# Contents

# Deformations of Galois Representations

*Gebhard Böckle*

## Introduction

These lecture notes give an introduction to deformations of Galois representations with an eye toward the application of this theory in the proof of the Serre conjecture [KW1, KW2] by Khare–Wintenberger. There exist several other surveys such as [DDT, Go, Ki7, Ma2]. We nevertheless hope that with the above scope in mind and by the arrangement and detail of the material presented we can add something useful to the existing literature. Clearly, we claim no originality in the material presented and all errors are to be blamed on the present author.

The idea of studying deformations of Galois representations on their own right goes back to the seminal article [Maz] of Mazur. Mazur's motivation was to give a conceptual if at the time conjectural framework for some discoveries of Hida [Hid] on ordinary families of Galois representations. It was the work of Wiles on Fermat's Last Theorem which made clear the importance of deformation theory developed by Mazur. The theory was a key technical tool in the proof [Wi2, TW] by Wiles and Taylor–Wiles of Fermat's Last Theorem.

Mazur's theory yields a universal deformation ring which can be thought of as a parameter space for all lifts of a given residual representation (up to conjugation). The ring depends on the residual representation and on supplementary conditions that one imposes on the lifts. If the residual representation is modular and the deformation conditions are such that the $p$-adic lifts satisfy conditions that hold for modular Galois representations, then one expects in many cases that the natural homomorphism $R \to T$ from the universal ring $R$ to a suitably defined Hecke algebra $T$ is an isomorphism. The proof of such isomorphisms, called $R = T$ theorems or modularity theorems, is at the heart of the proof of Fermat's Last Theorem. It expresses the fact that all $p$-adic Galois representations of the type described by $R$ are modular and, in particular, that they arise from geometry.

Many refinements of Wiles' methods have since been achieved and the theory has been vastly generalized to various settings of automorphic forms. $R = T$ theorems lie at the basis of the proof of the Taniyama–Shimura conjecture by Breuil, Conrad, Diamond and Taylor; the Sato–Tate conjecture by Clozel, Harris, Shepherd-Barron and Taylor; and the already mentioned Serre conjecture. The proof of Fermat's Last Theorem was also the first strong evidence to the conjecture of Fontaine and Mazur [FM]. The conjecture asserts that if a $p$-adic Galois representation satisfies certain local conditions that hold for Galois representations which arise from geometry, then this representation occurs in the $p$-adic étale cohomology of a variety over a number field. In fact, it is a major motivation for the formulation of the standard conditions on deformation functors. These conditions should (mostly) be local and reflect a geometric condition on a representation. Due to work of Emerton and independently Kisin [Ki6], there has been much progress on the Fontaine–Mazur conjecture over $\mathbb{Q}$.

The present notes are based on an advanced course given jointly with Laurent Berger at the CRM Barcelona. The course provided basic material on $p$-adic Hodge theory and deformation theory of Galois representations, motivated by the proof of the Serre conjecture by Khare and Wintenberger. The lectures by Berger focused on $p$-adic Hodge theory [Ber2] and our part on deformation theory.

The contents of our lectures are as follows: Lecture 1 recalls the foundations of Mazur's theory of deformations of Galois representations with some additional material added from the work of Kisin. Lecture 2 introduces pseudo-representations and studies their deformations. Pseudo-representations are functions that have the formal properties of traces of representations. They are important because completely reducible representations can be recovered from their traces. Moreover, $p$-adic Galois representations are often given in terms of traces of Frobenius automorphisms, i.e., as a pseudo-representation. The representation itself is not directly accessible.

Lecture 3 considers universal deformations of a mod $p$ representation of the absolute Galois group of a finite extension of the field $\mathbb{Q}_\ell$ for $\ell \neq p$. The corresponding theory of $p$-adic Galois representations is well understood in terms of Weil–Deligne representations. It will turn out that also the universal deformation can be given a natural description in terms of such parameters (or rather inertial Weil–Deligne types). This leads to conditions for deformation functors of a residual mod $p$ representation at places not above $p$. Weil–Deligne representations are naturally linked to $p$-adic Galois representations 'arising' from geometry: for instance, one may consider the Galois representation on the $p$-adic Tate module of an elliptic curve (or an abelian variety) over a number field and restrict this to a decomposition group at a prime $v$ above $\ell$. If the curve has good reduction at $v$, by the criterion of Néron–Ogg–Shafarevich, the representation is unramified and vice versa; moreover the associated inertial Weil–Deligne type is trivial. If it has potentially good reduction, the representation is potentially unramified and the inertial Weil–Deligne type is non-trivial but has trivial monodromy operator. In the remaining case the representation is potentially unipotent and the monodromy operator is non-trivial. At the end of Lecture 3 (for technical reasons) and in Lecture 4, we consider the deformation theory of a mod $p$ Galois representation of the absolute Galois group of a finite extension of $\mathbb{Q}_p$. More precisely, we study some subfunctors of Mazur's functor that satisfy conditions which hold for representations arising from geometry. This is technically the by far most subtle part and we only work out some of the simplest cases. To formulate and study the resulting deformation functors, $p$-adic Hodge theory aka Fontaine theory are needed; see [Ber2]. It enables one to describe local conditions for deformations of 2-dimensional representations arising from

(a) finite flat group schemes, in ordinary and non-ordinary cases;

(b) crystalline Galois representations of low Hodge–Tate weights $(0, k)$, where $1 \leq k \leq p - 1$;

(c) weight 2 semistable non-crystalline Galois representations.

On the geometric side, such representations arise from

(a) the $p$-power torsion of an elliptic curve with good ordinary or supersingular reduction at $p$;

(b) $p$-adic Galois representations associated with a modular form of weight $k$, where $2 \leq k \leq p$;

(c) the $p$-adic Tate module of an elliptic curve with multiplicative reduction at $p$.

Lecture 5 ends the lecture series with the following result: the global universal deformation ring $R$ for 2-dimensional totally odd residual representations of the absolute Galois group of a totally real field with (suitable) geometric conditions at all primes, fixed determinant and ramification at most at a fixed finite set of places of the base field, has Krull dimension at least 1. Together with results of Taylor on potential modularity, covered in a lecture series by J.-P. Wintenberger during the advanced course, the lower bound in fact suffices in many cases to show that the $p$-power torsion elements form a finite ideal $I$ of $R$ such that $R/I$ is finite flat over $\mathbb{Z}_p$. This implies an important lifting result needed in the proof by Khare and Wintenberger. The result is also in line with the expectation that typically $R$ should be isomorphic to a Hecke algebra on a finite-dimensional space of $p$-adic modular forms, which is clearly finite flat over $\mathbb{Z}_p$ and thus of exact Krull dimension 1.

During the lecture series we also cover a number of technically important issues for the theory of deformations of Galois representations: framed deformations, deformation functors via groupoids on a category, pseudo-representations and their deformations, the completion of a deformation functor at closed points of its generic fiber, and resolutions of deformation functors. Some lectures have appendices that, for the convenience of the reader, recall technical terms needed in the main body. To give a sample: there are appendices on Schlessinger's axioms, formal schemes, finite flat group schemes, filtered $\varphi$-modules, etc.

Much of the current perspective on deformations of Galois representations is due to work of M. Kisin, as is clear to everyone familiar with the topic. Moreover we found his lecture notes [Ki7] very helpful in preparing the present lecture series. Several parts of our exposition follow closely his notes.

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# Notation

The following list can be regarded as a reference page for the notation. Throughout the notes it will be introduced step by step.

- $p$ will be a rational prime.
- $\mathbb{F}$ will denote a finite field of characteristic $p$ and $W(\mathbb{F})$ its ring of Witt vectors.
- $\mathcal{O}$ will denote the ring of integers of some $p$-adic field which is finite and totally ramified over $W(\mathbb{F})[1/p]$, so that $\mathcal{O}$ has residue field $\mathbb{F}$.
- $\mathfrak{Ar}_{\mathcal{O}}$ will denote the category of pairs $(A, \pi_A)$ where $A$ is a finite local Artinian $\mathcal{O}$-algebra with a surjective homomorphism $\pi_A \colon A \to \mathbb{F}$ and maximal ideal $\mathfrak{m}_A = \operatorname{Ker} \pi_A$.
- $\widehat{\mathfrak{Ar}}_{\mathcal{O}}$ will denote the category of pairs $(A, \pi_A)$ where $A$ is a complete Noetherian local $\mathcal{O}$-algebra with a surjective homomorphism $\pi_A \colon A \to \mathbb{F}$ and maximal ideal $\mathfrak{m}_A = \operatorname{Ker} \pi_A$.
- $G$ will denote a profinite group.
- $V_{\mathbb{F}}$ will be a (continuous) representation of $G$ over $\mathbb{F}$ with $d = \dim_{\mathbb{F}} V_{\mathbb{F}} < \infty$.
- $\mathrm{ad} = \operatorname{End}_{\mathbb{F}}(V_{\mathbb{F}}) \cong V_{\mathbb{F}} \otimes_{\mathbb{F}} V_{\mathbb{F}}^*$ is the adjoint representation of $V_{\mathbb{F}}$; it is again a $G$-module.

- $\mathrm{ad}^0 \subset \mathrm{ad}$ is the subrepresentation on trace zero matrices.
- $\psi\colon G \to \mathcal{O}^*$ will denote a fixed lift of $\det V_{\mathbb{F}}$.
- For an arbitrary field $K$, we let $\overline{K}$ denote a fixed algebraic closure and write $G_K = \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of $K$. We denote the $G_K$-representation $\varprojlim_n \mu_{p^n}(\overline{K})$ by $\mathbb{Z}_p(1)$.
- For any ring $A$ and any free finitely generated $A$-module $M$, we denote by $M^* = \mathrm{Hom}_A(M, A)$ its linear dual. If $M$ carries an $A$-linear action by $G$ then so does $M^*$.
- For $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ and a continuous representation $M$ of $G_K$ on a free finitely generated $A$-module, we define its Cartier dual $M^\vee$ as $\mathrm{Hom}_A(M, A(1))$, where $A(1) = A\widehat{\otimes}_{\mathbb{Z}_p}\mathbb{Z}_p(1)$.

In Lecture 5 the following notation pertaining to number fields will be relevant:

- $F$ will be a number field.
- $S$ will denote a finite set of places of $F$ —typically it will contain all places above $p$ and $\infty$.
- $G_{F,S}$ or simply $G_S$ will denote the Galois group of the maximal outside $S$ unramified extension of $F$ inside $\overline{F}$.
- For a place $v$ of $F$, we will denote by $G_{F_v}$ or simply $G_v$ the absolute Galois group of the completion of $F$ at $v$.
- For each place $v$ of $F$, we fix a homomorphism $F^{\mathrm{alg}} \hookrightarrow F_v^{\mathrm{alg}}$ extending $F \hookrightarrow F_v$, yielding a homomorphism $G_v \to G_F \to G_S$.

# Lecture 1

# Deformations of representations of profinite groups

Throughout this lecture series, $p$ will be a prime and $\mathbb{F}$ a finite field of characteristic $p$. The ring of Witt vectors of $\mathbb{F}$ will be denoted by $W(\mathbb{F})$. By $G$ we denote a profinite group and by $V_{\mathbb{F}}$ a finite $\mathbb{F}[G]$-module on which $G$ acts continuously. We set $d = \dim_{\mathbb{F}} V_{\mathbb{F}}$ and fix an $\mathbb{F}$-basis $\beta_{\mathbb{F}}$ of $V_{\mathbb{F}}$.

In the first two lectures, $G$ will mostly be arbitrary but subjected to a certain finiteness condition. Later on, $G$ will either be the absolute Galois group of a finite extension of $\mathbb{Q}_p$ or $\mathbb{Q}_\ell$ for some $\ell \neq p$, or a quotient of the absolute Galois group of a number field.

In this lecture we discuss basic definitions, notions and results. The material is fairly standard, although framed deformations are not treated in older surveys such as [Ma2]. We mainly follow Kisin's notation, as in [Ki7]. The lecture ends with a discussion on groupoids over a category, which can be thought of as an alternative means to describe deformation functors. This is taken from [Ki4, Appendix].

## 1.1 Deformation functors

Let $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ denote the category of complete Noetherian local $W(\mathbb{F})$-algebras with residue field $\mathbb{F}$, and $\mathfrak{Ar}_{W(\mathbb{F})}$ denote the full subcategory of finite local Artinian $W(\mathbb{F})$-algebras. The maximal ideal of $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ is denoted by $\mathfrak{m}_A$. Note that, via the $W(\mathbb{F})$-structure, the residue field $A/\mathfrak{m}_A$ of any $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ is canonically identified with $\mathbb{F}$.

Let $A$ be in $\mathfrak{Ar}_{W(\mathbb{F})}$. A *deformation of* $V_{\mathbb{F}}$ *to* $A$ is a pair $(V_A, \iota_A)$ such that

(a) $V_A$ is an $A[G]$-module which is finite free over $A$ and on which $G$ acts continuously, and

(b) $\iota_A$ is a $G$-equivariant isomorphism $\iota_A\colon V_A \otimes_A \mathbb{F} \xrightarrow{\cong} V_{\mathbb{F}}$.

A *framed deformation of* $(V_{\mathbb{F}}, \beta_{\mathbb{F}})$ *to* $A$ is a triple $(V_A, \iota_A, \beta_A)$, where $(V_A, \iota_A)$ is a deformation of $V_{\mathbb{F}}$ to $A$ and $\beta_A$ is an $A$-basis of $V_A$ which reduces to $\beta_{\mathbb{F}}$ under $\iota_A$.

One defines functors $D_{V_{\mathbb{F}}}, D^{\square}_{V_{\mathbb{F}}}\colon \mathfrak{Ar}_{W(\mathbb{F})} \to \mathbf{Sets}$ by setting, for $A \in \mathfrak{Ar}_{W(\mathbb{F})}$,

$$D_{V_{\mathbb{F}}}(A) = \{\text{isomorphism classes of deformations of } V_{\mathbb{F}} \text{ to } A\},$$

$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\text{isomorphism classes of framed deformations of } (V_{\mathbb{F}}, \beta_{\mathbb{F}}) \text{ to } A\},$$

and with the obvious extension to morphisms.

*Remarks* 1.1.1. (a) The fixed basis $\beta_{\mathbb{F}}$ identifies the vector space underlying $V_{\mathbb{F}}$ with $\mathbb{F}^d$ and thus allows us to view $V_{\mathbb{F}}$ as a representation $\bar{\rho}\colon G \to \mathrm{GL}_d(\mathbb{F})$. Then $D^{\square}_{V_{\mathbb{F}}}(A)$ is the set of continuous representations

$$\rho\colon G \longrightarrow \mathrm{GL}_d(A)$$

lifting $\bar{\rho}$. In terms of representations, $D_{V_{\mathbb{F}}}(A)$ is the set of such representations modulo the action by conjugation of $\mathrm{Ker}(\mathrm{GL}_d(A) \to \mathrm{GL}_d(\mathbb{F}))$.

(b) It is often useful to consider deformation functors on $\mathfrak{Ar}_{\mathcal{O}}$, where $\mathcal{O}$ is the ring of integers of a finite totally ramified extension of $W(\mathbb{F})[1/p]$, so that $\mathbb{F}$ is still the residue field of $\mathcal{O}$, and where $\mathfrak{Ar}_{\mathcal{O}}$ is the category of local Artinian $\mathcal{O}$-algebras with residue field $\mathbb{F}$. We shall do this in later lectures without further mentioning.

(c) In Section 1.6 we reformulate deformation functors in terms of groupoids over a category. This gives a different viewpoint on the theory and will be important for certain applications.

## 1.2 A finiteness condition

**Definition 1.2.1** (Mazur)**.** A profinite group $G$ satisfies the *finiteness condition* $\Phi_p$ if, for all open subgroups $G' \subset G$, the $\mathbb{F}_p$-vector space $\mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p)$ of continuous group homomorphisms is finite-dimensional.

By the Burnside basis theorem (see Exercise 1.8.1), the group $G'$ satisfies $\dim_{\mathbb{F}_p} \mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p) < \infty$ if and only if the maximal pro-$p$ quotient of $G'$ is topologically finitely generated.

**Examples 1.2.2.** The group $\mathrm{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p)$ is isomorphic to $\mathrm{Hom}_{\mathrm{cont}}(G'^{\mathrm{ab}}, \mathbb{F}_p)$. Thus class field theory shows that the following groups satisfy Condition $\Phi_p$:

(a) The absolute Galois group of a finite extension $\mathbb{Q}_p$.

(b) The Galois group $G_{F,S} = \mathrm{Gal}(F_S/F)$, where $F$ is a number field, $S$ is a finite set of places of $F$, and $F_S \subset \overline{F}$ denotes the maximal extension of $F$ unramified outside $S$.

Both of these examples will be important in later lectures.

## 1.3 Representability

**Proposition 1.3.1** (Mazur). *Assume that $G$ satisfies Condition $\Phi_p$. Then:*

(a) $D^{\square}_{V_{\mathbb{F}}}$ *is pro-representable by some* $R^{\square}_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$.

(b) *If* $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$ *then* $D_{V_{\mathbb{F}}}$ *is pro-representable by some* $R_{V_{\mathbb{F}}} \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$.

One calls $R^{\square}_{V_{\mathbb{F}}}$ the *universal framed deformation ring* and $R_{V_{\mathbb{F}}}$ the *universal deformation ring* of $V_{\mathbb{F}}$.

*Remarks* 1.3.2. (a) Recall that (pro-)representability (e.g., for $D^{\square}_{V_{\mathbb{F}}}$) means that there exists an isomorphism

$$D^{\square}_{V_{\mathbb{F}}}(A) \cong \mathrm{Hom}_{W(\mathbb{F})}(R^{\square}_{V_{\mathbb{F}}}, A)$$

which is functorial in $A \in \mathfrak{Ar}_{W(\mathbb{F})}$. This universal property implies that $R^{\square}_{V_{\mathbb{F}}}$ is unique up to unique isomorphism. Moreover the identity map in $\mathrm{Hom}(R^{\square}_{V_{\mathbb{F}}}, R^{\square}_{V_{\mathbb{F}}})$ gives rise to a universal framed deformation over $R^{\square}_{V_{\mathbb{F}}}$.

(b) Originally, Mazur only considered the functor $D_{V_{\mathbb{F}}}$. It describes representations lifting $V_{\mathbb{F}}$ up to isomorphism. The additional choice of basis is not a very interesting datum. However, the functor $D_{V_{\mathbb{F}}}$ is not always representable. A good way to remedy this problem is to rigidify the situation by adding a choice of basis to a given representation and thus to consider the functor $D^{\square}_{V_{\mathbb{F}}}$ instead. This is important for residual representations $V_{\mathbb{F}}$ of the absolute Galois group of a number field $F$, in the sense that $V_{\mathbb{F}}$ may have trivial centralizer as a representation of $G_F$ and yet the restriction of $V_{\mathbb{F}}$ to a decomposition group may no longer share this property.

(c) Without Condition $\Phi_p$, the universal ring $R^{\square}_{V_{\mathbb{F}}}$ still exists (as an inverse limit of Artinian rings), but it may no longer be Noetherian.

(d) Due to the canonical homomorphism $\mathbb{F} \hookrightarrow \mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}})$, it is justified to write "=" in $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$.

*Proof of Proposition* 1.3.1. We prove part (a). Suppose first that $G$ is finite, say with a presentation $\langle g_1, \ldots, g_s \mid r_1(g_1, \ldots, g_s), \ldots, r_t(g_1, \ldots, g_s)\rangle$. Define

$$\mathcal{R} = W(\mathbb{F})[X^k_{i,j} \mid i, j = 1, \ldots, d;\ k = 1, \ldots, s]/\mathcal{I},$$

where $\mathcal{I}$ is the ideal generated by the coefficients of the matrices

$$r_l(X^1, \ldots, X^s) - \mathrm{id}, \quad l = 1, \ldots, t,$$

with $X^k$ the matrix $(X^k_{i,j})_{i,j}$. Let $\mathcal{J}$ be the kernel of the homomorphism $\mathcal{R} \to \mathbb{F}$ defined by mapping $X^k$ to $\bar{\rho}(g_k)$ for $k = 1, \ldots, s$, with $\bar{\rho}$ as in Remark 1.1.1(a).

Then $R^{\square}_{V_{\mathbb{F}}}$ is the $\mathcal{J}$-adic completion of $\mathcal{R}$ and $\rho^{\square}_{V_{\mathbb{F}}}$ is the unique representation $G \to \mathrm{GL}_d(R^{\square}_{V_{\mathbb{F}}})$ mapping $g_k$ to the image of $X^k$ in $\mathrm{GL}_d(R^{\square}_{V_{\mathbb{F}}})$.

We may write any profinite group $G$ as a filtered inverse limit $G = \varprojlim_i G/H_i$ over some index set $I$ of open normal subgroups $H_i \subset \mathrm{Ker}(\bar{\rho})$. For each $i$ the above construction yields a universal pair $(R^{\square}_i, \rho^{\square}_i)$. By the universality of these pairs, one can form their inverse limit over the index set $I$. This yields

$$(R^{\square}_{V_{\mathbb{F}}}, \rho^{\square}_{V_{\mathbb{F}}}) = \varprojlim_i (R^{\square}_i, \rho^{\square}_i),$$

which clearly satisfies the required universal property. By definition, $R^{\square}_{V_{\mathbb{F}}}$ lies in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$. It remains to show that $R = R^{\square}_{V_{\mathbb{F}}}$ is Noetherian. Since $R$ is complete, it suffices to show that $\mathfrak{m}_R/(\mathfrak{m}_R^2, p)$ is finite-dimensional as a vector space over $\mathbb{F}$. It is most natural to prove the latter using tangent spaces. We refer to the proof of Lemma 1.4.3, where we shall see how Condition $\Phi_p$ is used.

The proof of part (b) in [Maz] uses Schlessinger's representability criterion (Theorem 1.7.2). Following Kisin, we shall indicate a different proof in Section 2.1. The following is a preview of Kisin's proof. Let $\widehat{\mathrm{PGL}}_d$ denote the completion of the group $\mathrm{PGL}_d$ over $W(\mathbb{F})$ along its identity section. Then $\widehat{\mathrm{PGL}}_d$ acts on the functor $D^{\square}_{V_{\mathbb{F}}}$ by conjugation and hence it acts on the formal scheme $\mathrm{Spf}\, R^{\square}_{V_{\mathbb{F}}}$. The condition $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$ ensures that this action is free, and the idea is to define $\mathrm{Spf}\, R_{V_{\mathbb{F}}} = \mathrm{Spf}\, R^{\square}_{V_{\mathbb{F}}}/\widehat{\mathrm{PGL}}_d$. □

## 1.4 The tangent space

Let $\mathbb{F}[\varepsilon] = \mathbb{F}[X]/(X^2)$ denote the ring of dual numbers. The set $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ is naturally isomorphic to $\mathrm{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}})$, as an element of $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ gives rise to an extension[1]

$$0 \longrightarrow V_{\mathbb{F}} \longrightarrow V_{\mathbb{F}[\varepsilon]} \longrightarrow V_{\mathbb{F}} \longrightarrow 0,$$

where we have identified $\varepsilon \cdot V_{\mathbb{F}}$ with $V_{\mathbb{F}}$, and, conversely, any extension of one copy of $V_{\mathbb{F}}$ by another $V_{\mathbb{F}}$ can be viewed as an $\mathbb{F}[\varepsilon]$-module, with multiplication by $\varepsilon$ identifying the two copies of $V_{\mathbb{F}}$. In particular, $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ is naturally an $\mathbb{F}$-vector space.

**Definition 1.4.1.** The $\mathbb{F}$-vector space $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ is called the *Zariski tangent space* of $D_{V_{\mathbb{F}}}$. (The same terminology will be used for $D^{\square}_{V_{\mathbb{F}}}$ and other deformation functors.)

*Remark* 1.4.2. Recall that, for any $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$, its (mod $p$) Zariski tangent space is the $\mathbb{F}$-vector space $\mathfrak{t}_A = \mathrm{Hom}_{W(\mathbb{F})}(A, \mathbb{F}[\varepsilon])$. Thus, if $D_{V_{\mathbb{F}}}$ is pro-representable, then the tangent spaces of $D_{V_{\mathbb{F}}}$ and of the universal ring representing $D_{V_{\mathbb{F}}}$ agree.

[1] By $\mathrm{Ext}^i$ we denote the continuous extension classes.

**Lemma 1.4.3.** (a) *Defining* $\mathrm{ad}V_{\mathbb{F}}$ *as the* $G$*-representation* $End_{\mathbb{F}}(V_{\mathbb{F}})$*, there is a canonical isomorphism*

$$D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) \xrightarrow{\cong} H^1(G, \mathrm{ad}V_{\mathbb{F}}). \tag{1.4.1}$$

(b) *If* $G$ *satisfies Condition* $\Phi_p$*, then* $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ *is a finite-dimensional* $\mathbb{F}$*-vector space.*

(c) *One has* $\dim_{\mathbb{F}} D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) = \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) + d^2 - h^0(G, \mathrm{ad}V_{\mathbb{F}})$.

*Remark* 1.4.4. The symbol $h^?(\ldots)$ always denotes $\dim_{\mathbb{F}} H^?(\ldots)$.

*Proof.* Part (a) is immediate from the isomorphism $\mathrm{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}}) \cong H^1(G, \mathrm{ad}V_{\mathbb{F}})$ proved in Exercise 1.8.4.

We now prove part (b), thereby completing the proof of Proposition 1.3.1(a). Let $G' = \mathrm{Ker}(\bar{\rho})$, which is an open subgroup of $G$. The inflation-restriction sequence yields the left exact sequence

$$0 \longrightarrow H^1(G/G', \mathrm{ad}V_{\mathbb{F}}) \longrightarrow H^1(G, \mathrm{ad}V_{\mathbb{F}}) \longrightarrow (\mathrm{Hom}(G', \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{ad}V_{\mathbb{F}})^{G/G'}.$$

The term on the left is finite because $G/G'$ and $\mathrm{ad}V_{\mathbb{F}}$ are finite. The term on the right is finite because of Condition $\Phi_p$ for $G$. Hence (b) is proved.

To prove part (c), fix a deformation $V_{\mathbb{F}[\varepsilon]}$ of $V_{\mathbb{F}}$ to $\mathbb{F}[\varepsilon]$. The set of $\mathbb{F}[\varepsilon]$ bases of $V_{\mathbb{F}[\varepsilon]}$ lifting a fixed basis of $V_{\mathbb{F}}$ is an $\mathbb{F}$-vector space of dimension $d^2$. Let $\beta'$ and $\beta''$ be two such lifted bases. Then there is an isomorphism of framed deformations

$$(V_{\mathbb{F}[\varepsilon]}, \beta') \cong (V_{\mathbb{F}[\varepsilon]}, \beta'')$$

if and only if there is an automorphism $1 + \varepsilon\alpha$ of $V_{\mathbb{F}[\varepsilon]}$, where $\alpha \in \mathrm{ad}V_{\mathbb{F}}$, which takes $\beta'$ to $\beta''$, so that $\alpha \in \mathrm{ad}V_{\mathbb{F}}{}^G$. Thus the fibers of

$$D^{\square}(V_{\mathbb{F}[\varepsilon]}) \longrightarrow D_{V_{\mathbb{F}}}(V_{\mathbb{F}[\varepsilon]})$$

are a principal homogeneous space under $\mathrm{ad}V_{\mathbb{F}}/(\mathrm{ad}V_{\mathbb{F}})^G$. □

**Definition 1.4.5.** Let $\varphi\colon D' \to D$ be a natural transformation of functors from $\mathfrak{Ar}_{W(\mathbb{F})}$ to **Sets**. The map $\varphi$ will be called *formally smooth* if, for any surjection $A \to A' \in \mathfrak{Ar}_{W(\mathbb{F})}$, the map

$$D'(A) \longrightarrow D'(A') \times_{D(A')} D(A)$$

is surjective.

