. Let $q_{v,\beta}$ denote the cardinality of the residue field of $A_{v,\beta}$.

Definition 2.3 *We let* $W_v := \mathbb{Z}_p \times \mathbb{Z}/(q_{v,\beta} - 1)$ *and* $S_v := \mathbb{C}_v^* \times W_v$, *which is a group under the obvious addition. Elements are denoted by* $s = (z, w, y)$. *The* v-adic exponentiation map *is defined as*

$$\begin{aligned}&\{\textit{fractional ideals of } A \textit{ prime to } \mathfrak{p}_v\} \times S_v \to \mathbb{C}_v^* :\\ &(I, s) \mapsto z^{\deg(I)} u_{v,0}(\beta(I^{s_1}))^y u_{v,1}(\beta(I^{s_1}))^w.\end{aligned}$$

As before, this map is bilinear if we use multiplication on ideals, addition on S_v and multiplication on $\mathbb{C}_v^*$. We caution the reader that the kind of exponentiation we use, i.e., v-adic or with respect to ∞, is only indicated by giving the domain of the exponent. Note that the definition of the v-adic exponentiation depends on the choice of the uniformizer π_∞, as well as on the choice of embedding β_v.

Definition 2.4 *For $j \in \mathbb{Z}$, we define*

$$s_{v,?}\colon \mathbb{Z} \to S_v : j \mapsto s_{v,j} := (1, j, j).$$

To have a more uniform notation, we also write $s = (z, w, y)$ for $s \in S_\infty$, where we identify S_∞ with $S_\infty \times \mathbb{Z}/\mathbb{Z}$, and use $s_{\infty,j}$ for s_j. For $v \neq \infty$ the image of $\mathbb{Z}$ under $j \mapsto s_{v,j}$ is dense in $1 \times \mathbb{Z}_p \times \mathbb{Z}/(q_{v,\beta} - 1)$.

With the above definitions in place, the following is trivial:

Proposition 2.5 *For any place $v \neq \infty$, $j \in \mathbb{Z}$ and fractional ideal I which is prime to $\mathfrak{p}_v$, one has*

$$\beta_v(I^{s_{\infty,j}}) = I^{s_{v,j}} \in \mathbb{C}_v.$$

One can also define an exponentiation for any place $v \neq \infty$ in an analogous way to Goss' definition for ∞, i.e., with exponents in $\mathbb{C}_v \times \mathbb{Z}_p$. This is the viewpoint taken in [21]. One simply obtains the restriction of the exponentiation defined here to the subgroup $\mathbb{C}_v^* \times \mathbb{Z}_p \times \{0\}$ of S_v. An advantage of Goss' definition is explained by the previous proposition which says that for ideals I prime to v, the element 'I^j' for $j \in \mathbb{Z}$ is independent of the place v.

The following two results explain how to recover Goss' definition of exponentiation of ideals from that given in [21]. From loc. cit., §10, we quote:

Proposition 2.6 *For fixed $y \in \mathbb{Z}/(q_{v,\beta}-1)$, the map $I \mapsto I^{(0,0,y)}$ defines a character $\chi_{v,y} : J \to k_v^*$ which via class field theory corresponds to a character, also denoted by $\chi_{v,y}$, of* $\mathrm{Gal}(K^{\mathrm{sep}}/K)$*, which is at most ramified at v and ∞. In particular, one has $I^{(z,w,y)} = I^{(z,w,0)}\chi_{v,y}(I)$.*

To relate the above characters to crystals, we introduce some notation. By class field theory, there are only finitely many characters $\chi : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to k_v^*$ which are unramified outside v, ∞. We define $G_v := \mathrm{Gal}(K^{\mathrm{sep}}/K)/(\cap \mathrm{Ker}(\chi))$ where the intersection is over all such χ. Thus G_v is the Galois group of some finite abelian extension of K. We define $\hat{G}_v := \mathrm{Hom}(G_v, k_v^*)$, so that the characters in $\hat{G}_v$ are in bijection with the above characters of the absolute Galois group.

The following is a special case of [16], Prop. 4.1.1, recast in our terminology:

Theorem 2.7 *Let $\tilde{k}$ be a finite extension of k and $v_1, \ldots, v_n$ finite places of C. Then there is a bijection between*

- *continuous characters $\chi : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \tilde{k}^*$ which are unramified outside $\infty, v_1, \ldots, v_n \in C$, and*
- *locally free τ-sheaves $\underline{\mathscr{M}} \in \mathbf{Coh}_\tau(\mathrm{Spec}\, A \smallsetminus \{\mathfrak{p}_{v_1}, \ldots, \mathfrak{p}_{v_n}\}, \tilde{k})$ of rank one on which τ is an isomorphism.*

Given a character χ, the corresponding τ-sheaf, denoted by $\underline{\mathscr{M}}_\chi$, is uniquely determined by the condition

$$L(\mathfrak{p}, \underline{\mathscr{M}}_\chi, T)^{-1} = 1 - \chi(\mathrm{Frob}_\mathfrak{p}) T^{d_\mathfrak{p}}$$

for all $\mathfrak{p} \in \mathrm{Max}(A) \smallsetminus \{\mathfrak{p}_{v_1}, \ldots, \mathfrak{p}_{v_n}\}$.

In particular, for any $\chi \in \hat{G}_v$ we obtain a corresponding locally free τ-sheaf $\underline{\mathscr{M}}_\chi$ on $\mathrm{Spec}\, A(v)$ over k_v.

2.2. The definition of global L-functions

For $x \in X^0$, let $\mathfrak{p}_x$ be its image in $\mathrm{Max}(A)$ (this uses that X is of finite type over k). Recall from the discussion after Definition 1.40 that for $\underline{\mathscr{F}} \in \mathbf{Crys}(X, A)$ and $x \in X^0$ one has $L(x, \underline{\mathscr{F}}, T)^{-1} \in A[T^{d_x}] \subset A[T^{d_{\mathfrak{p}_x}}]$.

Definition 2.8 *Let $(\underline{\mathscr{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(X, A))$. If for $s \in S_v$, the product*

$$\prod_{x \in X(v)^0} L(x, (\underline{\mathscr{F}}^\bullet), T)_{|T^{d_{\mathfrak{p}_x}} = \mathfrak{p}_x^{-s}}$$

converges, we denote it by $L^{(v)}(X, (\underline{\mathscr{F}}^\bullet), s)$ and call it the value of the v-adic L-function of $(\underline{\mathscr{F}}^\bullet)$ at s.

We write $\zeta_X^{(v)}(s)$ for $L^{(v)}(X, \underline{\mathbb{1}}_{X,A}, s)$, and call it the value at s of the ζ-function of X over A.

Unless we want to emphasize the base scheme X, we write $L^{(v)}((\underline{\mathscr{F}}^\bullet), s)$ for $L^{(v)}(X, (\underline{\mathscr{F}}^\bullet), s)$. The following proposition collects some basic properties of v-adic L-functions.

Proposition 2.9 *(a) If X is the finite disjoint union of locally closed subsets X_i, then $L^{(v)}(X, (\underline{\mathscr{F}}^\bullet), s) = \prod_i L^{(v)}(X_i, (\underline{\mathscr{F}}^\bullet), s)$ provided the terms on the right are convergent products.*

(b) For the product in the above definition to converge for a fixed s, it is sufficient that, for this s, the product converges for each individual $\underline{\mathscr{F}}^i$.

(c) For $X = \operatorname{Spec} A$ one has

$$\zeta^{(v)}_{\operatorname{Spec} A}(s) = \prod_{\mathfrak{p} \in \operatorname{Max}(A(v))} (1 - \mathfrak{p}^{-s})^{-1}.$$

(d) Let $i : X_{\mathrm{red}} \hookrightarrow X$ be the closed immersion of the induced reduced subscheme of X and define $f_{\mathrm{red}} := f \circ i : X_{\mathrm{red}} \to \operatorname{Spec} A$. Then $L^{(v)}(X, (\underline{\mathscr{F}}^\bullet), s) = L^{(v)}(X_{\mathrm{red}}, i^*(\underline{\mathscr{F}}^\bullet), s)$ on the domain of convergence.

Ignoring for now the problem of convergence of the above product, cf. Theorem 2.16, we give an alternative description of $L^{(v)}((\underline{\mathscr{F}}^\bullet), s)$. For $\mathfrak{p} \in \operatorname{Max}(A)$, let $i_\mathfrak{p} : X_\mathfrak{p} \to X$ be the pullback map corresponding to $\operatorname{Spec} k_\mathfrak{p} \to \operatorname{Spec} A$. Viewing $X_\mathfrak{p}$ as a scheme over k, we obtain the map

$$\mathbf{Crys}(X, A) \longrightarrow \mathbf{Crys}(X_\mathfrak{p}, A) : \underline{\mathscr{F}} \mapsto \underline{\mathscr{F}}_\mathfrak{p} := i_\mathfrak{p}^* \underline{\mathscr{F}}.$$

Theorem 1.45 implies that for each $\mathfrak{p} \in \operatorname{Max}(A)$

$$L(X_\mathfrak{p}, (\underline{\mathscr{F}}^\bullet_\mathfrak{p}), T) = \prod_{x \in X_\mathfrak{p}^0} L(x, (\underline{\mathscr{F}}^\bullet), T) \in 1 + T^{d_\mathfrak{p}} A[[T^{d_\mathfrak{p}}]]$$

is a rational function. We call it the *local L-factor of $(\underline{\mathscr{F}}^\bullet)$ at $\mathfrak{p}$*. In the case where $\underline{\mathscr{F}}$ is attached to a Drinfeld-module, rationality was first proved in [21]. The following formula is immediate.

Proposition 2.10 *On the domain of convergence of $L^{(v)}((\underline{\mathscr{F}}^\bullet), s)$, the product*

$$\prod_{\mathfrak{p} \in \operatorname{Max}(A(v))} L((\underline{\mathscr{F}}^\bullet_\mathfrak{p}), T)_{|T^{d_\mathfrak{p}} = \mathfrak{p}^{-s}}$$

converges, and both expressions take the same value.

Having attached a crystal to any Drinfeld-A-module $(\psi, \mathscr{L})$ and to any A-motive $(\underline{\mathscr{M}}, \mathrm{ch}_{\underline{\mathscr{M}}})$ on X, say each of rank r, we need to compare the v-adic L-function attached to these objects as defined by Goss, cf. [12], Ch. 8, [13], § 3, with our definition for the associated crystals. Let f be ch_ψ, respectively $\mathrm{ch}_{\underline{\mathscr{M}}}$ and $\underline{\mathscr{F}}$ the crystal represented by $\underline{\mathscr{M}}(\psi)$, respectively $\underline{\mathscr{M}}$, cf. Example 1.6.

Fix a closed point $x \in X$. For a place v' of A different from v and $\mathfrak{p}_x$, let $T_{v'}(x)$ be the v'-adic Tate-module of ψ, respectively $\underline{\mathscr{M}}$ at x, [12], Def. 4.10.9 and p. 154. This is a free $A_{v'}$-module of rank r which carries an action by $\operatorname{Gal}(k_x^{\mathrm{sep}}/k_x) \cong \hat{\mathbb{Z}}$. Let Frob_x be the Frobenius element of $\operatorname{Gal}(k_x^{\mathrm{sep}}/k_x)$. The following is essentially shown in [12], Prop. 5.6.9.

Proposition 2.11 *Let v be any place of K. With the above notation, one has*

$$\det(1 - T^{d_x}\tau_x^{d_x}|\mathscr{F}_x) = \det(1 - T^{d_x}\mathrm{Frob}_x|T_{v'}(x)) \quad \forall x \in X^0.$$

In particular $L(x, \underline{\mathscr{F}}, T)_{|T^{d_{\mathfrak{p}_x}} = \mathfrak{p}_x^{-s}} = \det(1 - \mathfrak{p}_x^{-s}\mathrm{Frob}_x|T_{v'}(x))^{-1}$ *for all* $s \in S_v$, $x \in X^0(v)$.

Let $L(\psi/X, s)$, respectively $L(\underline{\mathscr{M}}/X, s)$ be the L-functions as defined by Goss, cf. [12], p. 256, [13], Rem. 3.14. These are also defined as infinite products over the points of X^0. The above lemma immediately implies the following, where by writing an equality of L-functions we mean that the respective infinite products have the same domain of convergence and that on this domain their values agree.

Corollary 2.12 *Let v be any place of K.*

(a) For any Drinfeld-A-module $(\psi, \mathscr{L})$ on X of fixed rank, and with $f = \mathrm{ch}_\psi$, one has
$$L^{(v)}(X, \underline{\mathscr{M}}(\psi), s) = L^{(v)}(\psi/X, s).$$

(b) For any A-motive $(\underline{\mathscr{M}}, \mathrm{ch}_{\underline{\mathscr{M}}})$ on X of fixed rank, and with $f = \mathrm{ch}_{\underline{\mathscr{M}}}$, one has
$$L^{(v)}(X, \underline{\mathscr{M}}, s) = L^{(v)}(\underline{\mathscr{M}}/X, s).$$

This is a non-empty statement, as it is shown in [12] and [13] that the functions $L^{(v)}(\psi/X, s)$ and $L^{(v)}(\underline{\mathscr{M}}/X, s)$ have a large domain of convergence. Alternatively, one can appeal to Theorem 2.16 below.

2.3. A half plane of convergence

For $c \in \mathbb{R}_{\geq 0}$, let $D_v^*(c) := \{z \in \mathbb{C}_v : |z|_v > c\}$ be the punctured disc around the infinite point ∞_v of $\mathbb{P}^1(\mathbb{C}_v)$ of radius c. Furthermore, let $D_v(c) := D_v^*(c) \cup \{\infty_v\}$, $\bar{D}_v^*(c) := \{z \in \mathbb{C}_v : |z|_v \geq c\}$ and $\bar{D}_v(c) := \bar{D}_v(c) \cup \{\infty_v\}$.

Definition 2.13 *A subset $D_v^*(c) \times W_v$ of S_v with $c > 0$ is called a* half plane.

Before proving that the Euler product representing $L^{(v)}(X, (\underline{\mathscr{F}}^\bullet), s)$ converges on some half plane, we need some auxiliary results for $v = \infty$.

Lemma 2.14 *Let $X = \operatorname{Spec} R$ be affine. Assume that $(\mathscr{F}, \tau)$ is a τ-sheaf with $\mathscr{F} = \mathscr{O}_{X \times \operatorname{Spec} A}^{\oplus r}$. Let $\widetilde{\mathscr{F}} := \mathscr{O}_{X \times C}^{\oplus r}$. Then there exists an extension*

$$\tilde{\tau} \colon (\sigma_X \times \mathrm{id}_C)^* \widetilde{\mathscr{F}} \to \widetilde{\mathscr{F}} \otimes_{\mathscr{O}_C} \mathscr{O}_C(-m\infty)$$

of τ for a suitable $m \in \mathbb{N}$.

Proof: After fixing a basis for the free sheaf $\mathscr{F}$, we may represent $(\mathscr{F}, \tau)$ as a τ-module $(R \otimes A^r, \alpha(\sigma \times \mathrm{id}_{\operatorname{Spec} A}))$ for a unique choice of matrix $\alpha \in M_r(R \otimes A)$. We may view the entries of α as elements of $R \otimes K_\infty$. As there are only finitely many of them, one can find $m \in \mathbb{N}$ such that all entries are in $R \otimes \pi_\infty^{-m} A_\infty$. The lemma follows easily. ∎

Proposition 2.15 *Given $(\underline{\mathscr{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(X, A))$, there exists $m \in \mathbb{N}$ such that*

$$\forall x \in X^0 : L(x, (\underline{\mathscr{F}}^\bullet), T\pi_\infty^m) \in 1 + T^{d_x} A_\infty[[T^{d_x}]].$$

Proof: By Proposition 2.9(a), we may decompose the scheme X, as in the proof of Proposition 1.32, into a finite disjoint union of locally closed affine schemes which are regular under their reduced subscheme structure. If we prove the proposition on each such subscheme, the general assertion follows because there is only a finite number of such subschemes. Hence, using Theorem 1.27, we assume that X is affine and regular.

Using Lemmas 1.4 and 1.33, we may assume that all $\mathscr{F}^i$ are represented by free τ-sheaves. Because $(\underline{\mathscr{F}}^\bullet)$ is bounded, there is only a finite number of non-zero $\mathscr{F}^i$, and so we may assume that $(\underline{\mathscr{F}}^\bullet)$ is concentrated in degree zero and that $\underline{\mathscr{F}}^0$ is represented by a free τ-sheaf $\underline{\mathscr{F}}$ as in the previous lemma.

The lemma provides us with an extension

$$\tilde{\tau} \colon (\sigma_X \times \mathrm{id}_C)^* \widetilde{\mathscr{F}} \to \widetilde{\mathscr{F}} \otimes_{\mathscr{O}_C} \mathscr{O}_C(-m\infty).$$

Therefore $\pi^m \tilde{\tau}$ defines an A_∞-crystal $\underline{\mathscr{F}}_\infty$ on X. From the definition of the L-function of crystals, it follows that

$$L(x, \underline{\mathscr{F}}_\infty, T) = L(x, \underline{\mathscr{F}}, T\pi_\infty^m) \in 1 + T^{d_x} A_\infty[[T^{d_x}]],$$

independently of x, by comparing the two expressions over $K_\infty[[T]]$. ∎

Theorem 2.16 *For each $v \in C$ and each $(\underline{\mathscr{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(X, A))$ the Euler product defining $L^{(v)}((\underline{\mathscr{F}}^\bullet), s)$ is convergent on some half plane $D_v^*(c) \times W_v$ of S_v and the convergence is uniform on $\bar{D}_v^*(c') \times W_v$ for any $c' > c$.*

PROOF: As in the proof of the previous proposition, we may assume that $(\underline{\mathscr{F}}^\bullet)$ is concentrated in degree zero and $\underline{\mathscr{F}} = \underline{\mathscr{F}}^0$.

Suppose first that $v \neq \infty$. We claim that we may take $D_v^*(1) \times W_v$ as a half plane of convergence. For $x \in X^0$ write $L(x, \underline{\mathscr{F}}, T) = 1 + a_1 T^{d_x} + a_2 T^{2d_x} + \ldots$. As $L(x, \underline{\mathscr{F}}, T) \in A[[T]]$, all a_i satisfy $|a_i|_v \leq 1$. If $|z|_v > 1$ in $s = (z, w, y)$, then $|\mathfrak{p}_x^{-s}|_v = |z|_v^{-d_{\mathfrak{p}x}} < 1$. Therefore we compute

$$|1 - L(x, \underline{\mathscr{F}}, T)_{T^{d_{\mathfrak{p}x}} = \mathfrak{p}_x^{-s}}|_v = |a_1 \mathfrak{p}_x^{-sd_x/d_\mathfrak{p}} + a_2 \mathfrak{p}_x^{-s2d_x/d_\mathfrak{p}} + \ldots|_v \leq |z|_v^{-d_x}. \tag{1}$$

Recall that an infinite product $\prod_{j=1}^\infty (1 + b_j)$ converges in $\mathbb{C}_v$, if and only if for any $r > 0$, the number of b_j with $|b_j|_v > r$ is finite. As X is of finite type over k, for any $n \in \mathbb{N}$ the number of $x \in X^0$ such that $d_x \leq n$ is finite. This together with the estimate (1) proves the convergence of the product for $L^{(v)}(\underline{\mathscr{F}}, s)$. The assertion on uniform convergence is left as an easy exercise.

