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Algebraic Hecke characters and strictly compatible systems of abelian mod p Galois representations over global fields

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Abstract. In [10] (C R Acad Sci Paris Ser I Math 323(2):117–120, 1996), [11] (Math Res Lett 10(1):71–83 2003), [12] (Can J Math 57(6):1215–1223 2005), Khare showed that any strictly compatible systems of semisimple abelian mod p Galois representations of a number field arises from a unique finite set of algebraic Hecke characters. In this article, we consider a similar problem for arbitrary global fields. We give a definition of Hecke character which in the function field setting is more general than previous definitions by Goss and Gross and define a corresponding notion of compatible system of mod p Galois representations. In this context we present a unified proof of the analog of Khare's result for arbitrary global fields. In a sequel we shall apply this result to strictly compatible systems arising from Drinfeld modular forms, and thereby attach Hecke characters to cuspidal Drinfeld Hecke eigenforms.

1. Introduction

In [10–12], Khare studied strictly compatible systems of abelian mod p Galois representations of a number field. He proves that the semisimplification of any such arises from a direct sum of algebraic Hecke characters, as was suggested by the framework of motives. In particular, this shows that under minimal hypotheses, such as only knowing the mod p reductions, one can reconstruct a motive from an abelian strictly compatible system. Regarding the method, it is remarkable, that the association of the Hecke characters to the strictly compatible systems is based on fairly elementary tools from algebraic number theory. The first such association was based on deep transcendence results by Henniart [9] and Waldschmidt [16] following work of Serre [15]. They reconstruct the Hodge-Tate weights, i.e., the infinity types of the corresponding Hecke characters from each individual member of a strictly compatible p-adic system. Khare in turn uses every member of the strictly compatible system, but from each one only the information contained in the mod p reduction.

An application of the result of Khare is the association by elementary means of a set of Hecke characters to any strictly compatible system of abelian Galois representations over a number field. In [1], we had attached such a compatible system over a global function field to any cuspidal Drinfeld Hecke-eigenform via an Eichler–Shimura type isomorphism. A natural question posed by Goss and, in

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different form, by Serre, cf. [5, p. 414] was whether this strictly compatible system would arise from a set of Hecke characters of A_∞-type, see also [12, § 3]. After studying Khare's result, only an affirmative answer was conceivable. However, the type of Hecke character needed seemed not to be present in the literature. It turned out that the definitions by Goss [6], Gross [7] and others are modeled too closely on the number field case for our purposes. The reason, put simply, is that there is a unique way to embed the rational numbers $\mathbb{Q}$ into any number field, but there are many ways to embed the function field $\mathbb{F}_p(t)$ into a global field of characteristic p, even if one preserves some notion of 'infinite places'.

The aim of the present article is twofold. First we introduce the required type of Hecke character that arise from the strictly compatible systems from [1]. Our treatment includes the usual Hecke characters (of A_∞-type) for number fields, so that our Hecke characters are indeed natural generalizations of classical ones. Second, we prove our central theorem, Theorem 2.21: It asserts that *over any global field there is a bijection between n-element lists of (suitably defined) algebraic Hecke characters and (suitably defined) strictly compatibly systems of abelian semisimple n-dimensional mod p Galois representations*. Our method of proof is modeled at Khare's. In a sequel to the present article, [2], we will apply the theorem to attach a set of Hecke characters to the space of Drinfeld cusp forms of any given weight and level.

The article is structured as follows: This introduction is followed by a subsection on notation used throughout the article. Sect. 2 starts with a number of basic definitions for global fields of any characteristic: *Sets of Hodge-Tate weights*, *Hecke characters* and *strictly compatible systems of Galois representations*. We then recall how to attach such a strictly compatible system to any (finite list of) Hecke character(s) and we state our main result, Theorem 2.21, which is basically the converse to this attachment.

The subsequent Sect. 3 collects some preparatory material for the proof. As should not be unexpected in light of Khare's proof, we present yet another version of the result of Corrales–Rodrigáñez and Schoof. Moreover, we also need a result about the density of fixed points of a correspondence. In the number field case, this is an easy consequence of the Čebotarev density theorem; in the function field case our Theorem 3.5 completes an old result of MacRae [13] and establishes a conjecture of his. The final Sect. 4 gives a unified proof of the main theorem which works in the number and function field setting.

1.1. Notation

A number field is a finite extension of $\mathbb{Q}$. In this article, the term **function field** will mean a finite extension of the field of rational functions $\mathbb{F}(t)$ over some finite field $\mathbb{F}$. Global field will mean either number field or function field. For any global field E the following notation is used:

- By $E^{\mathrm{sep}} \subset E^{\mathrm{alg}}$ we denote fixed separable and algebraic closures of E and by G_E the absolute Galois group $\mathrm{Gal}(E^{\mathrm{sep}}/E)$.

- The set of all places of E is denoted by $\mathcal{P}_E$. By $\mathcal{P}_E^{\mathrm{ar}}$ we denote its archimedean places (of which there may be none) and by $\mathcal{P}_E^{\mathrm{na}}$ its non-archimedean ones.
- For a place $w \in \mathcal{P}_E$ we define E_w as the completion of E at w. For $w \in \mathcal{P}_E^{\mathrm{na}}$ by $\mathcal{O}_{E_w}$, or simply $\mathcal{O}_w$, we denote the valuation ring of E_w, by $\mathfrak{p}_w$ its maximal ideal, by $\mathbb{F}_w$ its residue field and by q_w the cardinality of $\mathbb{F}_w$. We often identify $w \in \mathcal{P}_E^{\mathrm{na}}$ with the corresponding normalized valuation at w, so that $w(E^*) = \mathbb{Z}$.
- For $S \subset \mathcal{P}_E^{\mathrm{na}}$ denote by $\mathcal{O}_{E,S} \subset E$ the subring $\{\alpha \in E \mid \forall w \in \mathcal{P}^{\mathrm{na}} \smallsetminus S : \alpha \in \mathcal{O}_w\}$.
- The ring of adeles of E is denoted by $\mathbb{A}_E$, its idele group by $\mathbb{A}_E^*$.
- If E is a function field, $\varphi_E : E \to E$ denotes the Frobenius endomorphism $x \mapsto x^p$. Over E^{alg} it is an isomorphism, and so is its i-fold iterate $\varphi^i_{E^{\mathrm{alg}}}$ for any $i \in \mathbb{Z}$. We write φ^i_E for the restriction of $\varphi^i_{E^{\mathrm{alg}}}$ to E; it maps onto E^{p^i} ($\subset E$ for $i \geq 0$).

From now on we fix a field characteristic p, i.e., p is either zero or a prime number. We define (K, ∞) to be $\mathbb{Q}$ with its real place, if $p = 0$, and to be an arbitrary global field of characteristic p with a choice $\infty \in \mathcal{P}_K$, if $p > 0$.

The letter L will always denote a finite extension of K contained in K^{alg}. By E, F we denote some global fields of characteristic p which may have no relation to K (if $p > 0$). All Galois representations considered in this article will be continuous representations of G_E or G_F and take values in GL_n over some completion or some residue field of L. We introduce some further notation:

- Places in L above ∞ are called **infinite places**, the other ones **finite places**. The set of all infinite places of L is denoted by $\mathcal{P}_L^{\infty}$ that of all finite places by $\mathcal{P}_L^{\mathrm{fin}}$.
- For $v \in \mathcal{P}_L$ and $\sigma : E \to L$ a homomorphism, $v \circ \sigma$ denotes the place in $\mathcal{P}_E$ under v.
- By $(E^*_{\mathrm{ar}})^o$ we denote the connected component of $E^*_{\mathrm{ar}} := \prod_{w \in \mathcal{P}_E^{\mathrm{ar}}} E^*_w$.
- If $p = 0$, we set $\mathbb{A}_E^{(0)} := \mathbb{A}_E^*$, if $p > 0$, we define $\mathbb{A}_E^{(0)}$ as the kernel of the degree homomorphism $\deg \colon \mathbb{A}_E^* \to \mathbb{Z}$, $(a_w) \mapsto \sum_w \deg w \cdot w(a_w)$.
- For an effective divisor $\mathfrak{m} = \sum_i n_i w_i$ of E where no w_i is real or complex, let $U_{\mathfrak{m}}$ denote the subgroup of those elements in $\prod_{w \in \mathcal{P}_E^{\mathrm{na}}} \mathcal{O}^*_{E_w}$ which are congruent to 1 modulo $\mathfrak{m}$, and define the **strict $\mathfrak{m}$-class group of** E as
$$\mathrm{Cl}_{\mathfrak{m}} := E^* \backslash \mathbb{A}_E^{(0)} / (U_{\mathfrak{m}} \times (E^*_{\mathrm{ar}})^o).$$
If $p = 0$ then $\mathrm{Cl}_{\mathfrak{m}} \cong I_{\mathfrak{m}} / P_{\mathfrak{m}}$, where $I_{\mathfrak{m}}$ is the group of fractional ideals of E which are prime to $\mathfrak{m}$, and $P_{\mathfrak{m}} \subset I_{\mathfrak{m}}$ the set of principal fractional ideals which have a generator in $E^* \cap (U_{\mathfrak{m}} \times (E^*_{\mathrm{ar}})^o)$.
- For $p > 0$ we write $\mathbb{Z}[1/p]$ for the ring of rational numbers which are integral away from p. For $p = 0$ we declare for convenience that $\mathbb{Z}[1/p] := \mathbb{Z}$.
- For any $m \in \mathbb{N} = \mathbb{Z}_{\geq 1}$ which is not a multiple of p we write $\zeta_m \in K^{\mathrm{alg}}$ for a primitive m-th root of unity.

2. Basic notions and the main theorem

The aim of the present section is to formulate our main result on strictly compatible systems of mod p Galois representations. For this we first introduce the necessary notions, which in the generality needed seem to be new, and we recall various well-known constructions in our more general setting.

2.1. Hodge-Tate characters

All number fields contain the field of rational numbers $\mathbb{Q}$ in a canonical way. This constitutes an important difference to the situation for function fields. There, no minimal field exists which is contained in all others. One way to obtain an analog of $\mathbb{Q}$ is to fix one function field and consider only extensions of it. Embeddings between such extensions (where here and later the term embedding is synonymous with field homomorphism) are then required to fix the chosen base field. For our purposes this is not the correct path to pursue, since this would deprive us of the very flexibility needed for the Hecke characters attached to Drinfeld modular forms, cf. [2]. Instead, we fix for all purposes a finite set of embeddings. Recall that E is a global field of characteristic $p \geq 0$.

Definition 2.1. A **Hodge-Tate set** is a finite subset $\Sigma \subset \mathrm{Emb}(E, K^{\mathrm{alg}})$ of embeddings $\sigma : E \hookrightarrow K^{\mathrm{alg}}$ which we take to be $\mathrm{Emb}(E, \mathbb{Q}^{\mathrm{alg}})$ for $p = 0$.

A **set of Hodge-Tate weights** $(\Sigma, (n_\sigma)_{\sigma\in\Sigma})$ consists of

(a) a Hodge-Tate set Σ,
(b) for each $\sigma \in \Sigma$ a number $n_\sigma \in \mathbb{Z}[1/p]$ (recall that $\mathbb{Z}[1/p] = \mathbb{Z}$ for $p = 0$).

A **Hodge-Tate character** is a homomorphism $\psi : E^* \to (K^{\mathrm{alg}})^*$ for which there exists a set of Hodge-Tate weights $(\Sigma, (n_\sigma))$ such that ψ is equal to

$$\psi^{\Sigma,(n_\sigma)} : E^* \to \left(K^{\mathrm{alg}}\right)^*, \alpha \mapsto \prod_{\sigma\in\Sigma} \sigma(\alpha)^{n_\sigma}.$$

In positive characteristic taking p-power roots is an automorphism of $(K^{\mathrm{alg}})^*$, and so $\psi^{\Sigma,(n_\sigma)}$ is well-defined.

In the function field case there is no canonical a priori choice for Σ. However, given a Hodge-Tate character ψ, Proposition 2.3 will yield a canonical pair $(\Sigma, (n_\sigma))$ representing ψ.

The field $L' := K \cdot \prod_{\sigma\in\Sigma} \sigma(E) \subset K^{\mathrm{alg}}$ is finitely generated and algebraic over K, and thus a finite extension of K. By L we usually denote a finite extension of K which contains L'. If $p = 0$ we take for L the Galois closure L' of E over $\mathbb{Q}$. If $p = 0$, then any Hodge-Tate character $\psi^{\Sigma,(n_\sigma)}$ takes its values in L, if $p > 0$ in L^{1/p^n} for some $n \in \mathbb{N}_0$.

Definition 2.2. A Hodge-Tate set Σ is called **Frobenius-reduced** or **F-reduced** if either $p = 0$, or if $p > 0$ and no two elements in Σ differ by composition with a power of $\varphi_{K^{\mathrm{alg}}}$.

Two Hodge-Tate sets Σ and Σ' are **equivalent** if either $p = 0$, or if $p > 0$ and if we have the equality of sets $\bigcup_{n\in\mathbb{Z}} \varphi^n_{K^{\mathrm{alg}}} \circ \Sigma = \bigcup_{n\in\mathbb{Z}} \varphi^n_{K^{\mathrm{alg}}} \circ \Sigma'$.

Two sets of Hodge-Tate weights are called **equivalent**, if they define the same Hodge-Tate character.

For $p > 0$, $(\Sigma, (n_\sigma))$ a set of Hodge-Tate weights and $(e_\sigma) \in \mathbb{Z}^\Sigma$, define $\Sigma' := \{\varphi_{K^{\mathrm{alg}}}^{e_\sigma} \circ \sigma \mid \sigma \in \Sigma\}$ and for any $\sigma' = \varphi_{K^{\mathrm{alg}}}^{e_\sigma} \circ \sigma \in \Sigma'$, define $n_{\sigma'} := n_\sigma p^{-e_\sigma}$. Then $\psi^{\Sigma,(n_\sigma)} = \psi^{\Sigma',(n_{\sigma'})}$. This procedure yields a canonical representative for any Hodge-Tate character:

Proposition 2.3. *Any set of Hodge-Tate weights is equivalent to a unique set of Hodge-Tate weights* $(\Sigma, (n_\sigma))$ *such that*

(a) Σ *is F-reduced,*
(b) if $p > 0$ *then all exponents* n_σ *lie in* $\mathbb{Z} \setminus p\mathbb{Z}$*.*

For this set of Hodge-Tate weights one may take $L = K \prod_{\sigma\in\Sigma} \sigma(E) \subset K^{\mathrm{alg}}$.