Essentially the same proof as that of Lemma 1.4.3(c) implies the following:

**Corollary 1.4.6.** *The natural transformation* $D^{\square}_{V_{\mathbb{F}}} \to D_{V_{\mathbb{F}}}$*,* $(V_A, \beta_A) \mapsto V_A$ *is formally smooth. Thus, if* $R_{V_{\mathbb{F}}}$ *is representable, then* $R^{\square}_{V_{\mathbb{F}}}$ *is a power series ring over* $R_{V_{\mathbb{F}}}$ *of relative dimension* $d^2 - h^0(G, \mathrm{ad}V_{\mathbb{F}})$*.*

*Remark* 1.4.7. The above corollary says that the singularities of the two local $W(\mathbb{F})$-algebras $R_{V_\mathbb{F}}$ and $R^\square_{V_\mathbb{F}}$ are in some sense equivalent, provided that $D_{V_\mathbb{F}}$ is representable. Even if $D_{V_\mathbb{F}}$ is not representable, there is a sense in which it has an intrinsic geometry. However, this is best formulated in terms of groupoids; cf. Section 1.6.

## 1.5 Presentations of the universal ring $R_{V_\mathbb{F}}$

By Remark 1.4.2 and Lemma 1.4.3 we have shown part (a) of the following result:

**Proposition 1.5.1.** *Suppose that $G$ satisfies Condition $\Phi_p$ and $R_{V_\mathbb{F}}$ is representable. Then:*

(a) $\dim \mathfrak{t}_{R_{V_\mathbb{F}}} = h^1(G, \mathrm{ad}V_\mathbb{F}) =: h$ *and so there is a surjection*

$$\pi\colon W(\mathbb{F})[[X_1, \ldots, X_h]] \longrightarrow R_{V_\mathbb{F}}.$$

(b) *For any $\pi$ as in* (a)*, the minimal number of generators of the ideal* $\mathrm{Ker}\,\pi$ *is bounded above by $h^2(G, \mathrm{ad}V_\mathbb{F})$. More precisely, given $\pi$, one has a canonical monomorphism*

$$(\mathrm{Ker}\,\pi/(p, X_1, \ldots, X_h)\,\mathrm{Ker}\,\pi)^* \longrightarrow H^2(G, \mathrm{ad}V_\mathbb{F}),$$

*where, for a vector space $V$, we denote its dual by $V^*$.*

For the proof of (b) we refer to [Maz] or [Bö1, Thm 2.4]. A similar proof is given in Lemma 5.2.2.

**Corollary 1.5.2.** *Assume that the hypotheses of Proposition* 1.5.1 *hold. Then, if $h^2(G, \mathrm{ad}V_\mathbb{F}) = 0$ —in this case, $V_\mathbb{F}$ is called* unobstructed*—, the ring $R_{V_\mathbb{F}}$ is smooth over $W(\mathbb{F})$ of relative dimension $h^1(G, \mathrm{ad}V_\mathbb{F})$.*

*Remarks* 1.5.3. (a) If $G = G_{F,S}$ for a number field $F$ and a finite set of places $S$ containing all places above $p$ and $\infty$, all of the scarce evidence is in favor of the following conjecture: if $\mathrm{End}_{\mathbb{F}[G]}(V_\mathbb{F}) = \mathbb{F}$, then $R_{V_\mathbb{F}}$ is a complete intersection and flat over $W(\mathbb{F})$ and of relative dimension

$$h^1(G, \mathrm{ad}V_\mathbb{F}) - h^0(G, \mathrm{ad}V_\mathbb{F}) - h^2(G, \mathrm{ad}V_\mathbb{F}).$$

For $S$ not containing all places above $p$, there are counterexamples [BC2].

(b) Let $f = \sum a_n q^n$ be a newform of weight $k \geq 2$, level $N$ and character $\omega$. Let $S$ be any finite set of places of $\mathbb{Q}$ containing the infinite place and all primes dividing $N$. Let $K$ be the number field over $\mathbb{Q}$ generated by all the $a_n$. Then, by work of Eichler, Shimura, Deligne and Serre, for any prime $\wp$ of $K$ one has a semisimple two-dimensional Galois representation

$$\rho_{f,\wp}\colon G_{\mathbb{Q},S\cup\{\wp\}} \longrightarrow \mathrm{GL}_2(\mathbb{F}_\wp)$$

over the residue field $\mathbb{F}_\wp$ of $K$ at $\wp$ associated to $f$ in a natural way. Let $V_\wp$ denote the corresponding continuous representation of $G_\wp := G_{\mathbb{Q},S\cup\{\wp\}}$. The representation $V_\wp$ is known to be absolutely irreducible for almost all $\wp$.

By work of Mazur for $k = 2$ and $f$ associated with an elliptic curve, and by Weston for general $f$ (cf. [Wes]), the following is shown. If $k \geq 3$, then $V_\wp$ (with respect to $G_\wp$) is unobstructed for almost all primes $\wp$ of $K$. If $k = 2$, then $V_\wp$ is unobstructed outside an exceptional set of density zero.

## 1.6 Groupoids over categories

Universal deformation rings can be very singular at their unique closed point. The standard way in algebraic geometry to resolve singularities are blow-ups along the singular locus. If we apply a formal blow-up to (the formal spectrum of) a universal ring along a subscheme containing its closed point, the resulting (formal) scheme may have many closed points. Since we deal with universal rings representing a functor, it is natural to look for other functors whose representing objects realize this blow-up. This means that we can no longer consider functors on Artin rings only. An approach, breaking with all traditions in the area, would be to reformulate the whole local theory developed so far in terms of schemes. Functors of which one hopes that they are representable (by a formal scheme) could then be described as stacks over the category of schemes. If we want to stay within the realm of rings —at least in the description of functors— then one has to reformulate the theory of stacks in terms of rings. The spectra of these rings should be thought of as giving coverings of the schemes that one should have in mind. This has been done successfully by Kisin. Instead of studying (pre-)stacks, which are categories (of schemes) fibered in groupoids, he considers categories (of rings with supplementary structures) cofibered in groupoids. While this introduces the right level of generality to describe resolutions of the functors one is interested in, the theory is still close to the original theory of functors on $\mathfrak{Ar}_{W(\mathbb{F})}$.

In the present section we shall give an outline of this, hoping that it will be useful for the interested reader who wishes to consult Kisin's work, e.g. [Ki4]. Moreover we shall make use of this in later parts of these lecture notes.

Let us first recall the definition of a groupoid category: a *groupoid category* is a category in which all morphisms are isomorphisms. However, it is not required that between any two objects there is a morphism. There can be many isomorphism classes —these are also referred to as the *connected components* of the groupoid, thinking of a category as a kind of graph. The set of endomorphisms of an object, which is the same as the set of its automorphisms, is a group under composition. The neutral element is given by the identity morphism of this object. One can easily show that the automorphism groups of any two objects which are connected are (non-canonically) isomorphic.

We shall now, following Kisin [Ki4], reformulate the theory of deformations of Galois representations in terms of groupoids over categories.

Fix a base category $\mathfrak{C}$ which in many applications will be $\mathfrak{Ar}_{W(\mathbb{F})}$. We consider a second category $\mathfrak{F}$ and a functor $\Theta\colon \mathfrak{F} \to \mathfrak{C}$, and we say that

- $\eta \in \mathrm{Ob}(\mathfrak{F})$ *lies above* $T \in \mathrm{Ob}(\mathfrak{C})$ if $\Theta(\eta) = T$, and
- $(\eta \xrightarrow{\alpha} \xi) \in \mathrm{Mor}_{\mathfrak{F}}$ *lies above* $(T \xrightarrow{f} S) \in \mathrm{Mor}_{\mathfrak{C}}$ if $\Theta(\alpha) = f$.

Each object $T$ together with the morphism $\mathrm{id}_T$ forms a subcategory of $\mathfrak{C}$. By $\mathfrak{F}(T) \subset \mathfrak{F}$ we denote the subcategory over this particular subcategory of $\mathfrak{C}$.

**Definition 1.6.1.** The triple $(\mathfrak{F}, \mathfrak{C}, \Theta)$ is a *groupoid over* $\mathfrak{C}$ (or, more officially, a *category cofibered in groupoids over* $\mathfrak{C}$) if the following hold:

(a) For any pair of morphisms $\eta \xrightarrow{\alpha} \xi$ and $\eta \xrightarrow{\alpha'} \xi'$ in $\mathfrak{F}$ over the same morphism $T \to S$ in $\mathfrak{C}$, there exists a unique morphism $\xi \xrightarrow{u} \xi'$ in $\mathfrak{F}$ over $\mathrm{id}_S$ such that $u \circ \alpha = \alpha'$.

(b) For any $\eta \in \mathrm{Ob}(\mathfrak{F})$ and any morphism $T \xrightarrow{f} S$ in $\mathfrak{C}$ with $\eta$ over $T$ there exists a morphism $\eta \xrightarrow{\alpha} \xi$ in $\mathfrak{F}$ over $f$.

In particular, for any $T$ in $\mathfrak{C}$, the category $\mathfrak{F}(T)$ is a groupoid, i.e., a category in which all morphisms are isomorphisms. It is natural to specify a groupoid by specifying for any $T$ in $\mathfrak{C}$ the category in $\mathfrak{F}$ over $T$, and for any morphism $T \xrightarrow{f} S$ in $\mathfrak{C}$ the class of morphisms above $f$, and we shall often do so.

*Remark* 1.6.2. Let $\Theta\colon \mathfrak{F} \to \mathfrak{C}$ be a functor and $\Theta^o\colon \mathfrak{F}^o \to \mathfrak{C}^o$ the induced functor between the opposite categories. Then $\Theta$ defines a category cofibered in groupoids over $\mathfrak{C}$ if and only if $\Theta^o$ defines a category fibered in groupoids over $\mathfrak{C}$. The latter structure is well known in the theory of stacks. This is no accident: in the theory of stacks, the base category is typically the category of schemes. Now the opposite category of affine schemes is the category of rings —and we may look at a subclass of schemes corresponding to a subclass of rings. Since one base category will be the ring category $\mathfrak{Ar}_{W(\mathbb{F})}$, it is natural to work with categories cofibered over it. Note also that stacks have to satisfy some gluing conditions. The corresponding opposite conditions are not imposed in the present (admittedly very simple) setting.

If for each $T \in \mathrm{Ob}(\mathfrak{C})$ the isomorphism classes of $\mathfrak{F}(T)$ form a set, we associate to the category $\mathfrak{F}$ over $\mathfrak{C}$ a functor $|\mathfrak{F}|\colon \mathfrak{C} \to \mathbf{Sets}$ by sending $T$ to the set of isomorphism classes of $\mathfrak{F}(T)$.

**Example 1.6.3.** Let $\mathfrak{C} = \mathfrak{Ar}_{W(\mathbb{F})}$. To the representation $V_{\mathbb{F}}$ of $G$ we associate a groupoid $\mathcal{D}_{V_{\mathbb{F}}}$ over $\mathfrak{C}$ as follows:

(a) For $A \in \mathfrak{Ar}_{W(\mathbb{F})}$, the objects of $\mathcal{D}_{V_{\mathbb{F}}}$ over $A$ are pairs $(V_A, \iota_A)$ in $D_{V_{\mathbb{F}}}(A)$.

(b) A morphism $(V_A, \iota_A) \to (V'_{A'}, \iota_{A'})$ over a morphism $A \to A'$ in $\mathfrak{Ar}_{W(\mathbb{F})}$ is an isomorphism class

$$\{\alpha\colon V_A \otimes_A A' \xrightarrow{\cong} V'_{A'} \text{ an isomorphism} \mid \iota_{A'} \circ \alpha = \iota_A\}/(A')^*.$$

In the terminology now introduced, the functor previously denoted by $D_{V_{\mathbb{F}}}$ would be the functor $|\mathcal{D}_{V_{\mathbb{F}}}|$. For simplicity, we shall often omit the absolute value signs, if no confusion is likely.

When $V_{\mathbb{F}}$ has non-trivial automorphisms, then so do the objects in the categories $D_{V_{\mathbb{F}}}(A)$. In this situation, the groupoid $D_{V_{\mathbb{F}}}$ captures the geometry of the deformation theory of $V_{\mathbb{F}}$ more accurately than its functor of isomorphism classes.

### 1.6.1 Representability of a groupoid $\Theta\colon \mathfrak{F} \to \mathfrak{C}$

For $\eta \in \mathrm{Ob}(\mathfrak{F})$, define the category $\widetilde{\eta}$ (the *category under* $\eta$) as the category whose objects are morphisms with source $\eta$ and whose morphisms from an object $\eta \xrightarrow{\alpha} \xi$ to $\eta \xrightarrow{\alpha'} \xi'$ are morphisms $\xi \xrightarrow{u} \xi'$ in $\mathfrak{F}$ such that $u \circ \alpha = \alpha'$. (We do not assume that $\xi$ and $\xi'$ lie over the same object of $\mathfrak{C}$ and so $u$ may not be an isomorphism.)

**Definition 1.6.4.** The groupoid $\mathfrak{F}$ over $\mathfrak{C}$ is called *representable* if there is an object $\eta$ in $\mathfrak{F}$ such that the canonical functor $\widetilde{\eta} \to \mathfrak{F}$ is an equivalence of categories.

In the same way as $\widetilde{\eta}$, one defines the category $\widetilde{T}$ for any $T \in \mathfrak{C}$. One has a natural commutative diagram of categories

$$\begin{array}{ccc} \widetilde{\eta} & \longrightarrow & \mathfrak{F} \\ \downarrow & & \downarrow \\ \widetilde{\Theta(\eta)} & \longrightarrow & \mathfrak{C}. \end{array} \tag{1.6.1}$$

Both $\widetilde{\eta}$ and $\widetilde{\Theta(\eta)}$ are groupoids over $\mathfrak{C}$ and the top horizontal and left vertical homomorphisms are homomorphisms of groupoids over $\mathfrak{C}$. Because of the axioms of a groupoid over a category, the left vertical homomorphism is an equivalence of categories.

If $\mathcal{F}$ is representable, say by $\eta$, the equivalence $\widetilde{\eta} \to \widetilde{\Theta(\eta)}$ implies that $\eta$, as well as $\Theta(\eta)$, are well-defined up to canonical isomorphism. One says that $\Theta(\eta)$ represents $\mathfrak{F}$ over $\mathfrak{C}$. Under the same hypothesis, any two objects of $\mathfrak{F}(\Theta(\eta))$ are canonically isomorphic and there is an isomorphism of functors

$$\mathrm{Hom}_{\mathfrak{C}}(T, -) \xrightarrow{\cong} |\mathfrak{F}|,$$

so that $T$ represents $|\mathfrak{F}|$ in the usual set theoretic sense. Conversely, if $|\mathfrak{F}|$ is representable and for any $T$ in $\mathfrak{C}$ any two isomorphic objects of $\mathfrak{F}(T)$ are related by a unique isomorphism, then $\mathfrak{F}$ is representable.

*Remark* 1.6.5. The groupoid of Example 1.6.3 is usually *not* representable. To have a representability result, one needs to extend it to the category $\widehat{\mathfrak{A}}_{\mathfrak{r}W(\mathbb{F})}$. This can be done canonically and is explained in [Ki4, A.7].

The main reason why, in some circumstances, one needs to introduce the language of groupoids, is that formation of fiber products is not compatible with the passage from a groupoid $\mathfrak{F}$ over $\mathfrak{C}$ to its associated functor $|\mathfrak{F}|$. This is a serious technical issue, since Definition 2.4.4 of relative representability depends on the formation of fiber products. We illustrate this with a simple example taken from [Ki4, A.6].

Following Example 1.6.3, we define the groupoid $D^{\square}_{V_{\mathbb{F}}}$ on $\mathfrak{C} = \mathfrak{A}\mathfrak{r}_{w(\mathbb{F})}$ as follows. An object over $A \in \mathfrak{A}\mathfrak{r}_{W(\mathbb{F})}$ is a triple $(V_A, \iota_A, \beta_A)$, where $(V_A, \iota_A) \in D_{V_{\mathbb{F}}}(A)$ and $\beta_A$ is an $A$-basis of $V_A$ mapping under $\iota_A$ to the basis $\beta_{\mathbb{F}}$ of $V_{\mathbb{F}}$. A morphism $(V_A, \iota_A, \beta_A) \to (V_A, \iota_A, \beta_{A'})$ over $A \to A'$ is an isomorphism $\alpha\colon V_A \otimes_A A' \xrightarrow{\cong} V_{A'}$ taking $\beta_A$ to $\beta_{A'}$. There is an obvious morphism of groupoids $D^{\square}_{V_{\mathbb{F}}} \to D_{V_{\mathbb{F}}}$.

Consider now the situation when the group $G$ is trivial and fix $\eta = (V_A, \iota_A) \in D_{V_{\mathbb{F}}}(A)$ for some $A \in \mathfrak{A}\mathfrak{r}_{W(\mathbb{F})}$. Then $\widetilde{\eta} \times_{D_{V_{\mathbb{F}}}} D^{\square}_{V_{\mathbb{F}}}$ can be identified with quadruples $(V'_{A'} \psi'_{A'}, \varphi\colon V_A \otimes_A A' \xrightarrow{\cong} V'_{A'}, \beta_{A'})$, where $(V'_{A'} \psi'_{A'}, \beta_{A'}) \in D^{\square}_{V_{\mathbb{F}}}(A')$ and morphisms over $\mathrm{id}_{A'}$ are isomorphisms of $V'_{A'}$ reducing to the identity of $V_{\mathbb{F}}$. It follows that this category is a principal homogeneous space for the formal group obtained by completing $\mathrm{PGL}_d/W(\mathbb{F})$ along its identity section. Hence $|\widetilde{\eta} \times_{D_{V_{\mathbb{F}}}} D^{\square}_{V_{\mathbb{F}}}|(A')$ is isomorphic to the kernel $\mathrm{Ker}(\mathrm{PGL}_d(A') \to \mathrm{PGL}_d(\mathbb{F}))$. On the other hand, $|D^{\square}_{V_{\mathbb{F}}}(A')|$ is a singleton and hence the same holds for $|\widetilde{\eta}| \times_{|D_{V_{\mathbb{F}}}|} |D^{\square}_{V_{\mathbb{F}}}|(A')$.

# 1.7 Appendix

## 1.7.1 Schlessinger's axioms

**Definition 1.7.1.** Let $D\colon \mathfrak{A}\mathfrak{r}_{W(\mathbb{F})} \to \mathbf{Sets}$ be a functor such that $D(\mathbb{F})$ is a point. For any $A, A', A'' \in \mathfrak{A}\mathfrak{r}_{W(\mathbb{F})}$ with morphisms $A' \to A$ and $A'' \to A$, we have a map

$$D\big(A' \times_A A''\big) \longrightarrow D(A') \times_{D(A)} D(A''). \tag{1.7.1}$$

The axioms of Schlessinger in [Sch] are as follows:

(H1) If $A'' \to A$ is small surjective, then (1.7.1) is surjective.

(H2) If $A'' \to A$ is $\mathbb{F}[\varepsilon] \to \mathbb{F}$, then (1.7.1) is bijective.

(H3) $\dim_{\mathbb{F}} D(\mathbb{F}[\varepsilon])$ is finite.

(H4) If $A'' \to A$ is small surjective and $A' = A''$, then (1.7.1) is bijective.

Note that $D(\mathbb{F}[\varepsilon])$ carries a natural structure of $\mathbb{F}$-vector space. An epimorphism $A'' \to A$ in $\mathfrak{A}\mathfrak{r}_{W(\mathbb{F})}$ is called *small surjective* if its kernel is a principal ideal which is annihilated by $\mathfrak{m}_{A''}$.

The following is one of the main theorems of [Sch]:

**Theorem 1.7.2** (Schlessinger)**.** *If $D$ satisfies* (H1) *to* (H4)*, then $D$ is pro-representable.*

## 1.8 Exercises

*Exercise* 1.8.1. Show that for a profinite group $G$ the following conditions are equivalent:

(a) For all open subgroups $G' \subset G$ the maximal pro-$p$ quotient of $G'$ is topologically finitely generated.

(b) For all open subgroups $G' \subset G$ the vector space $\dim_{\mathbb{F}_p} \operatorname{Hom}_{\mathrm{cont}}(G', \mathbb{F}_p)$ is finite.

(c) For all open subgroups $G' \subset G$ and finite continuous $\mathbb{F}[G]$-modules $M$ one has $\dim_{\mathbb{F}} H^1(G', M) < \infty$.

*Exercise* 1.8.2. Give a proof of Proposition 1.3.1 by verifying Schlessinger's axioms (see Definition 1.7.1).

*Exercise* 1.8.3. Show that the natural transformation $D^{\square}_{V_{\mathbb{F}}} \to D_{V_{\mathbb{F}}}$ is formally smooth.

*Exercise* 1.8.4. Show that $\operatorname{Ext}^1_G(V_{\mathbb{F}}, V_{\mathbb{F}}) \xrightarrow{\cong} H^1(G, \mathrm{ad} V_{\mathbb{F}})$.

*Exercise* 1.8.5. Show that the groupoid $D_{V_{\mathbb{F}}}$ of Example 1.6.3 is representable if $\operatorname{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) \cong \mathbb{F}$.

*Exercise* 1.8.6. Describe the groupoid corresponding to the functor $D^{\square}_{V_{\mathbb{F}}}$. What are its morphism sets? For $G$ the trivial group, show that $W(\mathbb{F})^d$ with its standard basis and the canonical homomorphism $W(\mathbb{F})^d \to \mathbb{F}^d$ represent $D^{\square}_{V_{\mathbb{F}}}$.

*Exercise* 1.8.7. Fill in the details of the remarks following Definition 1.6.4. In particular, show that if $\mathfrak{F}$ is representable then $\operatorname{Aut}(\eta) = \mathrm{id}$ for all $\eta \in \operatorname{Ob}(\mathfrak{F})$.

*Exercise* 1.8.8. Let $\Phi' \colon \mathfrak{F}' \to \mathfrak{F}$ and $\Phi'' \colon \mathfrak{F}'' \to \mathfrak{F}$ be morphisms of categories. Define $\mathfrak{F}' \times_{\mathfrak{F}} \mathfrak{F}''$ as the category whose objects over $T$ are triples $(\eta', \eta'', \theta)$, where $\eta' \in \operatorname{Ob}(\mathfrak{F}'(T))$, $\eta'' \in \operatorname{Ob}(\mathfrak{F}''(T))$ and $\theta$ is an isomorphism $\Phi'(\eta') \xrightarrow{\cong} \Phi''(\eta'')$ over $\mathrm{id}_T$, and whose morphisms $(\eta', \eta'', \theta) \longrightarrow (\xi', \xi'', \tau)$ above $T \to S$ are pairs $(\eta' \xrightarrow{\alpha'} \xi', \eta'' \xrightarrow{\alpha''} \xi'')$ over $T \to S$ such that the following diagram in $\mathfrak{F}$ commutes:

$$\begin{array}{ccc} \Phi'(\eta') & \xrightarrow{\alpha'} & \Phi'(\xi') \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\tau} \\ \Phi''(\eta'') & \xrightarrow{\alpha''} & \Phi''(\xi''). \end{array}$$

For example, if $\mathfrak{F}' \to \mathfrak{F}$ is a morphism of groupoids over $\mathfrak{C}$ and $\xi \in \mathfrak{F}$, one can form $\mathfrak{F}'_{\xi} := \mathfrak{F}' \times_{\mathfrak{F}} \tilde{\xi}$.

Let now $S$ be a scheme. Then using the construction in 1.6.1 we may consider an $S$-scheme $X$ as a groupoid $\tilde{X}$ over $S$-schemes. Suppose that $X \to Y$ and $X' \to Y$ are morphisms of $S$-schemes. Show that there is an isomorphism

$$\tilde{X} \times_{\tilde{Y}} \tilde{X}' \xrightarrow{\cong} \widetilde{X \times_Y X'}.$$

# Lecture 2

# Deformations of pseudo-representations

We start this lecture by giving a proof of the representability of $D_{V_{\mathbb{F}}}$ under the hypothesis that $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, following [Ki7, Lecture III]. Then we turn to pseudo-representations and their deformations. Pseudo-representations as considered here were introduced in [Tay1]. Their deformation theory was first studied in [Nys] and [Rou]. The treatment of the deformation theory here is taken from [Ki7].

The material is not directly needed in relation to the proof of Serre's conjecture, but it is foundational and deserves further attention. Pseudo-representations are useful when a representation is not absolutely irreducible. They appeared first in work of Wiles (in somewhat different form; see Appendix 2.7.2). The present definition goes back to Taylor [Tay1]. In both instances, they were used in the construction of $p$-adic Galois representations by a patching argument which relied on the existence of a sequence $f_m$ of mod $p^m$ modular forms such that $f_m \equiv f_{m+1} \bmod p^m$ for all $m$. More relevant in relation to deformation theory is their use in the construction of $p$-adic families of Galois representations in the work of Bellaïche–Chenevier [BC1], Buzzard [Buz] or Coleman–Mazur [CM]. Pseudo-representations also play an important role in Kisin's work [Ki6] on the Fontaine–Mazur conjecture. If the dimension is larger than the characteristic, pseudo-representations do not behave well. We shall not discuss a recent variant introduced by Chenevier [Che], which works well in all characteristics.

In the appendix to this chapter, we provide a short introduction to formal schemes and recall the definition of pseudo-representations in the sense of Wiles.

## 2.1 Quotients by group actions

Quotients by finite (formal) group actions are often representable, and indeed there are general results which guarantee this in certain situations. In this section we assume that $G$ satisfies Condition $\Phi_p$. Our first aim is the proof of Proposition 1.3.1(b) from Lecture 1, whose proof had been postponed.

