

MOD ℓ REPRESENTATIONS OF ARITHMETIC FUNDAMENTAL GROUPS, I: AN ANALOG OF SERRE'S CONJECTURE FOR FUNCTION FIELDS

GEBHARD BÖCKLE and CHANDRASHEKHAR KHARE

Abstract

There is a well-known conjecture of Serre that any continuous, irreducible representation $\bar{\rho} : G_{\mathbf{Q}} \rightarrow \mathrm{GL}_2(\bar{\mathbf{F}}_{\ell})$ that is odd arises from a newform. Here $G_{\mathbf{Q}}$ is the absolute Galois group of \mathbf{Q} , and $\bar{\mathbf{F}}_{\ell}$ is an algebraic closure of the finite field \mathbf{F}_{ℓ} of ℓ elements. We formulate such a conjecture for n -dimensional mod ℓ representations of $\pi_1(X)$ for any positive integer n and where X is a geometrically irreducible, smooth curve over a finite field k of characteristic p ($p \neq \ell$), and we prove this conjecture in a large number of cases. In fact, a proof of all cases of the conjecture for $\ell > 2$ follows from a result announced (conditionally) by Gaitsgory in [G]. The methods are different.

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1. Introduction

Let X be a geometrically irreducible, smooth curve over a finite field k of characteristic p and cardinality q . Denote by K its function field and by \tilde{X} its smooth compactification, and set $S := \tilde{X} \setminus X$. Let $\pi_1(X)$ denote the arithmetic fundamental group of X .

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Thus $\pi_1(X)$ sits in the exact sequence

$$0 \rightarrow \pi_1(\overline{X}) \rightarrow \pi_1(X) \rightarrow G_k \rightarrow 0,$$

where \overline{X} is the base change of X to an algebraic closure of k and G_F denotes the absolute Galois group of any field F .

We study here mod ℓ representations of $\pi_1(X)$, that is, continuous, absolutely irreducible representations $\overline{\rho} : \pi_1(X) \rightarrow \mathrm{GL}_n(\mathbf{F})$ with \mathbf{F} a finite field of characteristic $\ell \neq p$. In this paper, we are mainly interested in an analog of (the qualitative part of) Serre's conjectures in [S] in the function field situation.

Let us fix once and for all an embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_\ell$. Then with respect to this embedding ι , and for any finite set T of places of X , there is a correspondence between n -dimensional ℓ -adic representations of $\pi_1(X \setminus T)$ with finite-order determinant and suitably ramified cuspidal eigenforms (or, equivalently, cuspidal automorphic representations with a newvector fixed by a suitable open compact subgroup of $\mathrm{GL}_n(\mathbf{A}_K)$) on $\mathrm{GL}_n(\mathbf{A}_K)$ with finite-order central character. This correspondence is the global Langlands correspondence for function fields due to Drinfeld (see [D1]) and Lafforgue (see [L]).

We call a residual representation $\overline{\rho}$ *automorphic* if it is isomorphic to the residual representation attached to (an integral model of) an n -dimensional continuous representation $\pi_1(X \setminus T) \rightarrow \mathrm{GL}_n(\overline{\mathbf{Q}}_\ell)$ that is associated to a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_K)$ in [D1] and [L] for some finite set of places T . An analog of Serre's conjecture in the function field setting is therefore that any absolutely irreducible residual representation $\overline{\rho}$ is automorphic. It is worth noting that unlike in the classical setting, here there are no local conditions that need to be imposed on $\overline{\rho}$ to expect it to be automorphic. In view of [L], this conjecture is equivalent to the assertion that any such $\overline{\rho}$ lifts to an ℓ -adic representation of $\pi_1(X \setminus T)$ of finite-order determinant for some finite subset T of X .

There is little known about Serre's original conjecture, while the analog that we study for function fields is more accessible because of the results in [D1] and [L]. The analog seems to be crucial for the applications of de Jong's conjecture by Drinfeld in [D2] to some purity conjectures of Kashiwara on perverse sheaves.

The main results of de Jong [dJ, Thms. 4.9 and 1.3 (ii)] directly imply that the function field analog of Serre's conjecture holds for $n \leq 2$ (for $n = 1$, it is a simple consequence of class field theory). This is strong evidence in favor of the analog. In fact, the main conjecture made in [dJ, Conj. 1.1] may be regarded as a refinement of the above analog and easily implies it.

In Theorem 2.4, we establish the analog in many more cases by producing suitable ℓ -adic liftings of $\overline{\rho}$. Our approach uses the Galois cohomological methods of

R. Ramakrishna in [R] and their further refinements by R. Taylor in [T]. The following is an important special case of Theorem 2.4.

THEOREM 1.1

Let X be a smooth, geometrically irreducible curve defined over a finite field k of characteristic p , and let $\bar{\rho} : \pi_1(X) \rightarrow \mathrm{SL}_n(\mathbf{F})$ be a representation with \mathbf{F} a finite field of characteristic $\ell \neq p$. Assume that

- (i) *$\bar{\rho}$ has full image, $|\mathbf{F}| \geq 4$, $\ell \nmid n$, and*
- (ii) *at any $v \in S$, the ramification is either tame or of order prime to ℓ .*

Then $\bar{\rho}$ lifts to a representation $\rho : \pi_1(X \setminus T) \rightarrow \mathrm{SL}_n(W(\mathbf{F}))$ with T a finite set of places of X and $W(\mathbf{F})$ the Witt vectors of \mathbf{F} . Hence $\bar{\rho}$ is automorphic.

What is mainly needed in the proof of Theorem 1.1 (and of Theorem 2.4) is that the adjoint representation $\mathrm{ad}^0(\bar{\rho})$ of $\bar{\rho}$ on the traceless matrices of $M_n(\mathbf{F})$ is irreducible and that $H^1(\mathrm{im}(\bar{\rho}), \mathrm{ad}^0(\bar{\rho}))$ is (almost) zero.

For $\ell > 2$, a proof of all cases of our analog of Serre's conjecture follows from the work of Gaitsgory (see [G]). The methods are completely different; and while Gaitsgory's work should prove the conjecture in totality for $\ell > 2$, our methods also apply in characteristic 2.

In [BK], a continuation of this paper, we study the conjecture of A. J. de Jong from [dJ], which is about deformations of representations of the type $\bar{\rho}$ studied in this paper. For this, we use the lifting result of the present article. In fact, proving de Jong's conjecture was the main motivation for this work. Our results toward de Jong's conjecture yield that in many cases, $\bar{\rho}$ arises from a cuspidal eigenform of level the conductor of $\bar{\rho}$, where by *arises from*, we mean that $\bar{\rho}$ is isomorphic to the reduction of the n -dimensional ℓ -adic representation (which might no longer have coefficients in Witt vectors) associated to the eigenform, thus proving results toward the analog of Serre's conjecture in its quantitative aspect.

2. Statement of the main result

Our main goal is to prove a general criterion for a residual representation to lift to a characteristic 0 representation which then gives a proof of Theorem 1.1, using Lafforgue's theorem. We start by first making all the necessary definitions to state a result, Theorem 2.4, that is more general but also more technical to state than Theorem 1.1. After stating Theorem 2.4, we first quickly derive Theorem 1.1 from it. Then in the following sections, following Ramakrishna (see [R]) and Taylor (see [T]), we give the proof of Theorem 2.4. For the general background on Galois cohomology of function fields, the reader is referred to [NSW, Chaps. 7 and 8].

Let us fix some notation. For a place v of \tilde{X} , denote by q_v the cardinality of the residue field at v . Let $G_v \supset I_v \supset P_v$ be the absolute Galois group of the

completion of K at v , its inertia, and wild inertia subgroup, respectively. We also choose an embedding $G_v \rightarrow G_K$. For any curve $X \subset \tilde{X}$, this yields morphisms $G_v/I_v \hookrightarrow \pi_1(X)$; and by $\text{Frob}_v \in \pi_1(X)$, we denote the corresponding Frobenius substitution at $v \in X$.

Let $\chi_\ell: \pi_1(X) \rightarrow \mathbf{F}_\ell^* \hookrightarrow \mathbf{F}^*$ be the mod ℓ cyclotomic character. For any $\mathbf{F}[\pi_1(X)]$ -module M , by $M(i)$, $i \in \mathbf{Z}$, we denote the twist of M by the i th tensor power of χ_ℓ , and by $M^* := \text{Hom}(M, \mathbf{F})$, we denote its dual representation. Note that M^* is the dual in the sense of representation theory, but it is *not* the \mathbf{G}_m -dual of M . We frequently use the abbreviation $h^i(\pi_1(X), M) := \dim_{\mathbf{F}} H^i(\pi_1(X), M)$ or a variation of this abbreviation with $\pi_1(X)$ replaced by some G_v .

Suppose now that $\bar{\rho}: \pi_1(X) \rightarrow \text{GL}_n(\mathbf{F})$ is a residual representation, where \mathbf{F} is some finite field of characteristic $\ell \neq p$. Then $M_n(\mathbf{F})$ is a $\pi_1(X)$ -module via the adjoint action composed with $\bar{\rho}$. We denote it by $\text{ad}(\bar{\rho})$. Its subrepresentation on the traceless matrices $M_n^0(\mathbf{F})$ of $M_n(\mathbf{F})$ is denoted by $\text{ad}^0(\bar{\rho})$. Via the perfect $\pi_1(X)$ -equivariant trace pairing

$$\text{ad}(\bar{\rho}) \times \text{ad}(\bar{\rho}) \rightarrow \mathbf{F} : (A, B) \mapsto \text{Trace}(AB),$$

the representation $\text{ad}(\bar{\rho})$ is self-dual. Because ℓ does not divide n , this pairing restricts to perfect pairings on the scalar matrices, and

$$\text{ad}^0(\bar{\rho}) \times \text{ad}^0(\bar{\rho}) \longrightarrow \mathbf{F} : (A, B) \mapsto \text{Trace}(AB) \quad (1)$$

on $\text{ad}^0(\bar{\rho})$. In particular, $\text{ad}^0(\bar{\rho}) \cong \text{ad}^0(\bar{\rho})^*$ as representations if $\ell \nmid n$.

To state the main technical theorem, we need to introduce some further notation. By E , we denote the *splitting field* of $\bar{\rho}$ over K , that is, the fixed field of $\bar{\rho}$ in a fixed separable closure K^{sep} of K . Let $\zeta_\ell \in K^{\text{sep}}$ be a primitive ℓ th root of unity.

Recall that a matrix $A \in \text{GL}_n(\mathbf{F})$ is called *regular* if $\dim_{\mathbf{F}} M_n(\mathbf{F})^A = n$, where A operates via the adjoint action, that is, via conjugation.

Definition 2.1

An *R-class* or *Ramakrishna-class* for $\bar{\rho}$ is the conjugacy class of an element $\sigma \in \text{Gal}(E(\zeta_\ell)/K)$ such that $A := \bar{\rho}(\sigma)$ is regular and one of the following two cases holds:

- (I) $\chi_\ell(\sigma) \neq 1$, and A has distinct simple roots $\lambda, \lambda' \in \mathbf{F}$ with $\lambda' = \chi_\ell(\sigma)\lambda$;
- (II) $\chi_\ell(\sigma) = 1$, and in the Jordan decomposition of A , there occurs at least one (2×2) -block with eigenvalue $\lambda \in \mathbf{F}$.

The conditions that a Galois automorphism is regular or that it satisfies (I) or (II) are invariant under conjugation, and so the definition makes sense. If $\ell = 2$, then only case (II) can occur.

Definition 2.2

We call a place v of X an R -place for $\bar{\rho}$ if the class of Frob_v in $\text{Gal}(E(\zeta_\ell)/K)$ is an R -class.

Note that if an R -class exists, then by the Čebotarev density theorem, there exist infinitely many R -places.

Following Ramakrishna, at suitably chosen R -places, we define local deformation problems of a particular type. In an inductive lifting procedure in favorable cases, this has two effects. First, a suitably defined global deformation problem has no global obstructions to lifting. Second, in the induction step, one may find sufficiently many 1-cocycles so that the lift can be deformed into another lift that is everywhere locally liftable.

To describe a sufficient condition for the first effect to happen, we denote by \bar{V} the space \mathbf{F}^n considered as a representation of $\pi_1(X)$ via $\bar{\rho}$, and we let σ be an R -class of type (II). The indecomposable summands of \bar{V} are denoted by \bar{V}_i , and $\text{ad}(\bar{\rho})_i$ ($\text{ad}^0(\bar{\rho})_i$) denotes the corresponding representation on (the trace zero matrices of) $\text{End}(\bar{V}_i)$, considered as a representation of $\langle \sigma \rangle$. Let λ_i be one of the eigenvalues of \bar{V}_i . We define

$$\text{ad}^0(\bar{\rho})_\sigma := \prod_{\substack{\text{mult}(\lambda_i)=2 \\ \lambda_i \in \mathbf{F}}} \text{ad}^0(\bar{\rho})_i. \quad (2)$$

Since $\bar{V} \cong \bigoplus_i \bar{V}_i$, there is a $\langle \sigma \rangle$ -equivariant homomorphism $\text{ad}^0(\bar{\rho}) \rightarrow \text{ad}^0(\bar{\rho})_\sigma$.

Definition 2.3

We say that $\bar{\rho}$ admits sufficiently many R -classes if there exists at least one R -class and if the following two restriction homomorphisms (composed with $\text{ad}^0(\bar{\rho}) \rightarrow \text{ad}^0(\bar{\rho})_\sigma$ at R -places) are injective:

$$\begin{aligned} & H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\bar{\rho})) \\ & \longrightarrow \prod_{\substack{[\sigma] \text{ an } R\text{-class} \\ \text{of type (II)}}} H^1(\langle \sigma \rangle, \text{ad}^0(\bar{\rho})_\sigma) \oplus \prod_{v \in S} H^1(\bar{\rho}(I_v), \text{ad}^0(\bar{\rho})), \\ & H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\bar{\rho})(1)) \\ & \longrightarrow \prod_{\substack{[\sigma] \text{ an } R\text{-class} \\ \text{of type (II)}}} H^1(\langle \sigma \rangle, \text{ad}^0(\bar{\rho})_\sigma(1)) \oplus \prod_{v \in S} H^1(\bar{\rho}(I_v), \text{ad}^0(\bar{\rho})(1)). \end{aligned}$$

The group $H^1(\langle \sigma \rangle, \text{ad}^0(\bar{\rho})_\sigma)$, as well as the homomorphism to it, does not depend on the choice of representative σ of the R -class $[\sigma]$.