The unique set of Hodge-Tate weights given by the proposition for a Hodge-Tate character will be called its **standard representative**. Note that for $p > 0$ condition (b) requires that all n_σ are non-zero. We do allow $\Sigma = \varnothing$ in that case.

Proof. By the remark preceding the proposition, the only assertion which requires proof is the uniqueness: Suppose that $(\Sigma, (n_\sigma)_{\sigma\in\Sigma})$ and $(\Sigma', (n'_\sigma)_{\sigma\in\Sigma'})$ are equivalent sets of Hodge-Tate weights, that Σ and Σ' are F-reduced, and, if $p > 0$, that all n_σ and n'_σ lie in $\mathbb{Z} \setminus p\mathbb{Z}$. We will show that the sets agree:

For this, let Σ'' be an F-reduced Hodge-Tate set equivalent to $\Sigma \cup \Sigma'$. For $p > 0$ set

$$n''_{\sigma''} := \begin{cases} p^i n_\sigma, & \text{if } \psi_E^{-i} \circ \sigma = \sigma'' \text{ for some } \sigma \in \Sigma \text{ and } \sigma'' \notin \varphi_{K^{\mathrm{alg}}}^{\mathbb{Z}} \circ \Sigma' \\ -p^j n'_{\sigma'}, & \text{if } \psi_E^{-j} \circ \sigma' = \sigma'' \text{ for some } \sigma' \in \Sigma' \text{ and } \sigma'' \notin \varphi_{K^{\mathrm{alg}}}^{\mathbb{Z}} \circ \Sigma \\ p^i n_\sigma - p^j n'_{\sigma'}, & \text{if } \psi_E^{-i} \circ \sigma = \sigma'' = \psi_E^{-j} \circ \sigma' \text{ for some } \sigma \in \Sigma \text{ and } \sigma' \in \Sigma' \\ 0, & \text{otherwise} \end{cases}$$

for all $\sigma'' \in \Sigma''$. For $p = 0$ set $n''_\sigma := n_\sigma - n'_\sigma$. Then $\psi^{\Sigma'',(n_{\sigma''})} = \psi^{\Sigma,(n_\sigma)} \left(\psi^{\Sigma',(n_{\sigma'})}\right)^{-1} = 1$. We need to show $n_{\sigma''} = 0$ for all $\sigma'' \in \Sigma''$. Consider the normalized valuation $v \in \mathcal{P}_L^{\mathrm{na}}$ and let w be the valuation $v \circ \sigma$. Then for any $\alpha \in E^*$ we find

$$\begin{aligned} 0 = v(1) = v\left(\psi^{\Sigma'',(n_{\sigma''})}(\alpha)\right) &= \sum_{\sigma''\in\Sigma''} n_{\sigma''}(v \circ \sigma'')(\alpha) \\ &= \sum_{w\in S_v} w(\alpha) \sum_{\sigma''\in\Sigma'': w = v\circ\sigma''} n_{\sigma''}. \end{aligned} \tag{1}$$

To deduce that the $n_{\sigma''}$ are all zero, we consider the set $\Xi_{\Sigma''} \subset \mathcal{P}_E$ defined as

$$\{w \in \mathcal{P}_E \mid \text{there exist distinct } \sigma, \sigma' \in \Sigma'' \text{ and } v \in \mathcal{P}_L : v \circ \sigma = v \circ \sigma' = w\}.$$

Below, in Proposition 3.4 for $p = 0$, and in Theorem 3.5 for $p > 0$, we shall prove that $\Xi_{\Sigma''}$ has density zero. Both results are independent of the rest of this article, and so there is no circular reasoning. As the reader might expect, Proposition 3.4 is an application of the Čebotarov density theorem.

We now argue as follows: For any $\sigma'' \in \Sigma''$ we have $\mathcal{P}_L \circ \sigma'' = \mathcal{P}_E$. Because $\Xi_{\Sigma''}$ has density zero, there exists $w \notin \Xi_{\Sigma''}$ and $v \in \mathcal{P}_L^{\mathrm{na}}$ such that $v \circ \sigma'' = w$. If $p > 0$ we choose an auxiliary place w' which does not lie in S_v. The well-known formulas for the $\mathbb{Z}$-rank of the S-units in a global field yield $\operatorname{rank} \mathcal{O}^*_{E,\{w\}} = \operatorname{rank} \mathcal{O}^*_E + 1$ if $p = 0$ and $\operatorname{rank} \mathcal{O}^*_{E,\{w,w'\}} = 1 > 0 = \operatorname{rank} \mathcal{O}^*_{E,\{w\}}$ if $p > 0$. In particular, if $p = 0$ there exists a $\{w\}$-unit $\alpha \in E^*$ which is not a unit, and if $p > 0$ there exists a $\{w, w'\}$-unit $\alpha \in E^*$ which is not a constant. Now formula (1) for this α yields $n_{\sigma''} = 0$. □

2.2. *Hodge-Tate sets as divisors*

For $p > 0$ Hodge-Tate sets can be interpreted in terms of divisors: Let Σ be a Hodge-Tate set and L an intermediate field for $K \subset K^{\mathrm{alg}}$ containing $\sigma(E)$ for all $\sigma \in \Sigma$. Let X and C_L denote the smooth projective geometrically irreducible curves with function fields E and L, respectively. Regarding the $\sigma \in \Sigma$ as morphisms $C_L \to X$, we obtain the reduced divisor

$$D_\Sigma := \bigcup_{\sigma \in \Sigma} \operatorname{Graph}(\sigma) \subset X \times C_L.$$

Conversely, let $L \subset K^{\mathrm{alg}}$ be a finite extension of K and let $D = \bigcup_{i \in I} D_i \subset X \times C_L$ be a reduced divisor with irreducible components D_i. Denote by L_i the function field of D_i and by $\sigma_i : E \hookrightarrow L_i$ the homomorphism corresponding to $D_i \to X$. For each $\tau \in \operatorname{Emb}_L(L_i, K^{\mathrm{alg}})$, we obtain a diagram $E \xhookrightarrow{\sigma_i} L_i \xhookrightarrow{\tau} K^{\mathrm{alg}} \hookleftarrow L$. The set

$$\Sigma^D := \{\tau \circ \sigma_i \, : \, i \in I, \tau \in \operatorname{Emb}_L(L_i, K^{\mathrm{alg}})\}$$

is a Hodge-Tate set. Correspondingly, we denote the finite extension $L \prod_{\tilde\sigma \in \Sigma^D} \tilde\sigma(E)$ of L by L^D. Under suitable hypotheses, the above constructions are mutually inverse. We leave the proof of the following elementary result to the reader.

Proposition 2.4. *Let $L \subset K^{\mathrm{alg}}$ be a finite extension of K. Then the assignments $\Sigma \mapsto D_\Sigma$ and $D \mapsto \Sigma^D$ define mutually inverse bijections between Hodge-Tate sets Σ such that $\sigma(E) \subset L$ for all $\sigma \in \Sigma$ and divisors $D \subset X \times C_L$ all of whose components are of degree 1 over C_L.*

For general D, a pair (D, L) contains slightly finer information than (Σ^D, L^D). As we shall see below, Hodge-Tate sets are more natural for Hecke characters, while the view point of divisors is better suited for strictly compatible systems of Galois representations.

For $i \in \mathbb{Z}$, let Φ^i_X denote the correspondence on $X \times X$ defined by φ^i_E.

Definition 2.5. A reduced divisor $D \subset X \times C_L$ is **F-reduced** if we cannot find irreducible components $D_1 \neq D_2$ of D and $i \in \mathbb{Z}$, such that $\Phi^i_X \circ D_1 = D_2$ (as divisors).

Reduced divisors D, D' are **equivalent** if $\bigcup_{i \in \mathbb{Z}} \Phi^i_X \circ D = \bigcup_{i \in \mathbb{Z}} \Phi^i_X \circ D'$ (as sets).

With this definition, the Hodge-Tate set Σ^D is F-reduced if and only if D is F-reduced.

By slight abuse of notation, we also regard a divisor $D \subset X \times C_L$ as the set

$$\{(w, v) \in \mathcal{P}_E \times \mathcal{P}_L \mid (w, v) \in D\}.$$

For $p = 0$ and L a number field, we introduce analogous notation and call

$$D_L := \{(w, v) \in \mathcal{P}_E \times \mathcal{P}_L \mid w|_{\mathbb{Q}} = v|_{\mathbb{Q}}\}.$$

a **divisor** (the **divisor for** L). Moreover for $\Sigma = \mathrm{Emb}(E, \mathbb{Q}^{\mathrm{alg}})$, we set $D_\Sigma := D_L$ where L is the Galois closure of E over $\mathbb{Q}$.

Definition 2.6. Let (D, L) be a pair consisting of a field L and $D = D_L$ for $p = 0$ and D a reduced divisor in $X \times C_L$ for $p > 0$. For every place v of L, we set

$$S_v := S_v^D := \{w \in \mathcal{P}_E \mid (w, v) \in D\} = \mathrm{pr}_1(D \cap (\mathcal{P}_E \times \{v\})).$$

For a Hodge-Tate set Σ, we also write S_v^Σ for $S_v^{D_\Sigma}$, so that $S_v^\Sigma = v \circ \Sigma$.

We define S_∞ $(= S_\infty^D)$ as the union of all S_v where v is an infinite place of L.

The sets S_v are all finite and so is S_∞. If E is a number field, then $S_\infty = \mathcal{P}_E^{\mathrm{ar}}$, and if v is above a prime $u \in \mathbb{N}$, then S_v is the set of all places of E above u. The proof of the following lemma is left to the reader.

Lemma 2.7. *Suppose $p > 0$. If $D, D' \subset X \times C_L$ are equivalent reduced divisors, then $(S_v^D)_{v \in \mathcal{P}_L} = (S_v^{D'})_{v \in \mathcal{P}_L}$. Consequently, if Σ and Σ' are equivalent Hodge-Tate sets, then $(S_v^\Sigma)_{v \in \mathcal{P}_L} = (S_v^{\Sigma'})_{v \in \mathcal{P}_L}$.*

The set S_v describes the places in $\mathcal{P}_E$ under a place v of L under the various $\sigma \in \Sigma$. It will also be useful to have a compact notation for places of L above some $w \in \mathcal{P}_E$. For any subset $S \subset \mathcal{P}_E$, we define

$$\Sigma^{-1}(S) := \{v \in \mathcal{P}_L \mid \exists \sigma \in \Sigma, v \circ \sigma \in S\}.$$

2.3. Algebraic Hecke characters

In this subsection, we introduce a general type of Hecke character. We shall see that in the number field case our definition agrees with the usual one. However in the function field setting our Hecke characters are more general than those previously considered in [6,7]. In Sect. 2.6 we shall attach to any such Hecke character a strictly compatible system of Galois representations. Under further hypothesis in the function field case, we can attach at the infinite places of L certain Größencharakters, as well.

Definition 2.8. An **algebraic Hecke character of** E is a continuous homomorphism

$$\chi : \mathbb{A}_E^* \longrightarrow \left(K^{\mathrm{alg}}\right)^*$$

for the discrete topology on K^{alg} such that the restriction $\chi|_{E^*}$ is a Hodge-Tate character.

We say that $(\Sigma, (n_\sigma)_{\sigma\in\Sigma})$ is a set of Hodge-Tate weights for χ, if $\chi|_{E^*} = \psi^{\Sigma,(n_\sigma)}$.

A place $w \in \mathcal{P}_E$ is said to be **finite for** χ, if the image under χ of E_w^* (extended by 1 to all other components) is finite.

The character χ is called **of** ∞**-type**, if $T_\infty := \Sigma^{-1}(S_\infty)$ agrees with $\mathcal{P}_L^\infty$, and if all $w \in S_\infty$ are finite for χ.

For simplicity and since no other Hecke characters occur in the main body of this article, we shall usually use the term **Hecke character** instead of **algebraic Hecke character**.

Remark 2.9. Suppose E is a number field. Then $K = \mathbb{Q}$, the field L is the Galois closure of E in $\mathbb{Q}^{\mathrm{alg}}$, and we have $S_\infty = \mathcal{P}_E^{\mathrm{ar}}$ and $T_\infty = \mathcal{P}_L^\infty = \mathcal{P}_L^{\mathrm{ar}}$. By continuity χ applied to E_{ar}^* has finite order. Hence any algebraic Hecke character is of ∞-type.

For a number field E, the set of Hecke characters of A_∞-type in the sense of Weil and the algebraic Hecke characters as defined above are the same, cf. [15, § 2.4, Exer.].

Remark 2.10. Suppose E is a function field, so that $p > 0$. Suppose in addition that we have chosen an embedding $K \hookrightarrow E$ and have set $\Sigma := \mathrm{Emb}_K(E, K^{\mathrm{alg}})$ and defined L as the Galois closure of E over K. Then, as the reader may verify, the definitions of an algebraic Hecke character in [7, § 1] and of a Hecke character of A_0-type in [6, § 1] are covered by the above definition.

Remark 2.11. Since χ is continuous the image of U_0 is finite, where 0 denotes the trivial divisor on E. Hence there exists an effective divisor $\mathfrak{m}$ such that $\chi(U_\mathfrak{m}) = \{1\}$. Clearly χ is also trivial on $(E_{\mathrm{ar}}^*)^o$, and restricted to E^* its values lie in L^* or in $(L^{1/p^n})^*$ for some $n \in \mathbb{N}$. Since $\mathrm{Cl}_\mathfrak{m} = E^*\backslash\mathbb{A}_E^{(0)}/(U_\mathfrak{m} \times (E_{\mathrm{ar}}^*)^o)$ is finite, the extension field $L(\chi(\mathbb{A}_E^{(0)}))$ is finite over L. By our conventions $\mathbb{A}_E^{(0)} = \mathbb{A}_E^*$ for number fields and $\mathbb{A}_E^*/\mathbb{A}_E^{(0)} \cong \mathbb{Z}$, for function fields, under the degree map. In either case $L(\chi(\mathbb{A}_E^*))$ is a finite extension of K.