**Theorem 2.1.1.** *Suppose* $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$. *Then* $D_{V_{\mathbb{F}}}$ *is representable.*

*Proof.* We saw that $D^{\square}_{V_{\mathbb{F}}}$ is representable by the formal scheme[1] $X_{V_{\mathbb{F}}} := \mathrm{Spf}\, R^{\square}_{V_{\mathbb{F}}}$, where $R^{\square}_{V_{\mathbb{F}}}$ is in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$. Let $\widehat{\mathrm{PGL}}_d$ denote the formal completion of the $W(\mathbb{F})$-group scheme $\mathrm{PGL}_d$ along its identity section, i.e., the formal neighborhood of $\mathrm{PGL}_d$ of the closed point $\mathrm{id} \in \mathrm{PGL}_d(\mathbb{F})$. The formal group $\widehat{\mathrm{PGL}}_d$ acts on the formal scheme $X_{V_{\mathbb{F}}}$:

$$\widehat{\mathrm{PGL}}_d \times X_{V_{\mathbb{F}}} \longrightarrow X_{V_{\mathbb{F}}},\ (g, x) \longmapsto gx.$$

The action can most easily be understood if the schemes involved are considered as functors on rings $A \in \mathfrak{Ar}_{W(\mathbb{F})}$: to every matrix $g$ in $\widehat{\mathrm{PGL}}_d(A) = \mathrm{Ker}(\mathrm{PGL}_d(A) \to \mathrm{PGL}_d(\mathbb{F}))$ and representation $\rho_A\colon G \to \mathrm{GL}_d(A)$ (given by $(V_A, \iota_A, \beta_A)$), one assigns $g\rho g^{-1}$. This action can be converted into the following equivalence relation:

$$\widehat{\mathrm{PGL}}_d \times X_{V_{\mathbb{F}}} \rightrightarrows X_{V_{\mathbb{F}}},\ \big(g, x\big) \longmapsto \big(x, gx\big).$$

A pair $(x, y) \in X \times X$ lies in the image of the relation if and only if $x$ and $y$ lie in the same $\widehat{\mathrm{PGL}}_d$-orbit.

By the hypothesis $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, the action of $\widehat{\mathrm{PGL}}_d$ on $X_{V_{\mathbb{F}}}$ is free. This implies that the induced map

$$\widehat{\mathrm{PGL}}_d \times X_{V_{\mathbb{F}}} \longrightarrow X_{V_{\mathbb{F}}} \times X_{V_{\mathbb{F}}},\ \big(g, x\big) \longmapsto (x, gx) \tag{2.1.1}$$

is a monomorphism as a functor of points, and thus a closed immersion of formal schemes; see Exercise 2.8.1.

Constructing $X_{V_{\mathbb{F}}}/\widehat{\mathrm{PGL}}_d$ as a formal scheme amounts to the same as constructing a formal scheme representing the above equivalence relation; indeed, both schemes parameterize orbits of the action of $\widehat{\mathrm{PGL}}_d$. To see that the latter is possible we need to recall a result from [SGA3].

Recall that $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ is the category of complete local Noetherian $W(\mathbb{F})$-algebras. Thus, the opposite category $(\widehat{\mathfrak{Ar}}_{W(\mathbb{F})})^o$ is equivalent to the category of formal Noetherian spectra of such $W(\mathbb{F})$-algebras with underlying space consisting of one point and residue field $\mathbb{F}$.

**Definition 2.1.2.** An *equivalence relation* $R \rightrightarrows X$ in $(\widehat{\mathfrak{Ar}}_{W(\mathbb{F})})^o$ is a pair of morphisms such that

(a) $R \to X \times X$ is a closed embedding, and

(b) for all $T \in (\widehat{\mathfrak{Ar}}_{W(\mathbb{F})})^o$ the subset $R(T) \subset (X \times X)(T)$ is an equivalence relation.

[1] See Appendix 2.7.1 for some background on formal schemes.

We have seen above that, for a group object $G$ in $(\widehat{\mathfrak{A}\mathfrak{r}}_{W(\mathbb{F})})^o$ and a free action $G \times X \to X$, the map

$$G \times X \rightrightarrows X, \ (g, x) \longmapsto (x, gx)$$

defines an equivalence relation.

**Definition 2.1.3.** A flat morphism $X \to Y$ in $(\widehat{\mathfrak{A}\mathfrak{r}}_{W(\mathbb{F})})^o$ is said to be a *quotient of* $X$ *by* $R$, and one also writes $Y = X/R$, if the embedding $R \to X \times X$ induces an isomorphism $R \cong X \times_Y X$.

**Theorem 2.1.4** ([SGA3, VIIb, Thm. 1.4])**.** *Let* $p_0, p_1 \colon R \rightrightarrows X$ *be an equivalence relation in* $(\widehat{\mathfrak{A}\mathfrak{r}}_{W(\mathbb{F})})^o$ *such that the first projection* $p_1 \colon R \to X$ *is flat. Then the quotient of* $X$ *by* $R$ *exists. It represents the functor on points defined by the equivalence relation. If* $X = \operatorname{Spf} B$ *and* $R = \operatorname{Spf} C$*, then* $X/R = \operatorname{Spf} A$*, where*

$$A = \{b \in B \mid p_0^*(b) = p_1^*(b) \text{ in } C\}.$$

Theorem 2.1.4 applied to the equivalence relation $\widehat{\mathrm{PGL}}_d \times X_{V_{\mathbb{F}}} \rightrightarrows X_{V_{\mathbb{F}}}$ completes the proof of Theorem 2.1.1. □

## 2.2 Pseudo-representations

Absolutely irreducible representations of finite groups are determined by their trace functions. A result of Carayol [Car] and Mazur [Maz] says that the analogous result holds also for deformations:

**Theorem 2.2.1** (Carayol, Mazur)**.** *Suppose that* $V_{\mathbb{F}}$ *is absolutely irreducible. If* $A$ *is in* $\mathfrak{A}\mathfrak{r}_{W(\mathbb{F})}$ *and* $V_A, V'_A \in D_{V_{\mathbb{F}}}(A)$ *are deformations such that* $\operatorname{Tr}(\sigma|V_A) = \operatorname{Tr}(\sigma|V'_A)$ *for all* $\sigma \in G$*, then* $V_A$ *and* $V'_A$ *are isomorphic deformations.*

*Proof.* The following proof is due to Carayol. Fix bases for $V_A$ and $V'_A$ and extend the resulting representations to $A$-linear maps

$$\rho_A, \rho'_A \colon A[G] \longrightarrow M_d(A).$$

We have to show that the bases can be chosen so that $\rho_A = \rho'_A$.

Let $\mathfrak{m}_A$ be the radical of $A \in \mathfrak{A}\mathfrak{r}_{W(\mathbb{F})}$, and $I = (a) \subset A$ be an ideal such that $\mathfrak{m}_A a = 0$. By induction on the length of $A$, we may assume that $\rho_A \equiv \rho'_A$ modulo $I$, and write $\rho_A = \rho'_A + \delta$, where for $\sigma \in A[G]$ the matrix $\delta(\sigma) \in M_d(I)$ has trace 0.

As $\rho_A$ and $\rho'_A$ are multiplicative, we find that, for $\sigma_1, \sigma_2 \in A[G]$,

$$\delta(\sigma_1\sigma_2) = \bar{\rho}(\sigma_1 v)\delta(\sigma_2) + \delta(\sigma_1 v)\bar{\rho}(\sigma_2). \tag{2.2.1}$$

If $\sigma_2 \in \operatorname{Ker}(\bar{\rho})$, we have that $\delta(\sigma_1\sigma_2) = \bar{\rho}(\sigma_1)\delta(\sigma_2)$ for all $\sigma_1 \in A[G]$; therefore $\operatorname{Tr}(\bar{\rho}(\sigma_1)\delta(\sigma_2)) = 0$ for all $\sigma_1 \in A[G]$. But by Burnside's theorem $\bar{\rho}(\mathbb{F}[G]) = M_d(\mathbb{F})$

as $\bar{\rho}$ is absolutely irreducible. Hence, $\mathrm{Tr}(X\delta(\sigma_2)) = 0$ for any $X \in M_d(\mathbb{F})$, so $\delta(\sigma_2) = 0$.

It follows that $\delta\colon M_d(\mathbb{F}) \to M_d(I) \cong M_d(\mathbb{F}) \otimes_{\mathbb{F}} I \cong M_d(\mathbb{F})$ is an $\mathbb{F}$-linear derivation on $M_d(\mathbb{F})$. Such a derivation is always inner; see e.g. [Wei, Lemma 9.2.1, Thm. 9.2.11]. Hence there exists $U \in M_d(I)$ such that $\delta(\sigma) = \bar{\rho}(\sigma)U - U\bar{\rho}(\sigma)$ and $\rho'_A = (1-U)\rho_A(1+U)$. □

The above result gives a clue that in many important cases also the representation theory of profinite groups is governed by traces. The idea of pseudo-representations, introduced by Wiles [Wi1] for odd two-dimensional representations and by Taylor [Tay1] for an arbitrary group, is to try to characterize those functions on $G$ which are traces and to study deformation theory via deformations of the trace functions.

**Definition 2.2.2.** Let $R$ be a (topological) ring. A (continuous) $R$-valued *pseudo-representation* of dimension $d$, for some $d \in \mathbb{N}_0$, is a continuous function $T\colon G \to R$ with the following properties:

(a) $T(\mathrm{id}) = d$ where $\mathrm{id} \in G$ is the identity element and $d!$ is a non-zero-divisor of $R$.

(b) For all $g_1, g_2 \in G$ one has $T(g_1g_2) = T(g_2g_1)$ ($T$ *is central*).

(c) $d \geq 0$ is minimal such that the following condition holds. Let $S_{d+1}$ denote the symmetric group on $d+1$ letters and let $\mathrm{sign}\colon S_{d+1} \to \{\pm 1\}$ denote its sign character. Then, for all $g_1, \ldots, g_{d+1} \in G$,

$$\sum_{\sigma \in S_{d+1}} \mathrm{sign}(\sigma)\, T_\sigma(g_1, \ldots, g_{d+1}) = 0,$$

where $T_\sigma\colon G^{d+1} \to R$ is defined as follows. Suppose that $\sigma \in S_{d+1}$ has cycle decomposition

$$\sigma = \left(i_1^{(1)}, \ldots, i_{r_1}^{(1)}\right) \ldots \left(i_1^{(s)}, \ldots, i_{r_s}^{(s)}\right) = \sigma_1 \ldots \sigma_s. \tag{2.2.2}$$

Then $T_\sigma(g_1, \ldots, g_{d+1}) = T\left(g_{i_1^{(1)}} \ldots g_{i_{r_1}^{(1)}}\right) \cdot \ldots \cdot T\left(g_{i_1^{(s)}} \ldots g_{i_{r_s}^{(s)}}\right)$.

*Remarks* 2.2.3. Let $T$ be a pseudo-representation of $G$ of dimension $d$.

(a) It is shown in [Rou, §2] that, if condition (c) holds for some $d$, then it holds for all $d' \geq d$. It is also shown in [Rou, Prop. 2.4] that conditions (b) and (c) for $d$ minimal imply $T(\mathrm{id}) = d$, provided that $R$ is a domain.

(b) The condition that $d!$ is a non-zero-divisor of $R$ is suggested by the work of Bellaïche and Chenevier [BC1]. In fact, they require that $d!$ be a unit of $R$. Being somewhat restrictive, this condition does avoid a number of pathologies. For instance, we shall make use of it in Lemma 2.3.4. In [BC1, Footnote 13] it is also observed that the condition $d! \in R^*$ is needed for Lemma 2.14, Lemma 4.1 and Theorem 5.1 in the article [Rou].

(c) In Taylor's work [Tay1] he is primarily interested in rings $R$ of characteristic zero. Then $d!$ is automatically a non-zero-divisor in $R$.

(d) In the recent preprint [Che], Chenevier replaces the notion of pseudo-representation by that of a *determinant* —a notion defined in [Che]. Its main advantage is that it requires no condition on $d!$, and hence it is a good notion over rings of any characteristic. The preprint [Che] also studies deformations of such and applies this theory to rigid analytic Galois representation arising from $p$-adic families of modular forms.

(e) It is often convenient to consider the $R$-linear extension $\widetilde{T}\colon R[G] \to R$ of a pseudo-representation. The relations in Definition 2.2.2(c) are then satisfied for all $(g_1, \ldots, g_{d+1}) \in R[G]^{d+1}$.

**Theorem 2.2.4** (Taylor, Rouquier)**.** (a) *If $\rho\colon G \to \mathrm{GL}_d(R)$ is a representation, then $\mathrm{Tr}\,\rho$ is a pseudo-representation of dimension at most d.*

(b) *Suppose $R$ is an algebraically closed field*[2] *of characteristic $\mathrm{Char}(R) > d$ or $\mathrm{Char}(R) = 0$. Then for any pseudo-representation $T$ of dimension $d$ there exists a unique semisimple representation $\rho\colon G \to \mathrm{GL}_d(R)$ with $\mathrm{Tr}\,\rho = T$.*

(c) *If $G$ is (topologically) finitely generated, then for every integer $d \geq 1$ there is a finite subset $S \subset G$, depending on $d$, such that a pseudo-representation $T\colon G \to R$ of dimension $d$ is determined by its restriction to $S$. (Recall that our hypotheses imply that $d!$ is a non-zero-divisor in $R$.)*

Except for the level of generality of part (b), the above theorem is due to Taylor; cf. [Tay1]. In op. cit., part (b) is only proved for algebraically closed fields of characteristic zero. Taylor's arguments are based on results of Procesi on invariant theory; see [Pro]. Part (b) as stated is from Rouquier [Rou, §4], who also gives a direct and self-contained proof of part (a) independent of the results in [Pro]. Below we follow Rouquier.

*Proof.* We only give the arguments for part (a). We let $T = \mathrm{Tr}\,\rho$ and define $\Theta\colon M_d(R) \to R$ as the map

$$\Theta(g_1, \ldots, g_{d+1}) = \sum_{\sigma \in S_{d+1}} \mathrm{sign}(\sigma) T_\sigma(g_1, \ldots, g_{d+1}).$$

We shall show that $\Theta \equiv 0$. It suffices to prove this for $G = \mathrm{GL}_d(R)$ and $\rho = \mathrm{id}$. Writing $R$ as a quotient of a domain $R'$ of characteristic 0, it suffices to prove the

[2] By [Rou, Thm. 4.2] it is necessary and sufficient to assume that $R$ is a field with trivial Brauer group. As an example, consider $D \neq K$ a division algebra over a $p$-adic field $K$ with $\mathcal{O}_D$ a maximal order and $G$ the group of units of $\mathcal{O}_D$. Then the reduced trace is a pseudo-representation $G = \mathcal{O}_D^* \to \mathcal{O}_K \subset K$ of dimension $d$ such that $d^2 = [D : K]$. However, the asserted representation in (b) only exists over a splitting field $L \supset K$ of $D$. For another hypothesis under which (b) holds, see Theorem 2.4.1 due to Nyssen and Rouquier.

result for $R'$; this case is then easily reduced to that where $R$ is an algebraically closed field of characteristic 0. This we assume from now on.

Let $V = R^d$ and define $W = V \otimes V^* = \operatorname{End}_R V$. Note that $\Theta$ is invariant under the action of $S_{d+1}$ (since, if one applies $T_\sigma$ to the permutation of $g_1, \dots, g_{d+1}$ under some $\xi \in S_{d+1}$, one obtains $T_{\xi^{-1}\sigma\xi}$). Hence, if we extend $\Theta$ to a multilinear map $\Theta\colon W^{\otimes(d+1)} \to R$, it is determined by its values on $\operatorname{Sym}^{d+1} W \subset W^{\otimes(d+1)}$. As we are in characteristic zero, a simple argument based on homogeneous polynomials of degree $d+1$ in $\dim W$ variables shows that as an $R$-vector space $\operatorname{Sym}^{d+1} W$ is spanned by the image of the diagonal map

$$\Delta\colon W \longmapsto W^{\otimes(d+1)},\ w \longmapsto w \otimes \cdots \otimes w.$$

Thus it suffices to show that $\Theta(\Delta(w)) = 0$ for all $w \in W$. As the semisimple elements in $\operatorname{Aut}_R(V)$ are Zariski dense in $W$, it is enough to show $\Theta(\Delta(w)) = 0$ for all semisimple $w \in W$. For small values of $d$, this can be verified explicitly. For general $d$, one has the following argument:

Choose a basis $\{e_1, \dots, e_d\}$ of $V$ in which $w$ is diagonal and consider the action of $\Xi := w \sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma)\sigma$ on $W^{\otimes(d+1)}$, where $S_{d+1}$ acts by permuting the factors and $w$ acts as $\Delta(w)$. We claim that $\operatorname{Tr}\Xi = \Theta(\Delta(w))$. Assuming the claim, we observe that obviously

$$\Big(\sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma)\sigma\Big)\Big(V^{\otimes(d+1)}\Big) \subset \bigwedge^{d+1} V = 0,$$

and so the proposition follows.

We now prove the claim: suppose that $w$ has diagonal entries $\lambda_1, \dots, \lambda_d$ with respect to $e_1, \dots, e_d$. The trace of $\Xi$ is then given by

$$\begin{aligned}
&\sum_{\underline{i}=(i_1,\dots,i_d)\in\{1,\dots,d\}^{d+1}} \Big\langle w\Big(\sum_{\sigma\in S_{d+1}} \operatorname{sign}(\sigma)\sigma\Big)\big(e_{i_1}\otimes\cdots\otimes e_{i_{d+1}}\big), \big(e_{i_1}\otimes\cdots\otimes e_{i_{d+1}}\big)\Big\rangle \\
&= \sum_{\sigma\in S_{d+1}} \operatorname{sign}(\sigma) \sum_{\underline{i}\in\{1,\dots,d\}^{d+1}} \Big\langle w\big(e_{\sigma(i_1)}\otimes\cdots\otimes e_{\sigma(i_{d+1})}\big), \big(e_{i_1}\otimes\cdots\otimes e_{i_{d+1}}\big)\Big\rangle,
\end{aligned}$$

where $\langle -, - \rangle$ is 1 if both entries are the same and zero otherwise. Thus if we write $\sigma$ in its cycle decomposition $\sigma_1 \dots \sigma_s$ as in (2.2.2) on page 24, the expression

$$\Big\langle \big(e_{\sigma(i_1)}\otimes\cdots\otimes e_{\sigma(i_{d+1})}\big), \big(e_{i_1}\otimes\cdots\otimes e_{i_{d+1}}\big)\Big\rangle$$

is non-zero (and thus equal to 1) exactly if the tuple $\underline{i}$ is constant on the support of each of the cycles $\sigma_k$. Moreover on each such support we can choose the value of $\underline{i}$ freely. Moreover if $(j_1, \dots, j_s)$ denotes the tuple of values on the $s$ supports of these cycles (cycles may have length one), then $w$ applied to $e_{\sigma(i_1)} \otimes \cdots \otimes e_{\sigma(i_{d+1})}$

results in multiplication by $\lambda_{j_1}^{|\sigma_1|} \cdot \ldots \cdot \lambda_{j_s}^{|\sigma_s|}$, where $|\sigma_k|$ is the length of the cycle. Summing over all $\underline{i}$ (for fixed $\sigma$) yields

$$\operatorname{Tr}\left(w^{|\sigma_1|}\right) \cdot \ldots \cdot \operatorname{Tr}\left(w^{|\sigma_s|}\right).$$

This expression clearly agrees with $\operatorname{Tr}_\sigma(w)$ and so the claim is shown. □

## 2.3 Deformations of pseudo-representations

Let $\tau_{\mathbb{F}}\colon G \to \mathbb{F}$ be a pseudo-representation. For $A$ in $\mathfrak{A}_{\mathfrak{r}W(\mathbb{F})}$ define

$$D_{\tau_{\mathbb{F}}}(A) = \big\{\text{pseudo-representations } \tau_A\colon G \longrightarrow A \text{ lifting } \tau_{\mathbb{F}}\big\}.$$

**Proposition 2.3.1.** *Suppose $G$ satisfies Condition $\Phi_p$. Then $D_{\tau_{\mathbb{F}}}$ is pro-representable by a complete local Noetherian $W(\mathbb{F})$-algebra $R_{\tau_{\mathbb{F}}}$.*

For an example of a universal pseudo-deformation, see Exercise 2.8.2. We note that there is no simple expression of the tangent space of the functor $D_{\tau_{\mathbb{F}}}$ similar to that given in Lemma 1.4.3 for $D_{V_{\mathbb{F}}}$. So interesting results on this can be found in [Bel].

For the proof of Proposition 2.3.1, we need some preparation:

**Definition 2.3.2.** For any pseudo-representation $T\colon G \to R$, define

$$\operatorname{Ker} T = \big\{h \in G \mid \forall g \in G : T(gh) = T(g)\big\}.$$

If we view $T$ as an $R$-linear map $\widetilde{T}\colon R[G] \to R$, then we set

$$\operatorname{Ker} \widetilde{T} = \big\{h \in R[G] \mid \forall g \in R[G] : T(gh) = 0\big\}.$$

**Lemma 2.3.3.** (a) *$\operatorname{Ker} T$ is a closed normal subgroup of $G$.*

(b) *$\operatorname{Ker} \widetilde{T}$ is an ideal of $R[G]$.*

(c) *If $R$ is finite, then $\operatorname{Ker} T$ is open in $G$.*

*Proof.* We leave parts (a) and (b) as exercises. Let us prove (c). For each $r \in R$, denote $U_r = \{g \in G \mid T(g) = r\}$. Since $T$ is continuous and $R$ is finite, the $U_r$ form a partition of $G$ by open subsets. Now note that the condition $T(gh) = T(g)$ for all $g \in G$ is equivalent to $U_{T(g)}h \subset U_{T(g)}$ for all $g \in G$. Thus

$$\operatorname{Ker} T = \bigcap_{r \in R} \big\{h \in G \mid U_r h \subset U_r\big\}.$$

The latter is clearly open in $G$ and this proves (c). □

Inspired by [Ki7, (2.2.3)], we show the following, where $\tau_{\mathbb{F}}$ is as in Proposition 2.3.1. For a profinite group $G$ and $m \in \mathbb{N}$, we denote by $G^m$ the closed subgroup generated by $\{g^m \mid g \in G\}$. It is clearly normal in $G$.

**Lemma 2.3.4.** *Set $G' = \operatorname{Ker} \tau_{\mathbb{F}}$. Then for any $A \in \mathfrak{Ar}_{W(\mathbb{F})}$ there exists a constant $m_A \in \mathbb{N}$ such that for all $\tau_A \in D_{\tau_{\mathbb{F}}}(A)$ one has $(G')^{p^{m_A}} \subset \operatorname{Ker} \tau_A$. In particular, if $H \subset G'$ denotes the closed normal subgroup such that $G'/H$ is the maximal pro-$p$ quotient of $G'$, then $\operatorname{Ker} \tau_A$ contains $H$.*

*Proof.* For any $g \in G'$, $h \in G$ and $i \geq 1$, one has $\widetilde{\tau}_A((g-1)^i h) \in \mathfrak{m}_A$. Taking $g_j = (g-1)^{i_j}$ and $g_{d+1} = (g-1)^{i_{d+1}} h$ with all $i_j \geq 1$ in Definition 2.2.2(c), and using the centrality of Definition 2.2.2(b), one finds that $d! \cdot \widetilde{\tau}_A((g-1)^i h) \in \mathfrak{m}_A^2$ for all $g \in G'$, $h \in G$ and $i \geq (d+1)$. Since $d!$ is a non-zero-divisor, we may cancel. Proceeding by induction yields

$$\widetilde{\tau}_A((g-1)^i h) \in \mathfrak{m}_A^{2^j} \text{ for all } g \in G',\ h \in G \text{ and } j \geq 1 \text{ and } i \geq (d+1)^j.$$

Since $A$ is Artinian, we can find $m \in \mathbb{N}$ such that $\widetilde{\tau}_A((g-1)^{p^m} h) = 0$ for all $g \in G'$ and $h \in G$. By enlarging $m$ if necessary we also assume that $p^m A = 0$. In particular, it follows that $\binom{p^{2m}}{i} = 0$ in $A$ for all $i < p^m$. But then binomial expansion yields

$$\widetilde{\tau}_A((g^{p^{2m}} - 1)h) = \widetilde{\tau}_A((((g-1)+1)^{p^{2m}} - 1)h) = \sum_{i=p^m}^{p^{2m}} \widetilde{\tau}_A((g-1)^i h) \binom{p^{2m}}{i} = 0,$$

where in the last step we use $\widetilde{\tau}_A((g-1)^i h) = 0$ for all $i \geq p^m$, $g \in G'$ and $h \in G$. The first part of the lemma follows with $m_A = 2m$, since $\operatorname{Ker} \tau_A$ is a closed normal subgroup of $G$. For the second part, observe that $H$ is a subgroup of $(G')^{p^{m_A}}$, since $G'/(G')^{p^{m_A}}$ is a finite $p$-group. Hence, by the first part, $H \subset \operatorname{Ker} \tau_A$. □

*Proof of Proposition* 2.3.1*.* Suppose first that $G$ is finite. Let $R_G$ be the quotient of $W(\mathbb{F})[X_g : g \in G]$ by the ideal $I$ generated by the relations $X_e - d$, $X_{gh} - X_{hg}$ for all $g, h \in G$, and the relations

$$\sum_{\sigma \in S_{d+1}} \operatorname{sign}(\sigma)\, X_{g_{i_1^{(1)}} \cdots g_{i_{r_1}^{(1)}}} \cdot \ldots \cdot X_{g_{i_1^{(s)}} \cdots g_{i_{r_s}^{(s)}}},$$

where each $\sigma$ is given in its cycle decomposition as in (2.2.2) on page 24. By the definition of $I$, mapping $X_{gH}$ to $\tau_{\mathbb{F}}(g)$ yields a well-defined homomorphism $R_G \to \mathbb{F}$ in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$. The completion $\widehat{R_G}$ of $R_G$ at the kernel of this homomorphism is the wanted universal ring; the corresponding universal deformation of $\tau_{\mathbb{F}}$ is the induced map

$$\tau_G \colon G \longrightarrow \widehat{R_G},\ g \longmapsto gH \longmapsto X_{gH}.$$

Let now $G$ be arbitrary. Write $G = \varprojlim_{i \in I} G/H_i$ for a basis of the identity by open normal subgroups $H_i$, $i \in I$. By the universality of the $(\widehat{R_{G/H_i}}, \tau_{R/H_i})$, they form an inverse system and their inverse limit is the wanted universal deformation $(\widehat{R_G}, \tau_G)$. It remains to see that under Condition $\Phi_p$ the ring $\widehat{R_G}$ is Noetherian. By the previous lemma, the elements in $D_\tau(\mathbb{F}[\varepsilon])$ factor via $R_{G/(G')^{p^m}}$ for some fixed $m \in \mathbb{N}$. Condition $\Phi_p$ implies that the group $G/(G')^{p^m}$ is finite, and hence $\dim_{\mathbb{F}} D_\tau(\mathbb{F}[\varepsilon]) < \infty$. □

Unlike for the case of deformations of representations, finding an exact formula for $\dim_{\mathbb{F}} D_\tau(\mathbb{F}[\varepsilon])$ seems difficult in general. For some recent partial results, see [Bel].

## 2.4 Deforming a representation $\bar\rho$ and the pseudo-representation $\operatorname{Tr}\bar\rho$

For absolutely irreducible representations we have the following:

**Theorem 2.4.1** (Nyssen–Rouquier). *Suppose that $G$ satisfies Condition $\Phi_p$ and that $\bar\rho\colon G \to \mathrm{GL}(V_{\mathbb{F}})$ is absolutely irreducible. Set $\tau_{\mathbb{F}} = \operatorname{Tr}\bar\rho$. Then there is an isomorphism of functors $D_{V_{\mathbb{F}}} \xrightarrow{\cong} D_{\tau_{\mathbb{F}}}$ on $\mathfrak{Ar}_{W(\mathbb{F})}$.*

This theorem does not require that $d!$ be invertible in $R$. However, note that $\bar\rho$ is given a priori.