Let now $v = \infty$. Then the local L-factors $L(x, \underline{\mathscr{F}}, T)$ are no longer in $A_\infty[[T]]$. However, if we choose m as in the previous proposition, then the function $L(x, \underline{\mathscr{F}}, T\pi_\infty^m)$ lies in $A_\infty[[T]]$ for all $x \in X^0$. Proceeding as above, it follows that $D_\infty^*(q_\infty^m) \times W_\infty$ is a half plane of convergence. ∎

Remark 2.17 Let us fix a place v of K. As in Definition 2.8, one may define a v-adic L-function for any complex in $\mathbf{D}^b(\mathbf{Crys}(X(v), \mathbb{C}_v))$. By results analogous to Lemma 2.14 and Proposition 2.15, one can show that the Euler product representing such an L-function is convergent on a half plane of S_v. If the coefficients are in the ring of integers of $\mathbb{C}_v$, the half plane contains $D_v^*(1) \times W_v$, and there is no need to refer to the above two results.

Thus, one may define a v-adic L-function for any complex in the derived category $\mathbf{D}^b(\mathbf{Crys}(X(v), B))$ where B is a subring of $\mathbb{C}_\infty$.

2.4. On the ground field k

Let k' be the constant field of K which clearly contains k. Via the structure map $f : X \to \operatorname{Spec} A$ composed with $\operatorname{Spec} A \to \operatorname{Spec} k'$, we see that X is naturally a variety over k' and not just over k.

Proposition 2.18 *Given a τ-sheaf $\underline{\mathscr{F}}$ on X over A relative to k (cf. remarks after Definition 1.1), there exists a τ-sheaf $\underline{\mathscr{F}}'$ on X over A relative to k', such that for any place v of C the v-adic L-functions of $\underline{\mathscr{F}}$ and $\underline{\mathscr{F}}'$ agree.*

Based on this proposition, we will from Section 3 on assume that k is the field of constants of K, i.e., that C is geometrically irreducible over k.

PROOF: Let $d = [k' : k]$. Then $\operatorname{Spec} k' \otimes_k A$ is the disjoint union of d copies of $\operatorname{Spec} A$. Let e be an idempotent of $A' := k' \otimes_k A$ which projects onto precisely one of these copies. By σ' the absolute Frobenius of X with respect to k' is denoted, i.e., $\sigma' = \sigma^d$.

Via $\mathscr{O}_X \otimes_k A \cong \mathscr{O}_X \otimes_{k'} A'$, we regard $\mathscr{F}$ as a sheaf over $X \times_{\operatorname{Spec} k'} \operatorname{Spec} A'$. Furthermore, the d-th iterate of τ gives us an $\mathscr{O}_X \otimes_{k'} A'$-linear endomorphism

$$(\sigma' \times \mathrm{id})^* \mathscr{F} \xrightarrow{\tau^d} \mathscr{F}.$$

Because τ^d is A'-linear, multiplication with the idempotent $1 \otimes e$ of $\mathscr{O}_X \otimes_{k'} A'$ commutes with τ^d, and we set

$$\underline{\mathscr{F}}' := ((1 \otimes e)\mathscr{F}, (1 \otimes e)\tau^d_{\mathscr{F}}).$$

It remains to verify that $\underline{\mathscr{F}}'$ satisfies the assertion of the proposition. Obviously, X has the same closed points as a variety over k or over k'. Thus it suffices to check that $\underline{\mathscr{F}}$ and $\underline{\mathscr{F}}'$ have the same local factors in the definition of the respective v-adic L-functions. Hence we may assume that $X = \operatorname{Spec} k_x$ for some finite extension k_x of k'. We may also work with K instead of A and $K' := k' \otimes_k K$ instead of A' as coefficients. So let M be a $k_x \otimes_k K$-module representing $\mathscr{F}$ and $\tau : M \to M$ the corresponding $\sigma \otimes \mathrm{id}$-linear endomorphism. Analogously one defines M', τ' for $\underline{\mathscr{F}}'$.

If τ is nilpotent, then M as well as M' have trivial v-adic L-functions. For arbitrary τ, consider the decreasing sequence of submodules $\tau^{ld_x}(M)$, $l \in \mathbb{N}$, of M. Because $k_x \otimes_k K$ is artinian, this sequence will become stationary. As the kernel of $M \to \tau^{ld_x}(M)$ is nilpotent, we may, by the above, assume that τ is an isomorphism. Furthermore if $\mathfrak{p}$ denotes the image of $\operatorname{Spec} k_x \to \operatorname{Spec} A$, we will assume that $\mathfrak{p} \neq \mathfrak{p}_v$ as otherwise again both v-adic L-functions will be trivial.

Write $K' = K_1 \times \ldots \times K_d$ where all K_i are isomorphic to K and where σ acts on K' by cyclically permuting the K_i. Correspondingly we write $M = M_1 \times \ldots \times M_d$, and we assume that e is the idempotent that is the identity on K_1 and zero on the other K_i. Relative to k, the local L-function of the τ-module M, written $L_k(x, M, T)$, is given as the inverse of $\det_{k_x \otimes_k K}(1 - T^{d_x} \tau^{d_x} | M)$, where the determinant is taken over the ring $k_x \otimes_k K$. As τ^{d_x}

fixes all the components M_i and as τ maps M_i isomorphically to M_{i+1}, it follows that $L_k(x,M,T)^{-1} = \det_{k_x\otimes_{k'}K_1}(1-T^{d_x}\tau^{d_x}|M_1)$. Therefore the v-adic L-function $L_k^{(v)}(M,s)$ relative to k satisfies

$$L_k^{(v)}(M,s)^{-1} = \det_{k_x\otimes_{k'}K_1}(1-\mathfrak{p}^{-sd_x/d_\mathfrak{p}}\tau^{d_x}|M_1).$$

Relative to k' we compute the corresponding data for $M' = M_1$ and $\tau' = \tau^d_{|M_1}$. We use d'_x and $d'_\mathfrak{p}$ to denote the degrees of x, respectively $\mathfrak{p}$ over k', and write subscripts k' at the L-functions to indicate that we work over k'. We find that $L_{k'}(x,M',T)^{-1} = \det_{k_x\otimes_{k'}K_1}(1-T^{d'_x}\tau'^{d'_x}|M_1)$ and hence

$$L_{k'}^{(v)}(M',s)^{-1} = \det_{k_x\otimes_{k'}K_1}(1-\mathfrak{p}^{-sd'_x/d'_\mathfrak{p}}\tau'^{d'_x}|M_1).$$

Since $d'_x/d'_\mathfrak{p} = d_x/d_\mathfrak{p}$ and $\tau^{d_x}_{|M_1} = \tau'^{d'_x}$, the assertion follows. ∎

2.5. Twisting L-functions by characters

Let $\tilde{k}$ be a finite extension of k and choose an embedding into the k-field $\mathbb{C}_v$.

Definition 2.19 *For* $(\underline{\mathscr{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(X,A))$ *and* $\chi\colon \mathrm{Gal}(K^{\mathrm{sep}}/K) \to \tilde{k}^*$ *a character which is unramified outside* v,∞*, we define*

$$L_\chi^{(v)}((\underline{\mathscr{F}}^\bullet),s) := \prod_{x\in X(v)^0} L(x,(\underline{\mathscr{F}}^\bullet_x),T)_{|T^{d_{\mathfrak{p}_x}}=\chi(\mathrm{Frob}_{\mathfrak{p}_x})\mathfrak{p}_x^{-s}}$$

to be the twist of the v-adic L-function of $(\underline{\mathscr{F}}^\bullet)$ by χ.

In the following discussion, we fix v, $\tilde{k}$ and χ. By going through the proof of Theorem 2.16, it is simple to see that $L_\chi^{(v)}$ has a half plane of convergence. However the proposition below will give us a more intrinsic way to see this.

Define $\tilde{A} := A\tilde{k}$ as a ring inside $\mathbb{C}_v$. Note that $\tilde{k} \to \tilde{A}$ and $A \to \tilde{A}$ are obviously flat. Via change of coefficients, $\_\_\otimes_{\tilde{k}}\tilde{A}$, we may view the crystal $\underline{\mathscr{M}}_\chi$ from Theorem 2.7 as a locally free crystal in $\mathbf{Crys}(\mathrm{Spec}\,A(v),\tilde{A})$, and $f(v)^*\underline{\mathscr{M}}_\chi$ as a locally free crystal of $\mathbf{Crys}(X(v),\tilde{A})$. Furthermore, the restriction $(\underline{\mathscr{F}}^\bullet)_{|X(v)}$ of $(\underline{\mathscr{F}}^\bullet)$ to $X(v)$ is an element of $\mathbf{D}^b(\mathbf{Crys}(X(v),\tilde{A}))$.

By Remark 2.17, the v-adic L-function of the $\tilde{A}$-crystal $(\underline{\mathscr{F}}^\bullet)_{|X(v)} \overset{L}{\otimes} f(v)^*\underline{\mathscr{M}}_\chi$ on $X(v)$ is defined and represented by an Euler product on some half plane of S_v. The characterization of $\underline{\mathscr{M}}_\chi$ given in Theorem 2.7 combined with Definitions 2.8 and 2.19 yields the following result.

Proposition 2.20 *Whenever* $L^{(v)}((\underline{\mathscr{F}^\bullet})_{|X(v)} \overset{L}{\otimes} f(v)^*\underline{\mathscr{M}}_\chi, s)$ *converges, the Euler product of* $L^{(v)}_\chi((\underline{\mathscr{F}^\bullet}), s)$ *converges and takes the same value.*

We now specialize the above discussion to the characters $\chi_{v,y}$ of $\hat{G}_v$, defined in Proposition 2.6. The quoted proposition and Theorem 2.7 yield the following, which except for the terminology is in [21], §10:

Proposition 2.21 *For* $(\underline{\mathscr{F}^\bullet}) \in \mathbf{D}^b(\mathbf{Crys}(X, A))$ *and* $s = (z, w, y)$ *in a suitable half plane one has*

$$\begin{aligned} L^{(v)}((\underline{\mathscr{F}^\bullet}), s) &= L^{(v)}_{\chi_{v,y}}((\underline{\mathscr{F}^\bullet}), (z, w, 0)) \\ &= L^{(v)}((\underline{\mathscr{F}^\bullet})_{|X(v)} \overset{L}{\otimes} f(v)^*\underline{\mathscr{M}}_{\chi_{v,y}}, (z, w, 0)). \end{aligned}$$

2.6. Reduction to $X = \operatorname{Spec} A$

Theorem 2.22 *Let* v *be a place of* K *and fix* $(\underline{\mathscr{F}^\bullet}) \in \mathbf{D}^b(\mathbf{Crys}(X, A))$. *If* $D^*_v(c) \times W_v$ *is a common half plane of convergence for the two* v*-adic* L*-functions* $L^{(v)}(X, (\underline{\mathscr{F}^\bullet}), s)$ *and* $L^{(v)}(\operatorname{Spec} A, Rf_!(\underline{\mathscr{F}^\bullet}), s)$, *then they agree on this half plane.*

Proof: According to Proposition 2.10, it suffices to show that

$$\prod_{\mathfrak{p} \in \operatorname{Max}(A(v))} L((\underline{\mathscr{F}^\bullet_\mathfrak{p}}), T)_{|T^{d_\mathfrak{p}} = \mathfrak{p}^{-s}} = \prod_{\mathfrak{p} \in \operatorname{Max}(A(v))} L((Rf_!\underline{\mathscr{F}^\bullet})_\mathfrak{p}, T)_{|T^{d_\mathfrak{p}} = \mathfrak{p}^{-s}}.$$

By base change, Theorem 1.36, we have $(Rf_!\underline{\mathscr{F}^\bullet})_\mathfrak{p} \cong Rf_!(\underline{\mathscr{F}^\bullet_\mathfrak{p}})$. The result now follows from the trace formula for L-functions of τ-sheaves, Theorem 1.45. ∎

Corollary 2.23 *Let* $(\underline{\mathscr{F}^\bullet})$ *be a complex in* $\mathbf{D}^b(\mathbf{Crys}(X, A))$. *Then there exists a complex* $(\underline{\mathscr{G}^\bullet}) \in \mathbf{D}^b(\mathbf{Crys}(\operatorname{Spec} A, A))$, *all of whose objects are locally free with the following property: Let* v *be a place of* K *and* $D^*_v(c) \times W_v$ *a half plane of convergence for the two* v*-adic* L*-functions* $L^{(v)}(X, (\underline{\mathscr{F}^\bullet}), s)$ *and* $L^{(v)}(\operatorname{Spec} A, (\underline{\mathscr{G}^\bullet}), s)$. *Then*

$$L^{(v)}(X, (\underline{\mathscr{F}^\bullet}), s) = L^{(v)}(\operatorname{Spec} A, (\underline{\mathscr{G}^\bullet}), s).$$

on $D^*_v(c) \times W_v$. *One may furthermore assume that* $(\underline{\mathscr{G}^\bullet})$ *is concentrated in degrees zero and one.*

PROOF: By Theorem 2.22 we may replace the complex $(\underline{\mathscr{F}}^\bullet)$ by $Rf_!(\underline{\mathscr{F}}^\bullet)$. As $\operatorname{Spec} A$ is smooth and affine, by Theorem 2.16 and Lemma 1.33, we may replace the latter complex by a bounded complex $(\underline{\mathscr{G}}^\bullet)$ of locally free A-crystals on $\operatorname{Spec} A$. If desired, one can replace $(\underline{\mathscr{G}}^\bullet)$ by the complex

$$\ldots \longrightarrow 0 \longrightarrow \bigoplus_{i\,\text{even}} \underline{\mathscr{G}}^i \xrightarrow{0} \bigoplus_{i\,\text{odd}} \underline{\mathscr{G}}^i \longrightarrow 0 \longrightarrow \ldots \blacksquare$$

2.7. *Meromorphy*

For $c > 0$, let $\mathrm{C}^{\mathrm{an}}(\bar{D}_v(c))$ denote the ring of power series $f = \sum_{n\geq 0} a_n z^{-n}$ over $\mathbb{C}_v$ which converge for $|z|_v \geq c$. This is a Banach space under the norm $||f||_c := \sup_{n\geq 0} |a_n|_v c^{-n}$. Similarly, for $c \geq 0$ we denote by $\mathrm{C}^{\mathrm{an}}(D_v(c))$ the ring of those power series that converge for $|z|_v > c$. Let $\{c_m\}$ be any strictly decreasing sequence which converges to c. Then $\mathrm{C}^{\mathrm{an}}(D_v(c))$ is a Fréchet space under the metric

$$\mathrm{dist}_{\{c_m\}}(f,g) := \sum_{m=1}^{\infty} 2^{-m} \frac{||f-g||_{c_m}}{1+||f-g||_{c_m}}.$$

This means $\mathrm{C}^{\mathrm{an}}(D_v(c))$ is a complete linear metric space with respect to $\mathrm{dist}_{\{c_m\}}$. A sequence $\{g_n\} \subset \mathrm{C}^{\mathrm{an}}(D_v(c))$ is a Cauchy-sequence if and only if this holds with respect to all norms $||\_\_||_{c_m}$. Different sequences $\{c_m\}$ will give equivalent metrics. If $c = 0$, we will usually use the sequence $c_m = q_v^{-m}$.

The following defines a metric dist_v on W_v. Let $|\_\_|_p$ be the valuation on $\mathbb{Z}_p$ such that $|p| = p^{-1}$ and define for $y \in \mathbb{Z}/(q_{v,\beta_v}-1)$ the symbol δ_y as zero if $y = 0$ and as 1 otherwise. For $(w_i, y_i) \in W_v$, $i = 1, 2$ we define

$$\mathrm{dist}_v((w_1, z_1), (w_2, y_2)) := \delta_{y_1-y_2} + |w_1 - w_2|_p.$$

By Theorem 2.16, the function $z \mapsto L^{(v)}(\underline{\mathscr{F}}, (z, w, y))$ represents a power series in z^{-1} which is convergent on $D_v^*(c)$ for some $c > 0$, independently of $(w, y) \in W_v$, and the coefficients of this power series vary continuously in (w, y). If we set $L^{(v)}(\underline{\mathscr{F}}, (\infty_v, w, y)) := 1$, we may regard $L^{(v)}(\underline{\mathscr{F}}, \_\_)$ as a continuous function $W_v \to \mathrm{C}^{\mathrm{an}}(D_v(c))$, where domain and range are the metric spaces defined above.

Remark 2.24 The obvious restriction map $\mathrm{C}^{\mathrm{an}}(D_v(c')) \to \mathrm{C}^{\mathrm{an}}(D_v(c))$ for $c' < c$ is injective, as analytic functions are uniquely determined by their power series expansion around ∞. This shows that $L^{(v)}(\underline{\mathscr{F}}, \_\_)$ is uniquely determined, if we know its restriction $W_v \to \mathrm{C}^{\mathrm{an}}(D_v(c))$ for some arbitrarily large c.

Definition 2.25 (Goss) *A continuous function* $f\colon W_v \to \mathrm{C}^{\mathrm{an}}(D_v(0))$ *is called* entire.

An entire function is called essentially algebraic*, if there exists a finite extension* $\tilde{K}$ *of* K *such that for all* $j \in \mathbb{N}_0$ *the functions* $f(-j,-j)$ *lie in* $\tilde{K}[z^{-1}]$.

The quotient of two entire functions is called meromorphic*. The quotient of two entire, essentially algebraic functions is called* essentially algebraic*.*

We say that $L^{(v)}(\underline{\mathscr{F}}, \_\_) : W_v \to \mathrm{C}^{\mathrm{an}}(D_v(c))$ *has an* entire*, respectively* meromorphic continuation to S_v *if there exists an entire, respectively meromorphic function* f *whose restriction to* $D_v(c)$ *agrees with* $L^{(v)}(\underline{\mathscr{F}}, \_\_)$.

Remark 2.24 shows that if $L^{(v)}(\underline{\mathscr{F}}, \_\_)$ has an entire continuation f to S_v, then the function f is unique. Thus often we will simply say that $L^{(v)}(\underline{\mathscr{F}}, \_\_)$ is entire, meromorphic, essentially algebraic, respectively. The definition of entireness, meromorphy, essential algebraicity, respectively, on any open (hence compact) subgroup of W_v is analogous and left to the reader.

Remark 2.26 In [12], Def. 8.5.12, one finds for $v = \infty$ the following definition of essential algebraicity: f is essentially algebraic if there exists a finite extension $\tilde{K}$ of K such that for all $j \in \mathbb{N}_0$ the functions $f(-j,-j)$ lie in $\tilde{K}[z^{-1}\pi_*^j]$. By our choice of π_∞, we know that $\pi_*^n \in K$ for some $n \gg 0$. Therefore $\tilde{K}$ is finite over K if and only if $\tilde{K}[\pi_*]$ is finite over K. Hence we may work with the above uniform definition of essential algebraicity for all places v.