By Remark 2.11, one can find an effective divisor $\mathfrak{m}$ of E and an open finite index subgroup U_{ar} of $\prod_{w\in\mathcal{P}_E^{\mathrm{ar}}} E_w^*$ such that χ is trivial on $U_\mathfrak{m} \times U_{\mathrm{ar}}$. Similar to the classical situation, any such pair $(\mathfrak{m}, U_{\mathrm{ar}})$ is called a **conductor** of χ. The maximal choice $(\mathfrak{m}_\chi, U_{\chi,\mathrm{ar}})$ in both components is called the **minimal conductor of** χ. The two components $\mathfrak{m}$ and U_{ar} we call the **non-archimedean** and **archimedean part** of the conductor of χ, respectively. The group $U_{\chi,\mathrm{ar}}$ always contains the connected component $(E_{\mathrm{ar}}^*)^o$.

The following goes back to Goss, cf. [6, p. 125] and carries over to our setting:

Proposition 2.12. *If K is a function field, then $\mathfrak{m}_\chi$ is squarefree.*

Proof. Suppose $\mathfrak{m}_\chi$ was not squarefree. Then there exists $w \in \mathcal{P}_E$ such that $2[w] \leq \mathfrak{m}_\chi$. The kernel of $\mathcal{O}_{E_w}^* \to \mathbb{F}_w^*$ is a pro-p group, and by our hypothesis on $\mathfrak{m}_\chi$ the image of this group in $(K^{\mathrm{alg}})^*$ is non-trivial. By continuity of χ the image is a finite p-group. We reach a contradiction, since $(K^{\mathrm{alg}})^*$ contains no elements of order p.

For later use we need to recall a simple criterion for the existence of Hecke characters with suitable sets of Hodge-Tate weights. □

Proposition 2.13. *Let Σ be an F-reduced Hodge-Tate set and let $S \subset \mathcal{P}_E^{\mathrm{na}}$ be finite. There exists a Hecke character χ with set of Hodge-Tate weights $(\Sigma, (n_\sigma))$ and such that χ is trivial on an open finite index subgroup of $\prod_{w \in S} E_w^*$, if and only if, the image of the group of S-units $\mathcal{O}_{E,S}^*$ under $\psi^{\Sigma,(n_\sigma)}$ is finite. In addition, if $\psi^{\Sigma,(n_\sigma)}(\mathcal{O}_E^*) = \{1\}$, then χ can be chosen to have trivial conductor.*

Proof. Suppose first that χ agrees with $\psi^{\Sigma,(n_\sigma)}$ on E^* and is trivial on an open finite index subgroup of $\prod_{w \in S} E_w^*$. Then χ maps the intersection of $\mathbb{A}_E^{(0)}$ with $U_S := \prod_{w \in \mathcal{P}_E^{\mathrm{na}} \smallsetminus S} \mathcal{O}_w^* \times \prod_{w \in S} E_w^*$ to a finite group, and so $\psi^{\Sigma,(n_\sigma)}(\mathcal{O}_{E,S}^*)$ is finite.

For the converse, let $U' \subset U_S$ be open and consider the short exact sequence

$$\begin{aligned} 0 \longrightarrow E^*/\left(U' \cap E^*\right) &\longrightarrow \mathbb{A}_E^{(0)}/\left((E_{\mathrm{ar}}^*)^o \times U' \cap \mathbb{A}_E^{(0)}\right) \\ &\longrightarrow E^* \backslash \mathbb{A}_E^{(0)}/\left((E_{\mathrm{ar}}^*)^o \times U' \cap \mathbb{A}_E^{(0)}\right) \longrightarrow 0. \end{aligned}$$

of abelian groups. By class field theory, the group on the right has finite order. By our hypothesis, we can find $U' \subset U_S$ open and of finite index, such that $\psi^{\Sigma,(n_\sigma)}$ is trivial on $U' \cap E^*$.

Since $(K^{\mathrm{alg}})^*$ is a divisible group there exists a character χ' extending $\psi^{\Sigma,(n_\sigma)}$ to the middle group. If $p = 0$ we are done. For $p > 0$ we consider the short exact sequence

$$0 \longrightarrow \mathbb{A}_E^{(0)}/\left(U' \cap \mathbb{A}_E^{(0)}\right) \longrightarrow \mathbb{A}_E^*/\left(U' \cap \mathbb{A}_E^{(0)}\right) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0,$$

which is clearly split. Therefore χ' extends to a character on $\mathbb{A}_E^*$, as desired. To see the final assertion, note that under the hypotheses there, we may choose $U' = U_\varnothing$, so that χ is trivial on $E_{\mathrm{ar}}^* \times U_0$. □

2.4. A basic formula

We will repeatedly make use of formulas similar to (1) in the proof of Proposition 2.3. The aim of this subsection is to give an axiomatic treatment of such a formula. The reason behind the usefulness of such a formula is that it relates a (multiplicatively written) $\mathbb{Z}$-linear combination of values of a Hecke character at Frobenius elements to values of the underlying Hodge-Tate character on E^*. So let us fix the following:

(a) A Hecke character χ with set of Hodge-Tate weights $(\Sigma, (n_\sigma))$ such that Σ is F-reduced and minimal conductor $(\mathfrak{m}_\chi, U_{\chi,\mathrm{ar}})$.
(b) For any $w \in \mathcal{P}_E^{\mathrm{na}}$, a uniformizer ϖ_w of E_w and the element

$$x_w := \left(1, \ldots, 1, \underbrace{\varpi_w}_{\text{at } w}, 1, \ldots, 1\right) \in \mathbb{A}_E^*$$

(c) If $p = 0$, for any $w \in \mathcal{P}_E^{\mathrm{na}}$ an element $\alpha \in E^*$ which is a $\{w\}$-unit, but not a unit, and such that at all places $w'' \in \mathcal{P}_E^{\mathrm{na}} \smallsetminus \{w\}$ the component of α lies in $U_{\mathfrak{m}_\chi}$.
(d) If $p > 0$, for any $w \neq w' \in \mathcal{P}_E^{\mathrm{na}}$ an element $\alpha \in E^*$ which is a $\{w, w'\}$-unit, but not a constant, and such that at all places $w'' \in \mathcal{P}_E^{\mathrm{na}} \smallsetminus \{w, w'\}$ the component of α lies in $U_{\mathfrak{m}_\chi}$.

Without the conditions on the components at the w'', the existence of α as in (c) and (d) was explained in the proof of Proposition 2.3. The additional condition can be achieved by replacing α by a suitable power — for instance the exponent of the (finite) strict class group $\mathrm{Cl}_{\mathfrak{m}_\chi}$. For the divisor of α we write $n[w]$ if $p = 0$ and $n(\deg w'[w] - \deg w[w'])$ if $p > 0$, and we assume without loss of generality that n is positive.

Let now v be a place of $\mathcal{P}_L^{\mathrm{na}}$ as well as its normalized valuation. We adopt the same convention for $w \in \mathcal{P}_E^{\mathrm{na}}$ (the ad hoc convention in the proof of Proposition 2.3 was different!). For a homomorphism $\sigma : E \hookrightarrow L$, the ramification index of v over $w'' = v \circ \sigma$ is denoted $e_{v/w'',\sigma}$. Note that the generic value of $e_{v/w'',\sigma}$ is the inseparable degree of $\sigma \in \Sigma$, and hence for $p > 0$ a power of p. Computing $\chi(\alpha)$ in two ways, using the Hecke character at the $x_{w''}$ and using the Hodge-Tate character on α, we obtain

$$\chi(x_w)^{n \deg w'} \chi(x_{w'})^{-n \deg w} = \chi(\alpha) = \prod_{\sigma \in \Sigma} \sigma(\alpha)^{n_\sigma} \tag{2}$$

for $p > 0$ and a similar but simpler formula for $p = 0$. Computing the v-valuation, canceling n and sorting the right hand side according to places in $\mathcal{P}_E$ yields

$$v(\chi(x_w)) = \sum_{\sigma \in \Sigma : v \circ \sigma = w} e_{v/w,\sigma} n_\sigma$$

for $p = 0$ and

$$\begin{aligned} &\deg w' \cdot v(\chi(x_w)) - \deg w \cdot v(\chi(x_{w'})) \\ &= \deg w' \cdot \sum_{\sigma \in \Sigma : v \circ \sigma = w} e_{v/w,\sigma} n_\sigma - \deg w \cdot \sum_{\sigma \in \Sigma : v \circ \sigma = w'} e_{v/w',\sigma} n_\sigma \end{aligned} \tag{3}$$

for $p > 0$. We record some particular cases of the two previous formulas:

Lemma 2.14. *(a) Suppose that neither w nor w' lie in the set*

$$\Xi_\Sigma = \{w \in \mathcal{P}_E \mid \textit{there are distinct } \sigma, \sigma' \in \Sigma \textit{ and a } v \in \mathcal{P}_L : v \circ \sigma = v \circ \sigma' = w\}.$$

Then the sums on the right contain at most one summand.

(b) If $\Sigma^{-1}(\{w\})$ and $\Sigma^{-1}(\{w'\})$ are disjoint, then in (3) at most one of the sums is non-zero.

2.5. Strictly compatible systems

In the following $L \subset K^{\mathrm{alg}}$ denotes a finite extension of K and D is either D_L if $p = 0$ or it is a reduced divisor on $X \times C_L$. Recall that E is a global field of characteristic $p \geq 0$. By S we denote a finite subset of $\mathcal{P}_E$ and by T a finite subset of $\mathcal{P}_L$ that contains $\mathcal{P}_L^{\mathrm{ar}}$. The following definition is adapted from [11]:

Definition 2.15. An L**-rational strictly compatible system** $\{\rho_v\}$ of n-dimensional mod v representations of G_E for D with **defect set** T and **ramification set** S consists of

(i) for each $v \in \mathcal{P}_L \smallsetminus T$ a continuous semisimple representation

$$\rho_v \colon G_E \to \mathrm{GL}_n(\mathbb{F}_v),$$

which is unramified outside $S \cup S_v^D$, and

(ii) for each place $w \in \mathcal{P}_E \smallsetminus S$ a monic polynomial $f_w \in L[t]$ of degree n,

such that for all $v \in \mathcal{P}_L \smallsetminus T$ and for all $w \in \mathcal{P}_E \smallsetminus (S \cup S_v^D)$ such that f_w is v-integral:

$$\mathrm{CharPol}_{\rho_v(\mathrm{Frob}_w)} \equiv f_w \pmod{\mathfrak{p}_v}.$$

We say that a place $w \in \mathcal{P}_E$ is **finite for** $\{\rho_v\}$ if there exists a finite extension E'_w of E_w, such that the image of $G_{E'_w}$ under any ρ_v is trivial

In an analogous way one defines the notion of an L-rational strictly compatible system of n-dimensional v-adic representations of G_E for D with defect set T and ramification set S. We omit a precise description of the obvious modification.

If $p = 0$, the defect set T will typically be the set $\mathcal{P}_L^{\mathrm{ar}}$. For $p > 0$ a typical T is given in (5) below. The set S usually contains S_∞. As we shall see below, if $\{\rho_v\}$ arises from a Hecke character χ based on a Hodge-Tate set Σ and if $D = D_\Sigma$, then S will contain the support of the minimal conductor of χ.

2.6. Strictly compatible systems from Hecke characters

Let χ be a Hecke character, let $(\Sigma, (n_\sigma)_{\sigma\in\Sigma})$ be the standard set of Hodge-Tate weights for $\chi|_{E^*}$, so that in particular Σ is F-reduced, and let $L \supset K$ be a finite extension containing $\sigma(E)$ for all $\sigma \in \Sigma$ as well as $\chi(\mathbb{A}_E^*)$.

Our aim is to attach to χ for almost all places $v \in \mathcal{P}_L$ a continuous abelian Galois representation

$$\rho_{\chi,v} \colon G_E \to L_v^*.$$

The representations $\rho_{\chi,v}$ will form a strictly compatible system. They are linked by the values taken on the Frobenius elements at unramified places. Under favorable circumstances (and always, if $p = 0$) we will attach a Größencharakter $\chi_{\mathrm{Gr},v}: \mathbb{A}_E^*/E^* \to L_v^*$ of the Weil group or ideal class group of E for the remaining places $v \in \mathcal{P}_L$. We follow [15, § 2] and [7].

Let us first recall the reciprocity law from class field theory: In both the number and the function field situation, one has a reciprocity homomorphism

$$\mathrm{rec}\colon \mathbb{A}_E^*/\overline{E^*(E_{\mathrm{ar}}^*)^o} \hookrightarrow G_E^{\mathrm{ab}}. \tag{4}$$

In the number field case rec is an isomorphism. In the function field case, rec has dense image and the restriction of rec to $\overline{E^*}\backslash\mathbb{A}_E^{(0)}$ is an isomorphism onto the kernel of $G_E^{\mathrm{ab}} \twoheadrightarrow G_{\mathbb{F}}^{\mathrm{ab}}$, where by $\mathbb{F}$ we denote the field of constants of E. Let $\mathrm{Frob}_{\mathbb{F}}$ denote the Frobenius automorphism on $\mathbb{F}^{\mathrm{alg}}$ fixing $\mathbb{F}$. Then the induced homomorphism from $\mathbb{A}_E^*/\mathbb{A}_E^{(0)} \cong \mathbb{Z}$ to $G_{\mathbb{F}}$ sends, suitably normalized, the element 1 to the topological generator $\mathrm{Frob}_{\mathbb{F}}$ of $G_{\mathbb{F}}$. Regarding continuous representations $G_E \to L_v^*$ we deduce:

Let v be in $\mathcal{P}_L^{\mathrm{na}}$. In the number field case, any continuous homomorphism from $\mathbb{A}_E^*/\overline{E^*(E_{\mathrm{ar}}^*)^o}$ to the topological group L_v^* induces a continuous homomorphism $G_E^{\mathrm{ab}} \to L_v^*$. In the function field situation, we require the (necessary and sufficient) additional hypothesis that there is an element in $\mathbb{A}_E^*$ of non-zero degree which maps to $\mathcal{O}_v^*$.