The theorem has the following consequences, the first of which is due to Carayol:

(a) Let $A$ be in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ and $V_A$ be a representation over $A$ with reduction $V_{\mathbb{F}}$ such that $V_{\mathbb{F}}$ is absolutely irreducible. Let $A_0 \subset A$ be the subring generated by $\operatorname{Tr}(V_A)(G)$. Then $V_A$ is defined over $A_0$, i.e., there exists $V_{A_0} \in D_{V_{\mathbb{F}}}(A_0)$ such that $V_A \cong V_{A_0} \otimes_{A_0} A$: by the above theorem, it suffices to prove the analogous statement for pseudo-representations. There it is trivial.

(b) Suppose that $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ and $\tau\colon G \to A$ is a pseudo-representation such that $\tau \bmod \mathfrak{m}_A$ arises from some $\bar\rho$ as in the theorem. Then there exists a representation $\rho\colon G \to \mathrm{GL}_d(A)$ whose trace is equal to $\tau$. It is unique up to isomorphism by Theorem 2.2.1.

(c) Lastly it gives another proof of the representability of the functor $D_{V_{\mathbb{F}}}$ in the case where $V_{\mathbb{F}}$ is absolutely irreducible.

The situation becomes more involved if the initial residual representation is no longer absolutely irreducible —which was one of the main reasons for introducing pseudo-representations. Suppose therefore that $\bar\rho\colon G \to \mathrm{GL}(V_{\mathbb{F}})$ is arbitrary and set $\tau_{\mathbb{F}} = \operatorname{Tr}\bar\rho$. One still has the canonical morphism of functors $D^{\square}_{V_{\mathbb{F}}} \to D_{\tau_{\mathbb{F}}}$, $V_A \mapsto \operatorname{Tr} V_A$.

Let us look at the following concrete example. Let $\chi_1, \chi_2 \colon G \to \mathbb{F}^\times$ be characters and $c_1, c_2 \in \mathrm{Ext}^1(\chi_2, \chi_1)$ —or rather $c_i \in Z^1(G, \chi_2\chi_1^{-1})$. Then

$$\begin{pmatrix} \chi_1 & c_1 + Tc_2 \\ 0 & \chi_2 \end{pmatrix}$$

is a representation $G \to \mathrm{GL}_2(\mathbb{F}[T])$, i.e., a family of representations over $\mathbb{A}^1_{\mathbb{F}}$. More naturally, one obtains a family of representations of $G$ over $\mathbb{P}(\mathrm{Ext}^1(\chi_2, \chi_1))$, the projectivization of $\mathrm{Ext}^1(\chi_2, \chi_1)$, which all have the pseudo-character $\chi_1 + \chi_2$. Note that the projectivized representation consists of a vector bundle of rank 2 over $\mathbb{P}(\mathrm{Ext}^1(\chi_2, \chi_1))$, which carries an action of a finite quotient of $G$ such that, over any sufficiently small affine $\mathrm{Spec}\, R \subset \mathbb{P}(\mathrm{Ext}^1(\chi_2, \chi_1))$, the action is isomorphic to a true representation $G \to \mathrm{GL}_2(R)$.

To fully express the relationship between the deformations of $V_{\mathbb{F}}$ and those of the pseudo-representations $\tau = \mathrm{Tr}\, V_{\mathbb{F}}$, it will be convenient to work with groupoids. The underlying category will however *not* be $\mathfrak{Ar}_{W(\mathbb{F})}$: the point is that, as we have seen above, the fiber over $V_{\mathbb{F}}$ of the natural transformation $D^{\square}_{V_{\mathbb{F}}} \to D_{\tau_{\mathbb{F}}}$ is no longer a single point!

Following Kisin, we consider the category $\mathfrak{Aug}_{W(\mathbb{F})}$. Its objects are morphisms $A \to B$ where $A$ is in $\mathfrak{Ar}_{W(\mathbb{F})}$ and $B$ is an $A$-algebra with no finiteness condition assumed. Morphisms $(A \to B) \to (A' \to B')$ are pairs of homomorphisms $A \to A'$ and $B \to B'$ which yield a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B'. \end{array}$$

So, even if $A = \mathbb{F}$, the second entry $B$ can be any $\mathbb{F}$-algebra, e.g. the coordinate ring of an affine subvariety of $\mathbb{P}(\mathrm{Ext}^1(\chi_2, \chi_1))$.

*Remarks* 2.4.2. There are several variants of the category $\mathfrak{Aug}_{W(\mathbb{F})}$.

(a) We may also consider $\widehat{\mathfrak{Aug}}_{W(\mathbb{F})}$. Its objects are pairs $(B, A)$ where $A \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ and $B$ is an $A$-algebra which is topologically complete with respect to the topology defined by $\mathfrak{m}_A B$.

(b) Another natural category to consider is that of pairs $(S, A)$ where $S$ is an $A$-scheme —or even the inverse limit category of it, as described in part (a). The point is that the pro-representing object of a groupoid fibered over $\mathfrak{Aug}_{W(\mathbb{F})}$ may not be an affine scheme. In the example above it is suggested that this scheme could be projective. Working therefore with schemes instead of rings, the universal object would still be within the category considered.

(c) For instance, in [Ki4] Kisin works with yet another definition of $\mathfrak{Aug}_{W(\mathbb{F})}$. The definition here emphasizes the underlying ring $A$ in $A \to B$. But for other purposes phrasing the deformation problem in such a relative way is unnecessary.

**Definition 2.4.3.** Fix a pseudo-representation $\tau_{\mathbb{F}}\colon G \to \mathbb{F}$. Define a groupoid $\mathbf{Rep}_{\tau_{\mathbb{F}}}$ over $\mathfrak{Aug}_{W(\mathbb{F})}$ by

$$\mathbf{Rep}_{\tau_{\mathbb{F}}}\big(A \to B\big) = \big\{\big(V_B, \tau_A\big) \mid \tau_A \in D_{\tau_{\mathbb{F}}}(A),\\ V_B \cong B^d \text{ a } G\text{-representation, } \operatorname{Tr}\big(V_B\big) = \tau_A)\big\}/\cong .$$

Similarly we define $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}(A \to B)$ as the groupoid over $\mathfrak{Aug}_{W(\mathbb{F})}$ with

$$\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}\big(A \to B\big) = \big\{\big(V_B, \beta_B, \tau_A\big) \mid \big(V_B, \tau_A\big) \in \mathbf{Rep}_{\tau_{\mathbb{F}}}\big(A \to B\big),\\ \beta_B \text{ a } B\text{-basis of } V_B\big\}/\cong .$$

Finally we extend $D_{\tau_{\mathbb{F}}}$ to a groupoid on $\mathfrak{Aug}_{W(\mathbb{F})}$ by setting

$$D_{\tau_{\mathbb{F}}}(A \to B) = D_{\tau_{\mathbb{F}}}(A).$$

We shall indicate in Remark 2.6.3 why it is desirable and useful to study the functor $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}$.

We introduce the following notion:

**Definition 2.4.4.** A morphism $\Phi\colon \mathfrak{F} \to \mathfrak{F}'$ of groupoids over $\mathfrak{C}$ is called *relatively representable* if for each $\eta \in \mathrm{Ob}(\mathfrak{F}')$ the 2-fiber product

$$\mathfrak{F}_\eta = \widetilde{\eta} \times_{\mathfrak{F}'} \mathfrak{F}$$

is representable.

Note that if $\mathfrak{F}'$ is representable and $\Phi\colon \mathfrak{F} \to \mathfrak{F}'$ is relatively representable, then $\mathfrak{F}$ is representable.

**Proposition 2.4.5.** *If $G$ satisfies Condition $\Phi_p$, then $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}$ is representable by a morphism of affine formal schemes* $\operatorname{Spf} R^{\square}_{\tau_{\mathbb{F}}} \to \operatorname{Spf} R_{\tau_{\mathbb{F}}}$ *which is formally of finite type.*

By Theorem 2.2.4(b) we know that $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}} \neq \varnothing$ only if $p > d$.

*Proof.* This is Exercise 2.8.6. □

Suppose that $\tau_{\mathbb{F}}$ is the trace of a semisimple representation $\bar{\rho}$, and so that $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}$ is non-empty. We give an explicit description of $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}$ above the point $x = \tau_{\mathbb{F}} \in D_{\tau_{\mathbb{F}}}(\mathbb{F})$.

**Proposition 2.4.6.** *The functor* $\widetilde{x}\times_{D_{\tau_{\mathbb{F}}}}\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}$ *is representable by the same* $\mathbb{F}$*-algebra as the functor on* $\mathbb{F}$*-algebras which to any such algebra* $S$ *assigns the set*

$$\{\rho\colon G\longrightarrow \mathrm{GL}_d(S)\mid \mathrm{Tr}\,\rho=\tau_{\mathbb{F}}\}.$$

*Any such representation* $\rho\colon G\to\mathrm{GL}_d(S)$ *factors via* $G/(G')^{p^d}$*, where* $G'$ *is the kernel of* $\bar{\rho}$*. Furthermore the semisimplification of* $\rho$ *is isomorphic to* $\bar{\rho}$*.*

*If* $G$ *is finite, then the ring* $R^{\square}_{\mathbb{F}}:=R^{\square}_{\tau_{\mathbb{F}}}\otimes_{R_{\tau_{\mathbb{F}}}}\mathbb{F}$ *representing* $\widetilde{x}\times_{D_{\tau_{\mathbb{F}}}}\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}$ *is given as follows. Let* $\mathcal{R}=W(\mathbb{F})[X^g_{i,j}\mid i,j=1,\dots,d;g\in G]$*. Then* $R_0=\mathcal{R}/\mathcal{I}$ *for* $\mathcal{I}\subset\mathcal{R}$ *the ideal generated by the elements* $\sum_{i=1}^d X^g_{i,i}-\tau_{\mathbb{F}}(g)$*,* $g\in G$*, together with the components of the* $d\times d$*-matrices* $r(X^{g_1},\dots,X^{g_s})-\mathrm{id}$*, where* $r(g_1,\dots,g_s)$ *ranges over all relations among the elements of* $G$ *and* $X^g=(X^g_{i,j})_{i,j=1,\dots,d}$*.*

## 2.5 Representable subgroupoids of $\mathbf{Rep}_{\tau_{\mathbb{F}}}$

The groupoid $\mathbf{Rep}_{\tau_{\mathbb{F}}}$ will not be representable in general. This is for instance the case in the situation of page 29. In this section we shall investigate a resolution of a particular subgroupoid of $\mathbf{Rep}_{\tau_{\mathbb{F}}}$ which will turn out to be representable. The material follows [Ki7, § 3.2]. Further details will appear in a planned future version of [Ki7]. In this section, we require for all $A\to B$ in $\mathfrak{Aug}_{W(\mathbb{F})}$ that the ring $B$ be of finite type over $A$.

We consider the following situation: suppose that for $i=1,\dots,s$ we are given pairwise distinct absolutely irreducible representations $\bar{\rho}_i\colon G\to\mathrm{GL}_{d_i}(\mathbb{F})$. We set $\tau_{\mathbb{F}}=\sum_{i=1}^s\mathrm{Tr}\,\bar{\rho}_i$. Let $\mathbf{Rep}'^{\square}_{\tau_{\mathbb{F}}}\subset\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}$ be the full subgroupoid over $\mathfrak{Aug}_{W(\mathbb{F})}$ on objects $(V_B,\beta_B,\tau_A)\in\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}(A\to B)$ such that the following holds: there exists an affine cover of $\mathrm{Spec}\,B/\mathfrak{m}_AB$ by $\mathrm{Spec}\,B_i$ such that

$$V_B\otimes_B B_i\sim\begin{pmatrix}\bar{\rho}_1 & c_1 & \cdots & \\ & \bar{\rho}_2 & \ddots & \vdots\\ & & \ddots & c_{s-1}\\ & & & \bar{\rho}_s\end{pmatrix}\tag{2.5.1}$$

with nowhere vanishing extension classes $c_i\in\mathrm{Ext}^1(\bar{\rho}_{i+1},\bar{\rho}_i)$ for all $i=1,\dots,s-1$.

*Remark* 2.5.1. The condition on the $c_i$ has the following consequence: since the $\bar{\rho}_i$ are absolutely irreducible and pairwise non-isomorphic, the centralizer of the matrix on the right of (2.5.1) is contained in the set of diagonal matrices which are scalar along the blocks $\bar{\rho}_i$. The non-triviality of $c_i$ implies that the scalar along $\bar{\rho}_i$ is the same as the scalar along $\bar{\rho}_{i+1}$. Therefore the centralizer of the representation on the left is precisely the set of scalar matrices. One deduces that the isomorphism in (2.5.1) is unique up to a scalar.

*Remark* 2.5.2. In (2.5.1) the diagonal blocks of the matrix on the right will always occur in the order indicated. This will be important in the sequel; see for instance Corollary 2.6.2.

**Theorem 2.5.3.** *There exists a locally closed formal subscheme, formally of finite type,* $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}} \to \operatorname{Spf} R_\tau$ *which represents* $\mathbf{Rep}'^{\square}_{\tau_{\mathbb{F}}}$ *on* $\widehat{\mathfrak{Aug}}_{W(\mathbb{F})}$.

*Idea of proof:* The special fiber of $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}$ is a scheme of finite type over $\mathbb{F}$. Any specialization $R^{\square}_{\tau_{\mathbb{F}}} \to k$ for $k$ a finite field containing $\mathbb{F}$ admits an isomorphism, as in (2.5.1), for at least one permutation of the $\bar\rho_i$ and without the non-vanishing requirement for the $c_i$. The condition that the chosen order occurs along the diagonal defines a closed subscheme of the special fiber $\operatorname{Spec} R_{\tau_{\mathbb{F}}} \otimes_{R_\tau} \mathbb{F}$. Similarly, one argues that the additional conditions $0 \neq [c_i] \in \operatorname{Ext}^1(\bar\rho_{i+1}, \bar\rho_i)$ under any such specialization define an open condition. Details will appear in the final version of [Ki7, §3.2]. □

In Remark 2.5.1 we observed that the action of $\mathrm{PGL}_d$ on the special fiber of $\mathbf{Rep}'^{\square}_{\tau_{\mathbb{F}}}$ is free. It easily follows that the conjugation action of $\widehat{\mathrm{PGL}}_d$ on $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$ is free. Even though $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$ is typically not a formal local scheme, the quotient $\mathfrak{X}'_{\tau_{\mathbb{F}}} := \mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}/\widehat{\mathrm{PGL}}_d$ still exists. The proof is formally similar to that of Theorem 2.1.1. However, here a theorem on representability of free group actions on formal schemes over Artin rings is needed. Such a result fits well the framework of Mumford's book on geometric invariant theory, but over Artin rings is not to be found there. An application of Schlessinger's criterion is not possible, as $\mathfrak{X}^{\square\prime}_{\tau_{\mathbb{F}}}$ is typically not local. Nevertheless, the result holds. One proof is due to B. Conrad; details will appear in the final version of [Ki7] by Kisin.

As was explained to us by Kisin, it is not so straightforward to define the functor which is represented by $\mathfrak{X}'_{\tau_{\mathbb{F}}}$. Over $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}} \to \operatorname{Spf} R_{\tau_{\mathbb{F}}}$ there is a universal object represented by this arrow: on $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$ we have a trivial vector bundle with a basis and a representation

$$\big(V_{\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}},\, \beta_{\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}},\, \rho_{\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}}\big);$$

on $\operatorname{Spf} R_{\tau_{\mathbb{F}}}$ we have the universal pseudo-representation $\tau^u$, and the morphism $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}} \to \operatorname{Spf} R_{\tau_{\mathbb{F}}}$ is induced from the pseudo-representation $\operatorname{Tr} \rho'^{\square}_{\tau_{\mathbb{F}}}$ on $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$. While $\widehat{\mathrm{PGL}}_d$ has a well-defined and free action on $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$, it does not act on the universal triple. Only the group $\widehat{\mathrm{GL}}_d$ acts on this triple. Since its center acts trivially on the base $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$, the quotient $V_{\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}}/\widehat{\mathrm{GL}}_d$ is a projective bundle over $\mathfrak{X}'_{\tau_{\mathbb{F}}}$ (if $\mathfrak{X}'_{\tau_{\mathbb{F}}}$ is local, it carries up to isomorphism a unique vector bundle of rank $d$ and one can take it as the quotient). In the global situation the Picard group of the special fiber of $\mathfrak{X}'_{\tau_{\mathbb{F}}}$ need not be trivial. Then it is not clear whether a vector bundle quotient should exist. Therefore one cannot expect that the universal object on $\mathfrak{X}'_{\tau_{\mathbb{F}}}$ is given by a vector bundle with a $G$-action up to isomorphism.

One natural way to bypass the above problem —the final version of [Ki7] might follow a different approach— is to consider projective bundles equipped with a $G$-action instead of vector bundles with a $G$-action, and to ensure that the projective action does lift locally to a linear one. This allows one to give a natural interpretation of $\mathfrak{X}'_{\tau_{\mathbb{F}}}$ as a formal moduli space as follows.

**Definition 2.5.4.** For $p > d = \sum d_i$, denote by $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ the groupoid on $\mathfrak{Aug}_{W(\mathbb{F})}$ which for any $(A \to B) \in \mathfrak{Aug}_{W(\mathbb{F})}$ is the category $\mathbf{Rep}'_{\tau_{\mathbb{F}}}(A \to B)$ whose objects are tuples $\big(\mathcal{P}_B, \mathcal{P}_{B,s-1}, \ldots, \mathcal{P}_{B,1}, \tau_A\big)$ satisfying the following conditions:

(a) $\mathcal{P}_B$ is a projective bundle on $\operatorname{Spec} B$ with a continuous action $G \to \operatorname{Aut}_B(\mathcal{P}_B)$.

(b) Setting $\mathcal{P}_{B,s} = \mathcal{P}_B \otimes_B B/\mathfrak{m}_A B$, the $\mathcal{P}_{B,i}$, $i = 1, \ldots, s$, are projective $G$-bundles on $\operatorname{Spec} B/\mathfrak{m}_A B$ of rank $d_1 + \cdots + d_i - 1$ and they form a flag of $G$-stable projective subbundles $\mathcal{P}_{B,1} \subset \mathcal{P}_{B,2} \subset \ldots \subset \mathcal{P}_{B,s}$ of $\mathcal{P}_{B,s}$.

(c) $\tau_A \colon G \to A$ is a pseudo-representation lifting $\operatorname{Tr} \bar\rho$.

(d) For any affine open $\operatorname{Spec} C \subset \operatorname{Spec} B/\mathfrak{m}_A$ over which, disregarding the $G$-action, $\mathcal{P}_{B,1} \subset \mathcal{P}_{B,2} \subset \ldots \subset \mathcal{P}_{B,s}$ is isomorphic to Proj of $C^{d_1} \subset C^{d_1+d_2} \subset \ldots \subset C^d$, the induced action of $G$ on

$$\bigoplus_{i=1,\ldots,s} C^{d_1+\cdots+d_i}/C^{d_1+\cdots+d_{i-1}} \text{ modulo scalars}$$

is conjugate under $\mathrm{GL}_{d_1}(C) \times \cdots \times \mathrm{GL}_{d_s}(C)$ to $\oplus_i \big(\bar\rho_i \otimes_{\mathbb{F}} C\big)$ modulo scalars.

(e) In the notation of (d), for any $i = 1, \ldots, s-1$ there is a well-defined extension class in

$$\operatorname{Ext}^1_{C[G]}(\bar\rho_{i+1} \otimes_{\mathbb{F}} C, \bar\rho_i \otimes_{\mathbb{F}} C)$$

and we assume that its specialization to any closed point $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} C$ is non-trivial.

(f) Let the notation be as in (d) and let $\operatorname{Spec} B_C \subset \operatorname{Spec} B$ denote the pullback of $\operatorname{Spec} C \subset \operatorname{Spec} B/\mathfrak{m}_A B$ —it is affine because $\mathfrak{m}_A B$ is nilpotent. Then under the above hypotheses one can show that

  (i) there exists a unique linear representation $\rho_C \colon G \to \mathrm{GL}_d(C)$ with $\det \rho_C = \det \bar\rho$ and attached projective representation equal to $G \to \operatorname{Aut}_C(\mathcal{P}_B \otimes_B C)$, and

  (ii) there exists a unique linear representation $\widetilde\rho_C \colon G \to \mathrm{GL}_d(B_C)$ with $\widetilde\rho_C \pmod{\mathfrak{m}_A B_C} = \rho_C$, $\det \widetilde\rho_C = \det \tau_A$ and attached projective representation equal to $G \to \operatorname{Aut}_{B_C}(\mathcal{P}_B \otimes_B B_C)$.

In addition to (a)–(e), we also require that $\operatorname{Tr} \widetilde\rho_C = \tau_A$ under $A \to B_C$.

The definition of morphisms on the so-defined objects is left to the reader.

*Remarks* 2.5.5. (i) By a projective bundle we mean Proj of a vector bundle. By a flag of projective bundles we mean that, Zariski locally on the base, there exists a flag of vector bundles (with all factors being again vector bundles) to which Proj associates a flag isomorphic to the given flag of projective bundles.

(ii) As we assume $p > d = \sum_i d_i$, the concept of pseudo-deformation is well behaved. By our hypotheses, $\bar{\rho}$ is multiplicity free in the terminology of [BC1]. Thus, by [Che, Remark 1.28] any pseudo-representation $\tau_A \colon G \to A$ for $A \in \widehat{\mathfrak{A}}_{\mathfrak{r}W(\mathbb{F})}$ and with $\tau_A \otimes_A \mathbb{F} = \operatorname{Tr} \bar{\rho}$ has a well-defined determinant $\det \tau_A \colon G \to A^*$.

(iii) It might be necessary to strengthen the condition $\operatorname{End}_{\mathbb{F}[G]}(\bar{\rho}_i) = \mathbb{F}$, which follows from the absolute irreducibility of $\bar{\rho}_i$, to the condition that the centralizer of the projective representation attached to $\bar{\rho}_i$ is $\mathbb{F}^* \subset \mathrm{GL}_{d_i}(\mathbb{F})$. This notion of projective absolute irreducibility is in general stronger than absolute irreducibility; cf. Exercise 2.8.8. If there exists a subgroup $G_i$ of $G$ such that $\bar{\rho}_i|_{G_i}$ is absolutely irreducible and such that the $d_i$-torsion of $\operatorname{Hom}(G_i, \mathbb{F}^*)$ is trivial, then $\bar{\rho}_i$ is projectively absolutely irreducible.

(iv) We leave the assertions in (f) as an exercise (perhaps a non-trivial one) to the reader. Observe however that from the existence and uniqueness of the local linear representations $\widetilde{\rho}_C$ one cannot deduce the existence of a global linear representation on some vector bundle: the uniqueness of $\widetilde{\rho}_C$ implies that the transition maps on the level of vector bundles are unique only up to units in $B_C^*$. Thus, one can only glue the local patches if the Picard group of the special fiber of $\mathfrak{X}'_{\tau_{\mathbb{F}}}$ is trivial.

The following result will, in some form, be fully explained in the final version of [Ki7].

**Theorem 2.5.6.** *The groupoid* $\mathbf{Rep}'_{\tau_{\mathbb{F}}} \to D_{\tau_{\mathbb{F}}}$ *is representable by the proper formal scheme* $\mathfrak{X}'_{\tau_{\mathbb{F}}}$ *over* $\operatorname{Spf} R_{\tau_{\mathbb{F}}}$.

*Sketch of proof:* Consider the universal object $\big(V_{\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}}, \beta_{\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}}, \rho_{\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}}\big)$ on $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$ together with the universal pseudo-representation $\tau^u$ on $\operatorname{Spf} R_{\tau_{\mathbb{F}}}$ and the universal filtration by sub-vector-bundles

$$V_{\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}} \otimes_{R_{\tau_{\mathbb{F}}}} \mathbb{F} = \bar{V}_s \supset \bar{V}_{s-1} \supset \ldots \supset \bar{V}_1$$

on $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}} \otimes_{R_{\tau_{\mathbb{F}}}} \mathbb{F}$ given by (2.5.1). These objects carry natural actions by $\widehat{\mathbb{G}}_m \subset \widehat{\mathrm{GL}}_d$. The central action of $\widehat{\mathbb{G}}_m$ takes us from vector bundles to projective bundles and a flag of such on the special fiber $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}} \otimes_{R_{\tau_{\mathbb{F}}}} \mathbb{F}$; the center $\widehat{\mathbb{G}}_m$ acts trivially on $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$. The free action of $\widehat{\mathrm{PGL}}_d$ on $\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$ yields the object

$$PV_{\mathfrak{X}'_{\tau_{\mathbb{F}}}} = \Big(V_{\mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}}, \bar{V}_{s-1}, \ldots, \bar{V}_1, \tau^u\Big)/\widehat{\mathrm{GL}}_d$$

in $\mathbf{Rep}'_{\tau_{\mathbb{F}}}(\mathfrak{X}'_{\tau_{\mathbb{F}}} \to \operatorname{Spf} R_{\tau_{\mathbb{F}}})$. Assuming assertion (ii) in Definition 2.5.4(f), it is not hard to see that $PV_{\mathfrak{X}'_{\tau_{\mathbb{F}}}}$ is the universal object for $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ over $\mathfrak{X}'_{\tau_{\mathbb{F}}} \to \operatorname{Spf} R_{\tau_{\mathbb{F}}}$, as follows. Let $PV_B$ be in $\mathbf{Rep}'_{\tau_{\mathbb{F}}}(A \to B)$. Choose an affine cover $\operatorname{Spec} B_i$ of $\operatorname{Spec} B$ such that on each $\operatorname{Spec} B_i$ we have a representation $\widetilde{\rho}_{C_i}$ as in Definition 2.5.4(f) for $PV_B \otimes_B B_i$. The $\widetilde{\rho}_{C_i}$ yield a unique morphism $\operatorname{Spec} B_i \to \mathfrak{X}'^{\square}_{\tau_{\mathbb{F}}}$. The induced morphisms $\operatorname{Spec} B_i \to \mathfrak{X}'_{\tau_{\mathbb{F}}}$ agree on overlaps since by construction they are unique. This shows the universality of $(PV_{\mathfrak{X}'_{\tau_{\mathbb{F}}}}, \mathfrak{X}'_{\tau_{\mathbb{F}}})$. $\square$

## 2.6 Completions of $\mathbf{Rep}'^{\square}_{\tau_{\mathbb{F}}}$

**Proposition 2.6.1.** *Let $x$ be in $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}(\mathbb{F} \to \mathbb{F})$ and let $V_{\mathbb{F}}$ be the corresponding representation of $G$.*

(a) *The canonical surjection $\widehat{R^{\square}_{\tau_{\mathbb{F}}x}} \longrightarrow\!\!\!\!\!\rightarrow R^{\square}_{V_{\mathbb{F}}}$ is an isomorphism.*

(b) *For $x \in \mathbf{Rep}'^{\square}_{\tau_{\mathbb{F}}}(\mathbb{F} \to \mathbb{F})$ the surjection from* (a) *induces an isomorphism $\widehat{\mathcal{O}_{\mathfrak{X}'_{\tau_{\mathbb{F}}}, x}} \twoheadrightarrow R_{V_{\mathbb{F}}}$.*

*Proof.* For part (a) observe that the completion of $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}$ at $x$ can be described as follows: it is given by the functor on $\mathfrak{A}_{\mathfrak{r}W(\mathbb{F})}$ which maps any $A \in \mathfrak{A}_{\mathfrak{r}W(\mathbb{F})}$ to

$$\{(\tau_A, \beta_A, \rho_A) \in \mathbf{Rep}^{\square}_{\tau_{\mathbb{F}}}(A \longrightarrow A) \mid \rho_A \in D^{\square}_{V_{\mathbb{F}}}(A)\}.$$

The datum $\tau_A$ is clearly superfluous and hence this functor is isomorphic to $D^{\square}_{V_{\mathbb{F}}}$. This proves (a). Part (b) follows from the construction of the rings as quotients under the same group action. $\square$

**Corollary 2.6.2.** *Let $E/W(\mathbb{F})[1/p]$ be a finite extension and $x\colon \mathbf{Rep}'_{\tau_{\mathbb{F}}} \to E$ a point*[3] *such that the corresponding $E$-valued pseudo-representation $x$ is absolutely irreducible. Then the map*

$$\mathbf{Rep}'_{\tau_{\mathbb{F}}} \longrightarrow \operatorname{Spf} R_{\tau_{\mathbb{F}}}$$

*is an isomorphism over a formal neighborhood of $\tau_x$.*

*Proof.* We first observe that $x$ is the only point of $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ lying over $\tau_x$. To see this, suppose that $x'$ is another such point. Denote by $V_x$ and $V_{x'}$ the corresponding $G$-representations. By the properness of $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ over $D_{\tau_{\mathbb{F}}}$, the points $x$ and $x'$ arise from $\mathcal{O}_E$-valued points, which in turn correspond to $G$-stable lattices $L_x \subset V_x$ and $L_{x'} \subset V_{x'}$. Since $V_x$ and $V_{x'}$ are absolutely irreducible with the same trace, they are isomorphic. We choose an isomorphism so that it induces a $G$-homomorphism $L_x \to L_{x'}$ whose reduction modulo $\mathfrak{m}_{\mathcal{O}_E}$ is non-zero. The representations $L_x/\mathfrak{m}_{\mathcal{O}_E} L_x$ and $L_{x'}/\mathfrak{m}_{\mathcal{O}_E} L_{x'}$ are both of the type described in (2.5.1). Since the order of the $\rho_i$ is fixed and the extension of $\rho_{I+1}$ by $\rho_i$ is never trivial (see Remark 2.5.2), the semisimplification of the image of $L_x/\mathfrak{m}_{\mathcal{O}_E} L_x$ is of the form $\bar{\rho}_i \oplus \cdots \oplus \bar{\rho}_s$ while the semisimplification of the image as a submodule of $L_{x'}/\mathfrak{m}_{\mathcal{O}_E} L_{x'}$ is of the form $\bar{\rho}_1 \oplus \cdots \oplus \bar{\rho}_j$. Since the $\rho_i$ are pairwise non-isomorphic,

[3] Talking about an $E$-valued point of a formal scheme over $W(\mathbb{F})$ is a slight abuse of notation. What is meant is the point on the generic fiber attached to the formal scheme; see Appendix 2.7.1. On this analytic space one considers the completion of the stalk at $E$.

and the image is non-trivial, we deduce that the morphism of the reductions is an isomorphism. It follows that $L_x \cong L_{x'}$.