For an L-function to be entire, we have the following simple criterion, essentially in terms of the coefficients of its Taylor expansion near ∞_v.

Proposition 2.27 *Let* $L(\underline{\mathscr{F}}, \_\_) : W_v \to \mathrm{C}^{\mathrm{an}}(D_v(c))$ *be given for some* $c > 0$. *Suppose we have a set* $M := \{(w_n, y_n) : n \in \mathbb{N}\}$ *which is dense in an open subgroup* U *of* W_v *such that*

(a) $||L(\underline{\mathscr{F}}, (\_\_, w_1, y_1))||_{q_v^{-m}}$ *is finite for all* $m \in \mathbb{N}$, *and*
(b) for each $m \in \mathbb{N}$, *there exists a constant* $C_m > 0$ *such that*

$$\begin{aligned} &||L(\underline{\mathscr{F}}, (\_\_, w_n, y_n)) - L(\underline{\mathscr{F}}, (\_\_, w_{n'}, y_{n'}))||_{q_v^{-m}} \\ &\leq C_m \operatorname{dist}_v((w_n, y_n), (w_{n'}, y_{n'})). \end{aligned}$$

Then $L^{(v)}(\underline{\mathscr{F}}, \_\_)$ *is entire on* U.

PROOF: For fixed m the two conditions above mean that we have a uniformly continuous map from a dense subset of U to the metric space $\mathrm{C}^{\mathrm{an}}(\bar{D}_v(q_v^{-m}))$. By a simple argument from the theory of metric spaces, this map has a unique continuous extension $f_m\colon U \to \mathrm{C}^{\mathrm{an}}(\bar{D}_v(q_v^{-m}))$.

Independently of m, the maps f_m and $L(\underline{\mathscr{F}}, \_\_)$ must agree when restricted to $D_v(\max\{c, q_v^{-m}\})$. Thus by Remark 2.24, the functions f_m patch, so that for each $(w, y) \in U$, one obtains a function $f(w, y) \in \mathrm{C}^{\mathrm{an}}(D_v(0))$ which when restricted to $\bar{D}_v(q_v^{-m})$ is $f_m(w, y)$. It remains to show that $f\colon U \to \mathrm{C}^{\mathrm{an}}(D_v(0))$ is continuous.

Let $\bar{w}_n$ be a sequence in U that converges to $\bar{w}$. We need to show that $f(\bar{w}_n) \to f(\bar{w})$. By the definition of the metric $\mathrm{dist}_{\{q_v^{-m}\}}$, it suffices to show this convergence for all norms $||\_\_||_{q_v^{-m}}$, i.e., that $f_m(\bar{w}_n) \to f_m(\bar{w})$ for each fixed m. This is clear from the construction of the f_m. ∎

We conclude our discussion of meromorphy by comparing the entireness of $\underline{\mathscr{F}}$ to that of its Frobenius twist $\underline{\mathscr{F}}^{(q)}$. For $l \in \mathbb{Z}$ and a power series $g = \sum a_n z^{-n}$, we define $g^{\sigma^l} := \sum a_n^{q^l} z^{-n}$. Because $\mathbb{C}_v$ is perfect, this also makes sense if $l < 0$. The map $g \mapsto g^{\sigma^l}$ is an isometric ring homomorphism from $\mathrm{C}^{\mathrm{an}}(\bar{D}_v(c))$ to $\mathrm{C}^{\mathrm{an}}(\bar{D}_v(c^{q^l}))$. With respect to suitable Fréchet metrics of the spaces involved, the analogous map from $\mathrm{C}^{\mathrm{an}}(D_v(c))$ to $\mathrm{C}^{\mathrm{an}}(D_v(c^{q^l}))$ is an isometry, too.

By $\mathrm{Cont}(\_\_, \_\_)$, we denote the set of continuous maps between topological spaces. For $f \in \mathrm{Cont}(W_v, \mathrm{C}^{\mathrm{an}}(\bar{D}_v(c)))$, we define

$$f^{(q)}\colon W_v \to \mathrm{C}^{\mathrm{an}}(\bar{D}_v(c^q)) : (w, y) \mapsto (f(w, y))^{\sigma}.$$

Thus we obtain a map

$$\mathrm{Cont}(W_v, \mathrm{C}^{\mathrm{an}}(\bar{D}_v(c))) \to \mathrm{Cont}(W_v, \mathrm{C}^{\mathrm{an}}(\bar{D}_v(c^q))),$$

and similarly $\mathrm{Cont}(W_v, \mathrm{C}^{\mathrm{an}}(D_v(c))) \to \mathrm{Cont}(W_v, \mathrm{C}^{\mathrm{an}}(D_v(c^q)))$.

Lemma 2.28 *Let $f = L(\underline{\mathscr{F}}, \_\_) : W_v \to \mathrm{C}^{\mathrm{an}}(D_v(c))$ for some $c > 0$. Then*

$$f^{(q)}(w, y) = L(\underline{\mathscr{F}}^{(q)}, (\_\_, qw, qy))\colon W_v \to \mathrm{C}^{\mathrm{an}}(D_v(c^q)).$$

PROOF: This is immediate from Definition 2.8 and Lemma 1.48. ∎

From the definition it is obvious, that $f^{(q)}(w,y)(z^q) = (f(w,y)(z))^q$. Combined with the previous lemma this shows:

Corollary 2.29 *For any place v of K, there exists a constant $c \in \mathbb{R}_{>0}$ such that $L^{(v)}(\underline{\mathscr{F}},(z,w,y))^q = L^{(v)}(\underline{\mathscr{F}}^{(q)},(z^q,qw,qy))$ on $D_v^*(c) \times W_v$.*

Proposition 2.30 *If $L^{(v)}(\underline{\mathscr{F}}^{(q)},s)$ is entire, meromorphic, essentially algebraic on $D_v^*(0) \times qW_v$, then so is $L^{(v)}(\underline{\mathscr{F}},s)$ on $D_v^*(0) \times W_v$.*

Proof: Let $f \in \mathrm{Cont}(W_v, \mathrm{C}^{\mathrm{an}}(D_v(c)))$ correspond to $L^{(v)}(\underline{\mathscr{F}},s)$ for c sufficiently large, and $g : W_v \to \mathrm{C}^{\mathrm{an}}(D_v(c^q))$ to $L^{(v)}(\underline{\mathscr{F}}^{(q)},s)$. By Lemma 2.28, we have $f^{(q)}(w,y) = g(qw,qy)$. Thus our assumptions imply that $f^{(q)}$ is entire, respectively meromorphic on W_v. Furthermore, if g is essentially algebraic on qW_v, then there exists a finite extension $\tilde{K}$ of K such that $f^{(q)}(-j,-j) = g(-qj,-qj)$ lies in $\tilde{K}[z^{-1}]$ for all $j \in \mathbb{N}_0$.

The operation $h \mapsto h^{\sigma^{-1}}$ preserves $\mathrm{C}^{\mathrm{an}}(D_v(0))$ as well as its subspace of polynomials, $\mathbb{C}_v[z^{-1}]$. Moreover, if a polynomials p_j belongs to $\tilde{K}[z^{-1}]$, then the polynomial $p_j^{\sigma^{-1}}$ belongs to $\tilde{K}^{1/q}[z^{-1}]$. Because K is finitely generated over k, the field $\tilde{K}^{1/q}$ is still a finite extension of K. Therefore f, which is the composite

$$W_v \xrightarrow{f^{(q)}} \mathrm{C}^{\mathrm{an}}(D_v(0)) \xrightarrow{h \mapsto h^{\sigma^{-1}}} \mathrm{C}^{\mathrm{an}}(D_v(0)),$$

has the desired property on W_v whenever g has it on qW_v. ∎

3. Drinfeld-Hayes modules

From now on, we will assume that k is the subfield of constants of K, i.e., that C is geometrically irreducible, cf. Proposition 2.18.

Let $\underline{\mathscr{C}} := (k[\theta] \otimes_k k[t], (t-\theta)(\sigma \otimes \mathrm{id}))$ denote the τ-sheaf on $\mathrm{Spec}\, k[\theta]$ over $k[t]$ associated to the Carlitz module. Then for any $j \in \mathbb{N}$

$$\zeta^{(v)}_{\mathrm{Spec}\, k[\theta]}((z,0,0) - s_{v,j}) = L(\underline{\mathscr{C}}^{\otimes j}, T)_{|T=z^{-1}},$$

independently of v. (That $\zeta^{(v)}_{\mathrm{Spec}\, k[\theta]}(s)$ has a meromorphic continuation to all of S_v is shown in [12], Thm. 8.9.2.) This is at the base of the following observation from [21],

$$L^{(v)}(\underline{\mathscr{F}},(z,0,0) - s_{v,j}) = L(\underline{\mathscr{F}} \otimes \underline{\mathscr{C}}^{\otimes j}, T)_{|T=z^{-1}}$$

for $A = k[t]$ and any crystal $\underline{\mathscr{F}}$, which relates special values of v-adic L-functions to L-functions of τ-sheaves. In this section, we will discuss similar results for general A.

3.1. The L-function of the Drinfeld-Hayes crystal $\underline{\mathscr{H}}_A$

Let $\mathrm{Cl}^+(A) := J/P^+$ denote the narrow class group of A and $h^+ := \operatorname{card}\mathrm{Cl}^+(A)$ the narrow class number. By O^+ we denote the integral closure of A in H^+. As H^+/K is unramified away from ∞, the extension O^+/A is everywhere unramified. Let $\xi : \operatorname{Spec} O^+ \to \operatorname{Spec} A$ be the corresponding covering map of schemes and $G := \mathrm{Gal}(H^+/K)$ its Galois group.

By [12], Prop. 7.2.20, every isomorphism class of rank one sign-normalized Drinfeld-A-modules defined over K^{alg}, can be realized by a Drinfeld-A-module $\psi_H : A \to O^+\{\tau\}$. (This is the first instance where we use that k is the constant field of A, as otherwise there are no rank one Drinfeld modules relative to τ.) Furthermore such a ψ_H has everywhere good reduction. For ψ_H as above and $\gamma \in G$ we define ψ_H^γ as the composite of ψ_H with the endomorphism on $O^+\{\tau\}$ obtained by having γ act on the coefficients. Then $\{\psi_H^\gamma : \gamma \in G\}$ is a complete set of representatives of isomorphism classes of rank one sign-normalized Drinfeld-Hayes modules, loc. cit., Thm. 7.4.8.

Proposition 3.1 *The sheaves $\mathscr{M}(\psi_H^\gamma)$ are locally free of rank one over O^+. The τ-sheaves $\xi_*\underline{\mathscr{M}}(\psi_H)$ and $\xi_*\underline{\mathscr{M}}(\psi_H^\gamma)$ are isomorphic for any $\gamma \in G$.*

Proof: The first part is immediate from Example 1.6, and so we now turn to the proof of the second. Applying ξ_* simply means that we regard $\underline{\mathscr{M}}(\psi_H)$ as a τ-module over $A\otimes A$. The action of γ on O^+ induces an $A\otimes A$-linear automorphism $\tilde\gamma$ on $O^+\otimes A\{\tau\}$, mapping $r\otimes a\tau^i$ to $\gamma(r)\otimes a\tau^i$.

As an $O^+\{\tau\}\otimes A$-module, $\underline{\mathscr{M}}(\psi_H)$ is isomorphic to the quotient of $O^+\{\tau\}\otimes A$ by the left ideal generated by $\{\psi_H(a)\otimes 1 - 1\otimes a : a \in A\}$. Under the action of $\tilde\gamma$, the latter set maps to $\{\psi_H^\gamma(a)\otimes 1 - 1\otimes a : a\in A\}$. Hence $\tilde\gamma$ induces an $A\otimes A$-linear isomorphism from $\xi_*\underline{\mathscr{M}}(\psi_H)$ to $\xi_*\underline{\mathscr{M}}(\psi_H^\gamma)$. ∎

From now on, we write $\underline{\mathscr{H}}_A$ for $\underline{\mathscr{M}}(\psi_H)$.

For $\mathfrak{p} \in \mathrm{Max}(A)$ let $\mathrm{Frob}_\mathfrak{p} \in G := \mathrm{Gal}(H^+/K)$ denote the corresponding Frobenius automorphism. Because G is abelian and O^+/A is unramified $\mathrm{Frob}_\mathfrak{p}$ is well-defined for all $\mathfrak{p}$. For $\mathfrak{P} \in \mathrm{Max}(O^+)$, let $G_\mathfrak{P}$ denote the corresponding decomposition subgroup of G, which is the subgroup generated by $\mathrm{Frob}_\mathfrak{p}$, where $\mathfrak{P}$ is above $\mathfrak{p}$. As $G_\mathfrak{P}$ only depends on $\mathfrak{p}$, we sometimes use the notation $G_\mathfrak{p}$. Via the isomorphism $G \cong \mathrm{Cl}^+(A)$ from class field theory, the element $\mathrm{Frob}_\mathfrak{p}$ corresponds to the ideal class $[\mathfrak{p}] \in \mathrm{Cl}^+(A)$ of $\mathfrak{p}$. Let $d_\mathfrak{P}$ denote the degree of $\mathfrak{P}$ over k, which is the same as the order of $[\mathfrak{p}]$.

Lemma 3.2 *For* $\mathfrak{p} \in \mathrm{Max}(A)$ *and* $\mathfrak{P} \in \mathrm{Max}(O^+)$ *a place above* $\mathfrak{p}$*, one has*

$$L(\mathfrak{P}, \underline{\mathscr{H}}_A, T)^{-1} = 1 - (\mathfrak{p}^{s_{\infty,1}} T^{d_\mathfrak{p}})^{\mathrm{card}\, G_\mathfrak{P}}.$$

Proof: By its definition, $L(\mathfrak{P}, \underline{\mathscr{H}}_A, T)^{-1} = \det(1 - T^{d_\mathfrak{P}} \tau^{d_\mathfrak{P}} | (i^*_\mathfrak{P} \underline{\mathscr{H}}_A) \otimes_A K)$. As is clear from Example 1.6(b), the pullback $i^*_\mathfrak{P} \underline{\mathscr{H}}_A$ is the crystal associated to $\psi_H \pmod{\mathfrak{P}}$. Because the order of the class $[\mathfrak{p}]$ is $\mathrm{card}\, G_\mathfrak{P}$, we can choose a positive generator $g_\mathfrak{p}$ in $\mathfrak{p}^{\mathrm{card}\, G_\mathfrak{P}}$, so that $g_\mathfrak{p} = \pi_*^{d_\mathfrak{p}\, \mathrm{card}\, G_\mathfrak{p}} \langle \mathfrak{p} \rangle^{\mathrm{card}\, G_\mathfrak{P}}$ and $\mathrm{sign}(g_\mathfrak{p}) = 1$. As $\psi_H \pmod{\mathfrak{P}}$ has supersingular reduction at $\mathfrak{p}$, and as it is sign-normalized, we have

$$\psi_H(g_\mathfrak{p}) \equiv \tau^{d_\mathfrak{p}\, \mathrm{card}\, G_\mathfrak{p}} \alpha(\mathrm{sign}(g_\mathfrak{p})) \equiv \tau^{d_\mathfrak{P}} \pmod{\mathfrak{P}},$$

for some $\alpha \in \mathrm{Gal}(k_\infty/k)$. Thus $\tau^{d_\mathfrak{P}} \otimes 1 = 1 \otimes g_\mathfrak{p}$ on $i^*_\mathfrak{P} \underline{\mathscr{H}}_A$, and we have

$$L(\mathfrak{P}, \underline{\mathscr{H}}_A, T)^{-1} = 1 - \Big((\pi_* T)^{d_\mathfrak{p}} \langle \mathfrak{p} \rangle \Big)^{\mathrm{card}\, G_\mathfrak{p}}. \quad \blacksquare$$

Let $\hat{G}$ denote the group of characters of G with values in $\mathbb{Z}[\zeta_{h^+}]$. We fix a prime ideal $\mathbf{P}$ of $\mathbb{Z}[\zeta_{h^+}]$ above p, and let $\tilde{k}$ denote the corresponding residue field. For a character $\chi \in \hat{G}$ let $\bar{\chi} \colon G \to \tilde{k}^*$ denote its reduction modulo $\mathbf{P}$. We extend ι_v to a map $\tilde{A} := \tilde{k}A \to \mathbb{C}_v$. The well-known and simple observation

$$\prod_{\chi \in \hat{G}} (1 - \chi(\mathrm{Frob}_\mathfrak{p}) T) = (1 - T^{\mathrm{card}\, G_\mathfrak{p}})^{\mathrm{card}\, G / \mathrm{card}\, G_\mathfrak{p}}$$

now yields

Corollary 3.3 *For any* $\mathfrak{p} \in \mathrm{Max}(A)$*:*

$$\prod_{\mathfrak{P} | \mathfrak{p}} L(\mathfrak{P}, \underline{\mathscr{H}}_A, T)^{-1} = \prod_{\chi \in \hat{G}} (1 - \bar{\chi}(\mathrm{Frob}_\mathfrak{p}) \mathfrak{p}^{s_{\infty,1}} T^{d_\mathfrak{p}}).$$

The following result is now an immediate consequence of the definition of $L^{(v)}$ and the above corollary:

Theorem 3.4 (Goss) *There exists a half plane of* S_v *on which the* v*-adic L-function* $L^{(v)}(\underline{\mathscr{H}}_A, s)$ *is given by*

$$\prod_{\mathfrak{p} \in \mathrm{Max}(A(v))} \prod_{\chi \in \hat{G}} \Big(1 - \bar{\chi}(\mathrm{Frob}_\mathfrak{p}) \mathfrak{p}^{s_{v,1-s}} \Big)^{-1}.$$

3.2. Crystals constructed from $\underline{\mathscr{H}}_A$

We define the following τ-sheaves over A on $\operatorname{Spec} A$:

$$\underline{\mathscr{P}} := \left(\Lambda^{h^+}\xi_*\underline{\mathscr{H}}_A\right)^{\otimes 2} \quad \underline{\mathscr{Q}}_j := \xi_*\left(\underline{\mathscr{H}}_A^{\otimes j}\right).$$

Had we chosen $\underline{\mathscr{M}}(\psi_H^\gamma)$ instead of $\underline{\mathscr{H}}_A$, then Proposition 3.1 shows that the resulting τ-sheaves are isomorphic and hence define the same crystal.

The proof of the following lemma is easy and left to the reader.