Define S as the union of $\mathcal{P}_E^{\mathrm{ar}}$ and the support of the minimal conductor of χ. If $K = \mathbb{Q}$, define $T := \mathcal{P}_L^{\mathrm{ar}}$. If $p > 0$, choose places $w, w' \in \mathcal{P}_E$ such that $\Sigma^{-1}(\{w\})$ and $\Sigma^{-1}(\{w'\})$ are disjoint, and define

$$T := \{v \in \mathcal{P}_L \mid w \notin S_v \text{ and } \chi(x_w) \notin \mathcal{O}_v^*\} \cup \{v \in \mathcal{P}_L \mid w' \notin S_v \text{ and } \chi(x_{w'}) \notin \mathcal{O}_v^*\}. \tag{5}$$

The definition of T in the function field case does not depend on the choice of $\{w, w'\}$:

Lemma 2.16. *Suppose $S_v \cap \{w, w'\} = \varnothing$. Then $\chi(x_w) \in \mathcal{O}_v^* \iff \chi(x_{w'}) \in \mathcal{O}_v^*$.*

The proof is immediate from formula (3) in Sect. 2.4, since by our hypothesis the right hand side of (3) is zero.

Let v be any place of L. Any $\sigma \in \Sigma$ induces a continuous homomorphism $\psi_{\sigma,v}\colon E_{v\circ\sigma}^* \to L_v^*$. Denote by π_w the projection $\mathbb{A}_E^* \to E_w^*$ and define the homomorphism

$$\chi_v := \chi \cdot \prod_{\sigma\in\Sigma} \left(\psi_{\sigma,v} \circ \pi_{v\circ\sigma}\right)^{-n_\sigma} : \mathbb{A}_E^* \to L_v^*.$$

The proof of the following elementary lemma, we leave to the reader.

Lemma 2.17. *The kernel of χ_v contains E^* and the subgroup of $U_{\mathfrak{m}_\chi} \times U_{\chi,\mathrm{ar}}$ of elements whose component at all places $w \in S_v$ is 1.*

Lemma 2.18. *Suppose v is a place of L not in T. Then χ_v defines a continuous character $\overline{\chi}_v : \mathbb{A}_E^*/\overline{E^*(E_{\mathrm{ar}}^*)^o} \longrightarrow \mathcal{O}_v^*$. The induced homomorphism $\rho_{\chi,v} : G_E \longrightarrow \mathcal{O}_v^*$ via the reciprocity homomorphism in (4) is characterized by*

$$\rho_{\chi,v}(\mathrm{Frob}_w) = \chi(x_w) \quad \textit{for all } w \in \mathcal{P}_E \smallsetminus (S \cup S_v).$$

Its conductor at places not in S_v is given by $(\mathfrak{m}_\chi, U_{\chi,\mathrm{ar}})$.

Proof. Recall that χ is continuous on $\mathbb{A}_E^*$ with respect to the discrete topology on $(K^{\mathrm{alg}})^*$. Since the characters $\psi_{\sigma,v}$ are continuous with respect to the v-adic topology on L_v^*, the character χ_v is continuous with respect to the adelic topology on $\mathbb{A}_E^*$ and the v-adic topology on L_v^* (a basis of open neighborhoods around 1 is given by the sets $1 + \mathfrak{m}_v^n$, $n \in \mathbb{N}$). By Lemma 2.17, the kernel of χ_v contains $E^*(E_{\mathrm{ar}}^*)^o$ and thus by continuity of χ_v the kernel contains $\overline{E^*(E_{\mathrm{ar}}^*)^o}$, proving the existence of $\overline{\chi}_v$.

Next, if $p = 0$, then under the reciprocity homomorphism (4) we have $\mathbb{A}_E^*/\overline{E^*(E_{\mathrm{ar}}^*)^o} \cong G_E^{\mathrm{ab}}$ and thus $\rho_{\chi,v}$ is defined and continuous. In particular, by compactness of G_E, the image of $\overline{\chi}_v$ must lie in $\mathcal{O}_v^*$. If $p > 0$, then by our choice of T there exists an idele of positive degree (either x_w or $x_{w'}$) whose image under χ_v lies in $\mathcal{O}_v^*$ (either $\chi(x_w)$ or $\chi(x_{w'})$). This implies that $\overline{\chi}_v$ has image in $\mathcal{O}_v^*$. By the discussion following (4), we see that $\overline{\chi}_v$ induces under rec a continuous homomorphism $\rho_{\chi,v} : G_E \longrightarrow \mathcal{O}_v^*$, as well.

Finally, assume that w is a place of E, not in $S \cup S_v$. By the definition of S and S_v and by Lemma 2.17, it follows that the ideles which at w lie in $\mathcal{O}_w^*$ and at all $\widetilde{w} \neq w$ are 1 lie in the kernel of χ and χ_v. By the compatibility of local and global class field theory, $\overline{\chi}_v$ is unramified at w and we have

$$\rho_{\chi,v}(\mathrm{Frob}_w) = \chi_v(x_w) = \chi(x_w).$$

The local to global compatibility of class field theory also implies the assertion on conductors and thus completes the proof of the lemma. $\square$

The following result on the system $\{\rho_{\chi,v}\}$ is an immediate consequence of Lemma 2.18. It is well-known if E is a number field, cf. [15, II.2.5].

Proposition 2.19. *The familiy $\{\rho_{\chi,v}\}_{v\in\mathcal{P}_L\smallsetminus T}$ forms an L-rational strictly compatible system of 1-dimensional representations of G_E for D_Σ with defect set T and ramification set S. The corresponding family of monic degree 1 polynomials is given by*

$$f_w(t) := t - \chi(x_w) \quad \textit{for all } w \in \mathcal{P}_E \smallsetminus S.$$

If some $w \in \mathcal{P}_E$ is finite for χ, it is finite for $\{\rho_{\chi,v}\}$.

Suppose now that χ is of ∞-type and let v be in $\mathcal{P}_L^\infty$. We will show that χ_v defines a **Größencharakter** in the sense of Hecke: We define $\mathcal{O}_E$ as the ring of integers of E if $p = 0$ and as the ring of elements of E with no poles outside S_∞ if $p > 0$. For a divisor $\mathfrak{m}$ denote by $I_\mathfrak{m}$ the set of fractional $\mathcal{O}_E$-ideals which are prime to the support of $\mathfrak{m}$. Also, denote by $(\mathbb{A}_E^\infty)^*$ the ideles over $\mathcal{P}_E \smallsetminus S_\infty$, by $U_\mathfrak{m}^\infty$ the intersection $U_\mathfrak{m} \cap (\mathbb{A}_E^\infty)^*$ and by E_∞ the product $\prod_{w\in S_\infty} E_w$. Since χ is of ∞-type, $U_\infty := \{x \in E_\infty^* \mid \chi(x) = 1\}$ is a finite index subgroup of E_∞^*

The group $I_{\mathfrak{m}_\chi}$ is isomorphic to the quotient of the restricted product $\prod'_{w\notin S} E_w^*$ modulo the compact open subgroup $\prod_{w\notin S}\mathcal{O}_{E_w}^*$. Extending $(\alpha_w)_{w\in\mathcal{P}_E\smallsetminus S}$ by 1 on $S\smallsetminus S_\infty$, the latter quotient injects into $(\mathbb{A}_E^\infty)^*/U_{\mathfrak{m}_\chi}^\infty$, yielding a monomorphism

$$I_{\mathfrak{m}_\chi}\hookrightarrow \left(\mathbb{A}_E^\infty\right)^*/U_{\mathfrak{m}_\chi}^\infty. \tag{6}$$

The right hand side injects into $E^*\backslash\mathbb{A}_E^*/U_{\mathfrak{m}_\chi}^\infty$. Observe that by its definition the character χ_v is trivial on $E^*U_{\mathfrak{m}_\chi}^\infty$. Composing the above maps defines therefore a homomorphism

$$\chi_{\mathrm{Gr},v}\colon I_{\mathfrak{m}_\chi}\longrightarrow L_v^*.$$

For $p=0$ the following result is well-known, cf. [15, II.2.7].

Proposition 2.20. *Suppose χ is of ∞-type. Then for each $v\in\mathcal{P}_L^\infty$, the character $\chi_{\mathrm{Gr},v}$ is a Hecke-type Größencharakter on $I_{\mathfrak{m}_\chi}$ such that for any $w\notin S_\infty\cup S$, one has $\chi_{\mathrm{Gr},v}(\mathfrak{p}_w)=\chi(x_w)$, independently of v, and for all $\alpha\in E^*\cap U_\infty$ with $\alpha\equiv 1 \pmod{\mathfrak{m}_\chi}$:*

$$\chi_{\mathrm{Gr},v}(\alpha\mathcal{O}_E)=\psi^{(\Sigma,(n_\sigma))}(\alpha).$$

Proof. Under (6) any prime ideal $\mathfrak{p}_w$ for $w\notin S_\infty\cup S$ is mapped to the idele x_w (modulo $U_{\mathfrak{m}_\chi}^\infty$). Now by definition $\chi_{\mathrm{Gr},v}(\mathfrak{p}_w)$ is equal to $\chi(x_w)$ for such w, and this proves the first assertion.

Suppose now that α lies in $E^*\cap U_\infty$ and satisfies $\alpha\equiv 1 \pmod{\mathfrak{m}_\chi}$. Then the value of the principal fractional ideal $\alpha\mathcal{O}_E$ under $\chi_{\mathrm{Gr},v}$ is given by

$$\begin{aligned}\chi_{\mathrm{Gr},v}(\alpha\mathcal{O}_E)&=\chi_v\left((\alpha,1)\right)=\chi_v\left(\left(1,\alpha^{-1}\right)\right)=\chi\left(\left(1,\alpha^{-1}\right)\right)\prod_\sigma\sigma(\alpha^{-1})^{-n_\sigma}\\&=\prod_\sigma\sigma\left(\alpha\right)^{n_\sigma},\end{aligned}$$

where pairs are considered as elements in $(\mathbb{A}_E^\infty)^*\times E_\infty^*$. The computation is justified, since α, and hence α^{-1} lie in U_∞. The term on the right is $\psi^{(\Sigma,(n_\sigma))}(\alpha)$, and this completes the proof. □

2.7. Statement of the main theorem

Let χ be a Hecke character of E. Proposition 2.19 shows that χ gives rise to an L-rational strictly compatible abelian system $\{\rho_{\chi,v}\}$ of v-adic Galois representations. As we observed, the $\rho_{\chi,v}$ take values in $\mathcal{O}_v^*$. Hence their reductions to $\mathbb{F}_v$ form an L-rational strictly compatible abelian system $\{\bar\rho_{\chi,v}\}$ of mod v Galois representations in the sense of Definition 2.5. Our main result states that the above construction yields all v-adic and all mod v strictly compatible abelian systems:

Theorem 2.21. *Let $\{\rho_v\}_{v\in\mathcal{P}_L\smallsetminus T}$ be an L-rational strictly compatible* **abelian** *system of n-dimensional mod v representations of G_E for a reduced divisor $D\subset\mathcal{P}_E\times\mathcal{P}_L$. Then there exist Hecke characters $(\chi_i)_{i=1,\dots,n}$ for Σ^D such that for all $v\in\mathcal{P}_{L^D}$ not above T the mod v Galois representation of G_E attached to $\oplus_{i=1}^n\chi_i$ is isomorphic to the semisimplification ρ_v^{ss} of ρ_v. The character list $(\chi_i)_{i=1,\dots,n}$ is unique up to permutation.*

If $w\in\mathcal{P}_E$ is finite for $\{\rho_v\}$, then it is finite for all χ_i. If the set T_∞ for Σ^D agrees with $\mathcal{P}_L^\infty$ and if all places in S_∞ are finite for $\{\rho_v\}$, then all χ_i are of ∞-type.

Remark 2.22. In the number field case, the strictly compatible systems considered by Khare and the A_∞-type Hecke characters in the sense of Weil are the strictly compatible systems and Hecke characters we consider. Not unexpectedly, the above theorem recovers the results of Khare, since except for some special results on function fields, we closely follow his method of proof, cf. also Remark 2.9 and Definition 2.5.

Any v-adic strictly compatible system gives rise to a mod v system. Since semi-simple strictly compatible systems with the same polynomials f_w are conjugate, the following is immediate:

Corollary 2.23. *Let $\{\rho_v\}$ be an L-rational strictly compatible* **abelian** *system of n-dimensional v-adic representations of G_E for a reduced divisor $D\subset\mathcal{P}_E\times\mathcal{P}_L$. Then there exist Hecke characters $(\chi_i)_{i=1,\dots,n}$ for Σ^D such that for all $v\in\mathcal{P}_{L^D}$ not above T the mod v Galois representation of G_E attached to $\oplus_{i=1}^n\chi_i$ is isomorphic to the semisimplification ρ_v^{ss} of ρ_v. The character list $(\chi_i)_{i=1,\dots,n}$ is unique up to permutation.*

If $w\in\mathcal{P}_E$ is finite for $\{\rho_v\}$, then it is finite for all χ_i. If the set T_∞ for Σ^D agrees with $\mathcal{P}_L^\infty$ and if all places in S_∞ are finite for $\{\rho_v\}$, then all χ_i are of ∞-type.

3. Preparations

3.1. On a result of Corrales-Rodrigáñez and Schoof

In [3] Corrales-Rodrigáñez and Schoof, and later Khare in [11], consider the following type of question, which in some form was first posed by Erdös: Suppose G is a finitely generated subgroup of L^*, for L a number field. What can be deduced for an element $x\in L^*$ if it is known that the reduction $x \pmod v$ (or a power of x with some restrictions on the exponent) lies in $G \pmod v$ for a set of places v of density 1? Is x itself in G, is a power of x in G? Following the methods of [3] and [11], we obtain a slight generalization of the results proved therein for a similar type question and for a general global field L. Via a different method, Erdös' question (and some variants) were also proved in [14].