Next, let $\widehat{R_{\tau_{\mathbb{F}}}}_{\tau_x}$ be the completion of $R_{\tau_{\mathbb{F}}}$ at $\tau_x$ and $\widehat{\mathcal{O}_{\mathfrak{X}'_{\tau_{\mathbb{F}}},x}}$ be that of $\mathfrak{X}'_{\tau_{\mathbb{F}}}$ at $x$. Denote by $V_{\mathbb{F}}$ the mod $\mathfrak{m}_{\mathcal{O}_E}$ reduction of $L_x$. We need to show that the canonical homomorphism

$$\widehat{R_{\tau_{\mathbb{F}}}}_{\tau_x} \longrightarrow \widehat{\mathcal{O}_{\mathfrak{X}'_{\tau_{\mathbb{F}}},x}} \cong \widehat{\mathcal{O}_{\mathfrak{X}'_{\tau_{\mathbb{F}}},V_{\mathbb{F}}}}_x \overset{2.6.1(b)}{\cong} \widehat{R_{V_{\mathbb{F}}}}_x$$

in $\widehat{\mathfrak{A}_{\mathfrak{r}}}_E$ is an isomorphism. This follows from the theorem of Carayol and Mazur (Theorem 2.2.1), which can easily be adapted to the case of residue field $E$ instead of $\mathbb{F}$. □

*Remark* 2.6.3. Suppose $\bar{\rho}$ is reducible and is the reduction of a $p$-adic representation attached to a modular form $f$. If one wants to study mod $p$ congruences of $f$, then one is interested in all modular forms $f'$ whose attached $p$-adic Galois representation has a mod $p$ reduction $\bar{\rho}'$ whose semisimplification is equal to the semisimplification of $\bar{\rho}$. From this perspective it is natural to consider the deformation space of the pseudo-representation $\tau_{\mathbb{F}} = \operatorname{Tr}\bar{\rho}$.

Next, suppose that the $p$-adic representation $V$ attached to $f$ is absolutely irreducible; for cusp forms this is a natural hypothesis. Suppose further that $V$ is residually multiplicity free. Then we can always find a lattice $L \subset V$ whose mod $p$ reduction $V_{\mathbb{F}}$ satisfies condition (2.5.1) for a suitable ordering of the irreducible constituents of $\bar{\rho}^{\mathrm{ss}}$. The results in this section show that infinitesimally near $V$ the universal pseudo-representation space, the universal deformation of $V_{\mathbb{F}}$ and the completion of $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ at $V$ agree. Moreover, the completion of $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ at $V_{\mathbb{F}}$ is isomorphic to $R_{V_{\mathbb{F}}}$.

Suppose now that $\mathfrak{X}'_{\tau_{\mathbb{F}}}$ has large dimension (as a formal scheme). Then we expect $R_{\tau_{\mathbb{F}}}$ to be highly singular at its closed point: each closed point $V_{\mathbb{F}}'$ of $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ has a different universal ring $R_{V_{\mathbb{F}}'}$; in $R_{\tau_{\mathbb{F}}}$ all these rings are glued together at the special fiber, while irreducible representations $p$-adic deformations lie in only one of the spaces $\operatorname{Spec} R_{V_{\mathbb{F}}'}$. Thus, $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ appears to be a partial desingularization of a part of $R_{\tau_{\mathbb{F}}}$: partial because the rings $R_{V_{\mathbb{F}}'}$ which occur for $V_{\mathbb{F}}'$ in the special fiber of $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ by completion of $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ at $V_{\mathbb{F}}'$ may themselves still be singular. But by gluing them one expects to create a more difficult singularity. Note that in the particular case in which the semisimplification of $\bar{\rho}$ consists of two (non-isomorphic) summands only, one can in fact regard $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ as a partial desingularization of all of $R_{\tau_{\mathbb{F}}}$. In this case, any $p$-adic representation $V$ contains a lattice whose reduction orders the two irreducible constituents in any given order.

To recapitulate the above, in order to study all mod $p$ congruences of a residually reducible representation, one needs to consider $R_{\tau_{\mathbb{F}}}$. However, it is to be expected that its geometry is highly singular at the closed point. A formal scheme with a less singular special fiber that sees many congruences is $\mathfrak{X}'_{\tau_{\mathbb{F}}}$. However, in general it will not contain all representations with congruent mod $p$ Frobenius eigenvalue systems. This can only be guaranteed if the $\bar{\rho}^{\mathrm{ss}}$ is a direct sum of exactly two absolutely irreducible non-isomorphic representations. An alternative, but also highly singular candidate that would in its generic fiber see all $p$-adic representations which have congruent mod $p$ Frobenius eigenvalue systems is the versal hull that arises from the deformation functor of $\bar{\rho}^{\mathrm{ss}}$. Here the reason is that, at least after enlarging the coefficient field, any $p$-adic representation $V$ with residually reducible mod $p$ reduction contains a lattice whose mod $p$ reduction is semisimple.

A second reason for choosing $\mathbf{Rep}'_{\tau_{\mathbb{F}}}$ instead of $D_{\tau_{\mathbb{F}}}$ is that the former groupoid allows it to impose local conditions quite easily in the case where $G$ is the absolute Galois group of a global field. This can be done as in the case of $D_{V_{\mathbb{F}}}$, which we shall discuss in the following three sections. For $D_{\tau_{\mathbb{F}}}$ it is perhaps slightly more difficult to impose and study local conditions.

## 2.7 Appendix

### 2.7.1 Formal schemes

In this appendix we recall the definition of a Noetherian formal scheme. In the following, we fix a Noetherian ring $R$ and an ideal $\mathfrak{a}$ of $R$. We assume that $R$ is $\mathfrak{a}$-adically complete, i.e., that the canonical homomorphism

$$R \longrightarrow \widehat{R}_{\mathfrak{a}} = \varprojlim_n R/\mathfrak{a}^n$$

to the $\mathfrak{a}$-adic completion of $R$ is an isomorphism.

We define a topological space $\operatorname{Spf} R$ (with respect to $\mathfrak{a}$), which will soon also be given a structure sheaf:

- A prime ideal $\mathfrak{p}$ of $R$ is called *open* (with respect to $\mathfrak{a}$) if $\mathfrak{p}$ contains $\mathfrak{a}$.
- The underlying set of the topological space $\operatorname{Spf} R$ consists of the open prime ideals of $R$, so that it is in bijection with $\operatorname{Spec} R/\mathfrak{a}$.
- The topology on $\operatorname{Spf} R$ is the topology induced from the bijection between $\operatorname{Spf} R$ and $\operatorname{Spec} R/\mathfrak{a}$.

For instance, if $R \in \mathfrak{A}_{\mathfrak{r}W(\mathbb{F})}$ and $\mathfrak{a} = \mathfrak{m}_R$ is the maximal ideal of $R$, then $\operatorname{Spf} R$ consists of a single point.

To define a structure sheaf on $\operatorname{Spf} R$, let us recall the Zariski topology on $\operatorname{Spec} R/\mathfrak{a}$: for $f \in R$ denote by $\bar{f}$ its image in $\bar{R} := R/\mathfrak{a}$. Define $D(\bar{f})$ as the set of prime ideals $\bar{\mathfrak{p}}$ of $\bar{R}$ such that $\bar{f}$ is non-zero at $\bar{\mathfrak{p}}$. This set is in bijection with the set of open prime ideals $\mathfrak{p}$ of $R$ at which $f$ is non-zero (under reduction). The sets $D(\bar{f})$ define a basis for the topology on Spf.

Now, for $f \in R$ define

$$R\langle f^{-1}\rangle = \varprojlim R[f^{-1}]/\mathfrak{a}^n.$$

It is not difficult to verify that the assignment $D(\bar f) \mapsto R\langle f^{-1}\rangle$ defines a sheaf on $\operatorname{Spf} R$. (The main task is to verify the sheaf property by coverings of an open subset $D(\bar f)$ by sets $D(\bar f_i)$.)

Let us see that the sheaf defined above is locally ringed, i.e., that its stalks are local rings. Suppose for this that $x \in \operatorname{Spf} R$ corresponds to the open ideal $\mathfrak{p} = \mathfrak{p}_x$ of $R$. Then the stalk at $x$ is

$$\mathcal{O}_x = \varinjlim_{x \in D(\bar f)} R\langle f^{-1}\rangle.$$

It is a good exercise to show that $\mathcal{O}_x$ is a local ring with maximal ideal $\mathfrak{p}\mathcal{O}_x$. (To see the latter one needs that $\mathfrak{p}$ be finitely generated, which is true in our case since $R$ is Noetherian.)

**Definition 2.7.1.** The *formal scheme* $\operatorname{Spf} R$ of $R$ (with respect to $\mathfrak{a}$) is the locally ringed space $(X, \mathcal{O}_X)$ where $X = \operatorname{Spec} R/\mathfrak{a}$ as a topological space and the structure sheaf $\mathcal{O}_X$ is defined by $\mathcal{O}_X(D(\bar f)) = R\langle f^{-1}\rangle$, for all $f \in R$.

We ignore all subtleties necessary for the definition of non-Noetherian formal schemes.

**Example 2.7.2.** Let $X = \mathrm{PGL}_d/W(\mathbb{F})$. Its affine coordinate ring consists of the set of homogeneous rational functions of degree zero in the ring

$$R = W(\mathbb{F})\big[X_{i,j}, \det((X_{i,j})^{-1}) \mid i,j = 1,\dots,d\big].$$

Consider the morphism $\pi_{\mathrm{id}}\colon R \to \mathbb{F}$, $X_{i,j} \mapsto \delta_{i,j}$ corresponding to the identity element of $\mathrm{PGL}_d(\mathbb{F})$. The completion of $\mathrm{PGL}_d$ along the kernel of $\pi_{\mathrm{id}}$ is a Noetherian affine formal scheme, denoted by $\widehat{\mathrm{PGL}}_d$ in the proof of Theorem 2.1.1.

**Definition 2.7.3.** A *Noetherian formal scheme* is a locally topologically ringed space $(X, \mathcal{O}_X)$ such that each point $x \in X$ admits an open neighborhood $U$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to an affine formal scheme $\operatorname{Spf} R$, as defined above.

Morphisms of formal schemes are morphisms of topologically ringed spaces. So a morphism from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^{\#})$ where $f\colon X \to Y$ is a morphism of topological spaces and $f^{\#}\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a continuous homomorphism of sheaves, i.e., locally such a homomorphism is given as a ring homomorphism $A \to B$ mapping a power of the defining ideal $\mathfrak{a}$ of $A$ into the defining ideal $\mathfrak{b} \subset B$.

A particular construction of a formal scheme is the following. Let $X$ be a scheme and $\mathcal{I} \subset \mathcal{O}_X$ be an ideal sheaf. Then the completion of $X$ along $\mathcal{I}$ is a formal scheme whose underlying topological space is the closed subscheme $Z$ of $X$ defined by $\mathcal{I}$. On an affine cover of $Z$ one applies the construction indicated in page 38, and then one glues the so obtained formal affine schemes. We write $\widehat{X}$ for it, or $\widehat{X}^{\mathcal{I}}$ if the need arises to indicate the ideal sheaf. A formal scheme constructed in this way is called *algebraizable*.

**Example 2.7.4.** Let $R$ be in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$. Consider the closed immersion $\mathbb{A}^1_{\mathbb{F}} \to \mathbb{A}^1_R$ obtained by pulling back $\mathbb{A}^1_{\mathbb{F}}$ along $\operatorname{Spec}\mathbb{F} \to \operatorname{Spec} R$. Let $\widehat{\mathbb{A}}_R$ be the formal scheme obtained as completion along this closed immersion. Its underlying topological space is $\mathbb{A}_{\mathbb{F}}$. However, its structure sheaf can be quite enormous. For instance, its ring of global sections is $R\langle x\rangle = \varprojlim R[x]/\mathfrak{m}_R^n[x]$. To describe the latter ring, define $\operatorname{ht}(r) = \max\{i \in \mathbb{N} \mid r \notin \mathfrak{m}_R^i\}$. Then $R\langle x\rangle$ is the subring of $R[[x]]$ of series $\sum r_i x^i$ such that $\operatorname{ht}(r_i) \to \infty$ for $i \to \infty$.

**Example 2.7.5.** Let $R$ be in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ and consider the projective scheme $\mathbb{P}^1_R \cong \mathbb{P}^1_{\mathbb{Z}} \times_{\mathbb{Z}} \operatorname{Spec} R$ over $\operatorname{Spec} R$. The maximal ideal of $R$ defines an ideal sheaf on $\operatorname{Spec} R$ and via pullback an ideal sheaf on $\mathbb{P}^1_R$. The completion of $\mathbb{P}^1_R$ along this pullback is a formal scheme with a natural homomorphism to Spf $R$. It is the formal projective line over Spf $R$. It can be obtained by gluing two copies of $\widehat{\mathbb{A}}_R$ from the previous example along $\widehat{\mathbb{G}}_{m,R}$. Carry out the construction in detail to make sure that you fully understand the corresponding formal scheme and the morphism of formal schemes.

### The generic fiber of a formal scheme over $W(\mathbb{F})$

Given a formal scheme $\mathfrak{X}$ with a morphism to Spf $W(\mathbb{F})$, one can, following Berthelot, associate a rigid space over $W(\mathbb{F})[1/p]$ to it. The detailed construction can be found in [deJ, §7]. Let us give the idea for Spf $R$ with $R \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$. Suppose first that $R = W(\mathbb{F})[[X_1, \ldots, X_k]]$. Then

$$\begin{aligned} W(\mathbb{F})[1/p]\{X_1, \ldots, X_k\} \subset W(\mathbb{F})[[X_1, \ldots, X_k]] \otimes_{W(\mathbb{F})} W(\mathbb{F})[1/p] \\ \subset (W(\mathbb{F})[1/p])[[X_1, \ldots, X_k]], \end{aligned}$$

where on the left we have the Tate algebra over $W(\mathbb{F})[1/p]$, i.e., the ring of those power series which converge on the closed disc $\mathcal{O}^k_{\mathbb{C}_p}$ of radius one. The ring in the middle consists of power series whose coefficients have uniformly bounded norm. These converge on the "open" unit disc of dimension $k$ of radius one, i.e., on

$$\mathring{\mathcal{O}}^k_{\mathbb{C}_p} = \{(x_1, \ldots, x_k) \in \mathcal{O}^k_{\mathbb{C}_p} \mid |x_i| < 1 \text{ for } i = 1, \ldots, k\}.$$

It is a rigid analytic space. The affinoid discs of radius $1 - \frac{1}{n}$ around 0 form an admissible cover.

A general $R \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ can be written as

$$R = W(\mathbb{F})[[X_1, \ldots, X_k]]/(f_1, \ldots, f_m).$$

Then the zero locus of the functions $f_j$, $j = 1, \ldots, m$, defines a rigid analytic subspace of the open unit disc. This will be the rigid analytic space $(\mathrm{Spf}\, R)^{\mathrm{rig}}$ associated to Spf $R$. As a next example, one could work out the case of $\widehat{\mathbb{P}^1}_R$ and show that the associated rigid analytic space is the projective line over $\mathring{\mathcal{O}}^k_{\mathbb{C}_p}$.

**Functors on formal schemes**

Representability of functors is also an important question for formal schemes. Schlessinger's representability criterion (Theorem 1.7.2) —or the theorem of Grothendieck behind it— can be regarded as a theorem on the representability of formal schemes: Schlessinger's criterion studies the pro-representability of a covariant functor $\mathfrak{Ar}_{W(\mathbb{F})} \to \mathbf{Sets}$ by an object in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$.

Formally passing to opposite categories, we obtain a functor from $(\mathfrak{Ar}_{W(\mathbb{F})})^o$ to $\mathbf{Sets}^o$. The category $(\mathfrak{Ar}_{W(\mathbb{F})})^o$ is the category of formal Artin schemes on one point over $\operatorname{Spf} W(\mathbb{F})$ with residue field $\operatorname{Spf} \mathbb{F}$. In this sense, Schlessinger's criterion provides necessary and sufficient conditions for a functor on such formal Artin schemes (to **Sets**) to be representable by a Noetherian formal scheme in $(\widehat{\mathfrak{Ar}}_{W(\mathbb{F})})^o$.

## 2.7.2 Pseudo-representations according to Wiles

The first occurrence of pseudo-representations in the theory of Galois representations was in the work of Wiles [Wi1] for 2-dimensional odd Galois representations of the absolute Galois group of a number field. His definition appears to be different from Definition 2.2.2. In the presence of a complex conjugation whose image is $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, Wiles' definition can easily seen to be equivalent to Definition 2.2.2 for $d = 2$.

**Definition 2.7.6.** A *pseudo-representation in the sense of Wiles* consists of continuous functions $a, d\colon G \to R$ and $x\colon G \times G \to R$ such that for all $g, g', h, h' \in G$ one has

(a) $a(gh) = a(g) + a(h) + x(g,h), \quad d(gh) = d(g) + d(h) + x(h,g),$

$$\begin{aligned} x(gh, g'h') = a(g)a(h')x(h,g') + a(h')d(h)x(g,g') + a(g)d(g')x(h,h') \\ + d(h)d(g')x(g,h'). \end{aligned}$$

(b) $a(1) = d(1) = 1$, $x(1,h) = x(g,1) = 1$ for all $g, h \in G$.

(c) $x(g,h)x(g',h') = x(g,h')x(g',h)$.

**Proposition 2.7.7.** *Suppose that* $\rho\colon G \to \mathrm{GL}_2(R)$, $g \mapsto \left(\begin{smallmatrix} a(g) & b(g) \\ c(g) & d(g) \end{smallmatrix}\right)$ *is a continuous representation. Then*

(a) $(a, d, x(g,h) = b(g)c(h))$ *forms a pseudo-representation in the sense of Wiles.*

(b) *If there exists* $c \in G$ *such that* $\rho(c) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, *then the pseudo-representation from* (a) *depends only on the trace of* $\rho$, *because*

$$\begin{aligned} a(g) &= \frac{1}{2}\big(\operatorname{Tr}\rho(g) - \operatorname{Tr}\rho(gc)\big), \\ d(g) &= \frac{1}{2}\big(\operatorname{Tr}\rho(g) + \operatorname{Tr}\rho(gc)\big), \\ x(h,g) &= a(gh) - a(g)a(h). \end{aligned}$$

Note that part (a) is in fact the motivation for the axioms of a pseudo-representation in the sense of Wiles. They simply arise from the rules for multiplying two matrices.

## 2.8 Exercises

*Exercise* 2.8.1. (a) Show that $\widehat{\mathrm{PGL}}_d(A) = \mathrm{Ker}(\mathrm{PGL}_d(A) \mapsto \mathrm{PGL}_d(\mathbb{F}))$ for all $A \in \mathfrak{Ar}_{W(\mathbb{F})}$ —cf. Example 2.7.2.

(b) Verify that the morphism (2.1.1) on page 22 is a closed immersion by using the following criterion (which is actually not hard to prove): a morphism of affine formal schemes $\mathrm{Spf}\, A = X \to Y = \mathrm{Spf}\, B$ is a closed immersion if and only if it is a monomorphism (of functors); cf. [SGA3, VIIA.1.3].

(c) Verify that the quotient constructed in the proof of Theorem 2.1.1 does indeed represent $R_{V_\mathbb{F}}$.

*Exercise* 2.8.2. Let $p > 2$, let $G = \mathbb{Z}_p$ considered as an additive profinite group and let $\tau_\mathbb{F} \colon G \to \mathbb{F}$, $g \mapsto 2$ be the trivial 2-dimensional pseudo-representation.

(a) Show that the tangent space of $D_{\tau_\mathbb{F}}$ is at most 2-dimensional. (*Hint:* Deduce from condition (b) of Definition 2.2.2 for $d = 2$ that any $\tau_2 \in D_{\tau_\mathbb{F}}(\mathbb{F}[\varepsilon])$ is determined by its values on 1 and 2; use the continuity of $\tau_2$.)

(b) Consider $\rho \colon \mathbb{Z}_p \to \mathrm{GL}_2(W(\mathbb{F})[[X,Y]])$, $z \mapsto \begin{pmatrix} 1+X & 1+X \\ Y & 1+Y \end{pmatrix}^z$. By studying $\mathrm{Tr}(\rho)$, show that $W(\mathbb{F})[[X,Y]]$ is a quotient of the universal pseudo-representation ring for $\tau_\mathbb{F}$.

(c) Prove that $\mathrm{Tr}(\rho)$ is the universal pseudo-deformation of $\tau_\mathbb{F}$.

(d) Prove that $\rho$ is the universal deformation of $\bar\rho := \rho \bmod (X,Y)$ —despite the fact that the representability criterion of Proposition 1.3.1 fails

*Exercise* 2.8.3 ([Ki6, §1.4]). (a) Give an example where $V_\mathbb{F}$ is not absolutely irreducible and there exist non-isomorphic deformations $V_A, V'_A \in D_{V_\mathbb{F}}(A)$ with the same traces. (*Hint:* Consider two characters $\chi_1, \chi_2 \colon G \to \mathbb{F}^\times$ with $\dim_\mathbb{F} \mathrm{Ext}^1(\chi_2, \chi_1) > 1$.)

(b) Show that if $\chi_1, \chi_2 \colon G \to \mathbb{F}^\times$ are distinct characters such that $\mathrm{Ext}^1(\chi_2, \chi_1)$ is 1-dimensional and $V_\mathbb{F}$ is a non-trivial extension of $\chi_1$ by $\chi_2$, then the analogue of Carayol's theorem holds for $V_\mathbb{F}$: two deformations in $D_{V_\mathbb{F}}(A)$ are non-isomorphic if and only if their traces are different.

*Exercise* 2.8.4. Show that if $\Phi \colon \mathfrak{F} \to \mathfrak{F}'$ is a relatively representable morphism of groupoids over $\mathfrak{C}$, and $\mathfrak{F}'$ is representable, then so is $\mathfrak{F}$.

*Exercise* 2.8.5. A morphism of groupoids $\Phi \colon \mathfrak{F} \to \mathfrak{F}'$ over $\mathfrak{C}$ is formally smooth if the induced morphism of functors $|\Phi| \colon |\mathfrak{F}| \to |\mathfrak{F}'|$ is formally smooth, i.e., if for any surjective morphism $T \to S$ in $\mathfrak{C}$, the map

$$|\mathfrak{F}|(T) \longrightarrow |\mathfrak{F}|(S) \times_{|\mathfrak{F}'|(S)} |\mathfrak{F}'|(T)$$

is surjective. Show that $\Phi\colon \mathfrak{F} \to \mathfrak{F}'$ is formally smooth if and only if for every $\eta \in \mathfrak{F}'$ the morphism $|\mathfrak{F}_\eta| \to |\eta|$ is formally smooth.

*Exercise* 2.8.6. Prove Proposition 2.4.5. (*Strategy:* To establish relative representability, fix $A_0 \in \mathfrak{Ar}_{W(\mathbb{F})}$, $B_0$ some $A$-algebra and $\tau_0\colon G \to A_0$ a pseudo-representation lifting $\tau_{\mathbb{F}}$. This defines an element $\eta$ in $D_{\tau_{\mathbb{F}}}(A_0 \to B_0)$. Describe the groupoid $\mathbf{Rep}^{\square}_{\tau_{\mathbb{F}},\eta}$. Show that all representations described by it are representations of a fixed finite quotient $\overline{G}$ of the originally given group $G$ —the quotient depends on $\eta$. It suffices to consider the case $B_0 = A_0$. Then write down the universal object for $\overline{G}$ in a way similar to the proof of Proposition 2.3.1 or Proposition 1.3.1(a). The wanted formal scheme is obtained by an inverse limit of such situations. To see that the morphism is of finite type, it suffices to consider the case $A_0 = B_0 = R_{\bar{\eta}}/\mathfrak{m}^2_{R_{\bar{\eta}}}$.)

*Exercise* 2.8.7. (a) Show that for $A \in \mathfrak{Ar}_{W(\mathbb{F})}$ one has $\mathbb{A}^1_A \cong \widehat{\mathbb{A}^1_A}$ and $\mathbb{P}^1_A \cong \widehat{\mathbb{P}^1_A}$ as locally ringed topological spaces.