Lemma 3.5 *Let $\underline{\mathscr{F}}$, $\underline{\mathscr{G}}$ be τ-sheaves on X over A and $x \in X^0$. Write $L(x, \underline{\mathscr{G}}, T)^{-1} = \prod_{l=1}^r (1 - \alpha_l T^{d_x})$ with $\alpha_l \in K^{\mathrm{alg}}$. Suppose $\underline{\mathscr{G}}$ is locally free of rank r. Then the following hold:*

(a) The j-th exterior power $\Lambda^j\underline{\mathscr{G}}$ of $\underline{\mathscr{G}}$ is a locally free τ-sheaf of rank $\binom{r}{j}$ on X, and

$$L(x, \Lambda^j\underline{\mathscr{G}}, T)^{-1} = \prod_{1\le l_1<l_2<\ldots<l_j\le r} (1 - \alpha_{l_1}\alpha_{l_2}\ldots\alpha_{l_j}T^{d_x}).$$

(b) If $L(x, \underline{\mathscr{F}}, T)^{-1} = \prod_{m=1}^s (1 - \beta_m T^{d_x})$ with $\beta_m \in K^{\mathrm{alg}}$, then

$$L(x, \underline{\mathscr{F}} \otimes \underline{\mathscr{G}}, T)^{-1} = L(x, \underline{\mathscr{F}} \overset{L}{\otimes} \underline{\mathscr{G}}, T)^{-1} = \prod_{l=1}^r \prod_{m=1}^s (1 - \alpha_l\beta_m T^{d_x}).$$

(c) Suppose $X = \operatorname{Spec} A$ and $\underline{\mathscr{G}}$ has rank one. Let $x = \mathfrak{p} \in \operatorname{Max}(A)$ and write $L(\mathfrak{p}, \underline{\mathscr{G}}, T)^{-1} = 1 - \alpha_\mathfrak{p} T^{d_\mathfrak{p}}$. Then for s in a suitable half plane

$$L^{(v)}(\underline{\mathscr{F}} \otimes \underline{\mathscr{G}}^{\otimes j}, s) = \prod_{\mathfrak{p}\in\operatorname{Max}(A(v))} L(\underline{\mathscr{F}}_\mathfrak{p}, T)_{|T^{d_\mathfrak{p}} = \alpha_\mathfrak{p}^j \mathfrak{p}^{-s}}.$$

Lemma 3.6 *For an abelian group H with character group $\hat{H}$ define $\chi_H := \prod_{\chi\in\hat{H}} \chi$. If $H/2H$ has order two, let $\chi_{H,0}$ be the unique character on H of order 2. Otherwise, let $\chi_{H,0}$ be trivial. Then $\chi_H = \chi_{H,0}$.*

Proof: We first consider the case where H is cyclic of order n. Then $\hat{H}$ is also cyclic of order n. So let h_0 be a generator of H and χ_0 of $\hat{H}$ such that $\chi_0(h_0) = \zeta_n$. Then

$$\prod_{\chi\in\hat{H}} \chi(h_0) = \prod_{i=0}^{n-1} \zeta_n^i = \zeta_n^{\binom{n}{2}} = \begin{cases} 1 & n \text{ is odd} \\ -1 & n \text{ is even} \end{cases}$$

and the lemma follows.

Next, if $H = H_1 \times H_2$ is the product of two subgroups, we write correspondingly $\hat{H} = \hat{H}_1 \times \hat{H}_2$ where the characters in $\hat{H}_1$ are trivial on H_2 and vice versa. Then

$$\chi_H = \prod_{\chi \in \hat{H}} \chi = \prod_{\chi_1 \in \hat{H}_1, \chi_2 \in \hat{H}_2} \chi_1 \chi_2 = \chi_{H_1}^{\operatorname{card} H_2} \chi_{H_2}^{\operatorname{card} H_1}.$$

It is clear how to extend the above formula to an m-fold product of cyclic groups. The result follows easily. ∎

Applying the above lemmas to Corollary 3.3 implies the following:

Corollary 3.7 *For any* $\mathfrak{p} \in \operatorname{Max}(A)$*:*

$$\begin{aligned} L(\mathfrak{p}, \underline{\mathscr{P}}^{\otimes j}, T)^{-1} &= (1 - \mathfrak{p}^{s_{\infty,2jh^+}} T^{d_\mathfrak{p}}), \\ L(\mathfrak{p}, \underline{\mathscr{Q}}_j, T)^{-1} &= \prod_{\chi \in \hat{G}} (1 - \bar{\chi}(\mathrm{Frob}_\mathfrak{p}) \mathfrak{p}^{s_{\infty,j}} T^{d_\mathfrak{p}}). \end{aligned}$$

Corollary 3.8 *Let* $(\underline{\mathscr{F}}^\bullet)$ *be in* $\mathbf{D}^b(\mathbf{Crys}(X, A))$. *For* $s = (z, w, y)$ *in a suitable half plane of* S_v *one has*

(a)

$$L^{(v)}((\underline{\mathscr{F}}^\bullet) \overset{L}{\otimes} f^* \underline{\mathscr{P}}^{\otimes j}, s) = L^{(v)}((\underline{\mathscr{F}}^\bullet), s - s_{v,2jh^+}),$$

(b)

$$L^{(v)}((\underline{\mathscr{F}}^\bullet) \overset{L}{\otimes} f^* \underline{\mathscr{Q}}_j, s) = \prod_{\chi \in \hat{G}} L^{(v)}((\underline{\mathscr{F}}^\bullet)_{|X(v)} \overset{L}{\otimes} f(v)^* \underline{\mathscr{M}}_{\bar{\chi}^j}, s - s_{v,j}).$$

Proof: By Theorem 2.22, we may apply $Rf_!$ to the crystals whose L-functions we want to compare. Using Theorem 1.37 (the projection formula), we may assume that $X = \operatorname{Spec} A$ and $f = \mathrm{id}$. Furthermore we may assume that $(\underline{\mathscr{F}}^\bullet)$ is a τ-sheaf concentrated in degree zero. Then the previous corollary, Lemma 3.5 and Proposition 2.20 imply both assertions of the corollary. ∎

4. An algebraic proof of Goss' conjecture

Let $g_{X(v)} : X(v) \to \operatorname{Spec} k$ be the structure morphism. The following result is an immediate consequence of Definition 2.8, and the trace formula for τ-sheaves, Theorem 1.45.

Proposition 4.1 *For* $(\underline{\mathscr{F}}^\bullet) \in \mathbf{D}^b(\mathbf{Crys}(X, A))$*,* v *a place of* K *and* $|z|_v \gg 1$*:*

$$\begin{aligned} L^{(v)}(X, (\underline{\mathscr{F}}^\bullet), (z, 0, 0)) &= L(X(v), (\underline{\mathscr{F}}^\bullet), T)_{|T=z^{-1}} \\ &= L(\operatorname{Spec} k, Rg_{X(v)!}(\underline{\mathscr{F}}^\bullet), T)_{|T=z^{-1}}. \end{aligned}$$

In particular, the special value of $L^{(v)}$ 'at zero' can be expressed as the L-function of a τ-sheaf and is therefore a rational function in the variable z^{-1} over A.

Using the results of the previous section, we will obtain a similar presentation for the special values of global L-functions at all negative integers. This will allow us to derive some congruence properties, which in turn will show that all v-adic L-functions of crystals on arbitrary A-schemes (of finite type) have a meromorphic continuation to all of S_v.

We remind the reader that from Section 3 on, we have assumed that C is geometrically irreducible over k.

4.1. Special values at negative integers

Throughout this subsection, we assume that $X = \operatorname{Spec} A$ and that $\underline{\mathscr{F}}$ is a locally free crystal on X of rank r. Thus for a place v of K we have $X(v) = \operatorname{Spec} A(v)$. The open immersion $X(v) \to C$ will be denoted j_v. By C/K, we denote the base change of the curve C defined over k to K. For a coherent sheaf $\mathscr{G}$ on C/K, we abbreviate $h^i(\mathscr{G}) := \dim H^i(C/K, \mathscr{G})$ and define its Euler-Poincaré characteristic $\chi(\mathscr{G}) := h^0(\mathscr{G}) - h^1(\mathscr{G})$.

Definition 4.2 *For* $j \in \mathbb{N}_0$*, we define* $H_{v,j}(z) := L^{(v)}(\underline{\mathscr{F}}, (z, 0, 0) - s_{v,j})$ *and call it* the special value of $L^{(v)}(\underline{\mathscr{F}}, s)$ at $-j$.

As a consequence of Proposition 4.1, Corollary 3.8(a) and Theorem 1.38 we have:

Proposition 4.3 *Suppose* $\underline{\widetilde{\mathscr{F}}}_j$ *is a locally free* τ*-sheaf in* $\mathbf{Coh}_\tau(C, K)$ *representing* $\mathrm{j}_{v!}((\underline{\mathscr{F}} \otimes \underline{\mathscr{P}}^{\otimes j})_{|X(v)}) \otimes_A K$*. Then for* $j \in \mathbb{N}_0$ *the special value* $H_{v,2^h+j}(z)$ *is a polynomial in* $A[z^{-1}]$ *of degree at most* $h^1(\widetilde{\mathscr{F}}_j)$*.*

Because $(\underline{\mathscr{F}} \otimes \underline{\mathscr{P}}^{\otimes j})_{|X(v)} \otimes_A K$ is a locally free crystal on the affine regular scheme $\operatorname{Spec} A(v) \otimes_k K$, by Lemma 1.4 it may be represented by a free finite rank τ-sheaf. Using Lemma 1.30 it is easy to construct a locally free representative $\underline{\widetilde{\mathscr{F}}}$ of $\mathrm{j}_{v!}((\underline{\mathscr{F}} \otimes \underline{\mathscr{P}}^{\otimes j})_{|X(v)}) \otimes_A K$.

To construct representatives $\underline{\widetilde{\mathscr{F}}}_j$ with good bounds on $h^1(\widetilde{\mathscr{F}}_j)$, we will need to control $h^0(X, \mathscr{F} \otimes \mathscr{L})$ where $\mathscr{F}$ is a fixed locally free sheaf on C/K and $\mathscr{L}$ varies over the locally free sheaves of some negative degrees. Define for any $n \leq 0$ the quantity $\delta_n(\mathscr{F})$ as the maximum of zero and

$$\max\{h^0(X, \mathscr{F} \otimes \mathscr{L}) - h^0(X, \mathscr{F}) : \mathscr{L} \text{ is invertible with } \deg \mathscr{L} \leq n\}.$$

The $\delta_n(\mathscr{F})$ form a sequence of positive integers; they are zero for $n \leq -g$, and increasing on $[-g, 0]$. The following lemma gives better bounds on $\delta_n(\mathscr{F})$.

Lemma 4.4 *Let $\mathscr{F}$ be a locally free sheaf of rank r on C/K. Then for any $n \leq 0$*

$$\delta_n(\mathscr{F}) \leq r \max\{0, g+n\} \leq rg.$$

Proof: Define $m := -[-\max\{0, g+n\}/d_\infty]$, i.e., $m \geq 0$ is the smallest integer such that $md_\infty - \deg \mathscr{L} \geq g$ for any $\mathscr{L}$ with $\deg \mathscr{L} \leq n$. The Riemann-Roch theorem implies that there is a non-zero section in $\mathscr{L}^{-1}(m\infty)$, and therefore one has the following inclusions of sheaves

$$\mathscr{F} \otimes \mathscr{L} \hookrightarrow \mathscr{F}(m\infty) \hookleftarrow \mathscr{F}.$$

The left hand side yields the inequality $h^0(\mathscr{F} \otimes \mathscr{L}) \leq h^0(\mathscr{F}(m\infty))$. Moreover bounding h^0 of the cokernel of the right monomorphism yields the inequality $h^0(\mathscr{F}(m\infty)) \leq h^0(\mathscr{F}) + rmd_\infty$. Combining the two inequalities proves the desired result. ∎

For $j \in \mathbb{N}_0$, let $j = j_0 + j_1 q + j_2 q^2 + \ldots$ be its q-adic expansion and write $\bar{j}$ for the sum $j_0 + j_1 + j_2 + \ldots$ of the q-adic digits of j. For $\bar{j}$ as above, define $n_{\bar{j}} := [(2h^{+}\bar{j} + d_\infty)/(1-q)] < 0$.

We now come to one of the central results of this article.

Theorem 4.5 *Let $\underline{\widetilde{\mathscr{F}}}$ be a locally free τ-sheaf on C over K whose restriction to $X(v)$ represents $\underline{\mathscr{F}}_{|X(v)} \otimes_A K$. If $v \neq \infty$, assume further that $\underline{\widetilde{\mathscr{F}}}$ is nilpotent at v. Then*

$$\deg H_{v,2h^+j} \leq h^1(\widetilde{\mathscr{F}}) + r\Big(\Big[\frac{2h^{+}\bar{j}}{q-1}\Big] + d_\infty\Big) + \delta_{n_{\bar{j}}}(\widetilde{\mathscr{F}}). \tag{2}$$

Note that after Proposition 4.3, we explained how to construct $\underline{\widetilde{\mathscr{F}}}$ for a given crystal $\underline{\mathscr{F}}$. As $\bar{j}$ is of order $O(\log j)$, the above theorem and lemma imply:

Corollary 4.6 *The special values $H_{v,2jh^+}(z)$ are polynomials in $A[z^{-1}]$ whose degrees grow like $O(\log j)$.*

The main step in the proof of Theorem 4.5 is the following lemma.

Lemma 4.7 *There exists a locally free τ-sheaf $\underline{\widetilde{\mathscr{L}}}_j$ of rank one on C over K representing $\mathrm{j}_{\infty!}\underline{\mathscr{P}}^{\otimes j} \otimes_A K$ such that*

$$0 < \frac{2h^+\bar{j} + d_\infty}{q-1} \leq -n_{\bar{j}} \leq -\deg \widetilde{\mathscr{L}}_j \leq \left[\frac{2h^+\bar{j}}{q-1}\right] + d_\infty .$$

Remark 4.8 To prove only Corollary 4.6 and not Theorem 4.5, it suffices to construct $\underline{\widetilde{\mathscr{L}}}_j$ as in the lemma which satisfy the condition $-\deg \widetilde{\mathscr{L}}_j \leq O(\bar{j})$. The following simple construction will give such $\underline{\widetilde{\mathscr{L}}}_j$:

Using Lemma 1.30, one constructs a locally free τ-sheaf $\underline{\widetilde{\mathscr{L}}}$ on C over K which represents $\mathrm{j}_{\infty!}\underline{\mathscr{P}} \otimes_A K$. Define $\underline{\widetilde{\mathscr{L}}}_j := \underline{\widetilde{\mathscr{L}}}^{\otimes j_0} \otimes \ldots \otimes \underline{\widetilde{\mathscr{L}}}^{(q^s)\otimes j_s}$. As $\widetilde{\mathscr{L}}^{(q^i)}$ and $\widetilde{\mathscr{L}}$ have the same (negative) degree, and as $\underline{\widetilde{\mathscr{L}}}^{(q^i)}$ is nil-isomorphic to $\underline{\widetilde{\mathscr{L}}}^{\otimes q^i}$, the assertion follows easily.

PROOF: To prove Lemma 4.7, we write j in its q-adic expansion as above, and define $\underline{\mathscr{P}}_j := \underline{\mathscr{P}}^{\otimes j_0} \otimes \ldots \otimes (\underline{\mathscr{P}}^{(q^s)})^{\otimes j_s} \otimes_A K$. By Proposition 1.23, $\underline{\mathscr{P}}^{(q^l)}$ is nil-isomorphic to $\underline{\mathscr{P}}^{\otimes q^l}$, and hence $\underline{\mathscr{P}}_j$ to $\underline{\mathscr{P}}^{\otimes j} \otimes_A K$.

As k is the constant field of K, there is precisely one point of C/K which lies above ∞, which we also denote by ∞. Since $\mathscr{P}_j$ is a line bundle on the open curve $C/K \smallsetminus \{\infty\}$, there exists a line bundle on C/K extending it. Using Lemma 1.30 we can find a τ-sheaf $\underline{\widetilde{\mathscr{L}}}_j$ representing $\mathrm{j}_{\infty!}\underline{\mathscr{P}}_j$ whose underlying sheaf is locally free of rank one and whose restriction to $C/K \smallsetminus \{\infty\}$ agrees with $\underline{\mathscr{P}}_j$.

To compute the degree of $\widetilde{\mathscr{L}}_j$, we will analyze the cokernel $\widetilde{\mathscr{C}}_j$ defined by the short exact sequence

$$0 \longrightarrow (\sigma \times \mathrm{id})^* \widetilde{\mathscr{L}}_j \xrightarrow{\tau} \widetilde{\mathscr{L}}_j \longrightarrow \widetilde{\mathscr{C}}_j \longrightarrow 0 .$$

The first thing to note about this sequence is that $\deg(\sigma \times \mathrm{id})^* \widetilde{\mathscr{L}}_j = q \deg \widetilde{\mathscr{L}}_j$. Thus one has $(1-q)\deg \widetilde{\mathscr{L}}_j = \dim_K \widetilde{\mathscr{C}}_j$ and it suffices to obtain good bounds on $\dim_K \widetilde{\mathscr{C}}_j$. To investigate $\widetilde{\mathscr{C}}_j$, we first consider its restriction $\mathscr{C}_j$ to $\operatorname{Spec} A \otimes K$, which fits into the short exact sequence

$$0 \longrightarrow (\sigma \times \mathrm{id})^* \mathscr{P}_j \xrightarrow{\tau} \mathscr{P}_j \longrightarrow \mathscr{C}_j \longrightarrow 0 . \tag{3}$$

By the proof of Proposition 5.10, in particular by (c) on page 61, the support of $\mathscr{C}_1$ is concentrated on the point Ξ of $\operatorname{Spec}(A \otimes K)$ which corresponds to the multiplication map $A \otimes K \to K$. (The point Ξ arises from the diagonal of $\operatorname{Spec} A \times \operatorname{Spec} A$ after base change from A to K.) Furthermore, the quoted result shows that the dimension of the stalk of $\mathscr{C}_1$ at Ξ is $2h^+$.

The Frobenius twist maps the short exact sequence (3) for $j = 1$ to another short exact sequence, whose cokernel has the same K-dimension and is concentrated above $\sigma(\Xi)$. The definition of $\underline{\mathscr{P}}_j$ now implies that the K-dimension of $\mathscr{C}_j$ is $2h^+\bar{j}$.

It remains to consider the stalk of $\widetilde{\mathscr{C}}_j$ above ∞. Let π be a uniformizing parameter of A_∞ that lies inside K — such a π can be constructed using the Riemann-Roch theorem. Then the completion of the stalk of $\mathscr{O}_{C/K}$ at ∞ is isomorphic to $S := (k_\infty \otimes K)[[\pi]]$, and the completion of $\underline{\widetilde{\mathscr{L}}}_j$, is a τ-module of the form $(S, u\pi^n(\sigma \times \mathrm{id}))$ for some unit u of S and some element $n \in \mathbb{N}$. Note that $k_\infty \otimes K$ is a field, as C is geometrically irreducible. Furthermore $(\sigma \times \mathrm{id})(\pi) = \pi^q$.