To approach the question, choose T so that x and G are contained in $\mathcal{O}_{L,T}^*$. By the Dirichlet unit theorem the units $\mathcal{O}_{L,T}^*$ form a finitely generated abelian group whose maximal torsion free quotient has rank $\#T-1+\#\mathcal{P}_L^{\mathrm{ar}}$. Following [3], [10] we consider the question "does x^e lie in $G\mathcal{O}_{L,T}^{*u}$" for infinitely many u. For $p\nmid u$, this question can be reformulated via Kummer theory. To apply Kummer theory, one needs to pass from L to $L(\zeta_u)$, and hence it is important to understand the kernel of the homomorphism

$$\mathcal{O}^*_{L,T}/\mathcal{O}^{*u}_{L,T} \to \mathcal{O}^*_{L(\zeta_u),T}/\mathcal{O}^{*u}_{L(\zeta_u),T}. \tag{7}$$

For $p \not| u$, it is shown in [3, Lem. 2.1] that the kernel embeds into $H^1(\mathrm{Gal}(L(\zeta_u)/L), \mu_u)$ and that the latter group is trivial unless $4|u$ and $\zeta_4 \notin L$. The embedding is given by sending the class of an element $t \in L^*$ such that $t = s^u$ for some $s \in L(\zeta_u)^*$ to the 1-cocycle

$$c_t : \mathrm{Gal}(L(\zeta_u)/L) \to \mu_u, \sigma \mapsto \frac{\sigma(s)}{s}.$$

For $4|u$ and $\zeta_4 \notin L$ one easily deduces $H^1(\mathrm{Gal}(L(\zeta_u)/L), \mu_u) \cong \mathbb{Z}/(2)$ from the results of loc.cit. However, this case, i.e., where $4|u$ and $\zeta_4 \notin L$, is not relevant to us, since in the proofs of our applications one may simply adjoin ζ_4 to L if 4 divides u.

Let $L_u := L\left(\zeta_u, \sqrt[u]{\mathcal{O}^*_{L,T}}\right)$ and $H_u := \mathrm{Gal}(L_u/L(\zeta_u))$. Then for $p \not| u$ the pairing

$$\mathcal{O}^*_{L,T}/\mathcal{O}^{*u}_{L,T} \times H_u \to \langle\zeta_u\rangle, (\alpha, h) \mapsto \langle\alpha, h\rangle := \frac{h(\alpha^{1/u})}{\alpha^{1/u}}$$

is perfect by Kummer theory provided that (7) is injective; here $\alpha^{1/u}$ denotes any element of L^{alg} whose u-th power is α. Note that for $g \in \mathrm{Gal}(L_u/L)$ one has

$$\left\langle\alpha, ghg^{-1}\right\rangle = \frac{ghg^{-1}(\alpha^{1/u})}{\alpha^{1/u}} = g\left(\frac{hg^{-1}(\alpha^{1/u})}{g^{-1}\alpha^{1/u}}\right) = g(\langle\alpha, h\rangle).$$

Writing $g(\zeta_u) = \zeta_u^f$, we find $\langle\alpha, ghg^{-1}\rangle = \langle\alpha, h^f\rangle$. From this one deduces $ghg^{-1} = h^f$. Consequently any subgroup of H_u is normal in $\mathrm{Gal}(L_u/L)$. Note also that for places v' of $L(\zeta_u)$ (so that $u|(q_{v'} - 1)$) which are unramified for L_u/L, one has a compatible pairing

$$\mathbb{F}^*_{v'}/\mathbb{F}^{*u}_{v'} \times \mathrm{Gal}\left(\mathbb{F}_{v'}(\sqrt[u]{\mathbb{F}_{v'}})/\mathbb{F}_{v'}\right) \to \langle\zeta_u\rangle, \left(\alpha, \mathrm{Frob}^i_{v'}\right) \mapsto \alpha^{i(q_{v'}-1)/u}.$$

Let $H_{G,u} \subset H_{\mathfrak{p}}$ be the annihilator of $G\mathcal{O}^{*u}_{L,T}/\mathcal{O}^{*u}_{L,T}$, and let $\mathcal{S}^{G,u}_L \subset \mathcal{P}_L$ be the set of places v which are totally split in $L_{G,u} := L(\zeta_u)(\sqrt[u]{G})$ over L and unramified in L_u/L, i.e., those places of L for which $\mathrm{Frob}_v \in \mathrm{Gal}(L_u/L)$ is well-defined and lies in $H_{G,u}$.

Lemma 3.1. *Set* $G_{\mathrm{sat}} := \{y \in \mathcal{O}^*_{L,T} \mid \exists n \in \mathbb{N} : y^n \in G\}$ *and fix* $u \in \mathbb{N}$. *Let the notation be as above, and assume that* $p \not| u$ *and that* $\zeta_4 \in L$ *if* $4|u$. *Then the following hold:*

(a) The set $\mathcal{S}^{G,u}_L$ *has density* $\frac{1}{[L_{G,u}:L]}$.

(b) Suppose $x_1, \ldots, x_n \in \mathcal{O}^*_{L,T}$ generate a subgroup X. Then the set

$$\left\{ v \in \mathcal{S}_L^{G,u} \mid (x_1, \ldots, x_n) \bmod v \in (G\mathcal{O}^{*u}_{L,T})^{\times n} \bmod v \right\}$$

has density at most

$$\frac{1}{[L_{G,u} : L] \cdot \left[X G_{\mathrm{sat}} \mathcal{O}^{*u}_{L,T} : G_{\mathrm{sat}} \mathcal{O}^{*u}_{L,T} \right]},$$

and, if $X \subset G_{\mathrm{sat}}$, has density equal to

$$\frac{1}{\left[L_{G,u} : L \right] \cdot \left[X G G^{*u}_{\mathrm{sat}} : G G^{*u}_{\mathrm{sat}} \right]}.$$

Proof. Part (a) is immediate from the Čebotarov density theorem, since $H_{G,u}$ is normal in $\mathrm{Gal}(L_u/L)$. For (b), we introduce some notation: For $H \subset H_u$ and $W \subset \mathcal{O}^*_{L,T}/\mathcal{O}^{*u}_{L,T}$, we write $\mathrm{Ann}_H(W) \subset H$ for the set of those $h \in H$ annihilating all of W – and similarly mod v. Then for $x \in \mathcal{O}^*_{L,T}$ and $v \in \mathcal{S}_L^{G,u}$ one has the following chain of equivalences:

$$\begin{aligned}
&(x \bmod v) \in \left(G\mathcal{O}^{*u}_{L,T} \bmod v \right) \\
\iff\ & \mathrm{Ann}_{\langle \mathrm{Frob}_v \rangle} \left(G\mathcal{O}^{*u}_{L,T} \bmod v \right) = \mathrm{Ann}_{\langle \mathrm{Frob}_v \rangle} \left(\langle x, G \rangle \mathcal{O}^{*u}_{L,T} \bmod v \right) \\
\iff\ & \langle \mathrm{Frob}_v \rangle \cap \mathrm{Ann}_{H_u} \left(G\mathcal{O}^{*u}_{L,T} \right) = \langle \mathrm{Frob}_v \rangle \cap \mathrm{Ann}_{H_u} \left(\langle x, G \rangle \mathcal{O}^{*u}_{L,T} \right) \\
\overset{v \in \mathcal{S}_L^{G,u}}{\iff}\ & \mathrm{Frob}_v \in \mathrm{Ann}_{H_u} \left(\langle x, G \rangle \mathcal{O}^{*u}_{L,T} \right).
\end{aligned}$$

For the tuple $(x_1, \ldots, x_n)$ this implies

$$(x_1, \ldots, x_n) \bmod v \in \left(G\mathcal{O}^{*u}_{L,T} \right)^{\times n} \bmod v \iff \mathrm{Frob}_v \in \mathrm{Ann}_{H_u} \left(\langle X, G \rangle \mathcal{O}^{*u}_{L,T} \right)$$

Since under our hypotheses the Kummer pairing is perfect, the index of the subgroup $\mathrm{Ann}_{H_u}(\langle X, G \rangle \mathcal{O}^{*u}_{L,T})$ in $\mathrm{Ann}_{H_u}(G\mathcal{O}^{*u}_{L,T})$ is equal to the index of $G\mathcal{O}^{*u}_{L,T} \subset \langle X, G \rangle \mathcal{O}^{*u}_{L,T}$.

Suppose first that X is contained in G_{sat}. From the definition of G_{sat}, one deduces $G_{\mathrm{sat}} \cap G\mathcal{O}^{*u}_{L,T} = GG^{*u}_{\mathrm{sat}}$. The second isomorphism theorem for groups now yields

$$\begin{aligned}
XG\mathcal{O}^{*u}_{L,T}/G\mathcal{O}^{*u}_{L,T} &\cong X/\left(X \cap G\mathcal{O}^{*u}_{L,T} \right) \overset{X \subset G_{\mathrm{sat}}}{\cong} X/\left(X \cap GG^{*u}_{\mathrm{sat}} \right) \\
&\cong XGG^{*u}_{\mathrm{sat}}/GG^{*u}_{\mathrm{sat}}.
\end{aligned}$$

An application of the Čebotarov density theorem concludes the proof of (b) in the case $X \subset G_{\mathrm{sat}}$. The argument in the general case is similar. One observes that there always is a surjection

$$XG\mathcal{O}^{*u}_{L,T}/G\mathcal{O}^{*u}_{L,T} \twoheadrightarrow XG_{\mathrm{sat}}\mathcal{O}^{*u}_{L,T}/G_{\mathrm{sat}}\mathcal{O}^{*u}_{L,T}.$$

$\square$

Remark 3.2. Let the notation be as in Lemma 3.1. In particular let X be a finitely generated subgroup of $\mathcal{O}^*_{L,T}$, by $X_1, \ldots, X_n$ and consider the density δ of

$$\left\{v \in \mathcal{S}_L^{G,u} \mid (x_1, \ldots, x_n) \bmod v \in \left(G\mathcal{O}^{*u}_{L,T}\right)^{\times n} \bmod v\right\}.$$

Then Lemma 3.1(b) has the following consequences for this density:

If X is not a subset of G_{sat}, and so the image of X in the free and finitely generated abelian group $\mathcal{O}^*_{L,T}/G_{\text{sat}}$ is non-zero, then the product $\delta \cdot [L_{G,u} : L]$ tends to zero for $u \to \infty$.

If on the other hand X is a subset of G_{sat}, then the same product stays bounded below by $\frac{1}{[G_{\text{sat}}:G]}$ as u tends to ∞.

Corollary 3.3. *Let $G \subset L^*$ be finitely generated, n in $\mathbb{N}$, $c_1, \ldots, c_d$ in $(L^*)^{\times n}$.*

(a) Suppose $q \in \mathbb{N}$ is prime to p and $q \geq 2$, and that the following set has density one

$$\left\{v \in \mathcal{P}_L \mid \exists j \in \{1, \ldots, d\}\, \exists e_v \in \mathbb{N} \textit{ coprime to } q, \textit{ such that } c_j^{e_v} \in G^{\times n} \bmod v\right\}.$$

Then there exist $j \in \{1, \ldots, d\}$ and $e \in \mathbb{N}$ such that $c_j^e \in G^{\times n}$. If q is a prime and $q > d$, one can choose e coprime to q.

(b) Suppose that $c_j \notin G_{\text{sat}}^{\times n}$ for all $j \in \{2, \ldots, d\}$ and that the density of

$$\{v \in \mathcal{P}_L \mid \exists j \in \{1, \ldots, d\} \textit{ such that } c_j \in G^{\times n} \bmod v\}$$

is one. Then there exists a p-power e such that $c_1^e \in G^{\times n}$.

Proof. Choose T in such a way that the finitely generated group $G \subset L^*$ becomes a subset of $\mathcal{O}^*_{L,T}$ and moreover that all components of all c_j lie $\mathcal{O}^*_{L,T}$, as well. If p is different from 2, we enlarge L to $L(\zeta_4)$; thereby G and x remain the same and all hypotheses are preserved. For (a) let $u := q^m$ for some $m \gg 0$. Since the exponents e_v in the hypothesis are all prime to q the operation $c \mapsto c^{e_v}$ on the finite group $\mathcal{O}^*_{L,T}/\mathcal{O}^{*u}_{L,T}$ of exponent q is bijective. Therefore the sum over all j of the densities of the sets

$$\left\{v \in \mathcal{S}_L^{G,u} \mid (c_j \bmod v) \in \left(\left(G\mathcal{O}^{*u}_{L,T}\right)^{\times n} \bmod v\right)\right\} \tag{8}$$

is at least $\frac{1}{[L_{G,u}:L]}$. Choose j_0 such that the density of the set is at least $\frac{1}{d \cdot [L_{G,u}:L]}$. Denote by $X_j \subset G$ the subgroup generated by the entries of the tuple c_j. We apply Lemma 3.1 (b) to the X_j. Since we assume $m \gg 0$, by Remark 3.2 the set X_{j_0} must be contained in G_{sat}. This proves the first assertion of (a). Moreover in this case the density of the set (8) for j_0 is explicitly determined by Lemma 3.1 (b), and we deduce

$$\frac{1}{d} \leq \frac{1}{\left[X_{j_0} G G_{\text{sat}}^{*u} : G G_{\text{sat}}^{*u}\right]}.$$

Again since $m \gg 0$, this implies that the q-primary component of $X_{j_0}G/G$ is of cardinality bounded by d. This proves the remaining assertion of (a).

To prove (b), we repeat the proof of (a) for any prime $q \neq p$. Since for $j = 2, \dots, n$, the densities of the sets (8) multiplied by $[L_{G,u} : L]$ are arbitrarily small, the expression

$$\frac{1}{\left[X_1 G G_{\text{sat}}^{*u} : G G_{\text{sat}}^{*u}\right]}$$

must come arbitrarily close to 1. Thus the q-primary component of $X_1 G/G$ has cardinality one. This completes the proof. □

3.2. On a result of MacRae

MacRae [13] proved a result on the density of conjugate fixed points under a correspondence under certain hypotheses on the correspondence. He also made a precise conjecture what the optimal set of hypothesis should be. In this section we prove this conjecture. We begin by presenting the well-known analogue of MacRae's conjecture in the number field setting.