Our main result is the following.

THEOREM 2.4

Let X be a smooth, geometrically irreducible curve defined over a finite field k of characteristic $p \neq \ell$, and let $\bar{\rho} : \pi_1(X) \rightarrow \mathrm{GL}_n(\mathbf{F})$ be a continuous representation. Assume that

- (a) $\mathrm{ad}^0(\bar{\rho})$ is irreducible over $\mathbf{F}_\ell[\mathrm{im}(\bar{\rho})]$,
- (b) $\bar{\rho}$ has sufficiently many R -classes,
- (c) at all $v \in S$, the ramification is either tame or of order prime to ℓ .

Then $\bar{\rho}$ lifts to a representation $\rho : \pi_1(X \setminus T) \rightarrow \mathrm{GL}_n(W(\mathbf{F}))$, where

- (i) T is a finite set of places of X ,
- (ii) $\det \rho$ is the Teichmüller lift of $\det \bar{\rho}$,
- (iii) for $v \in S$, the conductors of ρ and $\bar{\rho}$ agree, and
- (iv) if $\bar{\rho}$ is tame at v , then $\rho(I_v) \xrightarrow{\cong} \bar{\rho}(I_v)$, that is, ρ is minimal at v .

Note that we do *not* need that $\mathrm{ad}^0(\bar{\rho})$ is absolutely irreducible. Note also that the condition that $\mathrm{ad}^0(\bar{\rho})$ is irreducible implies that ℓ does not divide n since in the case $\ell | n$, the representation $\mathrm{ad}^0(\bar{\rho})$ contains the trivial representation on scalar matrices as a nontrivial submodule.

As an application of Lafforgue's theorem, we find the following.

COROLLARY 2.5

Any $\bar{\rho}$ as in Theorem 2.4 is automorphic.

We have the following example for the existence of sufficiently many R -classes. Combined with Theorem 2.4, it completes the proof of Theorem 1.1. \square

PROPOSITION 2.6

Suppose $\bar{\rho} : \pi_1(X) \rightarrow \mathrm{SL}_n(\mathbf{F})$ is surjective, $\ell \nmid n$, $\ell \neq p$, and $|\mathbf{F}| \geq 4$. Then $\bar{\rho}$ admits sufficiently many R -classes.

Proof

Let us first show the injectivity of the restriction homomorphisms in Definition 2.3. If $|\mathbf{F}| > 5$ or $n > 2$, then by [CPS, Table 4.5, p. 185], we have $H^1(\mathrm{SL}_n(\mathbf{F}), M_n^0(\mathbf{F})) = 0$. In this case, it easily follows from, for example [B1, Sect. 5] that

$$H^1(\mathrm{Gal}(E(\zeta_\ell)/K), \mathrm{ad}^0(\bar{\rho})) = 0 \quad \text{and} \quad H^1(\mathrm{Gal}(E(\zeta_\ell)/K), \mathrm{ad}^0(\bar{\rho})(1)) = 0.$$

If $4 \leq |\mathbf{F}| \leq 5$ and $n = 2$, the condition $\ell \nmid n$ rules out the case $|\mathbf{F}| = 4$. In [T], it is shown for $n = 2$ and $\mathbf{F} = \mathbf{F}_5$ how to find an R -class σ such that

$$\mathbf{F} \cong H^1(\mathrm{Gal}(E(\zeta_\ell)/K), \mathrm{ad}^0(\bar{\rho})) \rightarrow H^1(\langle \sigma \rangle, \mathrm{ad}^0(\bar{\rho}))$$

is injective (in this particular case, one has $\mathrm{ad}^0(\bar{\rho})_\sigma = \mathrm{ad}^0(\bar{\rho})$). If χ_ℓ is trivial, the same class also works for $\mathrm{ad}^0(\bar{\rho})(1) = \mathrm{ad}^0(\bar{\rho})$. If χ_ℓ is nontrivial, then by [B1, Sect. 5], one has $H^1(\mathrm{Gal}(E(\zeta_\ell)/K), \mathrm{ad}^0(\bar{\rho})(1)) = 0$.

It remains to prove the existence of at least one R -class. For this, note that $\mathrm{SL}_n(\mathbf{F})$ has no abelian quotients; and therefore the morphism

$$\bar{\rho} \times \chi_\ell: \pi_1(X) \longrightarrow \mathrm{SL}_n(\mathbf{F}) \times \mathbf{F}^*$$

surjects onto $\mathrm{SL}_n(\mathbf{F}) \times \mathrm{im}(\chi_\ell)$. Since $\mathrm{SL}_n(\mathbf{F})$ contains matrices of type (II), the existence of an R -class is obvious. Furthermore, if $\ell \neq 2$ and if $\mathrm{im}(\chi_\ell)$ is nontrivial, then one may also find matrices of type (I). This completes the proof of Proposition 2.6. \square

3. Strategy of the proof of Theorem 2.4

Our method of producing lifts is essentially that of Ramakrishna (see [R]). However, we follow the more axiomatic treatment as presented in [T]. Let us fix from now on a representation $\bar{\rho}: \pi_1(X) \rightarrow \mathrm{GL}_n(\mathbf{F})$ that satisfies the conditions of Theorem 2.4, and let $n \geq 2$ since $n = 1$ is trivial by using Teichmüller lifts. In the following, we assume that $\mathrm{ad}^0(\bar{\rho})$ is irreducible over $\mathbf{F}_\ell[\mathrm{im}(\bar{\rho})]$ (and hence that $\ell \nmid n$). Also, define $\eta: \pi_1(X) \rightarrow W(\mathbf{F})^*$ as the Teichmüller lift of $\det \bar{\rho}$; and for any place v , define restrictions $\eta_v := \eta|_{G_v}$ and $\bar{\rho}_v := \bar{\rho}|_{G_v}$.

The strategy in [R] to produce lifts of $\bar{\rho}$ to $W(\mathbf{F})$ is to first consider all deformations of $\bar{\rho}$ which are representations of $\pi_1(X \setminus T)$ for some fixed finite subset T of R -places of X and which at the places in $S \cup T$ are allowed to have ramification of a very specific type only. Without loss of generality, we assume that $\bar{\rho}$ is ramified at the places in S , and we call these *residually ramified places* or, simply, *r-places*.

The type of ramification is most conveniently formulated in terms of suitable local lifting problems \mathcal{C}_v at places $v \in S \cup T$. In this formulation, the crucial requirement locally is that the versal hull (of the deformation problem described by \mathcal{C}_v) is smooth over the ring $W(\mathbf{F})$ of relative dimension $h^0(G_v, \mathrm{ad}^0(\bar{\rho}))$. In Sections 4 and 5, we define such \mathcal{C}_v for R - and r -places, respectively.

The global conditions on T and the types \mathcal{C}_v are made in such a way that one can inductively construct lifts of $\bar{\rho}$ to the rings $W_n(\mathbf{F})$ of Witt vectors of length n . They can be entirely formulated in terms of Galois cohomology. In this section, we recall the necessary background from [T] and give a proof of the main theorem, Theorem 2.4, pending on a key lemma whose proof is given in Section 6.

Let \mathcal{A} denote the category of complete Noetherian local $W(\mathbf{F})$ -algebras (R, \mathfrak{m}_R) with residue field \mathbf{F} and where morphisms are morphisms of local rings which are the identity on the residue field. By a *lift of determinant* η_v of $\overline{\rho}_v$, we mean a continuous representation $\rho: G_v \rightarrow \mathrm{GL}_n(R)$ for some $(R, \mathfrak{m}_R) \in \mathcal{A}$ such that $\rho \pmod{\mathfrak{m}_R} = \overline{\rho}_v$ and $\det \rho = \eta_v$.

Definition 3.1

Following [T], we call a pair (\mathcal{C}_v, L_v) , where \mathcal{C}_v is a collection of lifts of $\overline{\rho}_v$ of determinant η_v and L_v is a subspace of $H^1(G_v, \mathrm{ad}^0(\overline{\rho}))$, *(locally) admissible and compatible with η_v* if it satisfies the conditions P1–P7 of [T], where one has to replace \mathfrak{m} by \mathfrak{m}_R and $M_2(\mathfrak{m})$ by $M_n(\mathfrak{m}_R)$ in property P2.

Unlike in [T], we do regard the \mathcal{C}_v as a functor from Noetherian local rings R with fixed residue field \mathbf{F} to lifts of $\overline{\rho}$ to R . This makes a slight notational difference.

We will repeatedly assert that certain pairs (\mathcal{C}_v, L_v) satisfy conditions P1–P7. Condition P4 is typically the most difficult to verify, while the other ones are rather straightforward. Therefore, in proofs that verify Taylor’s conditions, we exclusively treat condition P4. For the convenience of the reader, we now state this condition; while for the other ones, we refer to [T].

- P4. Suppose for $i = 1, 2$, we are given rings $R_i \in \mathcal{A}$, ideals $I_i \subset R_i$, representations $\rho_i \in \mathcal{C}_v(R_i)$, and an isomorphism $\phi: R/I_1 \xrightarrow{\cong} R_2/I_2$ such that $\phi(\rho_1 \pmod{I_1}) = \rho_2 \pmod{I_2}$. Let $R \in \mathcal{A}$ be the subring of $R_1 \oplus R_2$ consisting of pairs with the same image in $R_1/I_1 \cong R_2/I_2$. Then $\rho_1 \oplus \rho_2$ lies in $\mathcal{C}_v(R)$.

Remark 3.2

To any pair (\mathcal{C}_v, L_v) satisfying P1–P7, there corresponds a deformation problem in the sense of Mazur (see [M]) which possesses a versal hull whose corresponding versal deformation ring is smooth over $W(\mathbf{F})$ of relative dimension $\dim L_v$. Conversely, to any smooth, versally representable deformation problem, one can define a pair (\mathcal{C}_v, L_v) that satisfies Taylor’s conditions P1–P7. If given such a deformation problem, then under this correspondence, the subspace L_v of $H^1(G_v, \mathrm{ad}^0(\overline{\rho}))$ corresponds to the dual of the tangent space of the versal deformation. In formula (5), we give the explicit description of L_v .

Heuristically, one expects $\dim L_v \leq h^0(G_v, \mathrm{ad}^0(\overline{\rho}))$ since, conjecturally, the versal deformation ring of all deformations of $\overline{\rho}_v$ with fixed determinant is a complete intersection, flat over $W(\mathbf{F})$ and of relative dimension $h^0(G_v, \mathrm{ad}^0(\overline{\rho}))$.

Suppose one is given a finite set $T \subset X$ and, for each $v \in S \cup T$, a locally admissible pair (\mathcal{C}_v, L_v) compatible with η_v .

Definition 3.3

A lift of type $(\mathcal{C}_v)_{v \in S \cup T}$ is a continuous representation $\rho : \pi_1(X \setminus T) \rightarrow \mathrm{GL}_n(R)$ for some $(R, \mathfrak{m}_R) \in \mathcal{A}$ such that $\rho \pmod{\mathfrak{m}_R} = \bar{\rho}$, $\rho|_{G_v} \in \mathcal{C}_v$ for all $v \in S \cup T$ and $\det \rho = \eta$.

To describe tangential conditions on the (deformation ring corresponding to the) above lifts, we need to fix some more notation. For v a place of \tilde{X} and any G_v -module M , the pairing $M \times M^* \rightarrow \mathbf{F}$, defined by evaluation, is obviously perfect. Tate local duality says that the induced pairing

$$H^1(G_v, M) \times H^1(G_v, M^*(1)) \longrightarrow H^2(G_v, \mathbf{F}(1)) \cong \mathbf{F} \quad (3)$$

is perfect as well, and one denotes for any \mathbf{F} -submodule $L \subset H^1(G_v, M)$ its annihilator under this pairing by $L^\perp \subset H^1(G_v, M^*(1))$. In the particular case of the subspace of unramified cocycles

$$H_{\mathrm{unr}}^1(G_v, M) := H^1(G_v/I_v, M^{I_v}) \subset H^1(G_v, M),$$

one finds $H_{\mathrm{unr}}^1(G_v, M)^\perp = H_{\mathrm{unr}}^1(G_v, M^*(1))$. In line with the usual notation $h^1(\dots)$, we abbreviate $h_{\mathrm{unr}}^1(G_v, M) := \dim_{\mathbf{F}} H_{\mathrm{unr}}^1(G_v, M)$.

The situation most interesting to us is $M = \mathrm{ad}^0(\bar{\rho})$. By (1), this module is self-dual, and so Tate local duality induces the perfect pairing

$$H^1(G_v, \mathrm{ad}^0(\bar{\rho})) \times H^1(G_v, \mathrm{ad}^0(\bar{\rho})(1)) \longrightarrow H^2(G_v, \mathbf{F}(1)) \cong \mathbf{F}. \quad (4)$$

For a finite subset T of $X (= \tilde{X} \setminus S)$ and a collection $(L_v)_{v \in S \cup T}$ of subspaces of $H^1(G_v, \mathrm{ad}^0(\bar{\rho}))$, one defines $H_{\{L_v\}}^1(S \cup T, \mathrm{ad}^0(\bar{\rho}))$ as the kernel of

$$H^1(\pi_1(X \setminus T), \mathrm{ad}^0(\bar{\rho})) \longrightarrow \bigoplus_{v \in S \cup T} H^1(G_v, \mathrm{ad}^0(\bar{\rho}))/L_v.$$

Ramakrishna's first observation is the following lemma.

LEMMA 3.4

Suppose one is given locally admissible pairs $(\mathcal{C}_v, L_v)_{v \in S \cup T}$ compatible with η such that

$$H_{\{L_v^\perp\}}^1(S \cup T, \mathrm{ad}^0(\bar{\rho})(1)) = 0.$$

Then there exists a lift of $\bar{\rho}$ to $W(\mathbf{F})$ of type $(\mathcal{C}_v)_{v \in S \cup T}$.

The proof is essentially that of [T, Lemma 1.2], and so we omit the details.