(b) Work out all details in Examples 2.7.4 and 2.7.5.

*Exercise* 2.8.8. Let $p$ be the characteristic of an algebraically closed field $k$. Fix $n > 4$ prime to $p$ and denote by $D_{2n}$ the dihedral group of order $2n$. Consider the representation $\rho\colon D_{2n} \to \mathrm{GL}_2(k)$ sending a generator of the rotations in $D_{2n}$ to $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ for $\zeta \in k$ a primitive $n$-th root of unity and a reflection to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Show that $\rho$ is absolutely irreducible but not projectively absolutely irreducible.

# Lecture 3

# Deformations at places not above $p$ and ordinary deformations

The material on generic fibers is from [Ki4, §2.3]. The results on deformations at $\ell \neq p$ can either be found in the lecture notes [Pil], in [KW2, §3.3] or in [Ki4, §2.6]. For the last section on ordinary deformations we refer to [Ki5, §2.4] or [KW2, §3.2].

In the appendix to this chapter we recall various topics used in the main body. We recall the canonical subgroups of the absolute Galois group of a local field, we present basic results on Galois cohomology, we give a short introduction to Weil–Deligne representations and we provide some basic results on finite flat group schemes.

## 3.1 The generic fiber of a deformation functor

A deformation functor $D$ may have difficult singularities at its closed point. If the functor is representable, this means that the corresponding universal ring $R$ is highly singular at its maximal ideal. However, in many concrete and important situations it turns out that the generic fiber of the formal scheme $\mathrm{Spf}\, R^{\mathrm{rig}}$ associated with $R$ (see Appendix 2.7.1) has no singularity or at most very mild singularities. Moreover, in the cases we have in mind, closed points on the generic fiber are of utmost interest: a closed point of $\mathrm{Spf}\, R^{\mathrm{rig}}$ is a $W(\mathbb{F})$-algebra homomorphism $R \to E$ for some finite extension $E$ of $W(\mathbb{F})[1/p]$. Thus if $R = R^{\square}_{V_{\mathbb{F}}}$, as in Lecture 1, such points are precisely the $p$-adic representations $G \to \mathrm{GL}_d(E)$ which possess a conjugate $G \to \mathrm{GL}_d(\mathcal{O}_E)$ whose mod $\mathfrak{m}_{\mathcal{O}_E}$-reduction is $\bar{\rho}$. A similar interpretation holds for the closed points of $(\mathrm{Spf}\, R_{V_{\mathbb{F}}})^{\mathrm{rig}}$. Using functors on Artin rings (over finite extensions of $\mathbb{W}[1/p]$), one cannot recover $\mathrm{Spf}\, R^{\mathrm{rig}}$. However, given any closed point $\xi$ on Spf $R^{\mathrm{rig}}$, starting from $D$ one can construct such a functor $D_\xi$ which describes the infinitesimal neighborhood of $\xi$ on the generic fiber. Moreover, in many concrete examples, this functor can be easily written down explicitly; cf. Theorem 3.1.2. In particular, this often gives a simple way to compute the tangent space at such a point, e.g. Remark 3.1.3, and to check for smoothness. Let me finish this introduction by giving one example why it should be simpler to work on the generic fiber. Suppose $G$ is a finite group. Then the representation theory of $G$ over a field of characteristic zero, as is any finite extension of $\mathbb{Q}_p$, is completely dominated by Wedderburn's theorem. It says that the abelian category of finite-dimensional representations is semisimple. On the other hand, if $p$ divides the order of $G$, then the category of finite $A[G]$-modules for any finite $\mathbb{Z}_p$-algebra is a rather complicated object.

The above observations regarding the generic fiber have been exploited crucially by Kisin in several instances, e.g. [Ki4]. In this section, we briefly recall Kisin's constructions and some basic results. We shall consistently work with groupoids.

## Groupoids for closed points on the generic fiber

Let $E$ be a finite extension of $W(\mathbb{F})[1/p]$ with ring of integers $\mathcal{O}_E$. Define $\mathfrak{Ar}_E$ as the category of finite, local $W(\mathbb{F})[1/p]$-algebras $B$ with residue field $E$. Since $B$ is a finite $W(\mathbb{F})[1/p]$-module, the residue homomorphism $\pi\colon B \to E$ is canonically split[1] and thus $B$ is an $E$-algebra in a canonical way.

For $B \in \mathfrak{Ar}_E$ denote by $\mathbf{Int}_B$ the category of finite $\mathcal{O}_E$-subalgebras $A \subset B$ such that $A[1/p] = B$. The morphisms in $\mathbf{Int}_B$ are the natural inclusions. The category $\mathbf{Int}_B$ is ordered by inclusion and filtering, i.e., any two objects are contained in a third one. For the (filtered) direct limit of the $A \in \mathbf{Int}_B$ one obtains

$$\varinjlim_{A\in\mathbf{Int}_B} A = \pi^{-1}(\mathcal{O}_E).$$

The limit is taken in the category of rings.

Define $\widehat{\mathfrak{Ar}}_{W(\mathbb{F}),\mathcal{O}_E}$ as the category of $W(\mathbb{F})$-algebra homomorphisms $A \to \mathcal{O}_E$, where $A$ lies in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$, as $A[1/p]$ need not be Artinian. In particular, $\mathbf{Int}_B$ may be regarded as a subcategory of $\widehat{\mathfrak{Ar}}_{W(\mathbb{F}),\mathcal{O}_E}$ for all $B \in \mathfrak{Ar}_E$. If $E$ is totally ramified over $W(\mathbb{F})[1/p]$, so that $\mathcal{O}_E$ has residue field $\mathbb{F}$, then the last assertion is true in the obvious sense. Otherwise one proceeds as follows. Denote by $\pi_{\mathcal{O}_E} : \mathcal{O}_E \to \mathcal{O}_E/\mathfrak{m}_{\mathcal{O}_E}$ the homomorphism to the residue field, which is a finite extension of $\mathbb{F}$ and define $\mathcal{O}' \subset \mathcal{O}_E$ as the inverse image $\pi_{\mathcal{O}_E}^{-1}(\mathbb{F})$. Then, given $(A \subset B) \in \mathbf{Int}_B$, the pair $(A \cap \pi^{-1}(\mathcal{O}') \subset B)$ lies in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F}),\mathcal{O}_E}$.

[1] Lacking a reference, here is an indication of proof. The key is that $E$ is algebraic over $W(\mathbb{F})[1/p]$; so, for $x \in E$, consider its minimal polynomial $f$ over the other field. Pick an arbitrary lift to $B$. Use the Newton method to find the unique lift which is a root of $f$. This defines a canonical lift.

Let $\mathfrak{F}$ be a groupoid over $\mathfrak{Ar}_{W(\mathbb{F})}$. Extend it canonically to $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ (so that $\mathfrak{F}$ has some continuity property with regard to inverse limits). Fix $\xi \in \mathfrak{F}(\mathcal{O}_E)$ and define a groupoid on $\widehat{\mathfrak{Ar}}_{W(\mathbb{F}),\mathcal{O}_E}$ by

$$\mathfrak{F}_{(\xi)}(A \xrightarrow{\alpha} \mathcal{O}_E) = \{\eta \in \mathfrak{F}(A) \mid \eta \longrightarrow \xi \text{ lies over } A \xrightarrow{\alpha} \mathcal{O}_E\}$$

for $(A \xrightarrow{\alpha} \mathcal{O}_E) \in \widehat{\mathfrak{Ar}}_{W(\mathbb{F}),\mathcal{O}_E}$, i.e., we consider deformations of $\xi$ to objects $A \to \mathcal{O}_E$ with $A$ still in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$. This groupoid gives rise to a groupoid over $\mathfrak{Ar}_E$, again denoted $\mathfrak{F}_{(\xi)}$, by setting

$$\mathfrak{F}_{(\xi)}(B) = \varinjlim_{A \in \mathbf{Int}_B} \mathfrak{F}_{(\xi)}(A)$$

for $B \in \mathfrak{Ar}_E$.

**Lemma 3.1.1.** *Suppose $\mathfrak{F}$ is pro-represented by a complete local $W(\mathbb{F})$-algebra $R$ and $\xi$ is given by $\alpha\colon R \to \mathcal{O}_E$. Then the groupoid $\mathfrak{F}_{(\xi)}$ on $\mathfrak{Ar}_E$ is pro-represented by the complete local $W(\mathbb{F})[1/p]$-algebra $\widehat{R_\xi}$ obtained by completing $R \otimes_{W(\mathbb{F})} E$ along*

$$I_\xi = \operatorname{Ker}(R \otimes_{W(\mathbb{F})} E \xrightarrow{\alpha \otimes_{W(\mathbb{F})} E} E)\ .$$

*Proof.* When discussing representability of groupoids, we observed that a groupoid is representable if and only if isomorphic objects over the identity are isomorphic via a unique isomorphism and the functor $|\mathfrak{F}|$ is representable (see the comments below Definition 1.6.4). Using this fact, the lemma is a simple exercise left to the reader. □

## Application to the generic fiber of $D_{V_\mathbb{F}}$ and $D^\square_{V_\mathbb{F}}$

To see a first example for the usefulness of $\mathfrak{F}_{(\xi)}$, we consider the case $\mathfrak{F} = D^\square_{V_\mathbb{F}}$ (or $\mathfrak{F} = D_{V_\mathbb{F}}$). We define two groupoids related to $\xi = (V_{\mathcal{O}_E}, \beta_{\mathcal{O}_E}) \in D^\square_{V_\mathbb{F}}(\mathcal{O}_E)$. Set $V_\xi = V_{\mathcal{O}_E} \otimes_{\mathcal{O}_E} E$, which is a continuous representation of $G$, and the basis $\beta_{\mathcal{O}_E}$ canonically extends to a basis $\beta_\xi$ of $V_\xi$. Define $D_{V_\xi}$ and $D^\square_{V_\xi}$ as groupoids on $\mathfrak{Ar}_E$ as follows. For any $B \in \mathfrak{Ar}_E$, let $D_{V_\xi}(B)$ denote the set of deformations of $V_\xi$ to a free $B$-module $V_B$ with a continuous $G_K$-action. Similarly, let $D^\square_{V_\xi}(B)$ denote the set of deformations of $(V_\xi, \beta_\xi)$ to a free $B$-module $V_B$ with a continuous $G_K$-action together with a choice of basis $\beta_B$ lifting $\beta_\xi$.

**Theorem 3.1.2.** *There are natural isomorphisms of groupoids over $\mathfrak{Ar}_E$*

$$D_{V_\mathbb{F},(\xi)} \longrightarrow D_{V_\xi} \ \textit{ and } \ D^\square_{V_\mathbb{F},(\xi)} \longrightarrow D^\square_{V_\xi}.$$

*Proof.* We sketch the proof in the first case (in fact, we shall only sketch the proof for the functors associated to the groupoids). We begin by defining the natural transformation. Let $B$ be in $\mathfrak{Ar}_E$. Then an element in $D_{V_\mathbb{F},(\xi)}(B)$ is an element in

$$\varinjlim_{A \in \mathbf{Int}_B} D_{(\xi)}(A),$$

where in turn an element of $D_{(\xi)}(A)$ is a continuous $G_K$-representation on a free $A$-module $V_A$ together with an isomorphism $V_A \otimes_A \mathcal{O}_E \cong V_{\mathcal{O}_E}$ under the homomorphism $A \to \mathcal{O}_E$ for $A \in \mathbf{Int}_B$. So an element of $D_{V_{\mathbb{F}},(\xi)}(B)$ is a direct system of $(V_A)_{A\in\mathbf{Int}_B}$ of such. We have observed earlier that $\varinjlim_{A\in\mathbf{Int}_B} A = \pi^{-1}(\mathcal{O}_E)$ for $\pi\colon B \to E$ the structure map of $B$. Hence $\left(\varinjlim_{A\in\mathbf{Int}_B} V_A\right) \otimes_{\pi^{-1}(\mathcal{O}_E)} B$ defines an element in $D_{V_\xi}(B)$.

To prove surjectivity, suppose $V_B \in D_{V_\xi}(B)$. Since $V_\xi$ arises from $V_{\mathcal{O}_\xi}$ via $-\otimes_{\mathcal{O}_E} E$, the representation $V_B$ contains a natural subrepresentation $V_{\pi^{-1}(\mathcal{O}_E)}$ on a free finitely generated $\pi^{-1}(\mathcal{O}_E)$-module. Since the $A \in \mathbf{Int}_B$ exhaust $\pi^{-1}(\mathcal{O}_E)$ and since the action of $G$ is continuous, and $B$ is Artinian, we can find $A \in \mathbf{Int}_B$ and a subrepresentation $V_A \subset V_{\pi^{-1}(\mathcal{O}_E)}$ which is free as an $A$-module and with a canonical homomorphism onto $V_{\mathcal{O}_E}$. This completes the proof of essential surjectivity. The proof of injectivity is left as an exercise; see [Ki4, (2.3.5)]. □

*Remark* 3.1.3. One has the isomorphism $D_{V_\xi}(E[\varepsilon]) \cong H^1(G, \mathrm{ad}V_\xi)$ for the tangent space of $D_{V_{\mathbb{F}},\xi}$, i.e., for that of $\operatorname{Spec} R_{V_{\mathbb{F}}}[1/p]$ at $\xi$.

## 3.2 Weil–Deligne representations

Let $F$ be a finite extension of $\mathbb{Q}_\ell$ with uniformizer $\pi$ and residue field $k$. Set $q = \#k$. Consider the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_F & \longrightarrow & G_F & \longrightarrow & \hat{\mathbb{Z}} \cong \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & I_F & \longrightarrow & W_F & \xrightarrow{n} & \mathbb{Z} \cong \langle \mathrm{Frob}_k \rangle & \longrightarrow & 1 \end{array}$$

defining the Weil group $W_F$, where the bottom row is the pullback of the top row along the inclusion on the right. We regard $n$ as a homomorphism $W_F \to \mathbb{Z}$ and fix an inverse image $\sigma \in W_F$ of $1 \in \mathbb{Z}$ (where 1 is identified with $\mathrm{Frob}_k$).

Consider a compatible system $\sqrt[p^n]{\pi}$ of $p$-power roots of $\pi$. This induces an isomorphism from $\mathbb{Z}_p$ to the Galois group of $\bigcup_n F(\zeta_{p^n}, \sqrt[p^n]{\pi})$ over $\bigcup_n F(\zeta_{p^n})$. The action of $G_F$ on this copy of $\mathbb{Z}_p$ is via the cyclotomic character and whence we write $\mathbb{Z}_p(1)$. The action of $I_F$ on this compatible system defines a surjective $G_F$-equivariant homomorphism

$$t_p\colon I_F \longrightarrow \mathbb{Z}_p(1).$$

From the above and standard results about tame ramification of local fields one deduces the isomorphism

$$(\widehat{G_F})^p \xrightarrow{\cong} \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$$

for the pro-$p$ completion of $G_F$ with the $\mathbb{Z}_p$ on the right being the pro-$p$ completion of $\widehat{\mathbb{Z}} = \mathrm{Gal}(\overline{k}/k)$. More explicitly, one can find $t \in \mathbb{Z}_p(1)$ and $s \in (\widehat{G_F})^p$ mapping to Frobenius in $\mathbb{Z}_p$, such that $(\widehat{G_F})^p$ is the pro-$p$-completion of

$$\langle s, t \mid sts^{-1} = t^q \rangle. \tag{3.2.1}$$

Let $E/\mathbb{Q}_p$ be a finite extension and $\rho$ a continuous representation of $G_F$ on a finite-dimensional $E$-vector space $V$.

**Theorem 3.2.1** (Grothendieck)**.** *There exists a unique nilpotent $N \in \mathrm{End}(V)$, the* logarithm of monodromy*, and a finite index subgroup $I_1$ of $I_F$ such that, for all $g \in I_1$,*

$$\rho(g) = \exp(t_p(g)N).$$

One can verify that for $g \in I_F$ and $n \in \mathbb{Z}$ one has $\rho'(\sigma^n g)N = q^n N \rho'(\sigma^n g)$.

*Proof.* We indicate the construction of $N$. By continuity of $\rho$, one can find a free $\mathcal{O}_E$-submodule $\Lambda \subset V$ with $V = \Lambda[1/p]$ which is preserved under $G_F$. Let $F' \supset F$ be the fixed field of the kernel of the representation $G_F \to \mathrm{GL}_d(\Lambda/2p\Lambda)$ induced from $\rho$. The kernel of $\mathrm{GL}_d(\Lambda) \to \mathrm{GL}_d(\Lambda/2p\Lambda)$ is a pro-$p$ group. Thus the action of $G_{F'}$ on $V$ is via its pro-$p$ completion. Denote by $A_s$ and $A_t$ the matrices of the images of $s, t$ from (3.2.1) for the field $F'$. The relation $sts^{-1} = t^q$ implies that $A_t$ and $(A_t)^q$ have the same eigenvalues. Thus, the finite set of eigenvalues is invariant under $\lambda \mapsto \lambda^q$, and so its elements must be roots of unity. Since $\overline{\mathbb{Z}}_p \to \overline{\mathbb{Z}}_p/2p\overline{\mathbb{Z}}_p$ is injective on roots of unity, all eigenvalues of $A_t$ must be one. Define $N$ as the logarithm of the nilpotent endomorphism $A_t - \mathrm{id}_V$. Then the assertion of the theorem holds for $I_1 = I_{F'}$. □

The above result yields immediately the following bijection.

**Corollary 3.2.2.** *There is a bijection between isomorphism classes of representations $\rho\colon G_F \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ and isomorphism classes of pairs $(\rho', N)$ (Weil–Deligne representations; cf. Appendix* 3.9.3*) such that*

(a) *$\rho'\colon W_F \to \mathrm{GL}_d(\overline{\mathbb{Q}}_p)$ is a continuous representation with the discrete topology on $\overline{\mathbb{Q}}_p$;*

(b) *$N \in M_d(\overline{\mathbb{Q}}_p)$ is nilpotent and satisfies $\rho'(\sigma^m g)N = q^m N \rho'(\sigma^m g)$ for all $g \in I_F$ and $m \in \mathbb{Z}$;*

(c) *$\rho'(\sigma)$ is bounded (see Appendix* 3.9.3*).*

*The bijection sets the pair $(\rho', N)$ in correspondence with $\rho$ if and only if for all $m \in \mathbb{Z}$ and $g \in I_F$ one has*

$$\rho\big(\sigma^m g\big) = \rho'\big(\sigma^m g\big) \exp\big(t_p(g)N\big).$$

*Remark* 3.2.3. (a) By the continuity of $\rho'$, it is clear that $\rho'(I_F)$ is finite.

(b) The representation $\rho'$ depends on the choice of $\sigma$. Its restriction to $I_F$ does (obviously) not.

**Definition 3.2.4.** Given $\rho$, we call the pair $(\rho'_{|I_F}, N)$ its *inertial WD-type*.

**Corollary 3.2.5.** *If $d = 2$, then either*

(a) $N = 0$ *(this happens if and only if $\rho(I_F)$ is finite; note that in this case $\rho$ is semisimple), or*

(b) $N$ *has rank* 1*. This happens if and only if $\rho(I_F)$ is infinite. Then $\rho$ is a non-split extension of some character $\lambda\colon G_F \to E^*$ by $\lambda(1)$, the twist of $\lambda$ by the cyclotomic character; cf. Corollary* 3.9.6.

**Definition 3.2.6.** If $V$ has dimension 2 and $N \neq 0$, we call $V$ a *representation of (twisted) Steinberg type*.

Let now $R$ be in $\widehat{\mathfrak{Ar}}_{\mathcal{O}}$ for $\mathcal{O}$ the ring of integers of a finite ramified extension of $W(\mathbb{F})[1/p]$. Let $V_R$ be a free finitely generated $R$-module carrying a continuous $R$-linear action by $G_F$. Let $\sigma \in G_F$ be as above. Denote by $V_{R[1/p]}$ the representation $V_R \otimes_R R[1/p]$.

**Theorem 3.2.7.** *There exists a unique nilpotent $N \in \operatorname{End}(V_{R[1/p]})$ and a finite index subgroup $I_1$ of $I_F$ such that, for all $g \in I_1$,*

$$\rho(g) = \exp\big(t_p(g)N\big).$$

*The assignment*

$$\rho'\colon W_F \longrightarrow \mathrm{GL}_d(V_{R[1/p]}),\ \sigma^n g \longmapsto \rho(\sigma^n g)\exp(-t_p(g)N)$$

*for $n \in \mathbb{Z}$ and $g \in I_F$ defines a continuous representation of $W_F$ where we regard $V_{R[1/p]}$ as a topological vector space with the discrete topology; in particular, $\rho'(I_F)$ is finite.*

*Proof.* The argument is basically the same as that for the proof of Theorem 3.2.1. Define $F'$ as the fixed field of the kernel of $G_F \to \mathrm{GL}_d(R[1/p]/2pR[1/p])$. One verifies that $\rho(t)-\mathrm{id}$ modulo the nilradical of $R[1/p]$ is nilpotent for $t$ any generator of $I_{F'}$. But then $\rho(t) - \mathrm{id}$ itself is nilpotent. Define, as before, $N = \log(\rho(t))$ and $I_1 = I_{F'}$. □

**Corollary 3.2.8.** *Suppose* $\operatorname{Spec} R[1/p]$ *is geometrically irreducible over $W(\mathbb{F})[1/p]$. Let $x, y \in X = \operatorname{Spec} R[1/p]$ be closed geometric points. Let $\rho_x$ and $\rho_y$ denote the representations on $V_x$ and $V_y$ obtained from $V_R$ by base change. Then $\rho'_x$ and $\rho'_y$ are isomorphic as representations of $I_F$.*

Another way to paraphrase the corollary is to say that on geometrically irreducible components of $(\operatorname{Spf} R)^{\mathrm{rig}}$ the representation $\rho'_{|I_F}$ is constant. This result will be applied in Theorem 3.3.1 to a universal deformation ring.

*Proof.* By the construction of $(\varphi', N)$ in the previous theorem, the representation $(\rho_x)'$ of the specialization at $x$ (or $y$) is the specialization of the representation $\rho'$ for $R[1/p]$. (It may however happen that $N$ for $\rho$ is non-zero while $N_x$ for $\rho_x$ it is zero.) Hence it suffices to consider the representation $\rho'$. As the image of $I_F$ under $\rho'$ is finite, we may regard $\rho'$ as a representation of the finite group $\overline{G} = I_F / \operatorname{Ker}(\rho'|_{I_F})$.

Since the nilradical of $R[1/p]$ is contained in the kernels of the specializations at $x$ and $y$, we may assume that $R[1/p]$ is reduced and hence a domain. Let $m$ be the exponent of $\overline{G}$. Let $E'$ be the extension of $W[1/p]$ obtained by adjoining all $m$-th roots of unity, and let $R' = R[1/p] \otimes_{\mathbb{W}[1/p]} E'$. Then $R'$ is an $E'$-algebra and by geometric irreducibility it is still an integral domain. We need to show that, for any two homomorphisms $x, y\colon R' \to \overline{\mathbb{Q}}_p$, the specializations $\rho'_x$ and $\rho'_y$ of $\rho'\colon \overline{G} \to \mathrm{GL}_d(R')$ are isomorphic. By the choice of $E'$ and ordinary character theory for representations of finite groups, it suffices to show that $\rho'_x$ and $\rho'_y$ have the same traces.

Now the $E'$-algebra structure of $R'$ is inherited by all specializations. But then it is obvious that under specialization the traces of $\rho'_x$ and $\rho'_y$ will be the same. □

**Example 3.2.9.** The following example shows that geometric irreducibility is necessary in the above corollary. Suppose that $\ell \equiv 1 \pmod p$ and let $F$ be $\mathbb{Q}_\ell$. By local class field theory, the abelianization $G_F^{\mathrm{ab}}$ has a tamely ramified quotient isomorphic to $\mathbb{F}_\ell^*$. Since $\ell \equiv 1 \pmod p$, it has a quotient of order $p$. Hence there is a surjective homomorphism $\pi\colon G_F \to \mathbb{Z}/(p)$, $g \mapsto \bar{\imath}(g)$ such that the fixed field of its kernel is totally and tamely ramified. Let $\varphi_p(X) = (X^p - 1)/(X - 1)$. Consider the representation

$$G_F \longrightarrow \mathrm{GL}_1\big(W(\mathbb{F})[X]/\big(\varphi_p\big(1+X\big)\big)\big),\ g \longmapsto \big(1+X\big)^{\bar{\imath}(g)}.$$

The ring $R = W(\mathbb{F})[X]/(\varphi_p(1+X))[1/p] \cong \mathbb{Q}_p(\zeta_p)$ is not geometrically irreducible over $\mathbb{Q}_p$. In fact, one has $p-1$ different embeddings $\mathbb{Q}(\zeta_p) \hookrightarrow \overline{\mathbb{Q}}_p$ over $\mathbb{Q}_p$. Clearly each embedding gives rise to a different representation of $I_F$ on $\mathrm{GL}_1(\overline{\mathbb{Q}}_p)$.

## 3.3 Deformation rings for 2-dimensional residual representations of $G_F$ and their generic fiber

We continue to denote by $F$ a finite extension of $\mathbb{Q}_\ell$ for some $\ell \neq p$. Let $V_\mathbb{F}$ be a $\mathbb{F}[G_F]$-module on which $G_F$ acts continuously. let $\mathcal{O}$ be the ring of integers of a finite extension of $W(\mathbb{F})[1/p]$, and let $\psi\colon G_F \to \mathcal{O}^*$ be a character whose reduction modulo $\mathfrak{m}_\mathcal{O}$ agrees with $\det V_\mathbb{F}$.

We define subgroupoids $D^{\psi}_{V_{\mathbb{F}}} \subset D_{V_{\mathbb{F}}}$ and $D^{\psi,\Box}_{V_{\mathbb{F}}} \subset D^{\Box}_{V_{\mathbb{F}}}$ over $\mathfrak{A}_{\mathfrak{r}W(\mathbb{F})}$ by

$$D^{\psi}_{V_{\mathbb{F}}}(A) = \{(V_A, \iota_A) \in D_{\mathbb{F}}(A) \mid \det\big(V_A\big) = \psi\},$$

$$D^{\psi,\Box}_{V_{\mathbb{F}}}(A) = \{(V_A, \iota_A, \beta_A) \in D^{\Box}_{\mathbb{F}}(A) \mid \det(V_A) = \psi\},$$

for $A \in \mathfrak{A}_{\mathfrak{r}W(\mathbb{F})}$. By showing that these functors are relatively representable as subfunctors, one deduces that $D^{\psi,\Box}_{V_{\mathbb{F}}}$ is pro-representable (by $R^{\psi,\Box}_{V_{\mathbb{F}}}$) and, if $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then also $D^{\psi}_{V_{\mathbb{F}}}$ is pro-representable (by $R^{\psi,\Box}_{V_{\mathbb{F}}}$).

Let $(\rho', N)$ denote the Weil–Deligne representation attached to $\rho^{\psi,\Box}_{V_{\mathbb{F}}}$ by Theorem 3.2.7 and assume that $\mathcal{O}$ contains all $p^m$-th roots of unity if $p^m$ is the maximal $p$-power divisor of the exponent of the finite group $\rho'(I_F)$ (so that Corollary 3.2.8 is applicable). The following is the main theorem concerning $D^{\psi,\Box}_{V_{\mathbb{F}}}$.