Proposition 3.4. *Let $\varphi_i : E \to L$, $i = 1, 2$, be two distinct field homomorphisms of number fields. Then the set $\{w \in \mathcal{P}_E \mid \exists v \in \mathcal{P}_L : v \circ \varphi_1 = w = v \circ \varphi_2\}$ has density zero.*

Proof. It clearly suffices to prove the result for L the Galois closure of E over $\mathbb{Q}$, and so we assume this from now on. By standard results of Galois theory, we may extend the φ_i to field automorphisms $\psi_i \in G := \mathrm{Gal}(L/\mathbb{Q})$. Applying ψ_1^{-1} to the situation, we can assume $\psi_1 = \mathrm{id}_L$. Our hypothesis now says that ψ_2 is not the identity on E. Let $H := \mathrm{Gal}(L/E) \leq G$. We shall prove the following claim, which clearly implies the assertion of the proposition: The set

$$\Xi := \{w \in \mathcal{P}_E \mid \exists \sigma \in G \setminus H, \exists v \in \mathcal{P}_L : v \text{ lies above } w \text{ and } \sigma(w)\}$$

has density zero.

Let w be a place of E, v a place of L above w and $\sigma \in G$. The set of all places above w is then $H \cdot v$, that of places above $\sigma(w)$ is $\sigma H \sigma^{-1} \cdot \sigma(v) = \sigma H \cdot v$. Hence there is a place above w and $\sigma(w)$ simultaneously if and only if $Hv \cap \sigma H v$ is non-empty. If $G_v \subset G$ denotes the decomposition group at v, this is equivalent to G_v intersecting $H\sigma H$ non-trivially. Therefore we have the following equivalence:

$$\exists \sigma \in G \setminus H, \exists v \in \mathcal{P}_L : v \text{ lies above } w \text{ and } \sigma(w) \Longleftrightarrow G_v \cap \bigcup_{\sigma \in G \setminus H} H\sigma H \neq \varnothing.$$

Clearly $\bigcup_{\sigma \in G \setminus H} H\sigma H = G \setminus H$, and so the latter condition is equivalent to $G_v \not\subset H$, or in other words to $[E_w : \mathbb{Q}_{p_w}] > 1$ for $p_w \in \mathcal{P}_{\mathbb{Q}}$ the place below w. It remains to estimate densities. Let r be in $\mathbb{R}_{>0}$. First observe that

$$\begin{aligned}\#\{w \in \mathcal{P}_E \mid w \in \Xi, N(w) \leq r\} &= \#\{w \in \mathcal{P}_E \mid [E_w : \mathbb{Q}_{p_w}] > 1, N(w) \leq r\} \\ &\leq [E : \mathbb{Q}] \cdot \#\{p \in \mathcal{P}_{\mathbb{Q}} \mid p \leq \sqrt{r}\}.\end{aligned}$$

Next we have

$$\#\{w \in \mathcal{P}_E \mid N(w) \le r\} \ge \#\{w \in \mathcal{P}_E \mid p_w \text{ is totally split in } L/\mathbb{Q}, N(w) \le r\}$$
$$= [E : \mathbb{Q}] \cdot \#\{p \in \mathcal{P}_\mathbb{Q} \mid p \text{ is totally split in } L/\mathbb{Q}, p \le r\}.$$

By the Čebotarev density theorem, the latter number behaves asymptotically for $r \to \infty$ like $[E : \mathbb{Q}]/[L : \mathbb{Q}] \cdot \#\{p \in \mathcal{P}_\mathbb{Q} \mid p \le r\}$. Gauss' formula for the distribution of prime numbers yields the claimed density result

$$\frac{\#\{w \in \mathcal{P}_E \mid w \in \Xi, N(w) \le r\}}{\#\{w \in \mathcal{P}_E \mid N(w) \le r\}} \xrightarrow{r\to\infty} 0.$$

□

To explain the function field analogue of Proposition 3.4, let us fix some notation: Let X be a smooth projective, geometrically connected curve over the finite field $\mathbb{F}_q$. Let $\pi_i : X \times X \to X$, $i = 1, 2$, be the canonical projections, and let D be an effective divisor on $X \times X$ such that both projections from D to X are finite. We think of D as a correspondence on X and define

$$\Xi_D := \{x \in |X| \mid \exists z \in |D| : \pi_1(z) = x = \pi_2(z)\},$$

where $|X|$ is the set of closed points of X.

Using intersections on the surface $X \times X$, one can describe Ξ_D as follows: Let Frob: $X \to X$ be the Frobenius endomorphism relative to $\mathbb{F}_q$ which is given by $f \mapsto f^q$ on rational functions on X. Let Φ^i be the graph of Frob^i or, equivalently, the i-fold iterate of the correspondence $\Phi := \Phi^1$. Note that Φ^0 is simply the diagonal Δ_X on $X \times X$. Using these powers of Φ, the set Ξ_D is the projection of $\cup_{i\in\mathbb{N}_0} D \cap \Phi^i$ along π_1 or, equivalently, along π_2 from $X \times X$ to X. For $i < 0$ we denote by Φ^i the transpose of Φ^{-i}.

Theorem 3.5. *Suppose that D does not contain any divisor of the form Φ^i, $i \in \mathbb{Z}$. Then Ξ_D has density zero.*

This result was conjectured in [13], where it was proved under the following additional hypothesis: (a) D is irreducible, and (b), if m and n are the degrees of D over X with respect to the projections π_i, $i = 1, 2$, then either m/n does not lie in $q^\mathbb{Z}$, for q the cardinality of the constant field of X, or $m = n = 1$ and D is different from Δ_X. The main improvement of our result above is that it no longer imposes any restrictions on the quotient m/n.

Decomposing D into irreducible divisors, the above theorem may be restated in terms of global function fields. The statement is essentially that of Proposition 2.5, were in addition one needs to assume that the φ_i are separable. However, the method of proof there can not be generalized. The intersection $\varphi_1(E) \cap \varphi_2(E) \subset L$ may be finite. But even if it is infinite and thus a function field, it may happen that one cannot reduce to a Galois theoretic situation; for instance if $[L : \varphi_1(E)] \ne [L : \varphi_1(E)]$. Our proof is a simple application of intersection theory on surfaces – a heuristic argument is given in Remark 3.6. MacRae's proof is quite different.

Proof of Theorem 3.5. Our aim is to give, for any $r \in \mathbb{N}$, a bound on the number of points in Ξ_D which are defined over $\mathbb{F}_{q^r}$: For a point $x \in |X|$ to be contained in $\Xi_D(\mathbb{F}_{q^r})$ it is necessary and sufficient that

(a) there exists $i \in \mathbb{N}_0$ such that $x \in \pi_1(D \cap \Phi^i)$, and
(b) $x \in \pi_1(\Delta_X \cap \Phi^r)$.

Using (b), we may in (a) replace Φ^i by Φ^{i+rt} for any $t \in \mathbb{Z}$. (This can easily be verified using explicit coordinates.) Thus we have

$$\Xi_D\left(\mathbb{F}_{q^r}\right) \subset \pi_1\left(\bigcup_{i=[(1-r)/2]}^{[r/2]} D \cap \Phi^i\right)\left(\overline{\mathbb{F}_q}\right).$$

This gives the estimate

$$\#\Xi_D\left(\mathbb{F}_{q^r}\right) \leq \sum_{i=[(1-r)/2]}^{[r/2]} \#\left(D \cap \Phi^i\right).$$

By our hypothesis all intersections $D \cap \Phi^i$, $i \in \mathbb{Z}$, are proper. Whence we may use the intersection pairing on $X \times X$ to give an upper bound for $\#(D \cap \Phi^i)$. As in [8], we write $\cdot$ instead of $\cap$ for this pairing, and denote by g_X the genus of X.

By [8, Exer. V.1.10] one has $\Phi^i \cdot \Phi^i = q^{|i|}(2 - 2g_X)$ for any $i \in \mathbb{Z}$. Let d_1 and d_2 denote the degrees of D to X with respect to the projections π_i, $i = 1, 2$. The degrees of the same projections of $D + \Phi^i$ to X are then given by $d_1 + 1$ and $d_2 + q^{-i}$, if $i < 0$, and by $d_1 + q^i$ and $d_2 + 1$, if $i \geq 0$. In either case, the product of these two degrees is therefore bounded by $(d_1 + d_2)(d_1 + d_2 + q^{|i|})$. Applying [8, Exer. V.1.10], yields

$$2D \cdot \Phi^i \leq -D^2 - (\Phi^i)^2 + 2(d_1 + d_2)\left(d_1 + d_2 + q^{|i|}\right).$$

Using the above expression for $(\Phi^i)^2$, we deduce that there is a constant $c \in \mathbb{Z}$ such that for all $i \in \mathbb{Z}$

$$2D \cdot \Phi^i \leq c + q^{|i|}(2(d_1 + d_2) + 2g_X - 2).$$

Plugging this into the estimate for $\Xi_D(\mathbb{F}_{q^r})$, one finds

$$\begin{aligned}\#\Xi_D(\mathbb{F}_{q^r}) &\leq \sum_{i=[(1-r)/2]}^{[r/2]} \left(c/2 + q^{|i|}(d_1 + d_2 + g_X - 1)\right).\\ &\leq \frac{cr}{2} + 2(d_1 + d_2 + g_X - 1)q^{r/2+1}.\end{aligned}$$

Thus clearly the upper density $\limsup_{r\to\infty} \#\Xi(\mathbb{F}_{q^r})/q^r$ of Ξ is zero as asserted.

□

Remark 3.6. Let us give the idea of the above proof in the simplest case $X = \mathbb{P}^1$ and where D is the graph of a function f/g on $\mathbb{P}^1$ for polynomials f, g in an indeterminate x, say of degrees d_1 and d_2. The $\mathbb{F}_{q^r}$-valued points of Ξ_D are the x- (or y-) coordinates that satisfy the equations $y = x^{q^i}$ and $y = (f/g)(x)$ for some $i \in \{0, 1, \ldots, r-1\}$ and $x^{q^r} = x$. Because of the latter equation, the first equation may be replaced by $y^{q^{r-i}} = x$ if convenient. In the above proof, we consider both pairs of equations each time for i in the range $0, \ldots, [r/2]$. Then $y = x^{q^i}$ and $y = (f/g)(x)$ leads to a polynomial equation in x of degree at most $d_1 + q^i$, while $y = (f/g)(x)$ and $y^{q^i} = x$ yields a polynomial equation in y of degree at most $d_2 + q^i$. Counting the solutions in both cases leads to the estimate we obtained in the above proof. Had we not interchanged the role of x and y for the $i > [r/2]$ the estimates obtained would have been much worse - and not sufficient to deduce the desired result.

Remark 3.7. One may wonder whether it is possible to deduce Theorem 3.5 by a proof similar to that of Proposition 3.4. This leads to the question of whether the intersection of any two global fields inside K^{alg} is again a global field. The answer is in the negative as the following example, communicated to us by H. Stichtenoth, explains:

Let k be any field of characteristic $p > 0$ and consider $F := k(x)$, $F_1 := k(y)$, for $y = z^p - z$, and $F_2 := k(z)$, for $z = x^2(x-1)$. Observe that the places $x = 0$ and $x = 1$ of F both lie above the place $y = 0$ of F_1 and also above the place $z = 0$ of F_2. We claim that $F_1 \cap F_2$ is finite over k. To prove this, it suffices to assume $k = k^{\mathrm{alg}}$ and to show that $F' := F_1 \cap F_2$ is equal to k. Assume otherwise. Then F' is a subfield of F of positive transcendence degree, and hence, by Lüroth's theorem, of the form $k(w)$ for some $w \in F \setminus k$. By a change of variables, we may assume that the places $y = 0$ of F_1 and $z = 0$ of F_2 are above the place $w = 0$ of F'. It is now a simple matter of computing the ramification degrees of $x = 0$ and of $x = 1$ above $w = 0$ via $F' \subset F_1 \subset F$ and $F' \subset F_2 \subset F$, to obtain a contradiction: If e_1 denotes the ramification degree of $y = 0$ above $w = 0$ and if e_2 denotes the ramification degree of $z = 0$ above $w = 0$, via the first sequence of fields the ramification degrees of $x = 0$ and $x = 1$ above $w = 0$ are both e_1, wheres via the second sequence of fields they are $2e_2$ and e_2, respectively; this is impossible since we cannot have $2e_2 = e_1 = e_2$ with $e_2 > 0$.

4. Proof of the main result

PROOF of Theorem 2.21. We may clearly replace D by an F-reduced divisor, since by Lemma 2.7 this does not alter $v \mapsto S_v = S_v^D$. In § 2.2 we explained how to attach to a pair (D, L) with F-reduced $D \subset \mathcal{P}_E \times \mathcal{P}_L$ pair a (Σ^D, L^D) with Σ^D an F-reduced Hodge Tate set. Since replacing L by a finite extension yields again a strictly compatible system, we assume that in fact $L = L^D$ and $D = D_\Sigma$ for an F-reduced Hodge-Tate set Σ. Also there is no loss of generality in assuming that the strictly compatible system $\{\rho_v\}$ is semisimple.