Remark 3.5

Mimicking the proofs of [DDT, Thms. 2.13, 2.14], one obtains for a $\pi_1(X \setminus T)$ -module M and subspaces $L_v \subset H^1(G_v, M)$ for $v \in S \cup T$ the formula

$$\frac{|H^1_{\{L_v\}}(S \cup T, M)|}{|H^1_{\{L_v\}}(S \cup T, M^*(1))|} = \frac{|H^0(\pi_1(X), M)|}{|H^0(\pi_1(X), M^*(1))|} \prod_{v \in S \cup T} \frac{|L_v|}{|H^0(G_v, M)|}.$$

In our situation, $M \cong M^* \cong \text{ad}^0(\bar{\rho})$, and the first quotient on the right-hand side of the above formula is clearly 1. Thus, by Remark 3.2, one expects the product on the right to have the value at most 1. Furthermore, this should happen precisely when $\dim L_v = h^0(G_v, \text{ad}^0(\bar{\rho}))$ for all $v \in S \cup T$. Therefore, if the hypothesis of Lemma 3.4 is satisfied, then one expects

$$\dim H^1_{\{L_v\}}(S \cup T, \text{ad}^0(\bar{\rho})) = 0.$$

In terms of deformation theory, compare Remark 3.2; this can be interpreted by saying that the universal deformation ring of type $(\mathcal{C}_v)_{v \in S \cup T}$ is smooth over $W(\mathbf{F})$ of relative dimension zero, that is, isomorphic to $W(\mathbf{F})$.

Note that the above formula also holds for $S \cup T = \emptyset$ even though the duality results in [NSW] are not proved in this case. The reason is that in this case, the right-hand side is 1; and because $H^0(\pi_1(X), \text{ad}^0(\bar{\rho})) = 0$, the left-hand side expresses that fact that the Euler-Poincaré characteristic of the unramified $\mathbf{F}[\pi_1(\tilde{X})]$ -module $\text{ad}^0(\bar{\rho})$ is zero.

We need to generalize slightly the concept of sufficiently many R -classes for the following result. Suppose we are given locally admissible $(\mathcal{C}_v, L_v)_{v \in S \cup T}$ that are compatible with η .

Definition 3.6

We say that $\bar{\rho}$ *admits sufficiently many R -classes for $(\mathcal{C}_v, L_v)_{v \in S \cup T}$* if there exists at least one R -class and if the following restriction homomorphisms (composed with $\text{ad}^0(\bar{\rho}) \rightarrow \text{ad}^0(\bar{\rho})_\sigma$ at R -places) are injective:

$$\begin{aligned} & H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\bar{\rho})) \cap H^1_{\{L_v\}}(S \cup T, \text{ad}^0(\bar{\rho})) \\ & \longrightarrow \prod_{\substack{\sigma \text{ an } R\text{-class} \\ \text{of type(II)}}} H^1(\langle \sigma \rangle, \text{ad}^0(\bar{\rho})_\sigma), \\ & H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\bar{\rho})(1)) \cap H^1_{\{L_v\}}(S \cup T, \text{ad}^0(\bar{\rho})(1)) \\ & \longrightarrow \prod_{\substack{\sigma \text{ an } R\text{-class} \\ \text{of type(II)}}} H^1(\langle \sigma \rangle, \text{ad}^0(\bar{\rho})_\sigma(1)). \end{aligned}$$

The main observation of Ramakrishna, if adapted to our situation, is the following key lemma.

LEMMA 3.7

Suppose one is given a finite set of places $T' \subset X$ and locally admissible $(\mathcal{C}_v, L_v)_{v \in S \cup T'}$ that are compatible with η and such that

$$\sum_{v \in S \cup T'} \dim L_v \geq \sum_{v \in S \cup T'} h^0(G_v, \text{ad}^0(\bar{\rho})).$$

If $\bar{\rho}$ admits sufficiently many R -classes for $(\mathcal{C}_v, L_v)_{v \in S \cup T'}$, then one can find a finite set of R -places $T \subset X$ and locally admissible $(\mathcal{C}_v, L_v)_{v \in T}$ compatible with η such that

$$H_{\{L_v^+\}}^1(S \cup T \cup T', \text{ad}^0(\bar{\rho})) = 0.$$

The proof of Lemma 3.7 is given in Section 6. Let us now explain how this gives a proof of Theorem 2.4.

In the following two sections, we define good local lifting problems at certain unramified primes and at ramified primes, where the ramification is either of order prime to ℓ or prime to p . Correspondingly, we obtain pairs (\mathcal{C}_v, L_v) satisfying properties P1–P7 of Taylor [T]. We then apply Lemma 3.7 with $T' = \emptyset$ and assume that $\bar{\rho}$ ramifies at all places of S . In order to do that, we also have to check that if $\bar{\rho}$ has sufficiently many R -classes, then this implies that $\bar{\rho}$ has sufficiently many R -classes for $(\mathcal{C}_v, L_v)_{v \in S}$, where the (\mathcal{C}_v, L_v) is defined below. Once this is shown, Theorem 2.4 follows easily from Lemmas 3.4 and 3.7. The full proof is given at the end of Section 6.

4. Local lifting problems at R -places

In this section, we define locally admissible pairs (\mathcal{C}_v, L_v) at R -places v compatible with the Teichmüller lift $\eta_v : G_v \rightarrow W(\mathbf{F})$ of $\det \bar{\rho}_v$ (see Proposition 4.4). So for the remainder of this section, we fix an R -place v and denote by σ the image of Frob_v in $\text{Gal}(E(\zeta_\ell)/K)$, so that $[\sigma]$ is an R -class. We also fix an eigenvalue $\lambda \in \mathbf{F}$ of $A := \bar{\rho}_v(\sigma)$, as required in the definition of an R -place.

The definition of \mathcal{C}_v at an R -place

Using the rational canonical form, we may assume that A is given in the form

$$A = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_r \end{pmatrix},$$

where each A_i is a square matrix of size n_i , the matrices A_i for $i > 1$ are in rational canonical form and act indecomposably, and the matrix A_1 has the following form depending on our two cases:

$$A_1 = \begin{pmatrix} \lambda \chi_\ell(\sigma) & 0 \\ 0 & \lambda \end{pmatrix}$$

in case (I) and

$$A_1 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

in case (II).

Note that in case (II), the A_i , $i \geq 1$, are in bijection with the irreducible representations \overline{V}_i used in the definition of formula (2). Because the A_i act indecomposably, the eigenvalues form a single Galois orbit and the Jordan canonical form of an A_i consists of identical blocks for each of the eigenvalues. Because A is regular, different A_i have distinct orbits of eigenvalues. Also, clearly, each A_i is again regular.

For $i = 2, \dots, r$ we define

$$\rho_{v,i}: G_v \longrightarrow \mathrm{GL}_{n_i}(R_{v,i})$$

as a lift representing the versal unramified deformation of $\overline{\rho}_{v,i}: G_v \rightarrow \mathrm{GL}_{n_i}(\mathbf{F})$, defined as the restriction of $\overline{\rho}_v$ to the i th block.

For the definition in case $i = 1$, let $\hat{\mathbf{Z}}$ be the profinite completion of \mathbf{Z} and $\hat{\mathbf{Z}}'$ be the prime-to- p completion of $\hat{\mathbf{Z}}$. Let s, t be topological generators of $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Z}}'$, respectively. For q' a power of p , and thus prime to ℓ , define $\overline{G}_{q'} := \hat{\mathbf{Z}}' \rtimes \hat{\mathbf{Z}}$, where the semidirect product is given (in multiplicative notation) by the condition $sts^{-1} = t^{q'}$. Then \overline{G}_{q_v} can be identified with the tame quotient of G_v in such a way that t is a generator of I_v/P_v and s is a lift of the Frobenius automorphism in G_v/I_v . Therefore we make the following convention.

CONVENTION 4.1

Whenever a representation factors via the tame quotient of G_v , we identify this tame quotient in the above way with \overline{G}_{q_v} .

Thus if $\overline{\rho}$ is unramified at v , then the images of Frob_v and of s in $\mathrm{Gal}(E(\zeta_\ell)/K)$ are the same; and so for each $j \in \mathbf{Z}$, the elements s and σ act in the same way on $\mathrm{ad}^0(\overline{\rho})(j)$.

By $\hat{\mu} \in W(\mathbf{F})$, we denote the Teichmüller lift of any element μ of \mathbf{F} , and we set δ to be zero in case (I) and 1 in case (II). We now define $R_{v,1} := W(\mathbf{F})[[x_{1,0}, x_{1,1}]]$ and

$\rho_{v,1} : G_v \twoheadrightarrow \overline{G}_{q_v} \longrightarrow \mathrm{GL}_2(R_{v,1})$ by

$$s \mapsto \begin{pmatrix} \widehat{\lambda} q_v(1 - x_{1,1}) & \delta \\ 0 & \widehat{\lambda}(1 - x_{1,1}) \end{pmatrix} \quad \text{and} \quad t \mapsto \begin{pmatrix} 1 & x_{1,0} \\ 0 & 1 \end{pmatrix}.$$

The necessary condition $\rho_{v,1}(s)\rho_{v,1}(t) = \rho_{v,1}(t)^{q_v}\rho_{v,1}(s)$ can be verified easily. Since $\chi_\ell(\sigma) \equiv q_v \pmod{\ell}$, we also have $\rho_{v,1}(s) \equiv A_1 \pmod{(\ell, x_{1,0}, x_{1,1})}$.

We now define

$$R_v := \widehat{\bigotimes}_{i=1}^r R_{v,i} / \left(\prod_{i=1}^r \det \rho_{v,i}(s) - \eta_v(s) \right)$$

with $\widehat{\bigotimes}$ formed over $W(\mathbf{F})$ and the corresponding representation $\rho_v : G_v \longrightarrow \mathrm{GL}_n(R_v)$ as $\oplus \rho_{v,i}$ (where the entries are taken modulo the ideal generated by $\prod_{i=1}^r \det \rho_{v,i}(s) - \eta_v(s)$).

To investigate the resulting representations, we first need a simple result on the individual $\rho_{v,i}$. For this, we denote by $\mathrm{ad}(\overline{\rho})_i$ the adjoint representations of the A_i and by the $\mathrm{ad}^0(\overline{\rho})_i$ its subrepresentation on trace zero matrices; that is, in case (II), they agree with those defined in (2). Then we have the following lemma.

LEMMA 4.2

Let i be in $\{2, \dots, r\}$. Then the versal deformation ring $R_{v,i}$ is smooth over $W(\mathbf{F})$ of dimension $h_{\mathrm{unr}}^1(G_v, \mathrm{ad}(\overline{\rho})_i) = n_i$. If $\ell \nmid n_i$, and if η_i is any lift of $\det \overline{\rho}_{v,i}$ to $W(\mathbf{F})$, then the versal deformation ring for unramified deformations of determinant equal to η_i is smooth of dimension $h_{\mathrm{unr}}^1(G_v, \mathrm{ad}^0(\overline{\rho})_i) = n_i - 1$.

Proof

Since ρ is unramified at v , one has $h_{\mathrm{unr}}^1(G_v, \mathrm{ad}(\overline{\rho})) = \dim M_n(\mathbf{F})^A$ and, similarly, $h_{\mathrm{unr}}^1(G_v, \mathrm{ad}(\overline{\rho})_i) = \dim M_n(\mathbf{F})^{A_i}$. The assertion now follows easily from the regularity of $A (= \overline{\rho}(\mathrm{Frob}_v))$. \square

COROLLARY 4.3

Assume that there exists an $i \geq 2$ such that ℓ does not divide n_i . Then R_v is smooth over $W(\mathbf{F})$ of relative dimension $n - 1 = h_{\mathrm{unr}}^0(G_v, \mathrm{ad}^0(\overline{\rho}))$.

Note that for $\ell \nmid n$, there always exists an $i \geq 2$ with $\ell \nmid n_i$.

Proof

By Lemma 4.2 and the definition of $R_{v,1}$, the ring $\widehat{\bigotimes}_{i=1}^r R_{v,i}$ is smooth of dimension n . Because the A_i have distinct sets of eigenvalues for different i , we have $h_{\mathrm{unr}}^0(G_v, \mathrm{ad}(\overline{\rho})) = \sum_i h_{\mathrm{unr}}^0(G_v, \mathrm{ad}(\overline{\rho})_i)$, which in turn implies $h_{\mathrm{unr}}^1(G_v, \mathrm{ad}(\overline{\rho})) =$

$\sum_i h_{\text{unr}}^1(G_v, \text{ad}(\overline{\rho})_i) = n$ since $h^1(\hat{\mathbf{Z}}, M) = h^0(\hat{\mathbf{Z}}, M)$. Moreover, if one of the n_i is not divisible by ℓ , then it is easy to see that $h_{\text{unr}}^1(G_v, \text{ad}^0(\overline{\rho})) = h_{\text{unr}}^1(G_v, \text{ad}(\overline{\rho})) - 1$. Let i_0 be the corresponding index.

We now prove the smoothness of R_v . Lemma 4.2 applied to i_0 says that there is a system of local coordinates of R_{v,i_0} such that $\det \rho_{v,i_0}(s) = \eta_{i_0}(s)(1+x)$, where x is one of these coordinates. (This also works for $i_0 = 1$.) If we regroup the defining relation of R_v , it therefore yields the relation

$$\eta_v(s) \prod_{i \neq i_0} \det \rho_{v,i}^{-1}(s) = \eta_{i_0}(s)(1+x),$$

and the variable x does not occur on the left-hand side. Thus the relation eliminates the variable x , which is one of the local coordinates in a suitable set of such for the ring $\hat{\otimes}_i R_{v,i}$. Because $\hat{\otimes}_i R_{v,i}$ is smooth over $W(\mathbf{F})$ of relative dimension n , so is R_v of relative dimension $n - 1$. \square

The following defines a pair (\mathcal{C}_v, L_v) compatible with η_v . The functor $\mathcal{C}_v: \mathcal{A} \rightarrow \mathbf{Sets}$ is given by

$$R \mapsto \mathcal{C}_v(R) := \left\{ \rho: G_v \rightarrow \text{GL}_n(R) \mid \exists \alpha \in \text{Hom}_{\mathcal{A}}(R_v, R), \right. \\ \left. \exists M \in 1 + M_n(\mathfrak{m}_R) : \rho = M(\alpha \circ \rho_v)M^{-1} \right\}.$$

Moreover, if $\rho_0: G_v \rightarrow \text{GL}_n(\mathbf{F}[\varepsilon]/(\varepsilon^2))$ denotes the trivial lift of $\overline{\rho}_v$, the subspace $L_v \subset H^1(G_v, \text{ad}^0(\overline{\rho}))$ is the set of 1-cocycles

$$\left\{ c: G_v \rightarrow \text{ad}^0(\overline{\rho}) : g \mapsto \frac{1}{\varepsilon}(\rho(g)\rho_0^{-1}(g) - I) \mid \rho \in \mathcal{C}_v(\mathbf{F}[\varepsilon]/(\varepsilon^2)) \right\}, \quad (5)$$

and $L_{v,\text{unr}} \subset H_{\text{unr}}^1(G_v, \text{ad}^0(\overline{\rho}))$ is the intersection $L_v \cap H_{\text{unr}}^1(G_v, \text{ad}^0(\overline{\rho}))$.