**Theorem 3.3.1.** *The following hold (where* component *always means of the generic fiber):*

(a) *The generic fiber* $(\mathrm{Spf}\, R^{\psi,\Box}_{V_{\mathbb{F}}})^{\mathrm{rig}}$ *is the union of finitely many smooth components of dimension* 3.

(b) *The restriction of* $\rho'_{I_F}$ *to any component is constant (in the sense of Corollary* 3.2.8*).*

(c) *The components are in bijection with the inertial WD-types which arise from* $p$*-adic representations of* $G_F$ *that possess a conjugate reducing to* $\bar\rho$.

(d) *There is at most one component, which we call* $C_N$*, whose inertial WD-type has non-trivial monodromy (at some point). This component occurs if and only if* $V_{\mathbb{F}}$ *is an extension of a mod* $p$ *character* $\bar\lambda$ *by* $\bar\lambda(1)$.

(e) *There is at most one component* $C^{\gamma}_{\mathrm{nr}}$ *whose inertial WD-type is of the form* $(\gamma\,\mathrm{id}, 0)$ *for some character* $\gamma$*. This component occurs if and only if a twist of* $V_{\mathbb{F}}$ *is unramified.*

(f) *The only generic components which can possibly intersect are* $C_N$ *and* $C^{\gamma}_{\mathrm{nr}}$.

(g) $\mathrm{Spf}\, R^{\psi,\Box}_{V_{\mathbb{F}}}$ *is covered by* $\mathrm{Spf}\,\mathcal{R}_i$ *for domains* $\mathcal{R}_i$ *which are in bijection to its generic components.*

(h) $\dim_{\mathrm{Krull}} R^{\psi,\Box}_{V_{\mathbb{F}}} = 4$.

*Remarks* 3.3.2. (a) If $\mathrm{End}_{F_F}(V_{\mathbb{F}}) = \mathbb{F}$, then, using that the tangent space dimension drops by three if we pass from framed deformations to deformations, it follows that $\dim_{\mathrm{Krull}} R^{\psi}_{V_{\mathbb{F}}}[1/p] = 0$. Hence, by generic smoothness, the ring $R^{\psi}_{V_{\mathbb{F}}}[1/p]$ is a product of fields.

(b) The theorem tells us that the natural and only possible subfunctors of $D^{\psi,\square}_{V_{\mathbb{F}}}$ are those given by selecting a finite number of components of the generic fiber, i.e., a finite number of inertial WD-types.

(c) The theorem makes no distinction between $p = 2$ and $p > 2$. The case $p > 2$ is somewhat simpler, in the sense that, independently of the global choice of determinant, one can read of from the residual representation whether the components $C_N$ or $C^{\gamma}_{\mathrm{nr}}$ appear. For $p = 2$, the added complication is that not every character $G_F \to \mathcal{O}^*$ with trivial reduction possesses a square root.

We shall in the following three sections indicate parts of the proof of the above theorem. For convenience we assume $p > 2$. For a complete proof, see [Pil]. We begin in Section 3.4 with a brief discussion of the very simple case of unramified representations (up to twist). This corresponds to (f) in the theorem above. In the subsequent section we treat, rather completely, the case of (twisted) Steinberg type lifts. This concerns part (e). Due to formal similarities, the case of Steinberg type deformations is also helpful for the investigation of ordinary deformations above $p$ later in Section 3.7. Section 3.6 indicates many of the steps toward the proof of Theorem 3.3.1.

## 3.4 Unramified deformations for $\ell \neq p$

Throughout Sections 3.4 to 3.6 we shall keep the hypotheses of the previous section.

**Proposition 3.4.1.** *Suppose $V_{\mathbb{F}}$ is unramified. Denote by $D^{\psi,\mathrm{nr},\square}_{V_{\mathbb{F}},\mathcal{O}}$ the subgroupoid of $D^{\psi,\square}_{V_{\mathbb{F}}}$ over $\mathfrak{A}_{\mathfrak{r}\mathcal{O}}$ of unramified framed deformations, i.e., of $(V_A, \iota_A\beta_A) \in D^{\psi,\square}_{V_{\mathbb{F}}}(A)$ such that $I_F$ acts trivially on $V_A$. Then this subgroupoid is representable by a ring $R^{\psi,\mathrm{nr},\square}_{V_{\mathbb{F}},\mathcal{O}} \in \widehat{\mathfrak{A}_{\mathfrak{r}\mathcal{O}}}$ which is smooth over $\mathcal{O}$ of relative dimension 3.*

The case where $V_{\mathbb{F}}$ is a twist of the trivial representation can easily be reduced to Proposition 3.4.1. We denote by $(R^{\psi,\mathrm{nr},\lambda,\square}_{V_{\mathbb{F}}}, \rho^{\psi,\mathrm{nr},\lambda,\square}_{V_{\mathbb{F}}})$ the pair

$$(R^{\psi,\mathrm{nr},\square}_{V_{\mathbb{F}}}, \lambda \otimes \rho^{\psi,\mathrm{nr},\lambda,\square}_{V_{\mathbb{F}}})$$

for some character $\lambda \colon G_F \to \mathcal{O}^*$.

*Proof.* Because $\Gamma_F := \mathrm{Gal}(F^{\mathrm{nr}}/F) \cong \widehat{\mathbb{Z}}$ is a free group topologically generated by the Frobenius automorphism $\sigma$, the functor $D^{\square}_{V_{\mathbb{F}}}$ is smooth: representations of $\Gamma_F$ are determined by the image of $\sigma$ and the only requirement for the image is that it lies in a compact subgroup, which is vacuous for images in $\mathrm{GL}_d(A)$, $A \in \mathfrak{A}_{\mathfrak{r}\mathcal{O}}$. Now for any surjection $A \to A'$ and representation to $A'$, one can lift the image of $\sigma$ to $A$ and with determinant equal to $\psi(\sigma)$. Alternatively, one can simply appeal to the fact that $H^2(\Gamma_F, \mathrm{ad}V_{\mathbb{F}}) = 0$.

By smoothness, the relative dimension of $R^{\psi,\mathrm{nr},\square}_{V_{\mathbb{F}},\mathcal{O}}$ can be computed from that of the tangent space:

$$\begin{aligned}\dim_{\mathbb{F}} D^{\psi,\mathrm{nr},\square}_{V_{\mathbb{F}},\mathcal{O}}(\mathbb{F}[\varepsilon]) &= \dim_{\mathbb{F}} D^{\psi,\mathrm{nr}}_{V_{\mathbb{F}},\mathcal{O}}(\mathbb{F}[\varepsilon]) + (d^2-1) - h^0(\Gamma_F, \mathrm{ad}V_{\mathbb{F}})\\ &= 3 + h^1(\Gamma_F, \mathrm{ad}V_{\mathbb{F}}) - h^0(\Gamma_F, \mathrm{ad}V_{\mathbb{F}}) = 3.\end{aligned}$$

The last equality uses that $\hat{\mathbb{Z}}$ is free so that $h^1(\ldots) = h^0(\ldots)$. One can also give a short direct argument proving that $D^{\psi,\mathrm{nr},\square}_{V_{\mathbb{F}},\mathcal{O}}(\mathbb{F}[\varepsilon])$ is a principal homogenous space under $\mathrm{Ker}(\mathrm{PGL}_2(\mathbb{F}[\varepsilon]) \to \mathrm{PGL}_2(\mathbb{F})) \cong \mathbb{F}^3$. □

## 3.5 Deformations of Steinberg type for $\ell \neq p$

In this section, we analyze following [Ki4, §2.6] the deformation functor for deformations where the monodromy $N$ is typically non-zero. Thus $V_{\mathbb{F}}$ has a basis $\beta_{\mathbb{F}}$ such that

$$\bar{\rho}\colon G_F \longrightarrow \mathrm{GL}_2(\mathbb{F}),\ g \longmapsto \begin{pmatrix} \bar{\lambda}(1)(g) & * \\ 0 & \bar{\lambda}(g) \end{pmatrix}. \tag{3.5.1}$$

*Remarks* 3.5.1. (a) Writing (1) indicates that $\bar{\lambda}$ is twisted by the mod $p$ cyclotomic character.

(b) Since $\ell \neq p$, the mod $p$ cyclotomic character is unramified and depending on $\ell$ it may be trivial.

(c) If the lift is of Steinberg type, then the character $\lambda$ lifting $\bar{\lambda}$ can be ramified, even if $\bar{\lambda}$ is not. This is possible precisely if the ramification subgroup of $G_F^{\mathrm{ab}}$ contains non-trivial $p$-torsion, i.e., if $q \equiv 1 \pmod p$.

After twisting by the inverse of $\bar{\lambda}$, we shall for the remainder of this section assume the following:

(a) $\dim V_{\mathbb{F}} = 2$.

(b) $\det V_{\mathbb{F}}$ is equal to the mod $p$ reduction of the cyclotomic character

$$\chi\colon G_F \longrightarrow \mathbb{Z}_p^*.$$

(c) $V_{\mathbb{F}}(-1)^{G_F} \neq 0$, i.e., there is a subrepresentation of dimension at least one (and exactly one unless $\chi \bmod p$ is trivial) on which $G_F$ acts via $\chi$ modulo $p$.

We now define the groupoid $L^{\chi,\square}_{V_{\mathbb{F}}}$ (resp. $L^{\chi}_{V_{\mathbb{F}}}$) over $\mathfrak{Aug}$ —see page 30 for the definition of the base category. The groupoid $L^{\chi,\square}_{V_{\mathbb{F}}}$ maps naturally to $D^{\chi,\square}_{V_{\mathbb{F}}}$ over $\mathfrak{Aug}$ and serves, as we shall see, as a smooth resolution of the subgroupoid of $D^{\chi,\square}_{V_{\mathbb{F}}}$ of deformations of Steinberg type. It can only be understood over $\mathfrak{Aug}_{W(\mathbb{F})}$ and not over $\mathfrak{Ar}_{W(\mathbb{F})}$; cf. Exercise 3.10.3. For $(A \to B) \in \mathfrak{Aug}_{W(\mathbb{F})}$, so that $A \in \mathfrak{Ar}_{W(\mathbb{F})}$ and $B$ is an $A$-algebra, the set of objects in $L^{\chi,\square}_{V_{\mathbb{F}}}$ over $(A \to B)$ is the set of tuples $(V_A, \iota_A, \beta_A, L_B)$, where

- $(V_A, \iota_A, \beta_A) \in D^{\chi,\square}_{V_{\mathbb{F}}}(A)$,
- $L_B \subset V_B := V_A \otimes_A B$ is a *B-line*, i.e., a submodule such that $V_B/L_B$ is a projective $B$-module of rank 1,
- $L_B \subset V_B$ is a subrepresentation on which $G_F$ acts via $\chi$.

**Lemma 3.5.2.** *The functor $L^{\chi,\square}_{V_{\mathbb{F}}}$ is represented by a projective algebraizable morphism, which we denote by*

$$\Theta_{V_{\mathbb{F}}} : \mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}} \longrightarrow \operatorname{Spf} R^{\chi,\square}_{V_{\mathbb{F}}}.$$

*Proof.* We abbreviate $\mathcal{R} = R^{\chi,\square}_{V_{\mathbb{F}}}$. Consider the projective space $\mathbb{P}^1_{\mathcal{R}}$. It classifies $\mathcal{R}$-lines $L$ inside $V_{\mathcal{R}} \cong (\mathcal{R})^{\oplus 2}$. (Since $\mathcal{R}$ is local, these lines, as well as the quotients by them, are free of rank one and not just projective.) Denote by $\widehat{\mathbb{P}^1}_{\mathcal{R}}$ the completion of the above space along its specialization under $\mathcal{R} \to \mathbb{F}$. This is a formal scheme. It classifies $\mathcal{R}$-lines of $V_{\mathcal{R}}$ over an $\mathbb{F}$-line of $V_{\mathbb{F}}$.

Let $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}$ be the closed subscheme of $\widehat{\mathbb{P}^1}_{\mathcal{R}}$ defined by the equations $gv - \chi(g)v$ for all $v$ in the universal line over $\widehat{\mathbb{P}^1}_{\mathcal{R}}$ and all $g \in G_F$. By formal GAGA it is a projective scheme over the formal scheme $\operatorname{Spf} \mathcal{R}$. Algebraizability is automatic from Grothendieck's existence theorem in formal geometry; cf. [Ill]. But it can also be shown directly, as in [KW2, Proof of Proposition 3.6]. From the construction of $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}$, one deduces its universal property. $\square$

**Lemma 3.5.3.** *$\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}$ is formally smooth over $W(\mathbb{F})$. Its generic fiber $(\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}})^{\mathrm{rig}}$ is connected.*

*Proof.* To prove smoothness, consider a surjective homomorphism $B \to B'$ with nilpotent kernel of algebras over $A \to A'$ in $\mathfrak{Ar}_{W(\mathbb{F})}$ and let $(V_{A'}, \beta_{A'}, L_{B'})$ be an object of $|L^{\chi,\square}_{V_{\mathbb{F}}}(A' \to B')|$. Note that we want to show that $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}$ is smooth over $W(\mathbb{F})$ and not over $D^{\chi,\square}_{V_{\mathbb{F}}}$. Hence it suffices to find $A \to \widetilde{A}$ in $\mathfrak{Ar}_{W(\mathbb{F})}$ with $\widetilde{A} \subset B$ and $\widetilde{A}$ mapping to $A'$ and $(V_{\widetilde{A}}, \beta_{\widetilde{A}}, L_B)$ lifting the above triple.

Set $V_{B'} = V_{A'} \otimes_{A'} B'$. Since $\mathbb{P}^1$ is smooth (over any base), the pair $(V_{B'}, L_{B'})$ lifts to the pair $(V_B, L_B)$ consisting of a free rank 2 module over $B$ together with a $B$-line. The next step is to show that the extension

$$0 \longrightarrow L_{B'} \longrightarrow V_{B'} \longrightarrow V_{B'}/L_{B'} \longrightarrow 0$$

lifts. This amounts to proving that the natural homomorphism

$$\operatorname{Ext}^1_{B[G_F]}(V_B/L_B, L_B(1)) \longrightarrow \operatorname{Ext}^1_{B'[G_F]}(V_{B'}/L_{B'}, L_{B'}(1)) \tag{3.5.2}$$

is surjective. Clearly the map $P := L_B \otimes (V_B/L_B)^* \to P' := L_{B'} \otimes (V_{B'}/L_{B'})^*$ is a surjective homomorphism of $B$-modules. Identifying the $\operatorname{Ext}^1$ with an $H^1$, the surjectivity of (3.5.2) follows from the next lemma.

**Lemma 3.5.4.** *For any $A \in \mathfrak{Ar}_{W(\mathbb{F})}$ and any $A$-module $M$ (which is not necessarily finitely generated), the natural homomorphism*

$$H^1(G_F, \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} M \longrightarrow H^1(G_F, M(1))$$

*is an isomorphism.*

*Proof.* Using the commutativity of $H^i$ with direct limits, it suffices to prove this for $A = M = \mathbb{Z}/(p^n)$. Then the latter homomorphism arises from the long exact cohomology sequence for $0 \to \mathbb{Z}_p \to \mathbb{Z}_p \to \mathbb{Z}/(p^n) \to 0$, which yields that the map is injective with cokernel $H^2(G_F, \mathbb{Z}_p(1))[p^n]$. By local Tate duality, the $H^2$ is the Pontryagin dual of $\mathbb{Q}_p/\mathbb{Z}_p$, i.e., $\mathbb{Z}_p$, which has no $p$-power torsion. □

To complete the proof of smoothness, we apply the following lemma:

**Lemma 3.5.5.** *Suppose that $(A \to B) \xrightarrow{\pi} (A' \to B')$ is a homomorphism in $\mathfrak{Aug}_{W(\mathbb{F})}$ such that $B \to B'$ is surjective with nilpotent kernel. Suppose also that $V_B$ is a continuous representation of $G$ on a free $B$-module and $V_{A'} \in D_{V_{\mathbb{F}}}(A')$ such that $V_{A'} \otimes_{A'} B' \cong V_B \otimes_B B'$. Then there exists a factorization of $\pi$ in $\mathfrak{Aug}_{W(\mathbb{F})}$,*

$$(A \to B) \longrightarrow (A_0 \to B) \longrightarrow (A' \to B'),$$

*and a $V_{A_0}$ in $D_{V_{\mathbb{F}}}(A_0)$ such that $V_{A_0} \otimes_{A_0} B \cong V_B$ and $V_{A_0} \otimes_{A_0} A' \cong V_{A'}$.*

*Proof.* We first observe that

$$\varinjlim_{\widetilde{A}} \widetilde{A} = A \times_{B'} B \supset \operatorname{Ker}(B \to B'),$$

where the limit ranges over all subrings $\widetilde{A} \in \mathfrak{Ar}$ of $A \times_{B'} B$ which contain the image of $A$ under the homomorphism $A \to A \times_{B'} B$ deduced from the universal property of $A \times_{B'} B$. The continuous representation $V_B$ restricts to a continuous representation $V_{A \times_{B'} B}$ and, by linearity, to a continuous representation $V_{\widetilde{A}}$ for all $\widetilde{A}$. We need to find $\widetilde{A}$ such that $V_{\widetilde{A}} \otimes_{\widetilde{A}} B = V_B$. Since $V_{A'} \otimes_{A'} B' = V_{B'}$, we have $V_{A \times_{B'} B} \otimes_{A \times_{B'} B} B \cong V_B$, and hence it suffices to find $\widetilde{A}_0$ such that $V_{\widetilde{A}_0} \otimes_{\widetilde{A}_0} (A \times_{B'} B) = V_{A \times_{B'} B} = \varinjlim_{\widetilde{A}} V_{\widetilde{A}}$. Choose $m'$ such that $\mathfrak{m}_{A'}^{m'} = 0$ and $m''$ such that $\operatorname{Ker}(B \to B')$ has $m''$-th power equal to zero. Then for $m = m' + m''$ one has $\mathfrak{m}_{\widetilde{A}}^m = 0$ for all $\widetilde{A}$. By the universal property of $\mathcal{R}$ (we equip $V_A$ with a basis which induces one on all $V_{\widetilde{A}}$), there exist (unique) compatible homomorphisms $\mathcal{R}/\mathfrak{m}_{\mathcal{R}}^m \to \widetilde{A}$ such that $V_{\mathcal{R}}$ induce $V_{\widetilde{A}}$, where the $\widetilde{A}$ range over the above direct system. Now take $A_0$ as the subring of $A \times_{B'} B$ generated by the images of $A \to A \times_{B'} B$ and $\mathcal{R}/\mathfrak{m}_{\mathcal{R}}^m \to A \times_{B'} B$. □

It remains to prove connectivity. By smoothness of $\mathcal{L}_{V_{\mathbb{F}}}^{\chi,\square}$ over $W(\mathbb{F})$, the idempotent sections of $\mathcal{L}_{V_{\mathbb{F}}}^{\chi,\square}[1/p]$, of $\mathcal{L}_{V_{\mathbb{F}}}^{\chi,\square}$ and of $\mathcal{L}_{V_{\mathbb{F}}}^{\chi,\square} \otimes_{W(\mathbb{F})} \mathbb{F}$ are in bijection. Next, as a scheme over the local scheme $\operatorname{Spf} \mathcal{R}$, the idempotents of $\mathcal{L}_{V_{\mathbb{F}}}^{\chi,\square}$ are in bijection with those of $Z := \mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}} \otimes_{\mathcal{R}} \mathbb{F}$; see [GD, III, 4.1.5]. The scheme $Z$ (which is not formal) is isomorphic to a closed subscheme of $\mathbb{P}^1$. Depending on whether the image of $\rho$ is scalar or not, this subscheme is all of $\mathbb{P}^1$ or a single point; cf. Exercise 3.10.3(a), and thus in either case it has no non-trivial idempotent sections. Finally one observes (see [Ki4, 2.4.1] based on [deJ, 7.4.1]) that one has a bijection between the idempotents of $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}[1/p]$ and those of $(\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}})^{\mathrm{rig}}$. □

**Lemma 3.5.6.** *Let $E/\mathbb{Q}_p$ be a finite extension, let $\xi$ be in $L^{\chi,\square}_{V_{\mathbb{F}}}(\mathcal{O}_E)$, and denote by $\xi$ also its image in $D^{\chi,\square}_{V_{\mathbb{F}}}(\mathcal{O}_E)$.*

*The morphism of groupoids $L^{\chi,\square}_{(\xi)} \to D^{\chi,\square}_{(\xi)}$ on $\mathfrak{Ar}_E$ is fully faithful. If the $E$-representation $V_\xi$ of $G_F$ corresponding to $\xi$ is indecomposable, then this morphism is an equivalence, $D^{\chi}_{(\xi)}$ is representable and its tangent space is 0-dimensional.*

Note that, on the level of functors, full faithfulness means that $|L^{\chi,\square}_{(\xi)}|$ is a subfunctor of $|D^{\chi,\square}_{(\xi)}|$.

*Proof.* We first prove full faithfulness. Since both functors are representable, the homomorphism sets of objects are singletons or empty. Suppose we have two objects of $L^{\chi,\square}_{(\xi)}$. It is clear that if there is a morphism between them, then there will be one between the images. What we need to rule out is the possibility that there is a homomorphism between the images but no homomorphism between the objects.

To see this, let $B$ be in $\mathfrak{Ar}_E$ and $(V_B, \beta_B)$ in $D^{\chi,\square}_{(\xi)}(B)$. We have to show that if $V_B$ admits a $B$-line $L_B$ such that $L_B(-1)$ is $G_F$-invariant, then $L_B$ is the unique such line. Since $\det_B V_B = \chi$, we have $\mathrm{Hom}_{B[G_F]}(B(1), V_B/L_B) = 0$ —we are in characteristic zero—, so that $\mathrm{Hom}_{B[G_F]}(B(1), V_B) = \mathrm{Hom}_{B[G_F]}(B(1), L_B)$, and the uniqueness of $L_B$ follows. (The point is simply that while the mod $p$ reduction of $\chi$ can be zero, $\chi$ itself is never trivial.)

Suppose further that $V_\xi$ is indecomposable. Then $V_\xi$ satisfies $\mathrm{End}_E(V_\xi) \cong E$, so that $D^{\chi}_{(\xi)}$ is representable. We have to show that any $V_B$ contains at least one $B$-line $L_B \subset V_B$ on which $G_F$ acts via $\chi$. For this, it suffices to show that the tangent space of $D^{\chi}_{(\xi)}$ is trivial, since then $V_B \cong V_\xi \otimes_E B$, for $E \to B$ the canonical splitting, is the trivial deformation which inherits the required $B$-line from $V_E$. However, the dimension of this tangent space is computed by

$$\dim_E H^1\big(G_F, \mathrm{ad}^0 V_\xi\big) = \dim_E H^0\big(G_F, \mathrm{ad}^0 V_\xi\big) + \dim_E H^0\big(G_F, \mathrm{ad}^0 V_\xi(1)\big) \overset{\text{Exer.}}{=} 0,$$

where the zero at the end follows from the indecomposability of $\xi$. □

For $\xi$ as above, define a groupoid over $\mathfrak{Ar}_E$ by defining $D^{\chi}_{V_\xi}(B)$ (resp. $D^{\chi,\square}_{V_\xi(B)}$) for $B \in \mathfrak{Ar}_E$ as the category whose objects are deformations of $V_\xi$ to $B$ with determinant $\chi$ (and in addition a basis lifting the given one.)

**Proposition 3.5.7.** *Let $\mathrm{Spf}\, R^{\chi,1,\square}_{V_{\mathbb{F}}}$ denote the scheme theoretic image of the morphism $\Theta_{V_{\mathbb{F}}}$ of Lemma 3.5.2. Then*

(a) $R^{\chi,1,\square}_{V_{\mathbb{F}}}$ *is a domain of dimension* 4 *and* $R^{\chi,1,\square}_{V_{\mathbb{F}}}[1/p]$ *is formally smooth over* $W(\mathbb{F})[1/p]$.

(b) *Let* $E/\mathbb{Q}_p$ *be a finite extension. Then a morphism* $\xi\colon R^{\chi,\square}_{V_{\mathbb{F}}} \to E$ *factors through* $R^{\chi,1,\square}_{V_{\mathbb{F}}}$ *if and only if the corresponding two-dimensional* $E$*-representation* $V_\xi$ *is an extension of* $E$ *by* $E(1)$.

One way to avoid formal schemes in the definition of $\operatorname{Spf} R^{\chi,1,\square}_{V_{\mathbb{F}}}$ proceeds as follows. Since $L^{\chi,\square}_{V_{\mathbb{F}}} \to \operatorname{Spf} R^{\chi,\square}_{V_{\mathbb{F}}}$ is relatively representable, for any Artinian quotient $A$ of $R^{\chi,\square}_{V_{\mathbb{F}}}$, the pullback of $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}$ to $A$ is a honest scheme, say $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}/A}$. The morphism $\Theta_{V_{\mathbb{F}}}$ of Lemma 3.5.2 induces a projective morphism of schemes $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}/A} \to \operatorname{Spec} A$. Its scheme theoretic image thus defines a closed subscheme of $\operatorname{Spec} A$, say $\operatorname{Spf} R^{\chi,1,\square}_{V_{\mathbb{F}}/A}$. These are schemes in $\mathfrak{Ar}_{W(\mathbb{F})}$ which form an inverse limit system and their inverse limit is precisely $\operatorname{Spf} R^{\chi,1,\square}_{V_{\mathbb{F}}}$.

*Proof.* The scheme $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}$ is smooth over $W(\mathbb{F})$ and connected. The ring $R^{\chi,1,\square}_{V_{\mathbb{F}}}$ is the ring of global sections of this scheme over $\operatorname{Spf} R^{\chi,\square}_{V_{\mathbb{F}}}$ and hence it must be a domain.

Since $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}$ is projective over $\operatorname{Spf} R^{\chi,\square}_{V_{\mathbb{F}}}$, it surjects onto $\operatorname{Spf} R^{\chi,1,\square}_{V_{\mathbb{F}}}$. If we invert $p$, then by the previous lemma $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}[1/p]$ is a closed subscheme of $\operatorname{Spec} R^{\chi,\square}_{V_{\mathbb{F}}}[1/p]$ and hence it is isomorphic to $\operatorname{Spec} R^{\chi,1,\square}_{V_{\mathbb{F}}}[1/p]$, which shows that the latter is formally smooth over $W(\mathbb{F})[1/p]$.

By construction, $R^{\chi,1,\square}_{V_{\mathbb{F}}}$ is $p$-torsion free. To compute its dimension it therefore suffices to compute the dimension of its generic fiber (and add 1). This we may do at an indecomposable $V_\xi$. By Lemma 3.5.6, the functor $D^{\chi}_{(\xi)}$ is representable by $\operatorname{Spf} E$. Hence $D^{\chi,\square}_{(\xi)}$ is formally smooth over $\operatorname{Spf} E$ of dimension 3.