Interchangeably, we regard the ρ_v as representations of G_E^{ab} and of $\mathbb{A}_E^*$ with the caveat that in the function field case the reciprocity map only has dense image, so that being continuous with respect to G_E^{ab} is more restrictive than with respect to $\mathbb{A}_E^*$. We assume that the ρ_v are given in diagonal form over some finite extension $\widetilde{\mathbb{F}}_v$ of $\mathbb{F}_v$. The support of the minimal conductor of ρ_v is determined by the sets S and S_v. In the function field case, due to Proposition 2.12 the minimal conductor of ρ_v is bounded by the effective divisor $[S]+[S_v]$. In the number field case, the most one can say is that the minimal conductor is bounded by $s_v[S]+[S_v]$ for some $s_v \in \mathbb{N}$ depending on v. Define N to be

$$N := 2 \prod_{w \in S \smallsetminus \mathcal{P}_E^{\mathrm{ar}}} \# \left(\mathbb{F}_w \times \mathbb{F}_w^*\right).$$

Consider $\alpha \in E^*$ and denote by $\mathrm{Div}(\alpha) = \sum_{i \in I} d_i[w_i]$ its divisor and by $\mathrm{Supp}\,\alpha = \{w_i : i \in I\}$ its support. We assume that all d_i are non-zero and that the $w_i \in \mathcal{P}_E^{\mathrm{na}}$ are pairwise distinct. Suppose also that no w_i lies in $S \cup S_v$. For simpler notation, we assume that S_v and S are disjoint. Corresponding to the disjoint union

$$\mathcal{P}_E = S_v \cup \{w_i \mid i \in I\} \cup S \cup (\mathcal{P}_E \smallsetminus (S_v \cup \{w_i \mid i \in I\} \cup S)),$$

we write ideles in $\mathbb{A}_E^*$ as quadruples, and we write $\underline{\alpha}$ for a constant tuple α. Then

$$\begin{aligned} 1 &= \rho_v \left((\underline{\alpha}, \underline{\alpha}, \underline{\alpha}, \underline{\alpha})^{N^{s_v}} \right) = \rho_v \left(\underline{\alpha}^{N^{s_v}}, \underline{\alpha}^{N^{s_v}}, \underline{1}, \underline{1} \right)) \\ &= \rho_v \left(\underline{\alpha}^{N^{s_v}}, \underline{1}, \underline{1}, \underline{1} \right)) \prod_{i \in I} \rho_v(\mathrm{Frob}_{w_i})^{d_i N^{s_v}}. \end{aligned}$$

For each place $w \in \mathcal{P}_E \smallsetminus S$, let L^w denote the splitting field over L of the polynomial $f_w \in L[t]$. Let $\underline{\lambda}_w = (\lambda_{i,w})_{i=1,\dots,n}$ denote the roots of f_w in L^w repeated according to their multiplicity. If τ is a permutation of $\{1, \dots, n\}$, then $\underline{\lambda}_{w,\tau}$ denotes the tuple $(\lambda_{\tau(i),w})_{i=1,\dots,n}$. Having the ρ_v in diagonal form means, that for each $v \in \mathcal{P}_L \smallsetminus T$ and each $w \in \mathcal{P}_E \smallsetminus (S \cup S_v)$ at which f_w is v-integral, there exists a permutation $\tau_{w,v}$ of $\{1, \dots, n\}$ such that $\rho_v(\mathrm{Frob}_{w_i})$ is the diagonal matrix with diagonal

$$\underline{\lambda}_{w,\tau_{w,v}} \pmod{v} = (\lambda_{\tau_{w,v}(1),w}, \dots, \lambda_{\tau_{w,v}(n),w}) \pmod{v}.$$

Hence

$$\rho_v \left(\underline{\alpha}^{-N^{s_v}}, \underline{1}, \underline{1}, \underline{1} \right)) = \left(\prod_{i \in I} \underline{\lambda}_{w_i,\tau_{w_i,v}}^{d_i} \right)^{N^{s_v}} \pmod{v}, \tag{9}$$

where multiplication and exponentiation is componentwise on each entry of the n-tuple.

If the ρ_v come from an n-tuple of Hecke characters, the expression of the right hand side of (9), without reduction modulo v, will have to agree with a power of the associated Hodge-Tate character evaluated at α, cf. formula (2). The following is a first approximation to this:

Lemma 4.1. *Let α be in E^* and let* $\mathrm{Div}(\alpha) = \sum_{i\in I} d_i[w_i]$ *be its divisor. Define*

$$G_\alpha := \langle \sigma(\alpha) \mid \sigma \in \Sigma \rangle \subset L^* \subset \left(\prod_{i\in I} L^{w_i} \right)^* .$$

Suppose $\mathrm{Supp}\,\alpha$ *is disjoint from S. Then there exist a tuple $\underline{\tau} = (\tau_i)_{i\in I}$ of permutations of* $\{1,\ldots,n\}$ *and $e_\alpha \in \mathbb{N}$ such that $\underline{\lambda}_{\alpha,\underline{\tau}} := \prod_{i\in I} \underline{\lambda}_{w_i,\tau_i}^{d_i}$ satisfies*

$$\left(\underline{\lambda}_{\alpha,\underline{\tau}}\right)^{e_\alpha} \in (G_\alpha)^{\times n}.$$

Since $\underline{\tau}$ admits only finitely many choices, in the sequel, we regard e_α as independent of $\underline{\tau}$.

Proof. Let $q \geq 2$ be relatively prime to pN. Let v be any place of $\mathcal{P}_L \smallsetminus T$ such that S_v contains none of the w_i and such that all f_{w_i} are v-integral. Then by formula (9), for any such v there exists a $\underline{\tau}$, such that

$$\underline{\lambda}_{\alpha,\underline{\tau}}^{N^{s_v}} = \rho_v\left(\left(\underline{\alpha}^{-N^{s_v}}, \underline{1}, \underline{1}, \underline{1}\right)\right) \pmod{v}$$

We claim that the order of the right hand side divides the order of $(G_\alpha \mod v) \subset \widetilde{\mathbb{F}}_v^*$. Since $\widetilde{\mathbb{F}}_v^*$ is cyclic, this proves $(\underline{\lambda}_{\alpha,\underline{\tau}}^{N^{s_v}} \mod v) \in ((G_\alpha)^{\times n} \mod v)$ for almost all $v \in \mathcal{P}_L$. The lemma then follows from Corollary 3.3(a).

To prove the claim, observe that for $w \in S_v$ we have $\alpha \in \mathcal{O}_w$ and $\rho_v|_{E_w^*}$ is trivial on $1 + \mathfrak{p}_w$. Hence the component $\underline{\alpha}^{-N^{s_v}}$ above S_v as an argument of ρ_v may be viewed as an element of $\prod_{w\in S_v} (\mathcal{O}_w/\mathfrak{p}_w)^*$. For $w = v\circ\sigma \in v\circ\Sigma = S_v$, the units $(\mathcal{O}_w/\mathfrak{p}_w)^*$ embed via σ into $\mathbb{F}_v^*$. It follows that the order of the tuple $\underline{\alpha}$ in $\prod_{w\in S_v} (\mathcal{O}_w/\mathfrak{p}_w)^*$ is the least common multiple of the orders of the $\sigma(\alpha) \mod v$. Since $\mathbb{F}_v^*$ is cyclic the latter is the order of $G_\alpha \pmod{v}$. □

Consider $v \in \mathcal{P}_L^{\mathrm{na}}$. The polynomials f_w lie in $L \subset L_v$. For instance via the Newton polygon of f_w for v, one can see that one has well-defined values $v(\lambda_{i,w})$, $i = 1,\ldots,n$. By the support $\mathrm{Supp}(\underline{\lambda}_w)$ of $\underline{\lambda}_w$ we mean the finite set of those $v \in \mathcal{P}_L^{\mathrm{na}}$ for which one of these values is non-zero. It will be useful to have some information on this support:

Lemma 4.2. *There exists a finite set $T' \subset \mathcal{P}_L$ such that for all $w \notin S$ we have*

$$\mathrm{Supp}(\underline{\lambda}_w) \subset T' \cup \Sigma^{-1}(\{w\}).$$

Proof. Fix $w_0 \in \mathcal{P}_E$ but not in S. Let w be any place of E not in $S \cup \{w_0\}$, and let $\alpha \in E^*$ have support $\{w, w_0\}$, i.e., divisor $d[w] + d_0[w_0]$ with $d, d_0 \neq 0$. By Lemma 4.1 we have

$$\left(\underline{\lambda}_{w,\tau}^{d} \underline{\lambda}_{w_0,\tau_0}^{d_0}\right)^{e_\alpha} = \prod_{\sigma\in\Sigma} \sigma(\alpha)^{\underline{n}_{\alpha,\sigma}},$$

for tuples $\underline{n}_{\alpha,\sigma} \in \mathbb{N}^n$ and permutations τ, τ_0 of $\{1,\ldots,n\}$. If we compute the valuation of this expression at any $v \notin \Sigma^{-1}(\{w, w_0\})$, the right hand side is zero. It follows that

$$\mathrm{Supp}(\underline{\lambda}_w) \subset \mathrm{Supp}(\underline{\lambda}_{w_0}) \cup \Sigma^{-1}(\{w_0\}) \cup \Sigma^{-1}(\{w\}).$$

So the lemma holds with $T' := \mathrm{Supp}(\underline{\lambda}_{w_0}) \cup \Sigma^{-1}(\{w_0\})$. □

Let $\Xi_\Sigma := \{w \in \mathcal{P}_E \mid \#\Sigma > \#\Sigma^{-1}(\{w\})\}$. By Proposition 3.4 for $p = 0$ and Theorem 3.5 for $p > 0$, the set Ξ_Σ has density zero. For $w \in \mathcal{P}_E \smallsetminus \Xi_\Sigma$ and $\sigma \in \Sigma$ there a unique $v_{w,\sigma} \in \mathcal{P}_L$ such that $v_{w,\sigma} \circ \sigma = w$. For any such w we define $(\underline{m}_{w,\sigma})_{\sigma\in\Sigma} \in (\mathbb{Q}^n)^\Sigma$ by

$$\underline{m}_{w,\sigma} := \frac{v_{w,\sigma}(\underline{\lambda}_w)}{e_{v_{w,\sigma}/w,\sigma}} \in \mathbb{Q}^n. \tag{10}$$

On $\mathbb{Q}^n$ we consider the natural permutation action by the group S_n.

Lemma 4.3. *Suppose $\alpha \in E^*$ has non-empty support disjoint from $\Xi_\Sigma \cup T' \circ \Sigma$. Suppose in addition that $\Sigma^{-1}(\{w\})$ and $\Sigma^{-1}(\{w'\})$ are disjoint for any $w \neq w'$ in* $\operatorname{Supp}(\alpha)$. *Then with the notation from Lemma 4.1, for all $i \in I$ and $\sigma \in \Sigma$ one has $e_\alpha \underline{m}_{w_i,\sigma} \in \mathbb{Z}^n$ and*

$$\underline{\lambda}_{\alpha,\underline{\tau}}^{e_\alpha} = \prod_{\sigma\in\Sigma} \sigma(\alpha)^{e_\alpha \tau_i(\underline{m}_{w_i,\sigma})} \tag{11}$$

Moreover for any w, w' in $\mathcal{P}_E^{\mathrm{na}} \smallsetminus (\Xi_\Sigma \cup T' \circ \Sigma)$, there exists $\tau_{w,w'} \in S_n$ such that

$$\tau_{w,w'}(\underline{m}_{w,\sigma}) = \underline{m}_{w',\sigma} \quad \forall \sigma \in \Sigma.$$

Proof. Lemma 4.1 yields $\underline{\lambda}_{\alpha,\underline{\tau}}^{e_\alpha} = \prod_{\sigma\in\Sigma} \sigma(\alpha)^{\underline{n}_{\alpha,\sigma,\underline{\tau}}}$ for suitable $\underline{n}_{\alpha,\sigma,\underline{\tau}} \in \mathbb{Z}^n$. Applying a valuation v of L yields

$$e_\alpha \sum_{i\in I} d_i v(\underline{\lambda}_{w_i,\tau_i}) = \sum_{\sigma\in\Sigma} \underline{n}_{\alpha,\sigma,\underline{\tau}} e_{v/v\circ\sigma,\sigma} v \circ \sigma(\alpha) \in \mathbb{Q}^n.$$

(Recall that the valuations corresponding to v and $v \circ w$ are normalized.) By our hypotheses on $\operatorname{Supp}(\alpha)$, for $v = v_{w_i,\sigma}$ this simplifies to

$$e_\alpha d_i v_{w_i,\sigma}(\underline{\lambda}_{w_i,\tau_j}) = e_{v_{w_i,\sigma}/w_i,\sigma} \underline{n}_{\alpha,\sigma,\underline{\tau}} d_i.$$

Canceling the d_i, the first assertion follows from the definition of the $\underline{m}_{w,\sigma}$.

The first part yields $\tau_i(\underline{m}_{w_i,\sigma}) = \tau_j(\underline{m}_{w_j,\sigma})$ whenever w_i, w_j are in the support of α. We also know that given any two places in $\mathcal{P}_E^{\mathrm{na}}$ there exists $\alpha \in E^*$ whose support consists of these two places. If we are given w, w' in $\mathcal{P}_E^{\mathrm{na}} \smallsetminus (\Xi_\Sigma \cup T' \circ \Sigma)$, we can find w'' in this set such that $\Sigma^{-1}(\{w''\})$ is disjoint from the two corresponding sets for w and w'. We now apply the first part to $\alpha, \alpha' \in E^*$ with support $\{w'', w\}$ and $\{w'', w'\}$ to conclude the proof. ◻

By reindexing all $\underline{\lambda}_w$ with $w \in \mathcal{P}_E^{\mathrm{na}} \smallsetminus (\Xi_\Sigma \cup T' \circ \Sigma)$, we assume from now on that we have $(\underline{m}_\sigma)_{\sigma\in\Sigma} \in (\mathbb{Q}^n)^\Sigma$ such that

$$(\underline{m}_\sigma)_{\sigma\in\Sigma} = (\underline{m}_{w,\sigma})_{\sigma\in\Sigma} \quad \forall w \in \mathcal{P}_E^{\mathrm{na}} \smallsetminus (\Xi_\Sigma \cup T' \circ \Sigma)$$

and that each τ_i fixes $(\underline{m}_\sigma)_{\sigma\in\Sigma}$, so that all components of (all) $\underline{\tau}$ lie in

$$\cap_{\sigma\in\Sigma} \operatorname{Stab}_{S_n}(\underline{m}_\sigma) \subset S_n.$$

Note however that $\underline{\tau}$ is not redundant on the left hand side of (11).

Our next aim is to uniformly bound the exponents e_α (for α as in Lemma 4.3). This requires the following lemma:

Lemma 4.4. *There exists $M \in \mathbb{N}$ such that any root of unity contained in any of the fields L^w is of order dividing M.*

Proof. Recall that L^w is generated by the roots of the monic degree n polynomial f_w. If $p > 0$, write f_w as $g_w(x^{p^i})$ with g_w separable, else set $g_w := f_w$. Then $\deg g_w = n/p^i$ and by Galois theory, the degree of the normal closure of L^w over L ist bounded by $(n/p^i)!p^i \leq n!$. It suffices therefore to prove the following claim: For any $r \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that for all $L' \supset L$ with $[L' : L] \leq r$ the roots of unity contained in L' have order dividing M.