PROPOSITION 4.4

- (i) $\dim_{\mathbf{F}} L_v = 1 + \dim_{\mathbf{F}} L_{v,\text{unr}} = n - 1$.
- (ii) The pair (\mathcal{C}_v, L_v) satisfies conditions P1–P7 of [T].

Proof of Proposition 4.4 (i)

Let us fix local coordinates $x_{i,j}$, $j = 1, \dots, n_i$ of the rings $R_{v,i}$, $i = 2, \dots, r$. We also enumerate them, so that the variable x in the proof of Corollary 4.3 is given by $x_{i_0,1}$.

First let c_0 be the 1-cocycle that arises from $R_v \rightarrow \mathbf{F}[\varepsilon]/(\varepsilon^2)$ by mapping the $x_{i,j}$, $j \geq 1$ and $(i, j) \neq (i_0, 1)$, to zero and $x_{1,0}$ to ε . (The image of $x_{i_0,1}$ is determined by

the $x_{i,j}$ with $j \geq 1$.) The corresponding element in L_v is easily seen to nonzero and ramified. Moreover, all cocycles obtained from an assignment where $x_{1,0}$ maps to zero are unramified. This shows $\dim L_v = \dim L_{v,\text{unr}} + 1$.

Let L'_v be the set of cocycles corresponding to the ring $R' := \hat{\otimes}_i R_{v,i}/(x_{1,0})$. The ring R' together with the representation $\oplus_i \rho_{v,i} \pmod{(x_{1,0})}$ is by its very construction, and by Lemma 4.2, a versal deformation ring. Hence the same is true for its smooth quotient $R'/(\prod_{i=1}^r \det \rho_{v,i}(s) - \eta_v(s) \cong R/(x_{1,0}))$. The latter is of relative dimension $n-2$ over $W(\mathbf{F})$, and its mod ℓ tangent space is dual to $L_{v,\text{unr}}$. So $\dim_{\mathbf{F}} L_{v,\text{unr}} = n-2$, as asserted. \square

To prove the second part of Proposition 4.4, we need the following lemma.

LEMMA 4.5

Let \tilde{R} be in \mathcal{A} and $\alpha, \alpha' \in \text{Hom}_{\mathcal{A}}(R_v, \tilde{R})$ such that there exists $M \in \text{GL}_n(\tilde{R})$ congruent to the identity modulo $\mathfrak{m}_{\tilde{R}}$ with $M(\alpha \circ \rho_v)M^{-1} = \alpha' \circ \rho_v$. Then $\alpha \circ \rho_v(s) = \alpha' \circ \rho_v(s)$, so that, in particular, M commutes with $\alpha \circ \rho_v(s)$.

Proof

We use the same local parameters for R_v as in the proof of Proposition 4.4(i). The matrix $\rho_v(s)$ has entries in the power series ring over $W(\mathbf{F})$ in the variables $x_{i,j}$, $j \geq 1$. By $\rho_v(s)_r$, we denote the part of $\rho_v(s)$ that is homogeneous of degree r , so that $\rho_v(s) = \sum_{r=0}^{\infty} \rho_v(s)_r$. The assertion $\dim_{\mathbf{F}} L_{v,\text{unr}} = n-2$ of Proposition 4.4(i) means precisely that the $n-2$ matrices $\frac{\partial}{\partial x_{i,j}} \rho_v(s)_1$ over all i, j with $j \geq 1$ and $(i, j) \neq (i_0, 1)$ form a basis of the vector space $L_{v,\text{unr}} \subset \text{ad}^0(\bar{\rho})/(s-1)\text{ad}^0(\bar{\rho})$.

Define $\tau := \alpha \circ \rho_v$, $\tau' := \alpha' \circ \rho_v$. Let $\tau_{(m)} := \tau \pmod{\mathfrak{m}_{\tilde{R}}^m}$, and introduce analogous abbreviations $\tau'_{(m)}$, $\alpha_{(m)}$, $\alpha'_{(m)}$, and $M_{(m)}$. By induction on m , we show below that $\alpha_{(m)}(x_{i,j}) = \alpha'_{(m)}(x_{i,j})$ for all i, j with $j \geq 1$. This clearly implies $\tau_{(m)}(s) = \tau'_{(m)}(s)$ for all m , and thus the proof of Lemma 4.5 is completed. It remains to give the inductive argument. The case $m=1$ is clear, and so we now carry out the induction step $m \mapsto m+1$.

By the induction hypothesis, $M_{(m)}$ commutes with $\tau_{(m)}(s)$. Because $\bar{\rho}_v(s)$ is regular, [B2, Lem. 5.6] implies that there exists a lift M' of $M_{(m)}$ to $\text{GL}_n(\tilde{R}/\mathfrak{m}_{\tilde{R}}^{m+1})$ which commutes with $\tau_{(m+1)}(s)$. By considering $M'^{-1}M_{(m+1)}$, we may thus assume $M_{(m+1)} = I + \Delta$ for some $\Delta \in M_n(\mathfrak{m}_{\tilde{R}}^m/\mathfrak{m}_{\tilde{R}}^{m+1})$. We also define elements $\delta_{i,j} := \alpha_{(m+1)}(x_{i,j}) - \alpha'_{(m+1)}(x_{i,j})$, which, by the induction hypothesis, lie in $\mathfrak{m}_{\tilde{R}}^{m+1}$. The expansion of $\rho_v(s)$ in homogeneous parts shows

$$\tau'_{m+1}(s) = \tau_{m+1}(s) + \rho_v(s)_1|_{x_{i,j}=\delta_{i,j}} \text{ in } \text{GL}_n(\tilde{R}/\mathfrak{m}_{\tilde{R}}^{m+1}),$$

and so the condition $M_{(m+1)}\tau_{(m+1)}(s)M_{(m+1)}^{-1} = \tau'_{(m+1)}(s)$ yields

$$\sum_{(i,j)} \delta_{i,j} \frac{\partial}{\partial x_{i,j}} \rho_v(s)_1 = \rho_v(s)_1|_{x_{i,j}=\delta_{i,j}} = \Delta \tau_{(m+1)}(s) - \tau_{(m+1)}(s) \Delta \quad (6)$$

in $M_n(\mathfrak{m}_R^m/\mathfrak{m}_R^{m+1})$. The right-hand side is a linear combination of coboundaries, that is, zero in $H^1(G_v, \text{ad}^0(\bar{\rho})) \otimes \mathfrak{m}_R^m/\mathfrak{m}_R^{m+1}$. The elements $\frac{\partial}{\partial x_{i,j}} \rho_v(s)_1$ are linearly independent in $H^1(G_v, \text{ad}^0(\bar{\rho}))$. Therefore both sides must vanish, and this concludes the induction step and the proof of Lemma 4.5. \square

Proof of Proposition 4.4(ii)

The only nontrivial condition that needs to be verified is P4. So we assume the setup given in the condition P4, as displayed explicitly in the third paragraph after Definition 3.1; that is, we have rings $R_1, R_2 \in \mathcal{A}$, lifts $\rho_i \in \mathcal{C}_v(R_i)$, ideals $I_i \in R_i$, and an identification $\phi: R_1/I_1 \xrightarrow{\cong} R_2/I_2$ under which $\rho_1 \pmod{I_1} = \rho_2 \pmod{I_2}$. We want to glue the ρ_i to an element $\rho \in \mathcal{C}_v(R)$ for

$$R := \{(r_1, r_2) \in R_1 \oplus R_2 : r_1 \pmod{I_1} = r_2 \pmod{I_2}\}.$$

So let $\alpha_i \in \text{Hom}_{\mathcal{A}}(R_v, R_i)$ and $M_i \in \text{GL}_n(R_i)$ such that $\rho_i = M_i(\alpha_i \circ \rho_v)M_i^{-1}$, $i = 1, 2$. We claim that there exist $\alpha \in \text{Hom}_{\mathcal{A}}(R_v, R)$ and $M \in \text{GL}_n(R)$ with $M \equiv I \pmod{\mathfrak{m}_R}$ such that $\rho := M(\alpha \circ \rho_v)M^{-1} = \rho_1 \oplus \rho_2$.

By conjugating ρ_1 by some lift of $M_2 \pmod{I_1}$ to R_1 , we may assume that $M_2 = I$. By Lemma 4.5, the matrix $M_1 \pmod{I_1}$ commutes with

$$(\alpha_1 \pmod{I_1}) \circ \rho_v(s) = (\alpha_2 \pmod{I_2}) \circ \rho_v(s).$$

Using [B2, Lem. 5.6], and the regularity of $A = \bar{\rho}_v(s)$, we may choose a lift $M'_1 \in \text{GL}_n(R_1)$ of $M_1 \pmod{I_1}$ which commutes with $\alpha_1 \circ \rho_v(s)$. We now replace M_1 by $\tilde{M}_1 := M'_1 M_1'^{-1}$ and α_1 by some $\tilde{\alpha}_1: R_v \rightarrow R_1$, which differs from α_1 at most on the variable x_0 and such that

$$\tilde{M}_1(\tilde{\alpha}_1 \circ \rho_v)\tilde{M}_1^{-1} = M_1(\alpha_1 \circ \rho_v)M_1^{-1}.$$

Defining $M := (\tilde{M}_1, I) \in \text{GL}_n(R)$ and $\alpha := (\alpha'_1, \alpha_2): R_v \rightarrow R$, condition P4 is verified. \square

On the local duality pairing

As before, we fix v and eigenvalue(s) λ (and $\lambda' = \chi_\ell(\sigma)\lambda$) and identify \bar{G}_{q_v} with the tame quotient of G_v , so that s maps to $\sigma \in \text{Gal}(E(\zeta_\ell)/K)$.

Observe first that by repeatedly applying the Leray-Serre spectral sequence to $G_v \supset I_v \supset P_v$ and $\mathrm{ad}^0(\bar{\rho})$, one obtains the short exact sequence

$$0 \rightarrow \mathrm{ad}^0(\bar{\rho})/(s-1)\mathrm{ad}^0(\bar{\rho}) \rightarrow H^1(\bar{G}_{q_v}, \mathrm{ad}^0(\bar{\rho})) \rightarrow (\mathrm{ad}^0(\bar{\rho})(-1))^s \rightarrow 0 \quad (7)$$

and isomorphisms $H^i(G_v, \mathrm{ad}^0(\bar{\rho})) \cong H^i(\bar{G}_{q_v}, \mathrm{ad}^0(\bar{\rho}))$ for all $i \geq 0$ since P_v acts trivially on $\mathrm{ad}^0(\bar{\rho})$. This allows us to regard $L_{v,\mathrm{unr}}$ as a subspace of $\mathrm{ad}^0(\bar{\rho})/(s-1)\mathrm{ad}^0(\bar{\rho}) = \mathrm{ad}^0(\bar{\rho})/(\sigma-1)\mathrm{ad}^0(\bar{\rho})$.

Moreover, the short exact sequence (7) can be given an explicit interpretation in terms of 1-cocycles representing cohomology classes. Namely, any 1-cocycle c of \bar{G}_{q_v} with values in $\mathrm{ad}^0(\bar{\rho})$ is uniquely determined by its values $c(s), c(t)$. These are subject to the conditions $c(s) \in \mathrm{ad}^0(\bar{\rho})$ and $c(t) \in (\mathrm{ad}^0(\bar{\rho})(-1))^s$, that is, $c(t) \in \mathrm{ad}^0(\bar{\rho})$ satisfies $sc(t) = 1/q_v c(t)$. Furthermore, the 1-coboundaries are precisely the 1-cocycles with $c(s) \in (s-1)\mathrm{ad}^0(\bar{\rho})$ and $c(t) = 0$.

For $\mathrm{ad}^0(\bar{\rho})(1)$, one has analogous results. Namely, $H^i(G_v, \mathrm{ad}^0(\bar{\rho})(1)) \cong H^i(\bar{G}_{q_v}, \mathrm{ad}^0(\bar{\rho})(1))$ for all $i \geq 0$, and there is the short exact sequence

$$0 \rightarrow \mathrm{ad}^0(\bar{\rho})(1)/(s-1)\mathrm{ad}^0(\bar{\rho})(1) \rightarrow H^1(\bar{G}_{q_v}, \mathrm{ad}^0(\bar{\rho})(1)) \rightarrow \mathrm{ad}^0(\bar{\rho})^s \rightarrow 0.$$

It identifies $L_{v,\mathrm{unr}}^\perp := L_v^\perp \cap H_{\mathrm{unr}}^1(G_v, \mathrm{ad}^0(\bar{\rho})(1))$ as a subspace of the module $\mathrm{ad}^0(\bar{\rho})(1)/(\sigma-1)\mathrm{ad}^0(\bar{\rho})(1)$. In the same way as sequence (7), it can be rephrased using cocycles.