Write $n = (q-1)l + n_0$ for some $l, n_0 \in \mathbb{N}_0$ with $0 < n_0 < q$. Then $(\pi^{-l}S, u\pi^{n_0}(\sigma \times \mathrm{id}))$ is a 'formal' τ-sheaf on S which contains $(S, u\pi^n(\sigma \times \mathrm{id}))$. We now replace $\underline{\widetilde{\mathscr{L}}}_j$ by $\underline{\widetilde{\mathscr{L}}}_j(-l\infty)$. The above local analysis at ∞ shows that this still represents $\mathrm{j}_{\infty!}\underline{\mathscr{P}}_j$. Furthermore we have $\dim_K \widetilde{\mathscr{C}}_j = 2h^+\bar{j} + n_0 d_\infty$ for some $n_0 \in [1, \ldots, q-1]$. The asserted inequalities now follow readily from $(1-q)\deg \widetilde{\mathscr{L}}_j = \dim_K \widetilde{\mathscr{C}}_j$. ∎

Proof of Theorem 4.5: Let $\underline{\widetilde{\mathscr{L}}}_j$ be as in the lemma. Then $\underline{\widetilde{\mathscr{F}}}_j := \underline{\widetilde{\mathscr{F}}} \otimes \underline{\widetilde{\mathscr{L}}}_j$ represents $\mathrm{j}_{v!}((\underline{\mathscr{F}} \otimes \underline{\mathscr{P}}^j)_{|X(v)} \otimes_A K)$, and Proposition 4.3 yields the estimate $\deg H_{v,2h^+j} \leq h^1(\widetilde{\mathscr{F}}_j)$. We rewrite the expression on the right using the Euler-Poincaré characteristic of coherent sheaves.

The change of the Euler-Poincaré characteristic of a locally free sheaf of rank r under twisting with a line bundle is given by adding r times the degree of the line bundle. Therefore we have

$$\begin{aligned} h^0(\widetilde{\mathscr{F}}_j) - h^1(\widetilde{\mathscr{F}}_j) &= \chi(\widetilde{\mathscr{F}}_j) = \chi(\widetilde{\mathscr{F}}) + r \deg \widetilde{\mathscr{L}}_j \\ &= h^0(\widetilde{\mathscr{F}}) - h^1(\widetilde{\mathscr{F}}) + r \deg \widetilde{\mathscr{L}}_j. \end{aligned}$$

By the previous lemma, we have $\deg \widetilde{\mathscr{L}}_j \leq n_{\bar{j}} < 0$. Reordering the terms and using the definition of $\delta_{n_{\bar{j}}}$ completes the proof of the theorem. ∎

Let C^+/K denote the curve corresponding to the function field H^+, let g^+ denote its genus, ξ the corresponding map $C^+ \to C$, and define the open subscheme $C^+(v)$ as $\xi^{-1}(X(v))$. We will use ξ also for the map $C^+(v) \to X(v)$. Furthermore, we define $\delta_n^+(\mathscr{F})$ for locally free sheaves $\mathscr{F}$ on C^+/K of rank r and $n \in -\mathbb{N}$ in analogy to δ_n. Again one has $\delta_n^+(\underline{\mathscr{F}}) \leq r \max\{0, g^+ + n\} \leq rg^+$, for a locally free τ-sheaf of rank r on C/K. Note that the Hurwitz genus formula yields

$$g^+ = h^+(g-1) + d_\infty + 1/2(h^+/d_\infty - h/d_\infty).$$

Define $n_{\bar{j}}^+ := [(h^+\bar{j} + hd_\infty)/(1-q)] < 0$. By arguing as above, however working with C^+ instead of C, one can obtain the following two results, which we state without proof:

Lemma 4.9 *There exists a locally free τ-sheaf $\underline{\widetilde{\mathscr{L}}}_j$ of rank one on C^+/K representing $\mathrm{j}_{\infty!}\underline{\mathscr{H}}_A^{\otimes j} \otimes_A K$ such that*

$$0 < \frac{h^+\bar{j} + hd_\infty}{q-1} \leq -n_{\bar{j}}^+ \leq -\deg \widetilde{\mathscr{L}}_j \leq \frac{2h^+\bar{j}}{q-1} + hd_\infty.$$

Theorem 4.10 *Let $\underline{\widetilde{\mathscr{F}}}$ be a locally free τ-sheaf on C^+ over K of rank r whose restriction to $C^+(v)$ represents $\xi^*(\underline{\mathscr{F}}_{|X(v)}) \otimes_A K$. For $v \neq \infty$, assume further that $\underline{\widetilde{\mathscr{F}}}$ is nilpotent at all places above v. Then*

$$\deg L^{(v)}(\underline{\mathscr{F}} \overset{L}{\otimes} \underline{\mathscr{Q}}_j, (z,0,0)) \leq h^1(\widetilde{\mathscr{F}}) + r\Big([h^+\bar{j}/(q-1)] + hd_\infty\Big) + \delta_{n_{\bar{j}}^+}^+(\widetilde{\mathscr{F}}). \tag{4}$$

In particular, $L^{(v)}((\underline{\mathscr{F}}^\bullet) \overset{L}{\otimes} \underline{\mathscr{Q}}_j, (z,0,0))$ is a polynomial in $A[z^{-1}]$ whose degree grows like $O(\log j)$.

Remarks 4.11 (a) If the 2-part of the class group of A is not a cyclic non-trivial group, one can extend estimate (2) to

$$\deg H_{v,jh^+}(z) \leq h^1(\widetilde{\mathscr{F}}) + r\Big([h^+\bar{j}/(q-1)] + d_\infty\Big) + \delta_{n'_{\bar{j}}}(\widetilde{\mathscr{F}}),$$

where $n'_{\bar{j}} = [(h^+\bar{j} + d_\infty)/(1-q)]$. This can be seen by working with the highest exterior power of $\xi_*\underline{\mathscr{H}}_A$ instead of $\underline{\mathscr{P}}$ and taking Lemma 3.6 into account. The results needed from Proposition 5.10 continue to hold with some obvious modifications.

(b) Recall that by Corollary 3.8, we have

$$L^{(v)}((\underline{\mathscr{F}}^\bullet) \overset{L}{\otimes} \underline{\mathscr{Q}}_j, s) = \prod_{\chi \in \hat{G}} L^{(v)}((\underline{\mathscr{F}}^\bullet)_{|\operatorname{Spec} A(v)} \overset{L}{\otimes} \underline{\mathscr{M}}_{\bar{\chi}^j}, s - s_{v,j}). \tag{5}$$

If all the $|\hat{G}| = h^+$ factors in this expressions are asymptotically of the same size, then one should expect that there exists a constant c such that

$$\deg H_{v,j}(z) \leq c + r\bar{j}/(q-1),$$

for all $j \in \mathbb{N}_0$, as can be seen by dividing the estimate (4) through h^+. Based on (5), in Theorem 4.15 we prove the weaker estimate $\deg H_{v,j}(z) \leq c + \bar{j}rh^+/(q-1)$. Let h' be twice the exponent of the group $\mathrm{Gal}(H^+/K)$, so that $h'|2h^+$. Then for all j divisible by h', all factors on the right hand side of formula (5) are identical. Using our estimate for $\delta^+_{n^+_{\bar{j}}}(\underline{\tilde{\mathscr{F}}})$, this proves

$$\deg H_{v,h'j}(z) \leq \frac{h^1(\underline{\tilde{\mathscr{F}}})}{h^+} + \frac{r}{h^+}\Big(\max\Big\{g^+, \frac{h^+\overline{h'j} + hd_\infty}{q-1}\Big\} + \tfrac{q-2}{q-1}hd_\infty\Big).$$

Example 4.12 Let $\underline{\mathscr{F}} = \mathbb{1}_{\mathrm{Spec}\,A,A}$ and define $\underline{\tilde{\mathscr{F}}}$ to be the τ-sheaf $\underline{\mathbb{1}}_{C,K}$ if $v = \infty$ and $\mathscr{I}_v\underline{\mathbb{1}}_{C,K}$ otherwise, where $\mathscr{I}_v = \mathscr{O}_C(-v)$. By the Riemann-Roch theorem $\delta_n(\underline{\tilde{\mathscr{F}}}) = 0$ for all $n < 0$. Hence Proposition 4.5 gives the estimate

$$\deg H_{2jh^+}(z) \leq d_v + g - 1 + [2h^+\bar{j}/(q-1)] + d_\infty.$$

In particular, if $v = \infty$ and $h^+ = 1$, i.e., $h = d_\infty = 1$, then by part (a) of the previous remark we find $\deg H_j(z) \leq g + [\bar{j}/(q-1)]$. For $A = k[t]$ this bound was obtained in [9], and it was probably one of Goss' motivations to look for logarithmic bounds on the degrees of special values. The results of loc. cit. also show that our estimate (2) is sharp. See also, [12], Rems. 8.12.1 and [23].

4.2. Constructing a meromorphic continuation

We abbreviate $h_\infty := 2h^+$ and $h_v := 2h^+(q_{v,\beta_v} - 1)$ for $v \neq \infty$. For a locally free A-crystal $\underline{\mathscr{F}}$ on $\mathrm{Spec}\,A$, define $b_{v,j} := L^{(v)}(\underline{\mathscr{F}}, (z, -jh_v, -jh_v))$ for $j \in \mathbb{N}$. From the definition of $L^{(v)}$ and Theorem 2.16 we see that there exists $c \in \mathbb{R}$, independently of j, such that

$$b_{v,j} = \prod_{\mathfrak{p}\in\mathrm{Max}(A(v))} L(\mathfrak{p}, \underline{\mathscr{F}}, T)_{|T^{d_\mathfrak{p}} = \mathfrak{p}^{-s}}\Bigg|_{s=(z,-jh_v,-jh_v)}$$

for $|z|_v > c$. For each $\mathfrak{p}$, the expression $L(\mathfrak{p}, \underline{\mathscr{F}}, T)$ is a power series in $A[[T^{d_\mathfrak{p}}]]$ with constant coefficient 1. As A is a Dedekind domain and whence has unique factorization of ideals, expanding the Euler product yields $b_{v,j} = \sum_{I\leq A} a_I I^{-s}|_{s=(z,-jh_v,-jh_v)}$ for unique elements $a_I \in A$ for each ideal I of A. Note that $a_I = 0$ if I is not relatively prime to $\mathfrak{p}_v$. Regarding the infinite valuation of the a_I, it is a consequence of Proposition 2.15 that there exists a constant $M \in \mathbb{N}$ such that $a_I \pi_\infty^{M \deg I} \in A_\infty$ for all $I \leq A$.

For each ideal I of A relatively prime to v, the element

$$g_{v,I} := \begin{cases} \langle I \rangle & \text{if } v = \infty, \\ u_{v,1}(I^{s_{\infty,1}}) & \text{otherwise} \end{cases}$$

is a 1-unit in $\mathbb{C}_v$. If $v \neq \infty$, then the order of the units of the residue field of $A_{v,\beta}$ divides h_v. Thus for any v, including $v = \infty$, and any ideal I of A which is prime to v, we have $I^{(1,-jh_v,-jh_v)} = g_{v,I}^{jh_v}$. Therefore, reordering the above expression for $b_{v,j}$ yields

$$b_{v,j} = \sum_{n=0}^{\infty} z^{-n} \sum_{I \leq A, \deg I = n} a_I g_{v,I}^{jh_v}.$$

Lemma 4.13 *There exists a constant $C_m > 0$ such that for all $l \in \mathbb{N}$, all $0 < k < p$ and $0 < j \leq p^l$*

$$||b_{v,j} - b_{v,j+kp^l}||_{q_v^{-m}} \leq C_m p^{-l}.$$

Proof: Using Corollary 4.6, we choose a constant $C \in \mathbb{N}$ such that $\deg b_{v,j} \leq C \log_p j$. Furthermore, we choose M such that $|a_I| \leq q_v^{M \deg I}$ for all ideals I of A. Then

$$b_{v,j} - b_{v,j+kp^l} = \sum_{n=0}^{C(l+1)} z^{-n} \sum_{I \leq A, \deg I = n} a_I g_{v,I}^{jh_v} \Big(1 - g_{v,I}^{kp^l h_v}\Big).$$

Because $g_{v,I}^{h_v}$ is in $U_1(K_v)$, it follows that $|1 - g_{v,I}^{kp^l h_v}|_v \leq q_v^{-p^l}$. Thus

$$||b_{v,j} - b_{v,j+kp^l}||_{q_v^{-m}} \leq (q_v^m)^{C(l+1)} (q_v^M)^{C(l+1)} q_v^{-p^l},$$

and the lemma follows if we choose $C_m := q^{\sup_l \{C(M+m+1)(l+1) - p^l\}}$. ∎

Lemma 4.14 *Given m, there exists $C_m > 0$ such that for all $j, j' \in \mathbb{N}$*

$$||b_{v,j} - b_{v,j'}||_{q_v^{-m}} \leq C_m |j - j'|_p.$$

PROOF: We claim that the constant C_m from the previous lemma suffices. First note that $||f+g||_c \leq \max\{||f||_c, ||g||_c\}$ for any $c > 0$ because $|\_\_|_v$ is ultrametric. In particular for all j, j', j'' one has

$$||b_{v,j} - b_{v,j'}||_c \leq \max\{||b_{v,j} - b_{v,j''}||_c, ||b_{v,j'} - b_{v,j''}||_c\}. \tag{6}$$

We consider an arbitrary pair j, j' and let p^l be the exact p-power divisor of $j - j'$. By (6), we may and will assume that j is the unique representative in $[1, p^l]$ that is congruent to j' modulo p^l. We write

$$j' = j + j_1 p^l + j_2 p^{l+1} + \ldots + j_s p^{l+s-1}$$

for integers $j_i \in [0, p-1]$. Again by (6), it follows that

$$\begin{aligned}&||b_{v,j} - b_{v,j'}||_{q_v^{-m}} \\ &\leq \max_{n'=0,\ldots,n-1} ||b_{v,j+j_1p^l+\ldots+j_{n'}p^{l+n'-1}} - b_{v,j+j_1p^l+\ldots+j_{n'+1}p^{l+n'}}||_{q_v^{-m}}.\end{aligned}$$

By the previous lemma the maximum is taken over positive numbers which are bounded by $C_m p^{-l}$, and the desired inequality follows. ∎

Theorem 4.15 *Let $\underline{\mathcal{F}}$ be a locally free A-crystal over* Spec A. *Then for any $v \in C$, the L-function $L^{(v)}(\underline{\mathcal{F}}, s)$, which is defined on some half plane of S_v, is an entire, essentially algebraic function on S_v. Furthermore, the degrees of the special values $H_{v,j}(z) \in \mathbb{C}[z^{-1}]$ grow like $O(\log j)$.*

PROOF: Choose $i \in \mathbb{N}$ such that q^i is larger then the p-part of h_v. By Proposition 2.27 and the previous lemma, the function $L^{(v)}(\underline{\mathcal{F}}, s)$ is entire on $q^i\mathbb{Z}_p \times \{0\} \subset W_v$ for any place v and any locally free τ-sheaf on $\underline{\mathcal{F}}$.

By the discussion above Proposition 2.20, one may view $\underline{\mathcal{F}} \otimes \underline{\mathcal{M}}_{\chi_{v,y}}$ as a locally free τ-sheaf in $\mathbf{Coh}_\tau(X(v), Ak_v)$. Arguing as above, its v-adic L-function must be entire on $q^i\mathbb{Z}_p \times \{0\} \subset W_v$. Proposition 2.21 now shows that $L^{(v)}(\underline{\mathcal{F}}, s)$ is entire on $q^i W_v$. As $\underline{\mathcal{F}}$ is locally free the same holds for $\underline{\mathcal{F}}^{(q^i)}$ as well, and the entireness of $L^{(v)}(\underline{\mathcal{F}}, s)$ follows from Proposition 2.30.

To show that $L^{(v)}(\underline{\mathcal{F}}, s)$ is essentially algebraic, let χ be any character of $G = \mathrm{Gal}(H^+/K)$. The above result shows that $L^{(v)}_\chi(\underline{\mathcal{F}}, s)$ is entire. By Corollary 3.8 we have

$$\prod_{\chi \in G} L^{(v)}_{\chi^j}(\underline{\mathcal{F}}, s_{v,-j}) = L^{(v)}(\underline{\mathcal{F}} \overset{L}{\otimes} \underline{\mathcal{Q}}_j, 0) \in A[T].$$

As the product of entire power series is a polynomial if and only all factors are polynomials, the $L^{(v)}_{\chi^j}(\underline{\mathscr{F}}, s_{v,-j})$ must be polynomials. By Theorem 4.10, it follows that their degrees grow like $O(\log j)$.

To complete the proof, we will now show that all coefficients of all special values lie in a finite extension of K. Recall that for $|z|_v \gg 1$, we have

$$L^{(v)}(\underline{\mathscr{F}}, s_{v,-j}) = \sum_{n=0}^{\infty} z^{-n} \sum_{I \leq A, \deg I = n} a_I I^{s_{v,-j}}.$$

This is a power series with coefficients in the finite extension $\mathbb{V}$ of K, independently of n. (The a_I are in A, but the $I^{s_{v,-j}}$ are only in $\mathbb{V}$.) Therefore the coefficients of all polynomials $H_{v,j}(z)$ must be in the finite extension $\mathbb{V}$ of K, and whence $L^{(v)}_{\chi}(\underline{\mathscr{F}}, s)$ is essentially algebraic. ∎

Corollary 4.16 *Let $(\underline{\mathscr{F}}^\bullet)$ be a bounded complex of A-crystals over an arbitrary A-scheme X of finite type. Then for any $v \in C$, the L-function $L^{(v)}((\underline{\mathscr{F}}^\bullet), s)$, defined on a half plane of S_v, has a meromorphic entirely algebraic continuation to S_v.*

Proof: This is an immediate consequence of the above Theorem and Proposition 2.23. ∎

4.3. Entireness

In Theorem 4.15, we did not just obtain meromorphy of v-adic L-functions but entireness. The following is a general criterion for a v-adic L-function to be entire. It generalizes the result given in [22].

Theorem 4.17 *Suppose X_{red} is an affine equi-dimensional Cohen-Macaulay variety of dimension e, cf. [15], III.7. Let v be a place of K, and X an A-scheme. If $\underline{\mathscr{F}} \in \mathbf{Crys}(X(v), A)$ is locally free, then $L^{(v)}(\underline{\mathscr{F}}, s)^{(-1)^{e-1}}$ is entire and essentially algebraic on S_v.*

Proof: By Proposition 2.9(d), we may assume that $X = X_{\mathrm{red}}$. Because $X(v)$ is the pullback of the affine open subscheme $\operatorname{Spec} A(v)$ of $\operatorname{Spec} A$ along the affine morphism $f : X \to \operatorname{Spec} A$, it is affine itself. As $X(v)$ remains Cohen-Macaulay, Corollary 1.47 together with Proposition 4.1 implies that

$$L^{(v)}(\underline{\mathscr{F}}, (z, 0, 0))^{(-1)^{e-1}} = L(X(v), \underline{\mathscr{F}}, T)^{(-1)^{e-1}}_{|T=z^{-1}} \in A[z^{-1}]$$

for any locally free $\underline{\mathscr{F}}$ on X. Corollary 3.8(a) then shows that

$$L^{(v)}(\underline{\mathscr{F}}, (z,0,0) - s_{v,2jh^+}))^{(-1)^{e-1}} \in A[z^{-1}] \quad \text{for all } j \in \mathbb{N}_0.$$

Using Corollary 2.23 one can find locally free A-crystals $\underline{\mathscr{G}}_m$, $m = 0, 1$, on $\operatorname{Spec} A$ such that

$$L^{(v)}(X, \underline{\mathscr{F}}, s)^{(-1)^{e-1}} L^{(v)}(\operatorname{Spec} A, \underline{\mathscr{G}}_0, s) = L^{(v)}(\operatorname{Spec} A, \underline{\mathscr{G}}_1, s).$$

By Corollary 4.6, the special values of the $L^{(v)}(\operatorname{Spec} A, \underline{\mathscr{G}}_m, s)$, $m = 0, 1$, at the negative integers $-2h^+j$ are polynomials in $A[z^{-1}]$ whose degrees grow like $O(\log j)$. As A is an integral domain, the degrees of the polynomials $L^{(v)}(\underline{\mathscr{F}}, (z,0,0) - s_{v,2jh^+})^{(-1)^{e-1}}$ grow like $O(\log j)$. The assertion now follows by an argument analogous to the proof of Theorem 4.15. ∎

Combining the above results with Corollary 2.12 yields Goss' conjecture:

Corollary 4.18 *Let $(\psi, \mathscr{L})$ be a Drinfeld-A-module on X and $(\underline{\mathscr{M}}, \mathrm{ch}_{\underline{\mathscr{M}}})$ an A-motive, each of fixed rank. Then the L-functions $L^{(v)}(\psi/X, s)$ and $L^{(v)}(\underline{\mathscr{M}}/X, s)$ are meromorphic and essentially algebraic. If furthermore X_{red} is an equi-dimensional Cohen-Macaulay variety of dimension e over k, then these L-functions raised to the power $(-1)^{e-1} \in \{\pm 1\}$ are entire.*

4.4. Local L-factors at places of bad reduction

We conclude this section with a somewhat informal discussion of two applications of the theory of τ-sheaves to questions that arise naturally in the context of L-functions of Drinfeld-A-modules or A-motives. The first is to local L-factors at places of bad reduction.