Suppose first $p = 0$ and let $s := [L : \mathbb{Q}]$. For a prime p we can have $\zeta_p \in L'$ only if $rs \geq p - 1$, and we can have $\zeta_{p^i} \in L'$ only if $rs \geq p^{i-1}$. Thus $M = \prod_{p \leq rs+1} p^{[\log_p(rs)]+1}$ works. For $p > 0$ denote by s the degree of the constant field of L over its prime field $\mathbb{F}_p$. If ζ_l lies in L', we must have

$$\begin{aligned} r \geq [L' : L] \geq [\mathbb{F}_{p^s}(\zeta_l) : \mathbb{F}_{p^s}] &\geq \frac{1}{s}[\mathbb{F}_p(\zeta) : \mathbb{F}_p] \\ &= \frac{1}{s}\min\{k \in \mathbb{N} \mid l \text{ divides } p^k - 1\}. \end{aligned}$$

Hence l divides $M := \prod_{i=1}^{rs}(p^i - 1)$. □

Lemma 4.5. *There exists $e \in \mathbb{N}$ such that $e\underline{m}_\sigma \in \mathbb{Z}^n$ for all σ* **and** *such that for all $\alpha \in E^*$ satisfying the hypotheses of Lemma 4.3 and all $\underline{\tau}$ as in Lemma 4.1, one has*

$$\underline{\lambda}^e_{\alpha,\underline{\tau}} = \prod_{\sigma \in \Sigma} \sigma(\alpha)^{e\underline{m}_\sigma}.$$

In particular, the expression on the left is independent of the possible $\underline{\tau}$ provided by Lemma 4.1

Proof. Let M be as in the previous lemma and let $\tilde{e} \in \mathbb{N}$ be such that $\tilde{e}\underline{m}_\sigma \in \mathbb{Z}^n$ for all $\sigma \in \Sigma$. Raising (11) to the power $\tilde{e}/e_\alpha$ yields

$$\underline{\lambda}^{\tilde{e}}_{\alpha,\underline{\tau}} = \zeta_{\alpha,\underline{\tau}} \prod_{\sigma \in \Sigma} \sigma(\alpha)^{e\underline{m}_\sigma}$$

for some root of unity $\zeta_{\alpha,\underline{\tau}}$ which depends on α and $\underline{\tau}$. The left hand side lies in L^w. After division by $\zeta_{\alpha,\underline{\tau}}$, the right hand side lies in L. Hence by the previous lemma $\zeta^M_{\alpha,\underline{\tau}} = 1$. Defining $e = M\tilde{e}$, the lemma is established. □

Lemma 4.6. *For all $v \notin T$ the divisor $[S_v]$ is a conductor for ρ^e_v with e as in Lemma 4.5. In particular there exists $s \in \mathbb{N}$, independently of v, such that $s[S]+[S_v]$ is a conductor for ρ_v.*

Proof. The second assertion is immediate from the first – and for $p > 0$ it follows independently and much simpler from Proposition 2.12. To prove the first assertion, consider the n-tuple of characters

$$\underline{\psi}: E^* \to ((K^{\mathrm{alg}})^*)^{\times n}, \beta \mapsto \prod_{\sigma \in \Sigma} \sigma(\beta)^{e\underline{m}_\sigma}.$$

Fix any α as in Lemma 4.3. Lemma 4.5 yields $\underline{\psi}(\alpha) = \underline{\psi}(\alpha\beta)$ for all β with trivial divisor, i.e. $\beta \in \mathcal{O}_E^*$. Hence by Proposition 2.13 all components of $\underline{\psi}$ extend to a Hecke character with trivial conductor.

Let $\{\rho_v''\}$ be the strictly compatible system of mod v Galois representations associated to $\underline{\psi}$. By construction of ρ_v'', Lemma 4.5 implies that for any $\alpha \in E^*$ satisfying the hypotheses of this lemma, the matrices $\rho_v^e(\underline{\alpha})$ and $\rho_v''(\underline{\alpha})$ are conjugate. Let $\mathfrak{m}$ be a divisor which is a common conductor for ρ_v'' and ρ_v^e. Since Ξ_D has density zero, ideles $\underline{\alpha}$ for α as in Lemma 4.5 generate the kernel of the canonical homomorphism $\mathrm{Cl}_{\mathfrak{m}} \to \mathrm{Cl}_0$. Hence the restrictions of ρ_v^e and ρ_v'' to this kernel are conjugate. We deduce that they have the *same* conductor. By construction, $[S_v]$ is a finite conductor of ρ_v'', and the proof is complete. $\square$

Lemma 4.7. *The tuples $\underline{m}_\sigma$ lie in $\mathbb{Z}[1/p]^n$.*

Proof. Suppose α satisfies the hypotheses of Lemma 4.3 and in addition the condition $\alpha \equiv 1$ modulo the conductor $s[S]$. For such α, formula (9) holds with $s_v = 0$. Raising (9) to the power e, as in the proof of Lemma 4.1 we deduce that

$$\left\{v \in \mathcal{P}_L \smallsetminus T \mid \exists \underline{\tau} \,:\, \underline{\lambda}^e_{\alpha,\underline{\tau}} \bmod v \in \left(G^e_\alpha\right)^{\times n} \bmod v\right\}$$

has density one in $\mathcal{P}_L$. On the other hand, it is a consequence of Lemma 4.5 that for an arbitrary $\underline{\tau}'$, either $\underline{\lambda}^e_{\alpha,\underline{\tau}'} = \underline{\lambda}^e_{\alpha,\underline{\tau}}$, or $\underline{\lambda}^e_{\alpha,\underline{\tau}'}$ does not lie in $(G_\alpha)^{\times n}_{\mathrm{sat}}$. We deduce from Corollary 3.3 that there is a p-power e' such that

$$\left(\underline{\lambda}^e_{\alpha,\underline{\tau}}\right)^{e'} \in \left(G^e_\alpha\right)^{\times n}.$$

Computing $\underline{m}_\sigma$ via (10) yields $ee'\underline{m}_\sigma \in \frac{1}{e_{v_{w,\sigma}/w,\sigma}}(e\mathbb{Z}^n)$. Since for generic w the index $e_{v_{w,\sigma}/w,\sigma}$ is the degree of inseparability of $\sigma : E \to L$, and thus a p-power, the assertion follows. $\square$

We now complete the proof of Theorem 2.21:

We partition $\{1, \ldots, n\}$ into sets $M_1, \ldots, M_t$, such that whenever i, i' are in the same M_j, the tuples $(m_{\sigma,i})_{\sigma\in\Sigma}$ and $(m_{\sigma,i'})_{\sigma\in\Sigma}$ agree, and if they are in different M_j, they do not agree.

As in the proof of Lemma 4.6, we can apply Proposition 2.13 to obtain Hecke characters χ_j, $j = 1, \ldots, t$, with χ_j having set of Hodge-Tate weights given by $(m_{\sigma,i})_{\sigma\in\Sigma}$ for any $i \in M_j$.

Define $\{\rho_v'\}$ to be the strictly compatible system attached to the sum of Hecke characters $\oplus_{j=1}^t(\chi_j)^{\#M_j}$. Let $\mathfrak{m}$ be a common conductor for all χ_j which moreover satisfies $s[S] \le \mathfrak{m}$. Restricted to the closed subgroup of G_E^{ab} that is defined as the image of $E^*U_{\mathfrak{m}}(E^*_{\mathrm{ar}})^o$, this system is isomorphic to $\{\rho_v'\}$ by the Čebotarov density theorem. (Both systems have the same traces on all w not in $\Xi_D \cup T' \circ \Sigma$.) Arranging the matrix representations of the ρ_v suitably, we may assume that the restrictions are equal. Hence the system $\{\rho_v\}$ can be written simultaneously as a direct sum of strictly compatible systems of sizes $\#M_j$, $j = 1, \ldots, t$. (At this point we may have to enlarge T by finitely many $v \in \mathcal{P}_L$.)

Thus it suffices to continue the proof for one of these systems. By twisting by the inverse of χ_j, we may furthermore assume that all $\underline{m}_\sigma$ are identically zero, so that the ρ_v are representations of $H := E^*\backslash\mathbb{A}_E^*/U_\mathfrak{m}(E_{\mathrm{ar}}^*)^o$. In the number field case the latter is a finite group, and as explained in [11], a result by Deligne and Serre, c.f. [4, § 8], then implies that the system $\{\rho_v\}$ arises from an Artin representation ρ_0. Finally any abelian Artin representation arises from a direct sum of characters of some strict $\mathfrak{m}$-class group and thus from a direct sum of Hecke characters.

Suppose from now on that E is a function field and let h be the exponent of its strict class group Cl_0. Fix $w_0 \in \mathcal{P}_E \smallsetminus \Xi_D$ such that $\Sigma^{-1}(w_0)$ is disjoint from T'. Let w be any other place with these properties and such that $\Sigma^{-1}(w)$ is disjoint from $\Sigma^{-1}(w_0)$—thus we only exclude places in a set of density zero. By the definition of h, we can find $\alpha \in E^*$ with divisor $h(\deg w[w_0] - \deg w_0[w])$. Then by Lemma 4.5 we can find a permutation τ of $\{1, \ldots, n\}$ such that

$$\underline{\lambda}_{w,\tau}^{he \deg w_0} = \underline{\lambda}_{w_0}^{he \deg w}.$$

After yet again enlarging T by a finite amount, we can uniquely partition the representations ρ_v into a direct sum $\oplus_i \rho_{v,i}$ of subrepresentations $\rho_{v,i}$ with $\dim \rho_{v,i}$ independent of v, such that

(a) there is a bijection between the indices i and the different eigenvalues λ_i of $\underline{\lambda}_{w_0}^{eh}$
(b) $\dim \rho_{v,i}$ is the multiplicity of λ_i^{eh}.
(c) $\rho_{v,i}(\mathrm{Frob}_{w_0}^{eh}) \equiv \lambda_i^{eh} \pmod{v}$.

There are obvious characters on $\langle \mathrm{Frob}_{w_0} \rangle \subset H$ mapping Frob_{w_0} to λ_i. Because the characters take values in the divisible group $(K^{\mathrm{alg}})^*$, we may extend them to Hecke characters χ_i with Hodge-Tate weight 0, i.e., with $\Sigma = \varnothing$. Twisting $\rho_{v,i}$ by χ_i^{-1}, and restricting our attention to a single i, we may assume that the strictly compatible system $\{\rho_v\}$ is trivial when restricted to $\langle \mathrm{Frob}_{w_0}^{eh} \rangle$.

Since the ρ_v are semisimple and abelian, they all factor via the maximal quotient of $H/\langle \mathrm{Frob}_{w_0}^{eh} \rangle$ of order prime to p. Let h' be its exponent. Then all $\lambda_w^{h'}$ reduce to 1 at almost all places v. Thus all λ_w lie in $(\mathbb{F}_p^{\mathrm{alg}})^*$. Therefore reduction mod v is the identity on these, and so under this identification all ρ_v *are* the same representation. This is a semisimple Artin representation ρ_0 which is a representation of some strict $\mathfrak{m}$-class group and hence a direct sum of Hecke characters.

At this point we have left aside finitely many exceptional $v \in \mathcal{P}_L$ which were not in the original set T. For these v note that if we have two strictly compatible systems which are conjugate for almost all v, then they are conjugate for all v at which they are defined. This completes the proof of the main assertion of Theorem 2.21.

The remaining assertions of Theorem 2.21 are rather obvious, since if some place w is finite for $\{\rho_{\chi,v}\}$, it is so for χ. A similar comment applies to *being of ∞-type*. $\square$

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References

[1] Böckle, G.: An Eichler–Shimura isomorphism over function fields between Drinfeld modular forms and cohomology classes of crystals. preprint (2004). http://www.iwr.uni-heidelberg.del~Gebhard.Boeckle/preprints.html
[2] Böckle, G.: Hecke characters and Drinfeld modular forms. Preprint (2008)
[3] Corrales-Rodrigáñez, C., Schoof, R.: The support problem and its elliptic analogue. J. Number Theory **64**(2), 276–290 (1997)
[4] Deligne, P., Serre, J.-P.: Formes modulaires de poids 1. Ann. Sci. École Norm. Sup. (4) **7**, 507–530 (1974)
[5] Goss, D.: The algebraist's upper half-plane. Bull. Am. Math. Soc. (N.S.) **2**(3), 391–415 (1980)
[6] Goss, D.: L-series of Grössencharakters of type A_0 for function fields. In p-adic Methods in Number Theory and Algebraic Geometry, vol. 133. Contemp. Math., pp. 119–139. AMS (1992)
[7] Gross, B.: Algebraic Hecke characters for function fields. Seminar on Number Theory, Paris 1980–1981, Progr. Math. **22**, 87–90
[8] Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics No. 52, Springer, New York (1977)
[9] Henniart, G.: Représentations l-adiques abéliennes. Seminar on Number Theory, Paris 1980/1981, Progr. Math., **22**, 107–126 (1982)
[10] Khare, C.: Compatible systems of mod p Galois representations. C. R. Acad. Sci. Paris Sér. I Math. **323**(2), 117–120 (1996)
[11] Khar., C.: Compatible systems of mod p Galois representations and Hecke characters. Math. Res. Lett. **10**(1), 71–83 (2003)
[12] Khare, C.: Reciprocity law for compatible systems of abelian mod p Galois representations. Canad. J. Math. **57**(6), 1215–1223 (2005)
[13] MacRae, E.R.: On the density of conjugate fixed points of a correspondence. Illinois J. Math. **18**, 66–78 (1974)
[14] Schinzel, A.: On power residues and exponential congruences. Acta Arith. **27**, 397–420 (1975)
[15] Serre, J.-P.: Abelian l-adic representations and elliptic curves. Revised reprint, Research Notes in Mathematics vol. 7. A K Peters, Ltd., Wellesley (1998)
[16] Waldschmidt, M.: Sur certains caractères du groupe des classes d'idèles d'un corps de nombres. Seminar on Number Theory, Paris 1980–1981, Progr. Math., **22**, 323–335 (1982)