Clearly, the subspace $L_{v,\mathrm{unr}}$ of $\mathrm{ad}^0(\bar{\rho})/(\sigma-1)\mathrm{ad}^0(\bar{\rho})$ only depends on the element $\sigma \in \mathrm{Gal}(E(\xi_\ell)/K)$ and the choice of λ and not on the place v . For the inclusion $L_{v,\mathrm{unr}}^\perp \subset \mathrm{ad}^0(\bar{\rho})(1)/(\sigma-1)\mathrm{ad}^0(\bar{\rho})(1)$, this is not immediate since it was defined using the pairing (4), which in turn was based on Tate local duality. Also, while $L_{v,\mathrm{unr}}$ is built out of diagonal blocks, as is apparent from the construction of (R_v, ρ_v) , in general, $L_{v,\mathrm{unr}}^\perp$ is not of such a form.

Remark 4.6

Leaving the details to the reader, the following is an example in which $L_{v,\mathrm{unr}}^\perp$ is not composed of diagonal blocks. Suppose χ_ℓ is of order $\ell-1$ and $\bar{\rho}_v \cong \bigoplus_{i=1}^{\ell-1} \mathbf{F}(\ell-i)$. Then with respect to the given diagonal block form of $\bar{\rho}_v$, the unramified cocycles in $H^1(G_v, \mathrm{ad}^0(\bar{\rho})(1))$ can be nonzero precisely at the entries $(1, 2), (2, 3), \dots, (l-2, l-1)$, and $(l-1, 1)$. The same therefore holds for cocycles from $L_{v,\mathrm{unr}}^\perp \subset H_{\mathrm{unr}}^1(G_v, \mathrm{ad}^0(\bar{\rho})(1))$ for any choice of λ .

For later use, we now show that $L_{v,\mathrm{unr}}^\perp \subset \mathrm{ad}^0(\bar{\rho})(1)/(\sigma-1)\mathrm{ad}^0(\bar{\rho})(1)$, too, only depends on σ and λ and not on v . This is done by expressing Tate local duality for the pairing (4) as well as the isomorphism $H^2(G_v, \mathbf{F}(1)) \cong \mathbf{F}$ in terms of cocycle representatives.

Let us first turn to $H^2(G_v, \mathbf{F}(1))$. Since $\hat{\mathbf{Z}}$ and $\hat{\mathbf{Z}}'$ are of cohomological dimension one, a Leray-Serre spectral sequence argument yields isomorphisms $H^2(G_v, \mathbf{F}(1)) \cong H^2(\overline{G}_{q_v}, \mathbf{F}(1)) \cong \mathbf{F}$. Recall (a) that elements of $H^2(\overline{G}_{q_v}, \mathbf{F})$ classify extensions of \overline{G}_{q_v} by \mathbf{F} , and (b) that elements of this cohomology group may be represented by normalized 2-cocycles, that is, maps

$$[\cdot, \cdot]: \overline{G}_{q_v} \times \overline{G}_{q_v} \rightarrow \mathbf{F},$$

which satisfy $[1, g] = [g, 1] = 0$ for all $g \in \overline{G}_{q_v}$, and

$$f[g, h] - [fg, h] + [f, gh] - [f, g] \quad \forall f, g, h \in \overline{G}_{q_v}.$$

Note that, in our situation, $f[g, h] = [g, h]$; but we decided to leave f in the notation to remind the reader of the condition of a normalized 2-cocycle also in the case of nontrivial coefficients.

Regarding the duality pairing (4), we have the following results.

LEMMA 4.7

In terms of normalized 2-cocycles, an isomorphism

$$H^2(\overline{G}_{q_v}, \mathbf{F}(1)) \xrightarrow{\cong} \mathbf{F}$$

is given by

$$[\cdot, \cdot] \mapsto \sum_{i=1}^{q_v^{\ell-1}-1} [t^i, t] + [t^{q_v^{\ell-1}}, s^{\ell-1}] - [s^{\ell-1}, t] \in \mathbf{F}.$$

We do not claim that the isomorphism we construct is the canonical one. But since any two isomorphisms only differ by multiplication with some element of \mathbf{F}^* , the choice of isomorphism does not affect the definition of the annihilator under local Tate duality.

LEMMA 4.8

With respect to the isomorphism of Lemma 4.7, the trace pairing

$$H^1(G_v, \text{ad}^0(\overline{\rho})) \times H^1(G_v, \text{ad}^0(\overline{\rho})(1)) \longrightarrow \mathbf{F}$$

is given explicitly in terms of 1-cocycles as follows. Let c_1 and c_2 be 1-cocycles of $H^1(\overline{G}_{q_v}, \text{ad}^0(\overline{\rho}))$ and $H^1(\overline{G}_{q_v}, \text{ad}^0(\overline{\rho})(1))$, respectively. Then the image of (c_1, c_2) under the pairing is given by

$$\text{Trace}(c_1(s)c_2(t) - c_2(s)c_1(t)) \in \mathbf{F},$$

unless $\ell = 2$ and $q_v \equiv 3 \pmod{4}$. In the latter case, it is $\text{Trace}(c_1(s)c_2(t) + c_2(s)c_1(t) + c_1(t)c_2(t))$.

COROLLARY 4.9

$L_{v,\text{unr}}$ and $L_{v,\text{unr}}^\perp$ only depend on σ and the choice of eigenvalue λ .

The proof of the corollary shows that for any $L_v \subset H^1(G_v, \text{ad}^0(\bar{\rho}))$ which only depends on σ , the modules $L_{v,\text{unr}}$ as well as $L_{v,\text{unr}}^\perp$ only depend on σ .

Based on the corollary, we introduce the following notation.

Definition 4.10

Suppose σ is the image of Frob_v in $\text{Gal}(E(\zeta_\ell)/K)$. Then we write

$$L_{\sigma,\text{unr}} \subset \text{ad}^0(\bar{\rho})/(\sigma - 1)\text{ad}^0(\bar{\rho}) \quad \text{and} \quad L_{\sigma,\text{unr}}^\perp \subset \text{ad}^0(\bar{\rho})(1)/(\sigma - 1)\text{ad}^0(\bar{\rho})(1)$$

for $L_{v,\text{unr}}$ and $L_{v,\text{unr}}^\perp$, respectively. If we also want to include the choice of λ in the notation, we write $L_{\sigma,\lambda,\text{unr}}$ and $L_{\sigma,\lambda,\text{unr}}^\perp$, respectively.

Proof of Corollary 4.9

On rings of characteristic ℓ , the definition of \mathcal{C}_v depends only on σ ; hence so does L_v . Then, clearly, the modules $L_{v,\text{unr}} = H_{\text{unr}}^1(G_v, \text{ad}^0(\bar{\rho})) \cap L_v \subset \text{ad}^0(\bar{\rho})/(\sigma - 1)\text{ad}^0(\bar{\rho})$ as well as

$$\bar{L}_v := (L_v + H_{\text{unr}}^1(G_v, \text{ad}^0(\bar{\rho}))) / H_{\text{unr}}^1(G_v, \text{ad}^0(\bar{\rho})) \subset (\text{ad}^0(\bar{\rho})(-1))^\sigma$$

only depend on σ . Since $H_{\text{unr}}^1(G_v, \text{ad}^0(\bar{\rho}))$ and $H_{\text{unr}}^1(G_v, \text{ad}^0(\bar{\rho})(1))$ are orthogonal under the trace pairing, Lemma 4.8 shows that $L_{v,\text{unr}}^\perp$ is the module of those unramified cocycles \tilde{c}_2 of G_v for which $\tilde{c}_2(s)$ lies in

$$\{m' \in \text{ad}^0(\bar{\rho})(1) \mid \text{Trace}(m \cdot m') = 0 \quad \forall m' \in \bar{L}_v\}.$$

Since the latter module only depends on σ , so does $L_{v,\text{unr}}^\perp$. □

Proof of Lemma 4.8

We assume that we have proved Lemma 4.7. In terms of 1-cocycles, the map

$$H^1(\bar{G}_{q_v}, \text{ad}^0(\bar{\rho})) \times H^1(\bar{G}_{q_v}, \text{ad}^0(\bar{\rho})) \rightarrow H^2(\bar{G}_{q_v}, \text{ad}^0(\bar{\rho}) \otimes \text{ad}^0(\bar{\rho})(1))$$

is given by mapping a pair (c_1, c_2) to the (normalized) 2-cocycle defined by $[f, g] := c_1(f) \otimes c_2(g)$. If we compose this with the map on cohomology induced from the trace map

$$\text{ad}^0(\bar{\rho}) \otimes \text{ad}^0(\bar{\rho})(1) \rightarrow \mathbf{F}(1) : A \otimes B \mapsto \text{Trace}(AB),$$

we obtain the (normalized) 2-cocycle defined by $[f, g] := \text{Trace}(c_1(f)c_2(g)) \in \mathbf{F}(1)$. By Lemma 4.7, it follows that the pair (c_1, c_2) is mapped to

$$\text{Trace} \left(\sum_{i=1}^{q_v^{\ell-1}-1} c_1(t^i)c_2(t) + c_1(t^{q_v^{\ell-1}})c_2(s^{\ell-1}) - c_1(s^{\ell-1})c_2(t) \right) \in \mathbf{F}.$$

Because c_1 restricted to $\hat{\mathbf{Z}}'$ is a homomorphism, we have $c_1(t^i) = ic_1(t)$. So the sum simplifies to

$$c_1(t)c_2(t) \sum_{i=1}^{q_v^{\ell-1}-1} i = c_1(t)c_2(t)q_v^{\ell-1}(q_v^{\ell-1}-1)/2.$$

As $q_v^{\ell-1} \equiv 1 \pmod{\ell}$, this sum is zero unless $\ell = 2$ and $q_v \equiv 3 \pmod{4}$. In the latter case, it is $c_1(t)c_2(t)$. For the same reason, the term $c_1(t^{q_v^{\ell-1}}) = q_v^{\ell-1}c_1(t) = c_1(t)$.

To complete the proof of Lemma 4.8, it now suffices to show that we may replace $c_2(s^{\ell-1})$ by $-c_2(s)$ (and, similarly, $c_1(s^{\ell-1})$ by $-c_1(s)$). An easy calculation shows that $\text{Trace}(c_1(t)(\sigma - 1)c_2(s)) = 0$. Also, we have $c_2(s^{\ell-1}) = (1 + \sigma + \dots + \sigma^{\ell-2})c_2(s)$. Combining the previous two observations, we find

$$\text{Trace}(c_1(t)c_2(s^{\ell-1})) = \text{Trace}(c_1(t)(\ell - 1)c_2(s)) = -\text{Trace}(c_1(t)c_2(s)),$$

as asserted. The argument for $c_1(s^{\ell-1})$ is analogous. \square

Proof of Lemma 4.7

By the Leray-Serre spectral sequence applied to

$$\hat{\mathbf{Z}}' \rtimes (\ell - 1)\hat{\mathbf{Z}} \subset \hat{\mathbf{Z}}' \rtimes \hat{\mathbf{Z}}$$

and the module $\mathbf{F}(1)$, we obtain an isomorphism

$$H^2(\hat{\mathbf{Z}}' \rtimes \hat{\mathbf{Z}}, \mathbf{F}(1)) \cong (H^2(\hat{\mathbf{Z}}' \rtimes (\ell - 1)\hat{\mathbf{Z}}, \mathbf{F}(1))^{\mathbf{Z}/(\ell-1)},$$

given by restriction. The point is that the action of $\hat{\mathbf{Z}}' \times (\ell - 1)\hat{\mathbf{Z}}$ on $\mathbf{F}(1)$ is trivial (the residue field of the corresponding local Galois extension has order $q_v^{\ell-1}$ and, hence, contains a primitive $(\ell - 1)$ th root of unity). Since both $H^2(\dots)$ -terms are isomorphic to \mathbf{F} , it is not necessary to take invariants on the right for there to be an isomorphism. So it suffices to show that the identification asserted in Lemma 4.7 is given by first restricting normalized 2-cocycles and then giving an isomorphism $H^2(\hat{\mathbf{Z}}' \rtimes (\ell - 1)\hat{\mathbf{Z}}, \mathbf{F}) \cong \mathbf{F}$.

For the latter, we use the interpretation in terms of extension classes (cf. [W, Sect. 6.6]). So let $[\cdot, \cdot]$ be a normalized 2-cocycle of $H^2(\hat{\mathbf{Z}}' \rtimes (\ell - 1)\hat{\mathbf{Z}}, \mathbf{F})$. Then

the corresponding extension G can be described as the group whose underlying elements are pairs (a, x) , $a \in \mathbf{F}$, $x \in \hat{\mathbf{Z}}' \rtimes \hat{\mathbf{Z}}$, and whose composition law is given by $(a, x)(b, y) = (a + x \cdot b + [x, y], xy)$. The cocycle is trivial if and only if the group G is split, and this in turn is equivalent to the existence of elements $a, b \in \mathbf{F}$ such that $\tilde{s} := (a, s^{\ell-1})$ and $\tilde{t} := (b, t)$ satisfy

$$\tilde{s}\tilde{t} = \tilde{t}^{q_v^{\ell-1}}\tilde{s}. \quad (8)$$

Using the composition law, one can compute both sides of (8) as elements in $\mathbf{F} \rtimes (\hat{\mathbf{Z}}' \rtimes (\ell-1)\hat{\mathbf{Z}})$. The components in with values in $\hat{\mathbf{Z}}' \rtimes (\ell-1)\hat{\mathbf{Z}}$ always agree. So let us denote the difference of the \mathbf{F} -component by $d([\cdot, \cdot])$ (since it depends on $[\cdot, \cdot]$). It is given by

$$d([\cdot, \cdot]) = \sum_{i=1}^{q_v^{\ell-1}-1} [t^i, t] + [t^{q_v^{\ell-1}}, s^{\ell-1}] - [s^{\ell-1}, t].$$

The assignment $[\cdot, \cdot] \mapsto d([\cdot, \cdot]) \in \mathbf{F}$ is \mathbf{F} -linear, and we have argued that it takes the value zero only if $[\cdot, \cdot]$ is a 2-coboundary. Therefore d induced an isomorphism

$$d: H^2(\hat{\mathbf{Z}}' \rtimes (\ell-1)\hat{\mathbf{Z}}, \mathbf{F}) \cong \mathbf{F} : [\cdot, \cdot] \mapsto d([\cdot, \cdot]).$$

Given a 2-cocycle for $\hat{\mathbf{Z}}' \rtimes \hat{\mathbf{Z}}$, restricting it to $\hat{\mathbf{Z}}' \rtimes (\ell-1)\hat{\mathbf{Z}}$ and applying d yields precisely the formula in Lemma 4.7, and so its proof is completed. \square

5. Local lifting problems at r -places

Regarding places at which $\bar{\rho}$ is ramified, one has the following results.