We fix a finite extension K' of K and a Drinfeld-A-module ψ over $\operatorname{Spec} K'$ or rank r. Let O' be the ring of integers of K' and C' the smooth projective curve over k corresponding to K'. It is well-known that at all but finitely many places of O' the Drinfeld-A-module ψ has good reduction. Let $\Sigma \subset \operatorname{Spec} O'$ be the exceptional set, let $O'_\Sigma \subset K'$ be the ring of regular functions on $\operatorname{Spec} O' \smallsetminus \Sigma$, and let ψ' be a Drinfeld-A-module on O'_Σ of rank r with generic fiber ψ.

Example 1.6(b) attaches functorially a τ-sheaf $\underline{\mathscr{M}}(\psi')$ to ψ' whose fiber at the generic point agrees with $\underline{\mathscr{M}}(\psi)$. Ignoring the places of Σ, we define

$$L^{(v)}_\Sigma(\psi, s) := L^{(v)}(\operatorname{Spec} O'_\Sigma, \underline{\mathscr{M}}(\psi'), s).$$

Theorem 4.15 implies that $L_{\Sigma}^{(v)}(\psi, s)$ is entire and essentially algebraic on S_v.

In analogy with the classical case of L-functions attached to abelian varieties over number fields, one would also like to have local L-factors at places in Σ. For this we need to quote some work of Gardeyn, [8], Prop. 1.13, at least in a simplified form for the specific situation at hand.

Proposition 4.19 (Gardeyn) *Let $\varnothing \neq U \subset C'$ be open and $\underline{\mathscr{F}} \in \mathbf{Coh}_\tau(U, A)$ be locally free. Then there exists a unique (up to isomorphism) locally free τ-sheaf $\underline{\mathscr{F}}^{\max}$ in $\mathbf{Coh}_\tau(C', A)$ with the following properties.*

(a) The restriction of $\underline{\mathscr{F}}^{\max}$ to U is isomorphic to $\underline{\mathscr{F}}$.
(b) If $\underline{\mathscr{F}}' \in \mathbf{Coh}_\tau(C', A)$ is locally free and satisfies (a), then there exists a unique monomorphism $\underline{\mathscr{F}}' \hookrightarrow \underline{\mathscr{F}}$ which is compatible with the isomorphism in (a).

The proof of the above proposition with K replacing A, which suffices for the discussion below, can be obtained by a construction similar to that of $\underline{\widetilde{\mathscr{L}}}_j$ in the proof of Lemma 4.7.

In the case we are interested, the above proposition yields a τ-sheaf $\underline{\mathscr{M}}(\psi')^{\max}$ in $\mathbf{Coh}_\tau(C', A)$ which is locally free of rank r. Using Remark 2.17, we define the *v-adic L-function of ψ* as

$$L^{(v)}(\psi, s) := L^{(v)}(\operatorname{Spec} O', \underline{\mathscr{M}}(\psi')^{\max}_{|\operatorname{Spec} O'}, s).$$

The main consequence of the above results, in particular of Theorem 4.15, for $L^{(v)}(\psi, s)$ is the following.

Corollary 4.20 *The v-adic L-function of ψ is entire and essentially algebraic on S_v.*

We still owe the reader a good reason for the above definition of $L^{(v)}(\psi, s)$, which is yet another result of Gardeyn, cf. [8] Thm. 4.12. Let $\mathfrak{p} \in \Sigma$ and choose a finite place v' of K which is not below $\mathfrak{p}$. Denote by $T_{v'}(\psi)$ be the v'-adic Tate-module of ψ, cf. [12], §4.10, considered as a Galois representation of the absolute Galois $G_{K_\mathfrak{p}}$ of $K_\mathfrak{p}$. Let $V_{v'}(\psi) := T_{v'}(\psi) \otimes_{A_{v'}} K_{v'}$ be the corresponding $K_{v'}[[G_{K_\mathfrak{p}}]]$-module. Denote by $I_\mathfrak{p}$ the inertia subgroup of $G_{K_\mathfrak{p}}$, by $\mathrm{Frob}_\mathfrak{p}$ the Frobenius endomorphism in $G_{K_\mathfrak{p}}/I_\mathfrak{p}$ and by $V_{v'}(\psi)_{I_\mathfrak{p}}$ the covariants of $V_{v'}(\psi)$ under $I_\mathfrak{p}$. Finally denote by $H^1_{v'}(\psi)$ the $A_{v'}$-dual of $T_{v'}(\psi)$ and by $H^1_{v'}(\psi)^{I_\mathfrak{p}}$ its submodule of $I_\mathfrak{p}$-invariants.

Proposition 4.21 (Gardeyn) *In the above situation one has*

$$\begin{aligned} L(\mathfrak{p}, \underline{\mathscr{M}}(\psi')^{\max}, T)^{-1} &= \det(1 - T^{d_\mathfrak{p}} \mathrm{Frob}_\mathfrak{p}^{-1} | V_{v'}(\psi)_{I_\mathfrak{p}}) \\ &= \det(1 - T^{d_\mathfrak{p}} \mathrm{Frob}_\mathfrak{p}^{-1} | H^1_{v'}(\psi)^{I_\mathfrak{p}}). \end{aligned}$$

This is precisely what one would expect from the classical situation.

The following tries to shed some light on the action of $\mathrm{Frob}_\mathfrak{p}$ on $V_{v'}(\psi)_{I_\mathfrak{p}}$. Let $\mathfrak{p}$ be as above and assume that the reduction $\bar{\psi}$ of ψ at $\mathfrak{p}$ is a Drinfeld module over $O'/\mathfrak{p}$ of rank $1 \le \bar{r} < r$. At first glance, one might expect that $L(\mathfrak{p}, \underline{\mathscr{M}}(\psi')^{\max}, T)$ agrees with $L(\mathfrak{p}, \underline{\mathscr{M}}(\bar{\psi}), T)$. However, quite the opposite is true and, considering the classical situation, this should not come unexpectedly. For example in the case of an elliptic curve $E/\mathbb{Q}$ with multiplicative reduction at a prime l, on has $L_p(E, T) = (1 \pm T)$, while at the same time, the action of Frobenius on the l^n-torsion points of the reduction at p, i.e. of the multiplicative group over k, is via the cyclotomic character.

To describe the action of $\mathrm{Frob}_\mathfrak{p}$ on $V_{v'}(\psi)_{I_\mathfrak{p}}$, we need the following result due to Drinfeld, cf. [4], which is analogous to Tate-uniformization: There exists a Drinfeld-module φ of rank $\bar{r}$ which has good reduction at $\mathfrak{p}$ and an exponential function $e_\psi^\mathfrak{p} \colon \mathbb{C}_\mathfrak{p} \to \mathbb{C}_\mathfrak{p}$ such that for any $a \in A$ the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & \Lambda_\mathfrak{p} \arrow[r] \arrow[d, "\varphi(a)"] & \mathbb{C}_\mathfrak{p} \arrow[r, "e_\psi^\mathfrak{p}"] \arrow[d, "\varphi(a)"] & \mathbb{C}_\mathfrak{p} \arrow[r] \arrow[d, "\psi(a)"] & 0 \\ 0 \arrow[r] & \Lambda_\mathfrak{p} \arrow[r] & \mathbb{C}_\mathfrak{p} \arrow[r, "e_\psi^\mathfrak{p}"] & \mathbb{C}_\mathfrak{p} \arrow[r] & 0. \end{tikzcd}$$

For every ideal I of A this induces a short exact sequence $0 \to \varphi[I] \to \psi[I] \to \Lambda_\mathfrak{p}/I\Lambda_\mathfrak{p} \to 0$ of $G_{K_\mathfrak{p}}$-modules, and therefore a short exact sequence

$$0 \longrightarrow V_{v'}(\varphi) \longrightarrow V_{v'}(\psi) \longrightarrow V_{v'}(\Lambda_\mathfrak{p}) \to 0$$

of $K_{v'}[[G_{K_\mathfrak{p}}]]$-modules. One can show that $V_{v'}(\psi)_{I_\mathfrak{p}} \cong V_{v'}(\Lambda_\mathfrak{p})_{I_\mathfrak{p}}$, while the Galois-action of $G_{K_\mathfrak{p}}$ on $\bar{\psi}$ is described by the action on $V_{v'}(\varphi)$. In particular, $L(\mathfrak{p}, \underline{\mathscr{M}}(\bar{\psi}), T)$ is unrelated to $L(\mathfrak{p}, \underline{\mathscr{M}}(\psi')^{\max}, T)$, affirming our remark above.

If desired one could also define an L-factor for the places above ∞, which is related to the action of the local Galois group at this place on the corresponding lattice in $\mathbb{C}_\infty$. The mystery that still remains is, what L-factors, if any, one should use at the place v. See [12], introduction to §9, for some discussion of this.

4.5. On trivial zeroes

Our second application will be to trivial zeros. We first recall the motivation that lead to their definition.

In the classical situation, e.g., for the L-function of a finite Hecke character over a number field, one has precise formulas for the order of vanishing at all negative integers. Once these are known, it is easy to get a hold of the leading term of the Taylor series of the L-function at negative integers, which is often an algebraic integer with arithmetic significance.

In the function field case one can make the following analogy. Suppose $L^{(v)}(s)$ is the v-adic L-series, say of some τ-sheaf, which is entire and essentially algebraic. At the negative integer j we have the function $f_j(z) := L^{(v)}((z,0,0)-s_{v,j})$ which is entire in z^{-1}. If we write f_j as a Taylor series in z^{-1} around $z=1$ and are interested in the leading term, the vanishing order of this series at $z=1$ is important. There is no general formula which describes this vanishing order, and only in some specific cases this has been computed explicitly, cf. [24]. However via congruences between $f_j(z)$ modulo $\mathfrak{p}_v$ and the mod p reduction of certain classical L-functions of characters for certain $L^{(v)}$, Goss is able to define a simple polynomial factor of f_j. It is the zeroes of this factor to which one refers as trivial zeros. For a more detailed discussion see [12], Ch. 8.

In some examples, we observed that these trivial zeros may also be explained using our cohomological viewpoint. We outline the idea for $v=\infty$ and $L^{(v)}(s) = \zeta^{(\infty)}_{\operatorname{Spec} A}(s)$. For this we define the functions $g_j(z) := L(\underline{\mathscr{H}}_A^{\otimes j}, z^{-1})$, $j \in \mathbb{N}_0$, which are polynomials in z^{-1}. By Corollary 3.8 these are related to $\zeta^{(\infty)}_{\operatorname{Spec} A}((z,0,0)-s_{v,j})$ via:

$$L(\underline{\mathscr{H}}_A^{\otimes j}, z^{-1}) = \zeta^{(\infty)}_{\operatorname{Spec} A}((z,0,0)-s_{v,j}) \cdot \prod_{\chi \in \hat{G} \smallsetminus \{1\}} L^{(\infty)}(\underline{\mathscr{M}}_{\bar{\chi}^j}, (z,0,0)-s_{v,j}). \tag{7}$$

Let C^+ be the smooth projective curve over k corresponding to H^+. In particular, the places of $C^+ \smallsetminus \operatorname{Spec} O^+$ are precisely those that map to the place ∞ of C under ξ, cf. Section 3. Using Proposition 4.19, we define the locally free τ-sheaf

$$\underline{\mathscr{H}}_{K,j}^{\max} := (\underline{\mathscr{H}}_A^{\otimes j})^{\max} \in \mathbf{Coh}_\tau(C^+, K).$$

Following the proof of Lemma 4.7, one can show that $\mathscr{H}_{K,j}^{\max}$ has negative degree. Theorem 1.45 then shows that $L(\underline{\mathscr{H}}_{K,j}^{\max}, z^{-1})$ is a polynomial in z^{-1}. Clearly we have

$$L(\underline{\mathscr{H}}_A^{\otimes j}, z^{-1}) = L(\underline{\mathscr{H}}_{K,j}^{\max}, z^{-1}) \cdot \prod_{v|\infty} \det_K(\mathrm{id} - z^{-1}\tau | (\underline{\mathscr{H}}_{K,j}^{\max})_v).$$

Hence the local L-factors of $\underline{\mathscr{H}}_{K,j}^{\max}$ at the places above ∞ give visibly rise to zeroes of $L(\underline{\mathscr{H}}_A^{\otimes j}, z^{-1})$. Some local computations for $\underline{\mathscr{H}}_A$ at the places above ∞ show that whenever j is divisible by $q-1$, each of the h places above ∞ contributes precisely one zero of $L(\underline{\mathscr{H}}_A^{\otimes j}, z^{-1})$.

Let h_0 be the least common multiple of $q-1$ and twice the exponent of the abelian group $\mathrm{Gal}(H^+/K)$. Then for all $j = lh_0$, $l \in \mathbb{N}$, formula (7) implies that

$$L(\underline{\mathscr{H}}_A^{\otimes j}, z^{-1}) = \Big(\zeta_{\operatorname{Spec} A}^{(\infty)}((z,0,0) - s_{v,j})\Big)^{h^+},$$

which yields at least one linear factor of $\zeta_{\operatorname{Spec} A}^{(\infty)}((z,0,0) - s_{v,j})$. Such a factor is also obtained by Goss' congruence calculations, and the results obtained above look very similar to [12], Rem. 8.13.10. In fact, we expect that the above gives a reinterpretation of the trivial zeroes obtained by Goss as L-factors arising from places above ∞. We plan to come back to this in future work.

We also compared our calculations with the results given by Thakur in [24] concerning the precise vanishing order of zeta-functions at negative integers near $z = 1$. This showed that the above method will not be able to recover Thakur's finer results.

5. An analytic proof of Goss' conjecture

Except for the theory of shtukas, cf. [12], Ch. 6, or [20], the concepts discussed in this section are all from [22], where L-functions of φ-sheaves are investigated analytically. Our main contribution to this is Theorem 5.11 below. We continue to assume that C is geometrically irreducible over k.

We define

$$X \hat{\otimes} A_v := \varinjlim_n X \times_{\operatorname{Spec} k} \operatorname{Spec} A_v/(\mathfrak{p}_v^n)$$

and correspondingly, $\mathscr{O}_X \hat{\otimes} A_v := \varprojlim_n \mathscr{O}_X \hat{\otimes}_k A_v/\mathfrak{p}_v^n$.

Definition 5.1 *A* v-adic φ-sheaf over A_v *on a scheme* X *is a pair* $\underline{\mathscr{E}} := (\mathscr{E}, \tau)$ *consisting of locally free sheaf* $\mathscr{E}$ *on* $X \hat{\otimes} A_v$ *and an* $\mathscr{O}_X \hat{\otimes} A_v$*-linear homomorphism*

$$(\sigma \times \mathrm{id})^* \mathscr{E} \xrightarrow{\tau} \mathscr{E}.$$

A v*-adic* φ*-sheaf is called lisse if* τ *is an isomorphism.*

Let $\underline{\mathscr{F}}$ be any locally free τ-sheaf $\underline{\mathscr{F}}$ on X with A- (or A_v-) coefficients. For $v \neq \infty$ the v-adic completion of $\underline{\mathscr{F}}$ gives rise to a v-adic φ-sheaf. For $v = \infty$ and $X = \operatorname{Spec} R$ affine, one may first replace $\underline{\mathscr{F}}$ by a nil-isomorphic free τ-module $((R \otimes A)^r, \tau)$, cf. Lemma 1.4, and then use Lemma 2.14 to see that $((R \otimes A_\infty)^r, \tau\pi_\infty^m)$ is a τ-sheaf on X over A_∞ for any $m \gg 0$. Via ∞-adic completion, this gives rise to a ∞-adic φ-sheaf attached to $\underline{\mathscr{F}}$ (in a non-unique way!). A more canonical way to attach an ∞-adic τ-sheaf to $\underline{\mathscr{F}}$ as above, is to work with ∞-adic τ-sheaves over $X\hat{\otimes}K_\infty$ and to first pass from $\underline{\mathscr{F}}$ to $\underline{\mathscr{F}} \otimes_A K$, and then to complete this ∞-adically, cf. [21], Rem. 7.2.

Given a v-adic φ-sheaf $\underline{\mathscr{E}}$ on any scheme X of finite type over k, one can attach an L-function $L(\underline{\mathscr{E}}, T) \in 1+TA_v[[T]]$ to it as in Definition 1.42. If X is furthermore an A-scheme, one may use Definition 2.8 to define its v-adic L-function $L^{(v)}(\underline{\mathscr{E}}, s)$, and as in Theorem 2.16 and the discussion preceding Remark 2.24, one can see that there exists a $c > 0$ such that $L^{(v)}(\underline{\mathscr{E}}, s)$ is a continuous function from W_v to $\mathrm{C}^{\mathrm{an}}(D_v(c))$. If $v \neq \infty$ and if $\underline{\mathscr{E}}$ arises via completion from a locally free τ-sheaf $\underline{\mathscr{F}}$ on X over A, then $L(\underline{\mathscr{F}}, T) = L(\underline{\mathscr{E}}, T)$, and the same equality holds for the corresponding v-adic L-functions if X is an A-scheme. If $v = \infty$, X is affine and $\underline{\mathscr{E}}$ is attached to $\underline{\mathscr{F}}$ via Lemmas 1.4 and 2.14 as above, then $L(\underline{\mathscr{E}}, T) = L(\underline{\mathscr{F}}, T\pi_\infty^m)$ and $L^{(\infty)}(\underline{\mathscr{E}}, (z, w, y)) = L^{(\infty)}(\underline{\mathscr{F}}, (z\pi_\infty^m, w, y))$ for some $m \in \mathbb{N}$. If one uses the construction $\underline{\mathscr{F}} \mapsto \underline{\mathscr{E}}$ from [21], Rem. 7.2, no fudge factor is needed.