PROPOSITION 5.1

Suppose that $\bar{\rho}(I_v)$ is of order prime to ℓ . Define the functor $\mathcal{C}_v: \mathcal{A} \rightarrow \mathbf{Sets}$ by

$$R \mapsto \{\rho: G_v \rightarrow \mathrm{GL}_n(R) \mid \rho \pmod{\mathfrak{m}_R} = \bar{\rho}_v, \rho(I_v) \cong \bar{\rho}(I_v), \det \rho = \eta_v\},$$

and define L_v as the corresponding subspace in $H^1(G_v, \mathrm{ad}^0(\bar{\rho}))$. Then (\mathcal{C}_v, L_v) satisfies the conditions P1–P7 of [T], the conductors of $\bar{\rho}_v$ and of any lift $\rho \in \mathcal{C}_v(R)$, $R \in \mathcal{A}$ agree, $L_v = H_{\mathrm{unr}}^1(G_v, \mathrm{ad}^0(\bar{\rho}))$, and $\dim L_v = h^0(G_v, \mathrm{ad}^0(\bar{\rho}))$.

Proof

Except for the assertion on conductors, this is essentially [T, Exam. E1], and so we only prove the latter part. For any ring $R \in \mathcal{A}$ and $\rho \in \mathcal{C}_v(R)$, let $V_\rho(R)$ denote the $R[G_v]$ -module defined by ρ . The kernel of $\mathrm{GL}_n(R) \rightarrow \mathrm{GL}_n(\mathbf{F})$ is a pro- ℓ group and thus prime to p . Therefore the Swan conductors of ρ and $\bar{\rho}_v$ are the same. The module

$V_\rho(R)$ is free over R , and thus the difference of the valuations of the conductors of the two representations is given by

$$\text{rank}_R V_\rho(R)^{I_v} - \dim_{\mathbf{F}} V_{\bar{\rho}_v}(\mathbf{F})^{I_v}. \quad (9)$$

Since G_v acts on both representations via the same quotient \bar{T}_v , which is prime to ℓ , there is a natural equivalence between $R[\bar{T}_v]$ -representations, which are free and finite over R , and $\mathbf{F}[\bar{T}_v]$ -representations given by reduction modulo \mathfrak{m}_R . In particular, both categories are semisimple, and thus expression (9) is well defined. Furthermore, this implies that the number of trivial components contained in $V_\rho(R)$, as an \bar{T}_v -module, is the same as that of $V_{\bar{\rho}_v}(\mathbf{F})$ and, hence, that the difference (9) is zero, as asserted. \square

PROPOSITION 5.2

Suppose that $\bar{\rho}_v$ is at most tamely ramified and that $h^0(G_v, \text{ad}^0(\bar{\rho})) < h^0(G_v, \text{ad}(\bar{\rho}))$. Then there exists a pair (\mathcal{C}_v, L_v) that satisfies conditions P1–P7 of [T] with $L_v = H_{\text{unr}}^1(G_v, \text{ad}^0(\bar{\rho}))$ and $\dim L_v = h^0(G_v, \text{ad}^0(\bar{\rho}))$, is compatible with η_v , and is such that the conductors of $\bar{\rho}_v$ and any lift $\rho \in \mathcal{C}_v(R)$, $R \in \mathcal{A}$, agree.

The proof of Proposition 5.2 occupies the remainder of this section.

Remark 5.3

One can construct examples that show that the condition $\dim_{\mathbf{F}} \text{ad}^0(\bar{\rho})^{G_v} < \dim_{\mathbf{F}} \text{ad}(\bar{\rho})^{G_v}$ is necessary. The latter is automatically satisfied if $\ell \nmid n$.

Remark 5.4

If $\bar{\rho}(I_v)$ is of order prime to ℓp , we may apply either Proposition 5.1 or Proposition 5.2 to obtain a pair (\mathcal{C}_v, L_v) . The pairs so obtained do have similar properties; and, in fact, in Remark 5.10, we explain why the two are isomorphic.

We need a number of preparations for the proof of Proposition 5.2. Since all representations that intervene factor via the tame quotient \bar{G}_{q_v} of G_v , we fix for the rest of this section the usual (topological) generators s, t of \bar{G}_{q_v} satisfying the relation $sts^{-1} = t^{q_v}$.

For $B \in \text{GL}_n(W(\mathbf{F}))$, we denote by V the corresponding $(W(\mathbf{F})[\Theta])$ -module on $W(\mathbf{F})^n$ by having Θ act via B . Let $\mathbf{Q}_{\mathbf{F}}$ denote the fraction field of $W(\mathbf{F})$. We say that $B \in \text{GL}_n(W(\mathbf{F}))$ is a *minimal lift* of its reduction $\bar{B} \in \text{GL}_n(\mathbf{F})$ if $V = \bigoplus_i V_{i,s} \otimes_{W(\mathbf{F})} V_{i,u}$, where the $V_{i,\cdot}$ are $(W(\mathbf{F})[\Theta])$ -modules such that:

- (i) on $V_{i,u}$, the matrix representing Θ is $W(\mathbf{F})$ -conjugate to a regular unipotent matrix in Jordan form;

- (ii) on $V_{i,s}$, the characteristic polynomial of Θ is irreducible, and its roots are Teichmüller lifts of elements in $\overline{\mathbf{F}}$.

LEMMA 5.5

Any $\overline{B} \in \mathrm{GL}_n(\mathbf{F})$ has a minimal lift to $\mathrm{GL}_n(W(\mathbf{F}))$.

Proof

Let $\overline{V} := \mathbf{F}^n$ be the $\mathbf{F}[\Theta]$ -module $\overline{V} := \mathbf{F}^n$ obtained by having Θ act as \overline{B} . We choose a decomposition $\overline{V} \cong \oplus \overline{V}_i$ into indecomposable submodules \overline{V}_i with respect to the action of Θ . On \overline{V}_i , the action of Θ decomposes into commuting semisimple and unipotent parts defined over \mathbf{F} . For instance, by considering Jordan normal forms over $\overline{\mathbf{F}}$, one shows that, correspondingly, one has $\overline{V}_i \cong \overline{V}_{i,s} \otimes_{\mathbf{F}} \overline{V}_{i,u}$, where $\overline{V}_{i,s}$ is a semisimple representation of Θ and $\overline{V}_{i,u}$ is a unipotent representation of Θ . Because \overline{V}_i is indecomposable, the characteristic polynomial of Θ on $\overline{V}_{i,s}$ is irreducible over \mathbf{F} . For the same reason, the action of Θ on $\overline{V}_{i,u}$ is by a regular unipotent matrix. So we may assume that the operation of Θ on $\overline{V}_{i,s}$ is given by a companion matrix whose characteristic polynomial is irreducible over \mathbf{F} and that its operation on $\overline{V}_{i,u}$ is given by a single Jordan block with eigenvalue 1.

We now lift Θ on $\overline{V}_{i,u}$ to a single Jordan block with eigenvalue 1 over $W(\mathbf{F})$ and Θ on $\overline{V}_{i,s}$ to a companion matrix with eigenvalues the Teichmüller lifts of those of Θ on $\overline{V}_{i,s}$. The corresponding representations $V_{i,u}$ and $V_{i,s}$ combine to give a representation of $W(\mathbf{F})[\Theta]$ on $V = \bigoplus_i V_{i,s} \otimes_{W(\mathbf{F})} V_{i,u}$ which has all the required properties. Therefore the matrix representing this Θ is a minimal lift of \overline{B} . \square

To formulate some further auxiliary results, suppose B is a minimal lift of $\overline{B} := \overline{\rho}(t)$, and define $\mathcal{M} := M_n(W(\mathbf{F}))/\{AB - B^{q_v}A \mid A \in W(\mathbf{F})\}$. Then we have the following lemma.

LEMMA 5.6

- (i) The module \mathcal{M} is flat over $W(\mathbf{F})$.
(ii) There exists a lift $\rho_0: G_v \twoheadrightarrow \overline{G}_v \longrightarrow \mathrm{GL}_n(W(\mathbf{F}))$ with $\rho_0(t) = B$.

Proof

In the case when $\overline{\rho}(I_v)$ is an ℓ -group, part (i) was shown during the proof of [B1, Proposition 3.2]. This is used below.

For the general case, let Θ and the $V_{i,\cdot}$ be as in the definition of minimal lift of B . It is not difficult to see from condition (ii) that we may assume that Θ on $V_{i,s}$ is given as a companion matrix $B_{i,s}$. Now let \mathbf{F}' be a finite extension of \mathbf{F} which contains all eigenvalues of \overline{B} . Then, clearly, over $W(\mathbf{F}')$ the companion matrices $B_{i,s}$ may

be diagonalized. Moreover, this diagonalization procedure commutes with reduction modulo ℓ . Since the base change $_ \otimes_{W(\mathbf{F})} W(\mathbf{F}')$ is faithfully flat, for the proof of 5.6, we from now on assume that \mathbf{F} contains all the eigenvalues of \overline{B} .

To proceed with the proof of Lemma 5.6, we require a normal form for the pair of matrices $\overline{A}, \overline{B}$. Observe first that the relation $\overline{A} \overline{B} \overline{A}^{-1} = \overline{B}^{q_v}$ implies that the operation $x \mapsto x^{q_v}$ acts on the set of eigenvalues of \overline{B} . Second, since these eigenvalues lie in \mathbf{F} , the matrix \overline{B} can be brought into Jordan normal form over \mathbf{F} . The reader may now easily fill in the details for the following result.

LEMMA 5.7

Let d denote the number of orbits under $x \mapsto x^{q_v}$ among the eigenvalues of \overline{B} , and choose representatives $\mu_i \in \mathbf{F}$, $i = 1, \dots, d$, for the orbits. By m_i , we denote the length of the orbit of μ_i ; and by $\hat{\mu}_i$, we denote the Teichmüller lift of μ_i . Then:

- (i) $\mu_i^{q_v^{m_i}} = \mu_i$; and the elements $\mu_i^{q_v^j}$, $i = 1, \dots, d$, $j = 1, \dots, m_i$ are pairwise disjoint and form a complete list of the eigenvalues of $\overline{B} := \overline{\rho}(t)$; and
- (ii) with respect to a suitable basis, one has

$$B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_d \end{pmatrix}, \quad \text{where each } B_i = \begin{pmatrix} B_{i,1} & & 0 \\ & \ddots & \\ 0 & & B_{i,m_i} \end{pmatrix}$$

is a square matrix and, for fixed i , the $B_{i,j}$ can be written as $B_{i,j} = \hat{\mu}_i^{q_v^{j-1}} U_i$ for some unipotent U_i in Jordan form, independent of j .

Furthermore, if B is given as in (ii), then $\overline{A} := \overline{\rho}(s)$ takes the form

$$\overline{A} = \begin{pmatrix} \overline{A}_1 & & 0 \\ & \ddots & \\ 0 & & \overline{A}_d \end{pmatrix} \quad \text{with } \overline{A}_i = \begin{pmatrix} 0 & \overline{A}_{i,1} & 0 & \cdots & 0 \\ & 0 & \overline{A}_{i,2} & \ddots & \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & & 0 & \overline{A}_{i,m_i-1} \\ \overline{A}_{i,m_i} & 0 & \cdots & & 0 \end{pmatrix},$$

and the $\overline{A}_{i,j}$ satisfy the relation $\overline{A}_{i,j} U_i = U_i^{q_v} \overline{A}_{i,j}$.

We continue with the proof of Lemma 5.6. Since (i) is known in the case where $\rho(I_v)$ is an ℓ -group, there exist matrices $A_{0,i,j}$ over $W(\mathbf{F})$ that satisfy $A_{0,i,j} U_i = U_i^{q_v} A_{0,i,j}$ for all i, j and whose reduction modulo ℓ agrees with $\overline{A}_{i,j}$. Let A_0 be composed from the $A_{0,i,j}$ in the same way as \overline{A} is from the $\overline{A}_{i,j}$. Then A_0 is a lift to

$W(\mathbf{F})$ of \bar{A} such that $A_0 B A_0^{-1} = B^{q_v}$. (This proves Lemma 5.6 only under the further hypothesis that \mathbf{F} contains all eigenvalues of B .)

We now consider the exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow M_n(W(\mathbf{F})) \xrightarrow{A \mapsto AB - B^{q_v} A} M_n(W(\mathbf{F})) \rightarrow \mathcal{M} \rightarrow 0, \quad (10)$$

where $\mathcal{K} := \{A \in W(\mathbf{F}) : AB = B^{q_v} A\}$. To complete (i), we need to show that the generic rank of \mathcal{M} is the same as its special rank. Using the relation $A_0 B A_0^{-1} = B^{q_v}$, the middle homomorphism in (10) is equivalently given by $A \mapsto A_0(A_0^{-1}AB - BA_0^{-1}A)$. Thus if we apply the isomorphism $M_n(W(\mathbf{F})) \rightarrow M_n(W(\mathbf{F})) : A \mapsto A_0^{-1}A$ to the middle terms in (10), we obtain the isomorphic exact sequence

$$0 \rightarrow \mathcal{K}' \rightarrow M_n(W(\mathbf{F})) \xrightarrow{A' \mapsto A'B - BA'} M_n(W(\mathbf{F})) \rightarrow \mathcal{M}' \rightarrow 0 \quad (11)$$

with kernel $\mathcal{K}' = \{A' \in M_n(W(\mathbf{F})) : A'B = BA'\} \cong \mathcal{K}$ and cokernel $\mathcal{M}' = M_n(W(\mathbf{F}))/\{A'B - BA' | A' \in W(\mathbf{F})\} \cong \mathcal{M}$. We need to prove that the generic and special ranks of \mathcal{M}' agree.

Counting dimensions in the exact sequence (10) and in the corresponding sequence for the reduction modulo ℓ , it suffices to show that the dimensions of $\mathcal{K}' \otimes_{W(\mathbf{F})} \mathbf{Q}_{\mathbf{F}}$ and of $\overline{\mathcal{K}'} := \{\bar{A}' \in M_n(\mathbf{F}) | \bar{A}' \bar{B} = \bar{B} \bar{A}'\}$ agree. Because the $B_{i,j}$ have distinct eigenvalues modulo ℓ , the matrices $\bar{A}' \in \overline{\mathcal{K}'}$ and $A' \in \mathcal{K}'$, respectively, have the same block form as B . So we may consider blocks for each pair i, j separately. Therefore it suffices to prove the assertion in the case where B is a single Jordan block with eigenvalue 1. This case was treated explicitly in the proof of [B1, Proposition 3.2]. The proof of (i) is now complete.

It remains to deduce (ii) from (i). Because \mathcal{M} is flat, the reduction modulo ℓ of the exact sequence (10) remains exact, and so the kernel of the reduction is $\mathcal{K}/\ell\mathcal{K}$. The matrix $\bar{A} = \bar{\rho}(s) \in M_n(\mathbf{F})$ lies in this kernel and is therefore the reduction modulo ℓ of a matrix $A \in \mathcal{K}$. Because A and B satisfy the same relations as s, t , the desired lift exists, and Lemma 5.6 is thus proved. \square

As a corollary to the above proof and with $A_0 \in \mathrm{GL}_n(W(\mathbf{F}))$ as in the proof, we record the following technical result, obtained by using base change and flatness.

COROLLARY 5.8

Let $B \in \mathrm{GL}_n(W(\mathbf{F}))$ be a minimal lift of $\bar{B} \in \mathrm{GL}_n(\mathbf{F})$. Then for any $R \in \mathcal{A}$, the submodules $\mathcal{K}(R) := \{A \in M_n(R) : AB = B^{q_v} A\}$ and $\mathcal{K}'(R) := \{A \in M_n(R) : AB = BA\}$ of $M_n(R)$ are free and direct summands of R -rank independent of R . Moreover, $\mathcal{K}'(R) = A_0^{-1} \mathcal{K}(R)$.

Proof of Proposition 5.2

For a subset $\{b_1, \dots, b_m\}$ of $\mathcal{K}(W(\mathbf{F}))$, which is specified below, and for indeterminates x_1, \dots, x_m , we define

$$\begin{aligned}\mathcal{S}_v &:= A_0 + \sum x_i b_i \in \mathrm{GL}_n(R_v), \mathcal{T}_v := B, \\ R_v &:= W(\mathbf{F})[[x_1, \dots, x_m]] / (\det \mathcal{S}_v - \eta_v(s)), \\ \rho_v: G_v &\twoheadrightarrow \overline{G}_{q_v} \cong \hat{\mathbf{Z}}' \rtimes \hat{\mathbf{Z}} \longrightarrow \mathrm{GL}_n(R_v): s \mapsto \mathcal{S}_v, t \mapsto \mathcal{T}_v,\end{aligned}$$

and the functor $\mathcal{C}_v: \mathcal{A} \rightarrow \mathbf{Sets}$ by

$$\begin{aligned}R \mapsto \mathcal{C}_v(R) &:= \left\{ \rho: G_v \rightarrow \mathrm{GL}_n(R) \mid \exists \alpha \in \mathrm{Hom}_{\mathcal{A}}(R_v, R), \right. \\ &\quad \left. \exists M \in 1 + M_n(\mathfrak{m}_R) : \rho = M(\alpha \circ \rho_v)M^{-1} \right\}\end{aligned}$$

Let $L_v \subset H^1(G_v, \mathrm{ad}^0(\overline{\rho}))$ be the subspace corresponding to the lifting problem \mathcal{C}_v ; that is, L_v is defined as in (5). As $\rho_v(t)$ does not deform, the subspace L_v lies inside

$$H_{\mathrm{unr}}^1(G_v, \mathrm{ad}^0(\overline{\rho})) \cong \mathrm{ad}^0(\overline{\rho})^t / (s-1)(\mathrm{ad}^0(\overline{\rho})^t).$$

We denote by $\bar{b}_1, \dots, \bar{b}_m \in \mathcal{K}(\mathbf{F})$ the reductions of the b_i modulo ℓ . So the elements $\overline{A}^{-1} \bar{b}_i$ lie in $\mathcal{K}'(\mathbf{F}) = \mathrm{ad}^0(\overline{\rho})^t$, and an explicit calculation shows that L_v is spanned by the images in $\mathrm{ad}^0(\overline{\rho})^t / (s-1)(\mathrm{ad}^0(\overline{\rho})^t)$ of those linear combinations $\sum y_i \overline{A}^{-1} \bar{b}_i$, $y_i \in \mathbf{F}$, which are consistent with the determinant condition $\det \mathcal{S}_v = \eta(s)$. The latter condition modulo (ℓ, \mathfrak{m}_R^2) means

$$\begin{aligned}\det \overline{A} &\stackrel{!}{=} \det \overline{A} \cdot \det \left(I + \overline{A}^{-1} \left(\sum y_i \bar{b}_i \right) \right); \text{ that is,} \\ 1 &= 1 + \sum y_i \mathrm{Trace}(\overline{A}^{-1} \bar{b}_i).\end{aligned}$$

Thus the above linear combinations satisfy $\sum y_i \mathrm{Trace}(\overline{A}^{-1} \bar{b}_i) = 0$.

Now we fix the choice of the b_i by taking them as a subset of $\mathcal{K}(W(\mathbf{F}))$, whose reduction modulo ℓ forms a basis of $\mathcal{K}(\mathbf{F}) / \{\bar{b} \overline{A} - \overline{A} \bar{b} : \bar{b} \in \mathcal{K}(\mathbf{F})\}$. The elements $\overline{A}^{-1} \bar{b}_i$ then form a basis of

$$\mathrm{ad}(\overline{\rho})^t / (s-1)(\mathrm{ad}(\overline{\rho})^t) = \mathcal{K}'(\mathbf{F}) / \{\bar{A} \bar{b} \overline{A}^{-1} - \bar{b} : \bar{b} \in \mathcal{K}'(\mathbf{F})\}.$$

To pass from $\mathrm{ad}(\overline{\rho})$ to $\mathrm{ad}^0(\overline{\rho})$, we use our assumption

$$h^0(G_v, \mathrm{ad}(\overline{\rho})) > h^0(G_v, \mathrm{ad}^0(\overline{\rho})).$$

Because $(\text{ad}(\bar{\rho})^{I_v})^{G_v/I_v} \cong (\text{ad}(\bar{\rho})^{I_v})_{G_v/I_v}$ as an \mathbf{F} -vector space, and similarly for $\text{ad}^0(\bar{\rho})$, we deduce that $(\text{ad}(\bar{\rho})')/(s-1)(\text{ad}(\bar{\rho})')$ properly contains $(\text{ad}^0(\bar{\rho})')/(s-1)(\text{ad}^0(\bar{\rho})')$. This in turn shows that any element of the module $(\text{ad}^0(\bar{\rho})')/(s-1)(\text{ad}^0(\bar{\rho})')$ can be obtained as a linear combination $\sum y_i \bar{A}^{-1} \bar{b}_i$, which satisfies $\sum y_i \text{Trace}(\bar{A}^{-1} \bar{b}_i) = 0$. This has two consequences. Firstly, we have $L_v = H_{\text{unr}}^1(G_v, \text{ad}^0(\bar{\rho}))$; secondly, the relation $\det \mathcal{S}_v = \eta(s)$ allows one to eliminate one of the variables x_i since this is possible tangentially, and so R_v is smooth of relative dimension $\dim L_v$ over $W(\mathbf{F})$.

Note also that the determinant of ρ_v is the Teichmüller lift of that of $\bar{\rho}_v$. For $\rho_v(t)$, this follows from the construction of B ; for $\rho_v(s)$, it follows from the definition of R_v . This implies the result in general since ρ_v is only tamely ramified.

Let us now verify properties P1–P7 of [T]. As expected, the only property that is nontrivial is P4. To verify it, suppose we are given rings $R_1, R_2 \in \mathcal{A}$, lifts $\rho_i \in \mathcal{C}_v(R_i)$, ideals $I_i \in R_i$, and an identification $R_1/I_1 \cong R_2/I_2$ under which $\rho_1 \pmod{I_1} \equiv \rho_2 \pmod{I_2}$. We need to show that (ρ_1, ρ_2) lies in $\mathcal{C}_v(R)$ for

$$R := \{(r_1, r_2) \in R_1 \oplus R_2 : r_1 \pmod{I_1} = r_2 \pmod{I_2}\}.$$

So let $\alpha_i \in \text{Hom}_{\mathcal{A}}(R_v, R_i)$ and $M_i \in \text{GL}_n(R_i)$ such that $\rho_i = M_i(\alpha_i \circ \rho_v)M_i^{-1}$, $i = 1, 2$. We claim that there exist $\alpha \in \text{Hom}_{\mathcal{A}}(R_v, R)$ and $M \in \text{GL}_n(R)$ with $M \equiv I \pmod{\mathfrak{m}_R}$ such that $(\rho_1, \rho_2) = M(\alpha \circ \rho_v)M^{-1}$. By conjugating ρ_1 by some lift of $M_2 \pmod{I_1}$ to R_1 , we may assume $M_2 = I$.

By an inductive argument, which is left to the reader, one can show the following auxiliary result.

LEMMA 5.9

Suppose $\tilde{R} \in \mathcal{A}$, \tilde{J} is a proper ideal of \tilde{R} , $A' \in A + \sum \beta_i b_i + M_n(\tilde{J})$ for some $\beta_i \in \tilde{R}$, and $A'B = B^{q_v}A'$. Then there exists $\beta'_i \in \tilde{R}$ with $\beta'_i - \beta_i \in \tilde{J}$ and $C \in I + M_n(\tilde{J})$ such that

$$A' = C \left(A + \sum \beta'_i b_i \right) C^{-1}.$$

Continuing with the proof of Proposition 5.2, observe that the condition

$$M_1(\alpha_1 \circ \rho_v)M_1^{-1} \pmod{I_1} = \alpha_2 \circ \rho_v \pmod{I_2} \quad (12)$$

applied to t implies that $M_1 \pmod{I_1}$ commutes with B . By Corollary 5.8, we can find a lift $\tilde{M}_1 \in M_n(R_1)$ of $M_1 \pmod{I_1}$ which commutes with B .

Because of (12) and the choice of \tilde{M}_1 , we can apply Lemma 5.9 to $A' := \tilde{M}_1(\alpha_1 \circ \rho_v(s))\tilde{M}_1^{-1}$. It yields $\alpha'_1 : R_v \rightarrow R_1$ and $C \in I + M_n(I_1)$ such that

$$C(\alpha'_1 \circ \rho_v(s))C^{-1} = \tilde{M}_1(\alpha_1 \circ \rho_v(s))\tilde{M}_1^{-1}$$

and $\alpha'_1 \pmod{I_1} = \alpha_2 \pmod{I_2}$. Define $\tilde{C} := M_1 \tilde{M}_1^{-1} C \in I + M_n(I_1)$. Then

$$\tilde{C}(\alpha'_1 \circ \rho_v) \tilde{C}^{-1} = M_1(\alpha_1 \circ \rho_v) M_1^{-1} = \rho_1.$$

Therefore, if we set $M := (\tilde{C}, I) \in I + M_n(\mathfrak{m}_R)$ and $\alpha := (\alpha'_1, \alpha_2): R_v \rightarrow R$, we have

$$(\rho_1, \rho_2) = M(\alpha \circ \rho_v) M^{-1},$$

and so the proof of P4, and hence of all the axioms of Taylor, is completed.

It remains to prove the assertion on the conductors. As in the proof of Proposition 5.1, the difference in conductors is given by

$$\dim_R V_\rho(R)^{I_v} - \dim_{\mathbf{F}} V_{\bar{\rho}_v}(\mathbf{F})^{I_v},$$

where the notation is analogous to that in the quoted proof. Since I_v is topologically generated by the single element t , whose image is the image of the matrix $B \in \mathrm{GL}_n(W(\mathbf{F}))$, this difference is given by

$$\dim_R \mathcal{H}'(R) - \dim_{\mathbf{F}} \mathcal{H}'(\mathbf{F}).$$

By Corollary 5.8, this difference is zero. This shows that the conductors of ρ and $\bar{\rho}_v$ agree, and Proposition 5.2 is thus proved. \square

Remark 5.10

Suppose now that the image of I_v under $\bar{\rho}$ is of order prime to ℓp . Let (ρ'_v, R'_v) be the versal deformation constructed in Proposition 5.1, and let (ρ_v, R_v) be the one constructed in the previous proof.

The representation ρ_v was constructed so that $\rho_v(t)$ was a minimal lift of \bar{B} . Because ℓ does not divide $\#\bar{\rho}(I_v)$, the matrix \bar{B} is completely reducible. So the $V_{i,u}$ in the definition of minimal lift are 1-dimensional. Therefore $\rho_v(t)$ is completely reducible and of finite order prime to ℓ .

The versality of (ρ'_v, R'_v) shows that there is a morphism $R'_v \rightarrow R_v$ that induces ρ_v from ρ'_v . Because it is an isomorphism on mod ℓ tangent spaces, the morphism is surjective. Since both rings are smooth of the same dimension, it must be bijective. This shows that the two deformations agree.

6. Proof of Theorem 2.4 and the key lemma

Proof of Lemma 3.7

We first prove the following claim.

Claim 1

There exists a finite set T_0 of R -places of type (II), and for each such R -place v a choice of eigenvalue λ_v , as in Definition 2.1, such that

$$H^1_{\{L_v\}}(S \cup T' \cup T_0, \text{ad}^0(\bar{\rho})) \cap H^1(\text{Gal}(E(\zeta_\ell)/K), M_n^0(\mathbf{F})) = 0, \quad (13)$$

$$H^1_{\{L_v^\perp\}}(S \cup T' \cup T_0, \text{ad}^0(\bar{\rho})(1)) \cap H^1(\text{Gal}(E(\zeta_\ell)/K), M_n^0(\mathbf{F})(1)) = 0. \quad (14)$$

We only give the proof of (13), the proof of (14) being analogous.