Important for the meromorphy properties of $L(\underline{\mathscr{E}}, T)$ and $L^{(v)}(\underline{\mathscr{E}}, s)$ are the concepts of α log-convergence and overconvergence and the corresponding uniform notions. We now recall these from [21], §3. For this, let W be any compact topological space. We assume first that $X = \operatorname{Spec} R$ is affine. Let $x_1, \ldots, x_n \in R$ be a set of generators over k.

Given any element x of $R\hat{\otimes}A_v$, one can find a sequence $(c_{\underline{n}}) \subset A_v$, $\underline{n} \in \mathbb{N}_0^m$ such that

$$x = \sum_{\underline{n} \in \mathbb{N}^m} \underline{x}^{\underline{n}} \otimes c_{\underline{n}}$$

where $\underline{x} = (x_1, \ldots, x_m)$ and for $\underline{n} = (n_1, \ldots, n_m)$ we write $\underline{x}^{\underline{n}}$ for $x_1^{n_1} \cdot \ldots x_m^{n_m}$. We say that $(c_{\underline{n}})$ represents x (w.r.t. $\underline{x}$).

Definition 5.2 *A family $(x(w))_{w \in W}$ of elements $x(w) \in R$ is called* uniformly α log-convergent *if for each $w \in W$ there exists a sequence $(c_{\underline{n}}(w))$ in A_v such that*

(a) $x(w)$ is represented by the sequence $(c_{\underline{n}}(w))$, and

(b)

$$\liminf_{|\underline{n}| \to \infty} \frac{\inf_{w \in W} \operatorname{ord}_v c_{\underline{n}}(w)}{\log_q |\underline{n}|} \geq \alpha,$$

where $|\underline{n}| = n_1 + \ldots + n_m$.

The family $(x(w))$ *is called* uniformly overconvergent *if* $(c_{\underline{n}}(w))$ *as above exists such that*

$$\liminf_{|\underline{n}|\to\infty} \frac{\inf_{w\in W} \operatorname{ord}_v c_{\underline{n}}(w)}{|\underline{n}|} > 0.$$

Remark 5.3 (a) The notions of being uniformly α log-convergent or overconvergent, respectively, are independent of the choice of a generating set of R over k, because if $x'_1, \ldots, x'_{n'}$ is another such, then the x_i can be expressed as polynomials in the $x'_{i'}$ and vice versa.

(b) It is obvious that an overconvergent element is α log-convergent for all $\alpha \in \mathbb{R}_{\geq 0}$. Also an α log-convergent element is β log-convergent for all $\beta \in [0, \alpha]$.

(c) If W is the one-element set $\{w_0\}$ and $x = x(w_0)$, we will simply say that x is α log-convergent, respectively overconvergent, if the one-element family satisfies the corresponding uniform notion. This convention will always, without further mentioning, be applied when specializing a uniform notion to a one-element set.

(d) A family of matrices $(B(w))_{w\in W}$ in $M_r(R\hat{\otimes}A_v)$ is called uniformly α log-convergent, respectively uniformly overconvergent, if each component gives rise to such a family in $R\hat{\otimes}A_v$.

If $\underline{\mathscr{E}}$ is a v-adic φ-sheaf such that $\mathscr{E}$ is free over $R\hat{\otimes}A_v$ of rank r, then with respect to any basis $\mathfrak{B}$ of $\mathscr{E}$ over $R\hat{\otimes}A_v$, the operation τ is represented by a unique matrix $B_{\mathfrak{B}} \in M_r(R\hat{\otimes}A_v)$.

Definition 5.4 *A family* $(\underline{\mathscr{E}}(w))_{w\in W}$ *of free* v*-adic* φ*-sheaves of rank* r *is called* uniformly α log-convergent *if one can choose a basis* $\mathfrak{B}(w)$ *for each* w *such that the family of matrices* $B_{\mathfrak{B}(w)}$ *is uniformly* α log-*convergent.*

The family is called uniformly overconvergent *if one can choose a family of basis* $(\mathfrak{B}(w))_{w\in W}$ *such that* $(B_{\mathfrak{B}(w)})_{w\in W}$ *is uniformly overconvergent.*

The generalization of the previous definition to an arbitrary base scheme X of finite type over k is as follows:

Definition 5.5 *Let* $(\underline{\mathscr{E}}(w))_{w\in W}$ *be a family of* v*-adic* φ*-sheaves on* X*. This family is called* uniformly α log-convergent *if there exists a finite affine cover* U_i *of* X *such that for each* i *the restriction of* $(\underline{\mathscr{E}}(w))_{w\in W}$ *to* U_i *is free and uniformly* α log-*convergent.*

Analogously, one defines the notion of a uniformly overconvergent family of v*-adic* φ*-sheaves.*

Remark 5.6 (a) Suppose that $X = \operatorname{Spec} R$ is affine and that $\underline{\mathscr{F}}$ is a φ-sheaf. By Lemma 1.4, which is Trick (2.2) of [21], there exists a free τ-sheaf $\underline{\mathscr{F}}'$ which is nil-isomorphic to $\underline{\mathscr{F}}$. With respect to some basis, one can represent $\tau_{\underline{\mathscr{F}}'}$ by a matrix over $R \otimes A$. As elements of $R \otimes A_v$ are overconvergent in $R\hat{\otimes}A_v$, the v-adic τ-sheaf attached to $\underline{\mathscr{F}}'$ is an overconvergent v-adic φ-sheaf $\underline{\mathscr{E}}$ whose v-adic L-function is the same as that of $\underline{\mathscr{F}}$, except for a fudge factor if $v = \infty$ — for $v = \infty$ see the discussion below Definition 5.1.

For φ-sheaves on general X, it seems unknown whether such an $\underline{\mathscr{E}}$ always exists. However if one only studies meromorphy properties of L-functions, the global existence of such an $\underline{\mathscr{E}}$ is not important as one can always write X as a finite disjoint union of locally closed affine subschemes X_i and, correspondingly, obtain the L-function of $\underline{\mathscr{F}}$ on X as the product of the L-functions of $\underline{\mathscr{F}}$ on all X_i.

(b) If $(\underline{\mathscr{E}}(w))_{w\in W}$ and $(\underline{\mathscr{E}}'(w'))_{w'\in W'}$ are both uniform α log-convergent, respectively overconvergent families, then so is their tensor product $(\underline{\mathscr{E}}(w) \otimes \underline{\mathscr{E}}'(w'))_{(w,w')\in W\times W'}$.

The importance of the above notions stems from the following theorem due to Taguchi and Wan, cf. [21], Thm. 5.2, and [22], Thm. 4.1:

Theorem 5.7 *Suppose that X is smooth affine equi-dimensional over k of dimension e and $(\underline{\mathscr{E}}(w))_{w\in W}$ is a uniformly α* log-*convergent family of v-adic φ-sheaves. Then $w \mapsto L(\underline{\mathscr{E}}(w), z^{-1})^{(-1)^{e-1}}$ is a continuous map from W to $\mathrm{C}^{\mathrm{an}}(D_v(q_v^{-\alpha}))$. If the family is uniformly overconvergent, then this assignment gives a continuous map $W \to \mathrm{C}^{\mathrm{an}}(D_v(0))$.*

Using Theorem 4.17, the proof of [22], Thm. 4.1, yields:

Corollary 5.8 *The assertion of the above theorem holds under the weaker assumption that X_{red} is affine, Cohen-Macaulay and equi-dimensional over k of dimension e.*

In [21], the above was applied to prove meromorphy for the v-adic L-function of a v-adic φ-sheaf in the case $A = k[t]$. The idea is, to p-adically interpolate $(\underline{\mathscr{P}}^{\otimes n})_{n\in\mathbb{N}}$ by a uniformly overconvergent family of φ-sheaves $(\underline{\widetilde{\mathscr{P}}}_v(w))_{w\in\mathbb{Z}_p}$ of rank 1. Then the global v-adic L-function of any v-adic φ-sheaf $\underline{\mathscr{E}}$ is the map

$$w \mapsto L(\underline{\mathscr{E}} \otimes \underline{\widetilde{\mathscr{P}}}_v(w), z^{-1}).$$

For $A = k[t]$, so that $\underline{\mathscr{P}}$ is simply the Carlitz τ-sheaf $\underline{\mathscr{C}}$, such a family is given explicitly in [21].

Recall that $h_v = 2h^+$ if $v = \infty$ and $h_v = 2h^+(q_{v,\beta_v} - 1)$ otherwise.

Definition 5.9 *For v a finite place, we define $\underline{\mathscr{P}}_v$ to be the v-adic φ-sheaf on* $\operatorname{Spec} A(v)$ *associated to $\underline{\mathscr{P}}^{\otimes h_v/h_\infty}$.*

The construction at ∞ requires more effort. However once $\underline{\mathscr{P}}_\infty$ is constructed, it is as simple to handle as all the other $\underline{\mathscr{P}}_v$. We need the following proposition whose proof will be given later.

Proposition 5.10 *There exists a line bundle $\mathscr{L}$ on* $\operatorname{Spec} A \times C$ *and a morphism*

$$\tilde{\tau}\colon (\sigma \times \mathrm{id})^* \mathscr{L} \to \mathscr{L}(h_\infty/d_\infty(\operatorname{Spec} A \times \{\infty\})),$$

which satisfies the following conditions:

(a) The restriction of $\underline{\mathscr{L}} := (\mathscr{L}, \tilde{\tau})$ to $\operatorname{Spec} A \otimes A$ *is isomorphic to $\underline{\mathscr{P}}$.*
(b) If $\mathscr{P}_\infty$ denotes the completion of $\mathscr{L}$ at ∞ and if we set $\tau_\infty := \pi_\infty^{h_\infty/d_\infty} \tilde{\tau}$, then $\underline{\mathscr{P}}_\infty$ is a lisse ∞-adic φ-sheaf.

Our main result concerning the $\underline{\mathscr{P}}_v$ is the following:

Theorem 5.11 *There exists a family $(\widetilde{\underline{\mathscr{P}}}_v(w))_{w \in \mathbb{Z}_p}$ of free v-adic φ-sheaves of rank 1 which is overconvergent and such that for each $n \in \mathbb{N}_0$ the local L-factors of $\underline{\mathscr{P}}_v^{\otimes n}$ and of $\widetilde{\underline{\mathscr{P}}}_v(-n)$ agree for all $\mathfrak{p} \neq \mathfrak{p}_v$.*

The proof will be an immediate consequence of Theorem 5.18. Before giving details, we will derive various consequences for the meromorphy and holomorphy of the v-adic L-functions of v-adic φ-sheaves:

Corollary 5.12 *Suppose X is a scheme of finite type over A and $\underline{\mathscr{E}}$ is a v-adic φ-sheaf on X which is α* log*-convergent, respectively overconvergent.*

(a) If X_{red} is affine Cohen-Macaulay and equi-dimensional of dimension e over k, then the function $L^{(v)}(\underline{\mathscr{E}}, s)^{(-1)^{e-1}}$ is a continuous map from W_v to $\mathrm{C}^{\mathrm{an}}(D_v(q_v^{-\alpha}))$, respectively to $\mathrm{C}^{\mathrm{an}}(D_v(0))$.
(b) For general X, the function $L^{(v)}(\underline{\mathscr{E}}, s)$ maps W_v continuously into the quotient field of $\mathrm{C}^{\mathrm{an}}(D_v(q_v^{-\alpha}))$, respectively that of $\mathrm{C}^{\mathrm{an}}(D_v(0))$, with the further property that there exists a neighborhood of ∞_v on which the function is holomorphic and takes values near 1.

By Remark 5.6 and Corollary 2.12, this gives a second proof of Goss' conjecture.

If in part (a), one assumes X to be smooth (or at least an affine complete intersection), then the corollary can be obtained without making use of any of the results of Sections 1 or 4. It is a consequence entirely of Theorem 5.11 and the work of Taguchi and Wan, [21] and [22].

PROOF: We only give the proof in the α log-convergent case and for $v \neq \infty$, the other cases being analogous. Note that we may clearly assume that $X = X_{\mathrm{red}}$, as passing from X to X_{red} does not affect $L^{(v)}(\underline{\mathscr{E}}, s)$.

Suppose first that $X = \operatorname{Spec} R$ is affine, Cohen-Macaulay and equidimensional of dimension e over k, and let $f\colon X \to \operatorname{Spec} A$ be the structure morphism. Theorem 5.11 implies that the family $(f^*\underline{\widetilde{\mathscr{P}}}_v(w))_{w\in\mathbb{Z}_p}$ is uniformly overconvergent on X. Thus $(\underline{\mathscr{E}} \otimes f^*\underline{\widetilde{\mathscr{P}}}_v(w))_{w\in\mathbb{Z}_p}$ is uniformly α log-convergent on X. Therefore by Corollary 5.8, the following map, which we denote by a, is continuous:

$$w \mapsto L(\underline{\mathscr{E}} \otimes f^*\underline{\widetilde{\mathscr{P}}}_v(w), z^{-1})^{(-1)^{e-1}} : \mathbb{Z}_p \to \mathrm{C}^{\mathrm{an}}(D_v(q_v^{-\alpha})).$$

As remarked, when defining the v-adic L-function of a v-adic φ-sheaf, there exists $c > 1$ such that $L^{(v)}(\underline{\mathscr{E}}, s)\colon W_v \to \mathrm{C}^{\mathrm{an}}(D_v(c))$ is continuous. The proof, which is essentially that of Theorem 2.16, shows that near ∞_v this function takes values near 1. Therefore also $L^{(v)}(\underline{\mathscr{E}}, s)^{(-1)^{e-1}}$ is a continuous function from W_v to $\mathrm{C}^{\mathrm{an}}(D_v(c))$. Hence the composition

$$\mathbb{Z}_p \xrightarrow{w\mapsto(h_v w,0)} W_v \xrightarrow{(w,y)\mapsto L^{(v)}(\underline{\mathscr{E}},(z,w,y))^{(-1)^{e-1}}} \mathrm{C}^{\mathrm{an}}(D_v(c)),$$

denoted a', is a continuous map $\mathbb{Z}_p \to \mathrm{C}^{\mathrm{an}}(D_v(c))$.

Theorem 5.11 together with Definition 5.9, Lemma 3.5 and Corollary 3.8 shows that the maps a and a' agree on $-\mathbb{N}_0$. As this set is dense in $\mathbb{Z}_p$, by continuity one has $a = a'$. Thus we have shown that $L^{(v)}(\underline{\mathscr{E}}, s)^{(-1)^{e-1}}$ when restricted to $h_v\mathbb{Z}_p \times \{0\}$ is a continuous map to $\mathrm{C}^{\mathrm{an}}(D_v(q_v^{-\alpha}))$. Now one can follow the proof of Theorem 4.15 to complete the proof of part (a).

For (b), we first note that one may pass from X to X_{red} without changing neither the associated v-adic L-function nor the property of being α log-convergent. In this case, we may break up X, as in the proof of Proposition 1.32, into locally closed smooth affine schemes X_i' over k so that $L^{(v)}(X, \underline{\mathscr{E}}, s) = \prod_i L^{(v)}(X_i', \underline{\mathscr{E}}, s)$. The assertion of (b) is now a direct consequence of (a). ∎

As a preparation for the proof of Theorem 5.11, we first need to establish some properties of the φ-sheaves $\underline{\mathscr{P}}_v$.

Lemma 5.13 *For a place $v \neq \infty$, the v-adic φ-sheaf $\underline{\mathscr{P}}_v$ is lisse over* $\operatorname{Spec} A(v)$.

PROOF: To show that $\tau_{\mathscr{P}_v}$ is an isomorphism, it suffices to do so over each fiber $\operatorname{Spec} k_{\mathfrak{p}}$ of $\operatorname{Spec} A(v)$, and it suffices to do this for an iterate of $\tau_{\mathscr{P}_v}$. From Corollary 3.7(a) we know that $\tau_{\mathscr{P}}^{d_{\mathfrak{p}}} = \mathfrak{p}^{s_{\infty,2h^+}}$ on the fiber at $\mathfrak{p}$. For any $\mathfrak{p}$ different from $\mathfrak{p}_v$, it follows that $\tau_{\mathscr{P}_v}^{d_{\mathfrak{p}}} = \mathfrak{p}^{s_{\infty,h_v}}$ is a 1-unit of K_v. Hence $\tau_{\mathscr{P}_v}$ is an isomorphism. ∎

PROOF of Proposition 5.10: Let $\psi_H : A \to O^+\{\tau\}$ be the Drinfeld-Hayes module from Section 3. To understand the behavior of $\underline{\mathscr{H}}_A$ near ∞, we will make use of the shtuka attached to ψ_H, which we construct following [12], Ch. 6. In the terminology of [20], this shtuka will be a pure Drinfeld-Anderson sheaf of rank 1 with pole ∞ and dimension 1 over $\operatorname{Spec} O^+$. This we push down to a family on $\operatorname{Spec} A$, and then take its highest exterior power. The result will be $(\mathscr{L}, \tau)$. It will then not be difficult to establish the properties claimed in the proposition.

For the first construction, one can follow [12], §6.2, Data B → Data A, where L is to be replaced by O^+, L_0 by k and R by A. One obtains sheaves $\mathscr{F}_i$, $i \in \mathbb{Z}$, on $\operatorname{Spec} O^+ \times C$ and monomorphisms

$$\beta_i \colon \mathscr{F}_i \longrightarrow \mathscr{F}_{i+1} \quad \text{and} \quad \alpha_i \colon (\sigma \times \mathrm{id})^* \mathscr{F}_i \longrightarrow \mathscr{F}_{i+1}.$$

Furthermore this construction shows that restricted to $\operatorname{Spec} O^+ \times \operatorname{Spec} A$, one has $\beta_i = \mathrm{id}_{\mathscr{H}_A}$ and $\alpha_i = \tau_{\mathscr{H}_A}$, independently of i.

Moreover, this construction is compatible with the base change map $i_{\mathfrak{P}} \colon \operatorname{Spec} O^+/\mathfrak{P} \to \operatorname{Spec} O^+$ for $\mathfrak{P} \in \operatorname{Max}(O^+)$. The results in [12], §6.2, are directly applicable to $i_{\mathfrak{P}}^* \mathscr{F}_i$, $i_{\mathfrak{P}}^* \beta_i$ and $i_{\mathfrak{P}}^* \alpha_i$. This yields the following:

(a) The sheaves $\mathscr{F}_i$ are locally free of rank one on $\operatorname{Spec} O^+ \times C$.
(b) Define $\gamma_\infty : \operatorname{Spec} O^+ \to \operatorname{Spec} O^+ \times C : s \mapsto (s, \infty)$, using $\operatorname{Spec} O^+ \to \operatorname{Spec} k_\infty \to C$. Then there is a locally free rank one sheaf $\mathscr{C}_{i+1}$ on $\operatorname{Spec} O^+$ such that $\operatorname{Coker}(\beta_i) \cong \gamma_{\infty *} \mathscr{C}_{i+1}$. In fact, by the construction alluded to above one can identify $\mathscr{C}_i$ with $\mathscr{O}_{\operatorname{Spec} O^+}$, independently of i.
(c) Let $\xi \colon \operatorname{Spec} O^+ \to \operatorname{Spec} A \hookrightarrow C$ be the map from Section 3 and define

$$\gamma_Z \colon \operatorname{Spec} O^+ \to \operatorname{Spec} O^+ \times C : s \mapsto (s, \xi(s)).$$

Then there is a locally free rank one sheaf $\mathscr{Z}_{i+1}$ on $\operatorname{Spec} O^+$ such that $\operatorname{Coker}(\alpha_i) \cong \gamma_{Z*} \mathscr{Z}_{i+1}$.
(d) The map $\beta_{i+d_\infty-1} \ldots \beta_{i+1} \beta_i \colon \mathscr{F}_i \longrightarrow \mathscr{F}_{i+d_\infty}$ identifies $\mathscr{F}_{i+d_\infty}$ with the sheaf $\mathscr{F}_i(\operatorname{Spec} O^+ \times \{\infty\})$.
(e) For each $\mathfrak{P} \in \operatorname{Max}(O^+)$, the Euler characteristic of $i_{\mathfrak{P}}^* \mathscr{F}_0$ is zero.

We define $\mathscr{G}_i$ to be $(\Lambda^{2h^+}(\xi \times \mathrm{id}_C)_*(\mathscr{F}_i \oplus \mathscr{F}_i))$. By functoriality, the β_i induce monomorphisms $\beta'_i \colon \mathscr{G}_i \to \mathscr{G}_{i+1}$, and the α_i monomorphisms $\alpha'_i : (\sigma \times \mathrm{id})^* \mathscr{G}_i \to \mathscr{G}_{i+1}$. The above description of $\mathscr{F}_i$ on $\operatorname{Spec} O^+ \times \operatorname{Spec} A$ yields that $\beta'_i = \mathrm{id}_{\mathscr{P}}$ and $\alpha'_i = \tau_{\mathscr{P}}$ on $\operatorname{Spec} A \times \operatorname{Spec} A$. It is now straightforward to translate the above properties to the $\mathscr{G}_i$ to obtain.

(a') The sheaves $\mathscr{G}_i$ are locally free of rank one over $\operatorname{Spec} A \times C$.

(b') The sheaf $\mathscr{C}'_{i+1}$ defined by the short exact sequence

$$0 \longrightarrow \mathscr{G}_i \xrightarrow{\beta'_i} \mathscr{G}_{i+1} \longrightarrow \mathscr{C}'_{i+1} \longrightarrow 0$$

is supported on $\operatorname{Spec} A \times \{\infty\}$. If $p_\infty \colon \operatorname{Spec} A \times C \to \operatorname{Spec} A$ denotes the canonical surjection, then $p_{\infty *}\mathscr{C}'_{i+1}$ is locally free on $\operatorname{Spec} A$ of rank $2h^+$.

(c') Let I be the kernel of $A \otimes A \to A$ and define γ'_Z as

$$\operatorname{Spec}(A \otimes A)/I^{h_\infty} \to \operatorname{Spec} A \otimes A \to \operatorname{Spec} A \times C.$$

Then there is a locally free rank one sheaf $\mathscr{K}'_{i+1}$ on $\operatorname{Spec}(A \otimes A)/I^{h_\infty}$ such that $\operatorname{Coker}(\alpha'_i) \cong \gamma'_{Z*}\mathscr{K}'_{i+1}$.

Lemma 5.14 *Suppose we are given a short exact sequence of sheaves*

$$0 \longrightarrow \mathscr{L}' \longrightarrow \mathscr{L} \longrightarrow \mathscr{Q} \longrightarrow 0 \tag{8}$$

on $\operatorname{Spec} A \times C$, *where* $\mathscr{L}'$, $\mathscr{L}$ *are line bundles,* $\mathscr{Q}$ *is supported on* $\operatorname{Spec} A \times \{\infty\}$ *and* $p_{\infty *}\mathscr{Q}$ *is locally free on* $\operatorname{Spec} A$ *of rank* d. *Then* $d_\infty | d$ *and under the left inclusion of (8), one has*

$$\mathscr{L}' = \mathscr{L}\Big(-\frac{d}{d_\infty}(\operatorname{Spec} A \times \{\infty\})\Big).$$

PROOF: We induct on the rank of $\mathscr{Q}$ over $\operatorname{Spec} A$. The assertion is local near ∞, so we choose an affine $\operatorname{Spec} A' \subset C$ which contains ∞. Let

$$0 \longrightarrow L' \longrightarrow L \longrightarrow Q \longrightarrow 0$$

be the short exact sequence of $A \otimes A'$-modules corresponding to the restriction of (8) to $\operatorname{Spec} A \otimes A'$, and let $\mathfrak{p}_\infty$ be the maximal ideal of A' that gives rise to ∞. If we tensor this short exact sequence with $k_\infty \cong A'/\mathfrak{p}_\infty$ over A', we obtain the right exact sequence

$$L' \otimes_{A'} k_\infty \longrightarrow L \otimes_{A'} k_\infty \longrightarrow Q \otimes_{A'} k_\infty \longrightarrow 0$$

on the Dedekind domain $A \otimes k_\infty$ (note that C is geometrically irreducible). The module $L \otimes_{A'} k_\infty$ is projective of rank one over $A \otimes k_\infty$. Therefore it is either isomorphic to its quotient $Q \otimes_{A'} k_\infty$, or the quotient has finite support. The latter is absurd, since Q is projective on A of rank d. It follows that the morphism $L' \otimes_{A'} k_\infty \to L \otimes_{A'} k_\infty$ is the zero map, and hence that $\mathfrak{p}_\infty L \supset L'$.

Define Q_1 by the short exact sequence $0 \to L' \to \mathfrak{p}_\infty L \to Q' \to 0$. Comparing this sequence to the short exact sequence with L as its middle term, the snake lemma yields the short exact sequence $0 \to Q' \to Q \to L/\mathfrak{p}_\infty L \to 0$, and hence Q' is projective over A of rank $d' = d - d_\infty$. By the induction hypothesis, we have $d_\infty | d'$ and $L' = \mathfrak{p}_\infty^{d'/d_\infty}(\mathfrak{p}_\infty L)$, and the lemma easily follows. ∎

By the lemma, the map β_i' identifies $\mathscr{G}_{i+1}$ with $\mathscr{G}_i(2\tilde{h}(\operatorname{Spec} A \times \{\infty\}))$ where $\tilde{h} = h^+/d_\infty$. By (c') above, the cokernel of

$$\alpha_i' \colon (\sigma \times \mathrm{id})^* \mathscr{G}_i \longrightarrow \mathscr{G}_{i+1} \cong \mathscr{G}_i(2\tilde{h}(\operatorname{Spec} A \times \{\infty\}))$$

has its support away from $\operatorname{Spec} A \times \{\infty\}$. So if we complete along $\operatorname{Spec} A \times \{\infty\}$ and specify $i = 0$, we obtain:

$$\pi_\infty^{2\tilde{h}} \alpha_0' \colon (\sigma \times \mathrm{id})^* (\mathscr{G}_0)_\infty \xrightarrow{\cong} (\mathscr{G}_0)_\infty .$$

Therefore with $\mathscr{L} := \mathscr{G}_0$ and $\tilde{\tau} := \alpha_0'$ the proposition follows. ∎

Lemma 5.15 *The sheaf $\mathscr{P}_v$ is free of rank one over* $\operatorname{Spec} A(v) \hat{\otimes} A_v$ *for any place v.*

Proof: Note that the radical of $A(v)\hat{\otimes} A_v$ contains $A(v)\hat{\otimes}\mathfrak{p}_v A_v$, cf. the characterization of the radical of a ring given in [17], p. 3. Thus by Nakayama's Lemma, [17], Thm. 2.2, it suffices to show that $\mathscr{P}_v$ modulo $A(v)\hat{\otimes}\mathfrak{p}_v A_v$ is free of rank one over $A(v) \otimes k_v$.

By Theorem 2.7, the reduction modulo $A(v)\hat{\otimes}\mathfrak{p}_v A_v$ of the lisse φ-sheaf $\underline{\mathscr{P}}_v$ corresponds to a Galois representation $\rho \colon \operatorname{Gal}(K^{\mathrm{sep}}/K) \to k_v^*$ such that for all $\mathfrak{p} \neq \mathfrak{p}_v$, the element $\rho(\mathrm{Frob}_\mathfrak{p})$ is the mod $\mathfrak{p}_v$-reduction of the eigenvalue of $\tau_{\mathscr{P}_v}^{d_\mathfrak{p}}$ acting on $i_\mathfrak{p}^* \mathscr{P}_v$. By Corollary 3.7 and the definition of h_v as $2h^+(q_{v,\beta_v} - 1)$, we have

$$\begin{aligned} \rho(\mathrm{Frob}_\mathfrak{p}) = \mathfrak{p}^{s_{v,h_v}} \pmod{\mathfrak{p}_v} \;&=\; (\mathfrak{p}^{s_{v,2h^+}} \pmod{\mathfrak{p}_v})^{(q_{v,\beta_v}-1)} \\ = (g_\mathfrak{p}' \pmod{\mathfrak{p}_v})^{(q_{v,\beta_v}-1)} \;&=\; 1, \end{aligned}$$

where $g_\mathfrak{p}' \in A$ is the unique positive generator of the ideal $\mathfrak{p}^{2h^+}$. Hence by the Cebotarov density theorem, ρ is the trivial representation. Via the correspondence in Theorem 2.7, the τ-sheaf $\underline{\mathbb{1}}_{\operatorname{Spec} A(v), k_v}$ also gives rise to the trivial Galois representation. Appealing to the uniqueness statement of Theorem 2.7, the result follows. ∎

Proposition 5.16 *There exists for each v an element $u_v \in A(v) \otimes A_v$ such that*

(a) $u_v - 1 \in A(v) \otimes \mathfrak{p}_v A_v$.

(b) The local L-factors of $\underline{\mathscr{P}}_v$ *and of* $(A(v)\hat{\otimes}A_v, u_v(\sigma \times \mathrm{id}))$ *agree for all* $\mathfrak{p} \neq \mathfrak{p}_v$.

Proof: We first consider the case $v \neq \infty$. Let $\mathscr{L}$ be a locally free sheaf on $\mathrm{Spec}(A \otimes A)$ such that $\mathscr{L} \oplus \mathscr{P}^{\otimes h_v/h_\infty}$ is free of rank r. Define $\underline{\mathscr{L}}$ as the τ-sheaf $(\mathscr{L}, 0)$. Let L_v be the $A(v)\hat{\otimes}A_v$ module corresponding to the completion of $\mathscr{L}$.

The previous lemma and the choice of $\mathscr{L}$ imply that reduction modulo $A(v)\hat{\otimes}\mathfrak{p}_v A_v$ yields

$$L_v \pmod{A(v)\hat{\otimes}\mathfrak{p}_v A_v} \oplus (A(v) \otimes k_v) \cong (A(v) \otimes k_v)^{\oplus r}.$$

As C is geometrically irreducible, $A(v) \otimes k_v$ is a Dedekind domain. Thus, by [3], § 7.4, the module $L_v \pmod{\mathfrak{p}_v}$ must be free of rank $r-1$. Arguing as in Lemma 5.15, this implies that L_v is free of rank $r-1$. Hence with respect to a suitable basis we may represent $\tau_{\mathscr{P}_v \oplus \mathscr{L}_v}$ as a diagonal matrix M'_v with entries $(u'_v, 0, 0, \ldots, 0)$ along the diagonal.

Furthermore, $\mathscr{L}$ was chosen so that $\mathscr{L} \oplus \mathscr{P}^{\otimes h_v/h_\infty}$ is free of rank r. Therefore, we can represent τ as an $r \times r$-matrix M with entries in $A \otimes A$. The same matrix also represents $\tau_{\mathscr{P}_v \oplus \mathscr{L}_v}$. Hence we can find $N \in \mathrm{GL}_r(A(v)\hat{\otimes}A_v)$ such that

$$N^{-1} M N^{\sigma} = M' = \begin{pmatrix} u'_v & 0 & \ldots \\ 0 & 0 & \ldots \\ \ldots & \ldots & \ldots \end{pmatrix},$$

where N^{σ} is obtained from N by acting with $\sigma \times \mathrm{id}$ on all entries. This implies that

$$N^{-1} M N = M' N^{-\sigma} N = \begin{pmatrix} c_{1,1} & \ldots & c_{1,n} \\ 0 & \ldots & 0 \\ \ldots & \ldots & \ldots \end{pmatrix}.$$

for suitable $c_{1,1}, \ldots, c_{1,n} \in A\hat{\otimes}A_v$. We define $u_v := c_{1,1}$ and note that

$$u_v = \mathrm{Tr}(M' N^{-\sigma} N) = \mathrm{Tr}(N^{-1} M N) = \mathrm{Tr}(M) \in A \otimes A \subset A(v) \otimes A_v.$$

Lemma 5.17 *For $\alpha \in M_r(A(v)\hat{\otimes}A_v)$ and $\beta \in \mathrm{GL}_r(A(v)\hat{\otimes}A_v)$, define v-adic φ-sheaves*

$$\underline{\mathscr{G}} := ((A(v)\hat{\otimes}A_v)^r, \alpha(\sigma\hat{\otimes}\mathrm{id})), \underline{\mathscr{G}'} := ((A(v)\hat{\otimes}A_v)^r, \alpha\beta^{-\sigma}\beta(\sigma\hat{\otimes}\mathrm{id})).$$

Then for any $\mathfrak{p} \neq \mathfrak{p}_v$ one has

$$L(\mathfrak{p}, \underline{\mathscr{G}}, T) = L(\mathfrak{p}, \underline{\mathscr{G}'}, T).$$

Proof: Fix $\mathfrak{p} \neq \mathfrak{p}_v$ and denote by $\bar{\alpha}$ the reduction of α modulo $\mathfrak{p}\hat{\otimes} A_v$. One defines $\bar{\beta}$ and $\bar{\beta}^\sigma$ analogously. Note that $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\beta}^\sigma$ are matrices over $k_\mathfrak{p}\hat{\otimes} A_v \cong k_\mathfrak{p} \otimes A_v$, as $k_\mathfrak{p}$ is finite over k. Furthermore note that in Definition 1.40 of the local L-factor there is no need to pass to K_v-coefficients whenever $\mathscr{F}_x$ is already free over $k_x \otimes A_v$. It follows that

$$L(\mathfrak{p}, \underline{\mathscr{G}}', T)^{-1} \stackrel{\text{def}}{=} \det_{A_v}(1 - T\bar{\alpha}\bar{\beta}^{-\sigma}\bar{\beta}) = \det_{A_v}(1 - T\bar{\beta}\bar{\alpha}\bar{\beta}^{-\sigma}),$$

where for the second equality we use that conjugation by the A_v-linear operator $\bar{\beta}$ does not change determinants computed over A_v.

Because β is an isomorphism, $\underline{\mathscr{G}}'' := ((A(v)\hat{\otimes} A_v)^r, \beta\alpha\beta^{-\sigma}(\sigma\hat{\otimes}\mathrm{id}))$ is isomorphic to $\underline{\mathscr{G}}$, so that both have the same local L-factor at $\mathfrak{p}$. But the inverse local L-factor of $\underline{\mathscr{G}}''$ at $\mathfrak{p}$ is $\det_{A_v}(1 - T\bar{\beta}\bar{\alpha}\bar{\beta}^{-\sigma})$, and hence the assertion of the lemma is shown. ∎

The lemma shows that $\underline{\mathscr{P}}_v \oplus \underline{\mathscr{L}}_v$ and

$$\left((A(v)\hat{\otimes} A_v)^r, \begin{pmatrix} c_{1,1} & \dots & c_{1,n} \\ 0 & \dots & 0 \\ \dots & \dots & \dots \end{pmatrix} (\sigma \times \mathrm{id}) \right)$$

have the same local L-factors. Therefore $\underline{\widetilde{\mathscr{P}}}_v(-1) := (A(v)\hat{\otimes} A_v, u_v(\sigma \times \mathrm{id}))$ and $\underline{\mathscr{P}}_v$ must have the same local L-factors. Following the proofs of Lemma 5.13 and Lemma 5.15, this shows that $\underline{\widetilde{\mathscr{P}}}_v(-1)$ is lisse and that its reduction modulo $A(v)\hat{\otimes}\mathfrak{p}_v A_v$ is isomorphic to $\mathbb{1}_{\operatorname{Spec} A, k_v}$. This finally shows that $u_v \equiv 1 \,(\mathrm{mod}\ A(v)\hat{\otimes}\mathfrak{p}_v A_v)$, and concludes the proof for $v \neq \infty$.

If $v = \infty$, we choose an affine subscheme $\operatorname{Spec} A'$ of C which contains ∞. Let $\mathfrak{p}_\infty$ be the maximal ideal of A' corresponding to ∞. The above considerations apply almost verbatim to the restriction of $\underline{\mathscr{L}}$ from Proposition 5.10 to $\operatorname{Spec} A \times C'$. We leave the details to the reader. ∎

Theorem 5.18 *The family $\underline{\widetilde{\mathscr{P}}}_v(w) := (A(v)\hat{\otimes} A_v, u_v^{-w}(\sigma \times \mathrm{id}))_{w \in \mathbb{Z}_p}$ is a uniformly overconvergent family of v-adic φ-sheaves such that $\underline{\widetilde{\mathscr{P}}}_v(-n)$ and $\underline{\mathscr{P}}_v^{\otimes n}$ have the same local L-factors.*

Proof: By the previous proposition and Lemma 3.5, only uniform overconvergence remains to be shown. Tracing through the definitions of uniform overconvergence, one can see that we need to verify that $(u_v^w)_{w \in \mathbb{Z}_p}$ is a uniformly overconvergent family of elements of $A\hat{\otimes} A_v$ in the sense of Definition 5.2. For this, we write $u_v - 1$ as a finite sum $\sum_{i=1}^{l} x_i \otimes b_i \in A(v) \otimes \mathfrak{p}_v A_v$. Without loss of generality, we may assume that the x_i form a set of generators of $A(v)$.

We use the binomial theorem to rewrite u_v^w as

$$\begin{aligned} u_v^w &= \sum_{n=0}^{\infty} \binom{w}{n} \Big(\sum_i x_i \otimes b_i \Big)^n \\ &= \sum_{n=0}^{\infty} \binom{w}{n} \sum_{i_1+\ldots+i_l=n} x_1^{i_1} \ldots x_l^{i_l} \otimes b_1^{i_1} \ldots b_l^{i_l} \binom{n}{i_1, \ldots, i_l} \\ &= \sum_{\underline{n} \in \mathbb{N}_0^l} \underline{x}^{\underline{n}} \otimes \Big(\binom{|\underline{n}|}{n_1, \ldots, n_l} \binom{w}{|\underline{n}|} b_1^{n_1} \ldots b_l^{n_l} \Big). \end{aligned}$$

With $c_{\underline{n}}(w) := \binom{|\underline{n}|}{n_1,\ldots,n_l}\binom{w}{|\underline{n}|} b_1^{n_1} \ldots b_l^{n_l}$, it follows immediately that

$$\frac{\mathrm{v}_v(c_{\underline{n}}(w))}{|\underline{n}|} \geq 1 \quad \text{independently of } w,$$

and the desired results follows. ∎

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