Let $\sigma_1, \dots, \sigma_s$ be representatives of the different R -classes of type (II). For each σ_i , let $\lambda_{i,j}$, $j = 1, \dots, m_i$, be the list of eigenvalues in \mathbf{F} of multiplicity 2 (in the characteristic polynomial). Pick unramified places $v_{i,j}$, $i = 1, \dots, s$, $j = 1, \dots, m_i$, such that $\text{Frob}_{v_{i,j}} = \sigma_i$ for all i, j . Let $(\mathcal{C}_{v_{i,j}}, L_{v_{i,j}})$ be the lifting problem defined in Section 4 for the pair $(v_{i,j}, \lambda_{i,j})$, and let $T_0 := \{v_{i,j} : i = 1, \dots, s, j = 1, \dots, m_i\}$. Note that the $L_{v_{i,j}, \text{unr}}$ are the space $L_{\sigma_i, \lambda_{i,j}, \text{unr}}$ of Definition 4.10. We consider the following commuting diagram:

$$\begin{array}{ccc} H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\bar{\rho})) \cap H^1_{\{L_v\}}(S \cup T', \text{ad}^0(\bar{\rho})) & \longrightarrow & \prod_{v \in T_0} H^1(\langle \sigma_v \rangle, \text{ad}^0(\bar{\rho})_{\sigma_v}) \\ \downarrow & & \downarrow \\ H^1_{\{L_v\}}(S \cup T', \text{ad}^0(\bar{\rho})) & \longrightarrow & \prod_{v \in T_0} H^1(G_v, \text{ad}^0(\bar{\rho}))/L_v. \end{array}$$

The kernel of the bottom row is $H^1_{\{L_v\}}(S \cup T' \cup T_0, \text{ad}^0(\bar{\rho}))$. By assumption, there are sufficiently many R -classes for $(\mathcal{C}_v, L_v)_{v \in S}$, and so the top horizontal arrow is injective. For each i , the image of the top left term in $\prod_{j=1, \dots, m_i} H^1(\langle \sigma_{v_{i,j}} \rangle, \text{ad}^0(\bar{\rho})_{\sigma_{v_{i,j}}})$ is diagonal. Therefore we may replace the top right term by $\prod_i H^1(\langle \sigma_i \rangle, \text{ad}^0(\bar{\rho})_{\sigma_i})$ and still retain the injectivity of the top horizontal map. Below we show that the induced right vertical arrow

$$\prod_i H^1(\langle \sigma_i \rangle, \text{ad}^0(\bar{\rho})_{\sigma_i}) \rightarrow \prod_{v \in T_0} H^1(G_v, \text{ad}^0(\bar{\rho}))/L_v \quad (15)$$

is injective. An easy diagram chase then shows that the intersection of $H^1_{\{L_v\}}(S \cup T' \cup T_0, \text{ad}^0(\bar{\rho}))$ and $H^1(\text{Gal}(E(\zeta_\ell)/K), \text{ad}^0(\bar{\rho}))$ is zero, as desired.

The injectivity of (15) may be verified on the morphisms

$$H^1(\langle \sigma_i \rangle, \text{ad}^0(\bar{\rho})_{\sigma_i}) \longrightarrow \prod_{j=1, \dots, m_i} H^1(G_{v_{i,j}}, \text{ad}^0(\bar{\rho}))/L_{v_{i,j}},$$

individually. The representation $\text{ad}^0(\bar{\rho})_{\sigma_i}$ itself is a direct sum of adjoint representations on (2×2) -blocks of matrices of trace zero. Furthermore, the image of $H^1(\langle \sigma_i \rangle, \text{ad}^0(\bar{\rho})_{\sigma_i})$ lies in $H_{\text{unr}}^1(G_{v_{i,j}}, \text{ad}^0(\bar{\rho}))/L_{v_{i,j}, \lambda_{i,j}, \text{unr}}$, and $H_{\text{unr}}^1(G_{v_{i,j}}, \text{ad}^0(\bar{\rho}))$ itself breaks up into a direct sum over pieces corresponding to the rational canonical form of $\bar{\rho}(\sigma_i)$. So one is reduced to consider a single (2×2) -block, and we may assume $n = 2$ and $m_i = 1$. But then $L_{v, \text{unr}} = 0$ ($v = v_{i,j}$) by Proposition 4.4, and the map

$$H^1(\langle \sigma_i \rangle, \text{ad}^0(\bar{\rho})) \longrightarrow H_{\text{unr}}^1(G_{v_{i,j}}, \text{ad}^0(\bar{\rho}))$$

is simply given by inflation and is thus injective. Note finally that (13) is preserved under adding further R -primes to T_0 . We have thus proved Claim 1.

By enlarging T' if necessary, we assume from now on that (13) and (14) hold with $T_0 = \emptyset$. We now induct on the dimension of $H_{\{L_v^\perp\}}^1(S \cup T', \text{ad}^0(\bar{\rho}))$ and assume that it contains a nonzero cocycle ϕ . By the formula in Remark 3.5 and our assumptions, the space $H_{\{L_v\}}^1(S \cup T', \text{ad}^0(\bar{\rho}))$ contains a nonzero cocycle ψ as well.

Claim 2

There exists $w \in X \setminus T'$ and an admissible pair (\mathcal{C}_w, L_w) compatible with η such that the following hold:

- (i) $n - 1 = \dim L_w = \dim L_{w, \text{unr}} + 1$,
- (ii) ϕ does not map to zero in $H^1(G_w, \text{ad}^0(\bar{\rho})(1))/L_w^\perp$, and
- (iii) the space $(H_{\text{unr}}^1(G_w, \text{ad}^0(\bar{\rho})) + L_w)/L_w$ lies in the image of the morphism

$$H_{\{L_v\}}^1(S \cup T', \text{ad}^0(\bar{\rho})) \rightarrow H^1(G_w, \text{ad}^0(\bar{\rho}))/L_w.$$

Suppose for a moment that we have proved Claim 2, and let $T'' := T' \cup \{w\}$. The argument given in [T, proof of Lem. 1.2] then shows, by using (i) and (iii), that

$$H_{\{L_v^\perp\}}^1(S \cup T', \text{ad}^0(\bar{\rho})(1)) = H_{\{L_v^\perp\}_{v \in S \cup T' \cup \{L_w^\perp + H_{\text{unr}}^1(G_w, \text{ad}^0(\bar{\rho})(1))\}}}^1(S \cup T'', \text{ad}^0(\bar{\rho})(1)),$$

and, by using (ii), that

$$H_{\{L_v^\perp\}}^1(S \cup T'', \text{ad}^0(\bar{\rho})(1)) \hookrightarrow H_{\{L_v^\perp\}_{v \in S \cup T' \cup \{L_w^\perp + H_{\text{unr}}^1(G_w, \text{ad}^0(\bar{\rho})(1))\}}}^1(S \cup T'', \text{ad}^0(\bar{\rho})(1))$$

is a proper containment, so that the proof of Lemma 3.7 is completed.

Claim 2 is clearly implied by the following.

Claim 3

There exists $w \in X \setminus T'$ and an admissible pair (\mathcal{C}_w, L_w) compatible with η such that

- (i) above holds and, furthermore, that
- (ii') ϕ does not map to zero in $H_{\text{unr}}^1(G_w, \text{ad}^0(\bar{\rho})(1))/L_{w, \text{unr}}^\perp$ and
- (iii') ψ does not map to zero in $H_{\text{unr}}^1(G_w, \text{ad}^0(\bar{\rho}))/L_{w, \text{unr}} (\cong \mathbf{F})$.

To prove Claim 3, note that conditions (13) and (14) imply that the cocycles ψ and ϕ restrict to nonzero homomorphisms

$$\phi: G_{E(\zeta_\ell)} \rightarrow (\mathrm{ad}^0(\bar{\rho}))(1) \quad \text{and} \quad \psi: G_{E(\zeta_\ell)} \rightarrow \mathrm{ad}^0(\bar{\rho}).$$

Let E_ϕ and E_ψ be the fixed fields of the respective kernels. Depending on whether the cyclotomic character χ_ℓ is trivial, they may or may not be equal. The induced morphisms on $\mathrm{Gal}(E_\phi/E(\zeta_\ell))$ and $\mathrm{Gal}(E_\psi/E(\zeta_\ell))$, respectively, are equivariant for $\mathrm{Gal}(E(\zeta_\ell)/K)$. Because $\mathrm{ad}^0(\bar{\rho})$ is irreducible as an $\mathbf{F}_\ell[\mathrm{im}(\bar{\rho})]$ -module, the morphisms ϕ, ψ are bijective. Thereby the group $M := \mathrm{Gal}(E_\psi E_\phi/E(\zeta_\ell))$ may be regarded as an $\mathbf{F}[\mathrm{Gal}(E(\zeta_\ell)/K)]$ -module that surjects onto $\mathrm{ad}^0(\bar{\rho})$ and $\mathrm{ad}^0(\bar{\rho})(1)$.

Now let $\sigma \in \mathrm{Gal}(E(\zeta_\ell)/K)$ represent an R -class, and denote by $\tilde{\sigma}$ a lift of σ to $\mathrm{Gal}(E_\psi E_\phi/K)$. The subspaces $L_{\mathrm{unr},\sigma}$ of $\mathrm{ad}^0(\bar{\rho})/(\sigma - 1)\mathrm{ad}^0(\bar{\rho})$ and $L_{\mathrm{unr},\sigma}^\perp$ of $\mathrm{ad}^0(\bar{\rho})(1)/(\sigma - 1)\mathrm{ad}^0(\bar{\rho})(1)$ from Definition 4.10 are of codimension one with respect to \mathbf{F} . Define $\tilde{L}_{\mathrm{unr},\sigma}$ and $\tilde{L}_{\mathrm{unr},\sigma}^\perp$ as the corresponding subspaces of codimension one in $\mathrm{ad}^0(\bar{\rho})$ and $\mathrm{ad}^0(\bar{\rho})(1)$, respectively. Each of the conditions

$$\psi(\xi) \in -\psi(\tilde{\sigma}) + \tilde{L}_{\mathrm{unr},\sigma} \quad \text{and} \quad \phi(\xi) \in -\phi(\tilde{\sigma}) + \tilde{L}_{\mathrm{unr},\sigma}^\perp,$$

for $\xi \in M = \mathrm{Gal}(E_\psi E_\phi/E(\zeta_\ell))$, determines a hyperplane in the \mathbf{F} -vector space M . As we assumed $|\mathbf{F}| > 2$, the join of these two hyperplanes cannot span all of M ; and, hence, there exists $\xi \in \mathrm{Gal}(E_\psi E_\phi/E(\zeta_\ell))$, which lies on neither. We fix such a ξ and define $\tilde{\xi} := \xi \tilde{\sigma}$.

Since $E_\psi E_\phi$ is Galois over K , by the Čebotarev density theorem, we can choose a place w in $X \setminus T'$ such that $\mathrm{Frob}_w = \tilde{\xi}$. Take \mathcal{C}_w and L_w as constructed in Section 4, so that by Proposition 4.4, condition (i) is satisfied and (\mathcal{C}_w, L_w) is compatible with η . Condition (ii') is satisfied since the image of ϕ in $H_{\mathrm{unr}}^1(G_w, \mathrm{ad}^0(\bar{\rho}))$ is given by the element $\phi(\mathrm{Frob}_w) = \phi(\tilde{\xi}) = \phi(\xi) + \phi(\tilde{\sigma}) \in \mathrm{ad}^0(\bar{\rho})/(\sigma - 1)\mathrm{ad}^0(\bar{\rho})$, which does not lie in $L_{\sigma, \mathrm{unr}}$. Condition (iii') is verified in the same way, and this completes the proof of Lemma 3.7. \square

Proof of Theorem 2.4

By enlarging X , if necessary, we may assume that $\bar{\rho}$ ramifies at all places of S . Using Propositions 5.1 and 5.2, there exist locally admissible pairs $(\mathcal{C}_v, L_v)_{v \in S}$ compatible with η for which one has $\dim L_v = h^0(G_v, \mathrm{ad}^0(\bar{\rho}))$ and such that the conductor (at v) of any lift of type \mathcal{C}_v is the same as that of $\bar{\rho}_v$.

We claim that if $\bar{\rho}$ admits sufficiently many R -classes, then it admits sufficiently many R -classes for $(\mathcal{C}_v, L_v)_{v \in S}$. We only verify the injectivity of the first

homomorphism in Definition 3.6. For this, consider the diagram

$$\begin{array}{ccc}
 H^1(\mathrm{Gal}(E(\zeta_\ell)/K), \mathrm{ad}^0(\bar{\rho})) & \longrightarrow & \prod_{v \in S} H^1(\bar{\rho}(I_v), \mathrm{ad}^0(\bar{\rho})) \\
 \downarrow & & \downarrow \\
 H^1_{\{L_v\}}(S, \mathrm{ad}^0(\bar{\rho})) & \longrightarrow & \prod_{v \in S} H^1(G_v, \mathrm{ad}^0(\bar{\rho}))/L_v.
 \end{array}$$

To prove the claim, it suffices to show that the right vertical arrow is injective. By Propositions 5.1 and 5.2, we have $L_v = H^1_{\mathrm{unr}}(G_v, \mathrm{ad}^0(\bar{\rho}))$ for $v \in S$; and thus by the inflation restriction sequence, the morphism $H^1(G_v, \mathrm{ad}^0(\bar{\rho}))/L_v \hookrightarrow H^1(I_v, \mathrm{ad}^0(\bar{\rho}))$ is a monomorphism. Therefore it suffices to show for each v that

$$H^1(\bar{\rho}(I_v), \mathrm{ad}^0(\bar{\rho})) \longrightarrow H^1(I_v, \mathrm{ad}^0(\bar{\rho}))$$

is injective. This is clear since it is an inflation map.

Having established the existence of sufficiently many R -classes for $\bar{\rho}$, we can apply Lemmas 3.4 and 3.7, and Theorem 2.4 follows readily. \square

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Böckle

Institut für Experimentelle Mathematik, Universität Duisburg-Essen, Campus Essen,
Ellernstrasse 29, D-45326 Essen, Germany; boeckle@exp-math.uni-essen.de

Khare

Department of Mathematics, University of Utah, 155 South 1400 East, Salt Lake City, Utah
84112, USA; shekhar@math.utah.edu; and School of Mathematics, Tata Institute of
Fundamental Research, Homi Bhabha Road, Mumbai 400 005, India; shekhar@math.tifr.res.in