The last assertion follows essentially from Lemma 3.5.6, as $\xi$ factors through $R^{\chi,1,\square}_{V_{\mathbb{F}}}$ if and only if it lifts to a (necessarily unique) point of $\mathcal{L}^{\chi,\square}_{V_{\mathbb{F}}}$, i.e., if and only if $V_\xi$ admits an $E$-line $L_E$ such that $L_E(-1)$ is $G_F$-invariant. □

Let $\mathcal{O}$ be the ring of integers of a finite extension $E$ of $W(\mathbb{F})[1/p]$. By twisting the rings in the previous proposition with any global characters $\lambda\colon G_F \to \mathcal{O}^*$, one obtains:

**Corollary 3.5.8.** *Define* $R^{\square}_{V_{\mathbb{F}},\mathcal{O}} = R^{\square}_{V_{\mathbb{F}}} \otimes_{W(\mathbb{F})} \mathcal{O}$. *Then there exists a unique quotient* $R^{\chi\lambda,\lambda,\square}_{V_{\mathbb{F}},\mathcal{O}}$ *of* $R^{\square}_{V_{\mathbb{F}},\mathcal{O}}$ *with the following properties:*

(a) $R^{\chi\lambda,\lambda,\square}_{V_{\mathbb{F}},\mathcal{O}}$ *is a domain of dimension* 4 *and* $R^{\chi\lambda,\lambda,\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p]$ *is formally smooth over* $\mathcal{O}$.

(b) *Let* $E/\mathbb{Q}_p$ *be a finite extension. Then a morphism* $\xi\colon R^{\lambda\chi,\square}_{V_{\mathbb{F}},\mathcal{O}} \to E$ *factors through* $R^{\chi\lambda,\lambda,\square}_{V_{\mathbb{F}},\mathcal{O}}$ *if and only if the corresponding two-dimensional* $E$*-representation* $V_\xi$ *is an extension of* $\lambda$ *by* $\lambda(1)$.

## 3.6 On the proof of Theorem 3.3.1

*Proof of Theorem* 3.3.1. Parts (b), (d) and (e) follow directly from Corollary 3.2.8 and our results in Sections 3.4 and 3.5, respectively. We now prove (f). Let $E$ be a finite extension field of $\mathcal{O}[1/p]$ and $\xi\colon R^{\psi,\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p] \to E$ be an $\mathcal{O}$-algebra homomorphism. The smoothness at $\xi$ in the generic fiber, i.e., the smoothness of $(R^{\psi,\square}_{V_{\mathbb{F}},\mathcal{O},})_\xi$, certainly holds if $H^2(G_F, \mathrm{ad}^0 V_\xi) = 0$ and in this case the tangent space dimension is

$$\dim_E H^1\bigl(G_F, \mathrm{ad}^0 V_\xi\bigr) + \dim_E \mathrm{ad}^0 V_\xi - \dim_E H^0\bigl(G_F, \mathrm{ad}^0 V_\xi\bigr) = 3.$$

Suppose now that $\xi$ is such that $H^2(G_F, \mathrm{ad}^0 V_\xi) \neq 0$ and recall that, by local duality,

$$\dim_E H^2\bigl(G_F, \mathrm{ad}^0 V_\xi\bigr) = \dim_E H^0\bigl(G_F, \mathrm{ad}^0(1)\bigr).$$

This last group is non-trivial if and only if $V_\xi$ is isomorphic to a sum of characters $\lambda \oplus \lambda(1)$. Moreover, in this case $\dim_E H^2(G_F, \mathrm{ad}^0 V_\xi) = 1$ and the tangent space has dimension 4.

Our particular point $\xi$ lies on the components

$$\operatorname{Spec} R^{\chi\lambda,\lambda,\square}_{V_{\mathbb{F}},\mathcal{O}}[1/p] \text{ and } \operatorname{Spec} R^{\psi,\mathrm{nr},\lambda,\square}_{V_{\mathbb{F}}}[1/p].$$

The containment in the second is obvious. For the containment in the first, observe that $\xi$ has deformations with $N$ non-trivial: take any non-zero class of $H^1(G_F, E(1))$ which is thus ramified and consider the corresponding deformation to $E[\varepsilon]$.

In fact, the two components identified are the only ones through $\xi$. By twisting the entire situation by $\lambda^{-1}$ we may assume that $\xi$ is an extension of $E$ by $E(1)$. Let $C$ be an irreducible component through it. If $C$ contains a point where $N$ is non-zero, then $C = C_N$, else $N = 0$ on all of $C$. Now the triviality of $\rho'_{|I_F}$ at $\xi$ implies that $\rho'_{|I_F}$ is zero on all of $C$ and hence $\rho$ is unramified on $C$. Thus $C$ is the other component we have already identified.

For the proof of (c) and (g) one defines deformation functors for all inertial WD-types that can occur. (Their classification is not difficult.) In several cases one needs to consider a resolution by adding the datum of an additional line such as in the analysis we gave in the (twisted) Steinberg case. In each case one directly shows that the functor is representable by a domain which proves the bijection in (c) between inertial WD-types and components. To see (g) one shows that any deformation is described by one of the functors so-defined. See [Pil] for details.

We now prove (a). The finiteness of the number of components is clear, since $R^{\psi,\square}_{V_{\mathbb{F}}}$ is Noetherian. By the proof of (f), all (closed) points on the generic fiber except for those in the intersection of $C_N$ and $C^\gamma_{\mathrm{nr}}$ are smooth and have tangent dimension 3. But we also know that $C_N$ and $C^\gamma_{\mathrm{nr}}$ are smooth of dimension 3. Hence (a) is proved.

The proof of (h) is now simple. By (g) the rings $\mathcal{R}_i$ are $p$-torsion free and thus their dimension is one more than the dimension of their generic fiber. Hence (h) follows from (a). □

## 3.7 Ordinary deformation at $p$

In this section, we let $K$ be a finite extension of $\mathbb{Q}_p$. We shall investigate the ordinary flat deformation ring of a two-dimensional representation of $G_K$ following [Ki5]. Let $\chi\colon G_F \to \mathbb{Z}_p^*$ denote the cyclotomic character. The deformation functor and its resolution are very similar to those used in the Steinberg case in Section 3.5. One key difference occurs in the proof of the smoothness of the resolution. In the Steinberg case this relied on the surjectivity of the homomorphism (3.5.2). Here we need a different argument. The first results of this section recall the necessary background for this. Then we closely follow the discussion in the Steinberg case.

Set $K^{\mathrm{nr}} = \overline{K}^{I_K}$ and $\Gamma_K = G_K/I_K = \mathrm{Gal}(K^{\mathrm{nr}}/K)$. Let $M$ be a discrete possibly infinite $\Gamma_K$-module over $\mathbb{Z}_p$ on which $p$ is nilpotent.

For any finite subrepresentation $M' \subset M$ consider the twist

$$M'(1) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1).$$

Since $\mathbb{Z}_p(1)$ arises from a $p$-divisible group over $\mathcal{O}_K$ and since $M'$ is unramified, the representation $M'(1)$ arises from a finite flat group scheme over $\mathcal{O}_K$ (we shall give some background on this in Appendix 3.9.4). A proof is given in Corollary 3.9.14. We now introduce the group $H^1_f(G_K, M'(1))$. It classifies, for any $n \in \mathbb{N}$ such that $M'$ is annihilated by $p^n$, extensions of $\mathbb{Z}/p^n$ by $M'(1)$ which arise from the generic fiber of a finite flat group scheme over $\mathcal{O}_K$. The group $H^1_f(G_K, M(1))$ is then defined as the direct limit $\varinjlim_{M'\subset M} H^1_f(G_K, M'(1))$, where $M'$ ranges over the finite $\Gamma_K$-submodules of $M$.

To define $H^1_f(G_K, M'(1))$, we first consider $H^1(G_K, M'(1))$. Restriction in cohomology yields a homomorphism

$$\mathrm{Res} : H^1(G_K, M'(1)) \longrightarrow H^1(I_K, M'(1))^{\Gamma_K}.$$

If $M'(1)^{I_K} = 0$, i.e., if $\chi \pmod p$ is non-trivial on $I_K$, then the inflation restriction sequence shows that the above map is an isomorphism – it is also an isomorphism if $M'$ is a free $\mathbb{Z}_p$-module. To obtain an alternative expression for $H^1(I_K, M'(1))$ suppose that $p^n$ annihilates $M'$. Then

$$\begin{aligned} H^1(I_K, M'(1)) &\cong H^1(I_K, \mu_{p^n}) \otimes_{\mathbb{Z}} M' \cong \varinjlim_{L/K^{\mathrm{nr}}} H^1(G_L, \mu_{p^n}) \otimes_{\mathbb{Z}} M' \\ &\overset{\text{Kummer}}{\cong} \varinjlim_{L/K^{\mathrm{nr}}} L^*/L^{*p^n} \otimes_{\mathbb{Z}} M' \cong \varinjlim_{L/K^{\mathrm{nr}}} L^* \otimes_{\mathbb{Z}} M' \cong (K^{\mathrm{nr}})^* \otimes_{\mathbb{Z}} M'. \end{aligned}$$

Denote this isomorphism by $\psi$ and define $H^1_f(G_K, M'(1))$ as the kernel of

$$H^1(G_K, M'(1)) \xrightarrow{\text{Res}} H^1(I_K, M'(1))/\psi^{-1}(\mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}} M'). \tag{3.7.1}$$

Observe that by Kummer theory one has $H^1_f(G_K, \mathbb{Z}/p\mathbb{Z}(1)) \cong \mathcal{O}^*_K/\mathcal{O}^{*p}_K$.

If $\chi$ is non-trivial on $I_K$, then $H^1_f(G_K, M'(1)) \cong (\mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}} M')^{\Gamma_K}$. In general, by [Ki5, Lem. 2.4.2] the group $H^1_f(G_K, M'(1))$ is isomorphic to

$$(\mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}_p} M_0)^{\Gamma_K}/\text{image}\big((\mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}_p} M_1)^{\Gamma_K} \to (\mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}_p} M_0)^{\Gamma_K}\big), \tag{3.7.2}$$

where $0 \to M_1 \to M_0 \to M' \to 0$ is a short exact sequence of $\mathbb{Z}_p[[\Gamma_K]]$-modules such that $M_0$ (and hence also $M_1$) is free and finitely generated over $\mathbb{Z}_p$.[2] To obtain (3.7.2), one first considers the long exact Tor-sequence for the resolution of $M'$ and $\mathcal{O}^*_{\bar{K}}$. Since $\mathcal{O}^*_{\bar{K}}$ is divisible and $M_0$ is free over $\mathbb{Z}_p$, it yields the short exact sequence $0 \to \text{Tor}_1(M, \mathcal{O}^*_{\bar{K}}) \to M_1 \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\bar{K}} \to M_0 \otimes_{\mathbb{Z}_p} \mathcal{O}^*_{\bar{K}} \to 0$. Using the Kummer sequence for $\mathcal{O}^*_{\bar{K}} \to \mathcal{O}^*_{\bar{K}}, x \mapsto x^{p^n}$ with $n$ such that $p^n M' = 0$, the Tor-term can be evaluated as $M'(1)$. One now obtains

$$\begin{aligned} \ldots \to &(\mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}} M_0)^{\Gamma_K} \to (\mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}} M_1)^{\Gamma_K} \\ &\to H^1(G_K, M'(1)) \to H^1(G_K, \mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}} M_1) \to \ldots \end{aligned}$$

from the long exact $G_K$-cohomology sequence. Kisin shows that

$$H^1(G_K, \mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}} M_1) \hookrightarrow H^1(I_K, \mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}} M_1)$$

is injective and that a class lies in $H^1_f(G_K, M'(1))$ if its image in $H^1(I_K, \mathcal{O}^*_{K^{\text{nr}}} \otimes_{\mathbb{Z}} M_1)$ is zero.

**Lemma 3.7.1.** *On discrete $\mathbb{Z}_p[\Gamma_K]$-modules $M$ on which $p$ is nilpotent, the functor $M \mapsto H^1_f(G_K, M(1))$ is right exact.*

*Proof.* The assertion is immediate from formula (3.7.2), since it suffices to verify right exactness for sequences of finite $\mathbb{Z}_p[\Gamma_K]$-modules. □

Suppose now that $V_{\mathbb{F}}$ is two-dimensional, flat and ordinary. As before, let $D^{\chi,\square}_{V_{\mathbb{F}}}$ be the full subgroupoid of $D^{\square}_{V_{\mathbb{F}}}$ consisting of those framed deformations $(V_A, \iota_A, \beta_A)$ such that $\det V_A \cong \chi$. Define the groupoid $D^{\text{ord},\chi,\square}_{V_{\mathbb{F}}}$ over $\mathfrak{Aug}_{W(\mathbb{F})}$ as follows: an object of $D^{\text{ord},\chi,\square}_{V_{\mathbb{F}}}$ over $(A \to B)$ is a quadruple $(V_A, \iota_A, \beta_A, L_B)$, where

(a) $(V_A, \iota_A, \beta_A) \in D^{\chi,\square}_{V_{\mathbb{F}}}(A)$;

[2] Note that given any short exact sequence $0 \to M_1 \to M_0 \to M' \to 0$ of $\mathbb{Z}_p$-modules, with $M_0$ free and finitely generated over $\mathbb{Z}_p$, one can always extend the $\Gamma_K$ action from $M'$ to a continuous action on $M_0$.

(b) $L_B \subset V_B := V_A \otimes_A B$ is a $B$-line, i.e., a projective $B$-submodule of rank 1 such that $V_B/L_B$ is projective;

(c) $L_B$ is $G_K$-stable subrepresentation and $I_K$ acts on $L_B$ via $\chi$;

(d) the extension class of $0 \to L_B \to V_B \to V_B/L_B \to 0$ in

$$\mathrm{Ext}^1_{B[G_K]}(V_B/L_B, L_B) \cong H^1(G_K, L_B \otimes_B (V_B/L_B)^*)$$

lies in $H^1_f(G_K, L_B \otimes_B (V_B/L_B)^*)$.

**Proposition 3.7.2.** *The functor $D^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ of groupoids over $\mathfrak{Aug}_{W(\mathbb{F})}$ is representable by a projective algebraizable morphism of formal schemes, which we denote by*

$$\Theta^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}} \colon \mathcal{L}^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}} \longrightarrow \mathrm{Spf}\, R^{\chi,\square}_{V_{\mathbb{F}}}.$$

*The morphism $\Theta^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ becomes a closed embedding after inverting $p$. The scheme $\mathcal{L}^{\mathrm{ord},\chi,\square}_{\xi}$ is formally smooth over $W(\mathbb{F})$.*

*Proof.* The proof of the first part is more or less the same as that of Lemma 3.5.2. For the second assertion one proceeds as in Lemma 3.5.6. For any closed point $\eta$ on the generic fiber of $R$, one proves that the completion of $\Theta^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ at $\eta$ is either zero or an infinitesimal isomorphism (by showing that the respective functor is fully faithful). Because the morphism $\Theta^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ is also projective, it must be a closed immersion.

It remains to verify formal smoothness. The proof proceeds as the proof of Lemma 3.5.3. The key input is Lemma 3.7.1. It provides the desired lifting of an extension $0 \to L_{B/I} \to V_{B/I} \to V_{B/I}/L_{B/I} \to 0$ over $B/I$ to an extension over $B$ for $B \to B/I$ surjective with nilpotent kernel. □

**Corollary 3.7.3.** *Define* $\mathrm{Spf}\, R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ *as the scheme theoretic image of $\Theta^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$. Then:*

(a) *If $E/W(\mathbb{F})[1/p]$ is a finite extension and $x\colon R^{\chi,\square}_{V_{\mathbb{F}}} \to E$ is an $E$-valued point, then $x$ factors through $R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ if and only if the two-dimensional $E$-representation of $G_K$ corresponding to $x$ is crystalline and has the form $\begin{pmatrix} \chi\eta & * \\ 0 & \eta^{-1} \end{pmatrix}$ with $\eta\colon G_K \to E^*$ an unramified character.*

(b) *$R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}[1/p]$ is formally smooth over $W(\mathbb{F})[1/p]$.*

(c) *$R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ is a domain unless $V_{\mathbb{F}} \cong \chi_1 \oplus \chi_2$ for distinct characters $\chi_i\colon G_K \to \mathbb{F}^*$ such that $\chi_1|_{I_K} = \chi_2|_{I_K} = \chi|_{I_K}$.*

*Proof.* Let $\mathcal{O}_E$ denote the ring of integers of $E$. Then $x$ arises from an $\mathcal{O}_E$-valued point and $V_x$ has the properties listed in part (a) if and only if $x$ lifts to an $\mathcal{O}_E$-valued point of $\mathcal{L}^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$. By the previous proposition and the valuative criterion for properness, the $\mathcal{O}_E$-valued points of Spec $R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ and of $\mathcal{L}^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ are in bijection. Taking into account the relation between flat deformations and crystalline representations with weights in $\{0,1\}$, cf. Proposition 4.2.1, this proves part (a). Part (b) follows directly from the previous proposition.

The arguments used to prove connectivity in Lemma 3.5.3 can be applied to part (c). Thus the number of connected components of $R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ is in bijection with the number of connected components of $\mathcal{L}^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}} \otimes_{R^{\chi,\square}_{V_{\mathbb{F}}}} \mathbb{F}$. The latter is a subscheme of $\mathbb{P}^1_{\mathbb{F}}$. If $V_{\mathbb{F}}$ is not semisimple, it is a point, and if $V_{\mathbb{F}}$ is scalar then it is all of $\mathbb{P}^1$. If $V_{\mathbb{F}}$ is semisimple but the characters after restriction to inertia are different, then again the subscheme is a point. In the remaining case, it consists of two points. □

**Proposition 3.7.4.** *Let $E$ be a finite totally ramified extension of $W(\mathbb{F})[1/p]$ with ring of integers $\mathcal{O}$. Let $\psi\colon G_K \to \mathcal{O}^*$. Consider now all groupoids over $\mathfrak{Ar}_{\mathcal{O}}$ (or $\mathfrak{Aug}_{\mathcal{O}}$). Then there exists a quotient $R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ of $R^{\square}_{V_{\mathbb{F}}}$ such that:*

(a) *If $E'/E$ is a finite extension and $x\colon R \to E'$ is an $E'$-valued point, then $x$ factors through $R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ if and only if the two-dimensional $E'$-representation of $G_K$ corresponding to $x$ is crystalline and has the form $\left(\begin{smallmatrix} \chi\psi\eta & * \\ 0 & \eta^{-1} \end{smallmatrix}\right)$ with $\eta\colon G_K \to E^*$ an unramified character.*

(b) *$R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}[1/p]$ is formally smooth over $W(\mathbb{F})[1/p]$ of relative dimension equal to $3 + [K : \mathbb{Q}_p]$.*

(c) *$R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ is a domain unless $V_{\mathbb{F}} \cong \chi_1 \oplus \chi_2$ for distinct characters $\chi_i\colon G_K \to \mathbb{F}^*$ such that $\chi_1|_{I_K} = \chi_2|_{I_K} = \chi|_{I_K}$.*

*Proof.* We only give the proof for $p > 2$. Twisting $V_{\mathbb{F}}$ with a square root of $\psi$ reduces us to the case $\psi = 1$ treated in the previous corollary. The only claim that remains to be proved is that on the dimension of $R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}[1/p]$. The computation will be indicated later; cf. Proposition 4.2.4. For ordinary crystalline deformations, so that $V_x$ is an extension of two one-dimensional crystalline representations, the computation is particularly simple. □

## 3.8 Complements

The methods of the previous section on flat (hence weight 2) ordinary deformations may be generalized to ordinary deformations of arbitrary weight. Except for the computation of the Ext-group describing extensions of a twist of the $(k-1)$-th power of the cyclotomic character by an unramified character, there are few changes. One again uses an auxiliary functor $L^{?,\square}_{V_{\mathbb{F}}}$. Therefore we only describe the setting and the result —note however that the computation of the $\mathrm{Ext}^1$-groups and in particularly the surjectivity analogues to Lemmas 3.5.4 and 3.7.1 is in certain cases quite involved.

We fix $K/\mathbb{Q}_p$ an unramified finite extension. We suppose $V_{\mathbb{F}} \cong \begin{pmatrix} \chi_1 & * \\ 0 & \chi_2 \end{pmatrix}$ with $\chi_2$ for a basis $\beta_{\mathbb{F}}$ of $V_{\mathbb{F}}$. We let $\psi\colon G_K \to \mathcal{O}^*$ be an unramified character with $\psi^{1/2}$ well-defined. Let $R^{k,\psi,\square}_{V_{\mathbb{F}}}$ be the universal ring for framed deformations of determinant $\psi\chi^{k-1}$ with $\chi$ the $p$-adic cyclotomic character.

**Theorem 3.8.1** ([KW2, §3.2])**.** *Suppose $2 \le k \le p$ is an integer or that $k = p+1$, $K = \mathbb{F}_p$ and $p > 2$. Then:*

(a) *$R^{k,\psi,\square}$ has a quotient $R^{\mathrm{ord},k,\square}_{V_{\mathbb{F}}}$ for $k > 2$; $R^{\mathrm{ord},\mathrm{flat},\square}_{V_{\mathbb{F}}}$ for $k = 2$ if $V_{\mathbb{F}}$ is flat and $R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ for $k = 2$ if $V_{\mathbb{F}}$ is non-flat, such that, for all $E/\mathcal{O}[1/p]$ finite and for all $x\colon R^{k,\psi,\square}_{V_{\mathbb{F}}} \to E$, the following equivalences hold:*

  (i) *If $k > 2$, then $x$ factors through $R^{\mathrm{ord},k,\square}_{V_{\mathbb{F}}}$ if and only if $V_x \sim \begin{pmatrix} \widetilde{\chi}_1 & * \\ 0 & \widetilde{\chi}_2 \end{pmatrix}$ with $\widetilde{\chi}_2$ unramified.*

  (ii) *If $k = 2$ and $V_{\mathbb{F}}$ is flat, then $x$ factors through $R^{\mathrm{ord},\mathrm{flat},\square}_{V_{\mathbb{F}}}$ if and only if $V_x \sim \begin{pmatrix} \widetilde{\chi}_1 & * \\ 0 & \widetilde{\chi}_2 \end{pmatrix}$ and $V_x$ is flat.*

  (iii) *If $k = 2$ and $V_{\mathbb{F}}$ is non-flat, then $x$ factors through $R^{\mathrm{ord},\chi,\square}_{V_{\mathbb{F}}}$ if and only if $V_x \sim \begin{pmatrix} \chi\psi^{1/2} & * \\ 0 & \psi^{1/2} \end{pmatrix}$.*

  *Note that, as a quotient of $R^{k,\psi,\square}$, one automatically has $\det V_x = \psi\chi^{k-1}$.*

(b) *Except for $k > 2$ and $\chi_1\chi_2^{-1} = \chi$, the rings $R^{\mathrm{ord},?,\square}_{V_{\mathbb{F}}}$ just defined are formally smooth over $\mathbb{Q}_p$ and of relative dimension $3 + [K : \mathbb{Q}_p]$, unless they are $0$.*

(c) *The number of components of $R^{\mathrm{ord},?,\square}_{V_{\mathbb{F}}}$ is given as follows:*

  (i) *Case $? = k > 2$: the number is $2$ if $\chi_1$ is unramified and $* = \mathrm{triv}$, else it is $1$.*

  (ii) *Case $? = \mathrm{flat}$: the number is $2$ if $\chi_1$ is unramified and $* = \mathrm{triv}$, else it is $1$.*

  (iii) *Case $? = \chi$: the number is one.*

## 3.9 Appendix

### 3.9.1 The canonical subgroups of the absolute Galois group of a local field

Let $k$ be the residue field of the finite extension $F \supset \mathbb{Q}_\ell$. The extension $F^{\mathrm{nr}} \supset F$ is Galois and one has

$$\mathrm{Gal}\left(F^{\mathrm{nr}}/F\right) \cong \mathrm{Gal}\left(\overline{k}/k\right) \cong \widehat{\mathbb{Z}},$$

where the first isomorphism is the canonical one arising from reduction.

The absolute Galois group $G_F$ of $F$ admits two canonical subgroups. First there is the *inertia subgroup* $I_F$ of $F$, which is the kernel of the surjective homomorphism $G_F \twoheadrightarrow \mathrm{Gal}(F^{\mathrm{nr}}/F)$, i.e., one has the short exact sequence

$$1 \longrightarrow I_F \longrightarrow G_F \longrightarrow \mathrm{Gal}\left(\overline{k}/k\right) \longrightarrow 1.$$

Second, by the structure theory of $I_F$, the pro-$\ell$ Sylow subgroup $P_F$ of $I_F$ is a closed normal subgroup, the *wild ramification subgroup* of $F$. The quotient $I_F^t := I_F/P_F$ is the *tame quotient* of $I_F$; it is isomorphic to $\prod_{q\neq\ell}\mathbb{Z}_q$ where the product is over all rational primes $q \neq \ell$. Thus, one has a short exact sequence

$$1 \longrightarrow P_F \longrightarrow I_F \longrightarrow \prod_{q\neq\ell}\mathbb{Z}_q \longrightarrow 1.$$

The subgroup $P_F$ is also normal in $G_F$: it is *the* pro-$\ell$ Sylow subgroup of $I_F$.

Suppose now that $\ell \neq p$. In order to focus on representations of $G_F$ into $\mathrm{GL}_d$ over rings in $\mathfrak{Ar}_{W(\mathbb{F})}$ it is convenient to introduce the subgroup $P_{F,p}$ of $I_F$ as follows. From the structure theory of $I_F$ it is apparent that there is a surjective homomorphism $t_p\colon I_F \twoheadrightarrow \mathbb{Z}_p$. Introduce $P_{F,p}$ as the kernel, so that one has the exact sequence

$$1 \longrightarrow P_{F,p} \longrightarrow I_F \longrightarrow \mathbb{Z}_p \longrightarrow 1.$$

The quotient $\mathbb{Z}_p$ carries an action of $\mathrm{Gal}(F^{\mathrm{nr}}/F)$ via the cyclotomic character; this is immediate from the Kummer isomorphism

$$\mathbb{Z}_p \cong \mathrm{Gal}\Big(\bigcup_n F^{\mathrm{nr}}\big(\sqrt[p^n]{p}\,\big)/F^{\mathrm{nr}}\Big).$$

### 3.9.2 Galois cohomology

We recall some results on the cohomology of $G_F$ for $F \supset \mathbb{Q}_\ell$ finite, where $\ell$ can be equal to or different from $p$; see [NSW]. For $\ell \neq p$ we set $[F:\mathbb{Q}_p] = 0$ —for $\ell = p$ it is, as usual, the extension degree of $F$ over $\mathbb{Q}_p$. Let $E$ be either a finite extension of $\mathbb{F}_p$ or $\mathbb{Q}_p$, and let $V$ be a continuous representation of $G_F$ on a finite-dimensional $E$-vector space. Recall that $V^\vee = \mathrm{Hom}(V, E(1))$. We write $h^i(F,V) = \dim_E H^i(G_F, V)$.

**Theorem 3.9.1** (Tate)**.** *The groups $H^i(G_F,V)$ are finite-dimensional $E$-vector spaces for $i \in \mathbb{Z}$ and zero for $i \notin \{0,1,2\}$. Moreover:*

$$h^0(F,V) - h^1(F,V) + h^2(F,V) = -[F:\mathbb{Q}_p]\dim_E V, \text{ and } h^2(F,V) = h^0(F,V^\vee).$$

**Corollary 3.9.2.** *Suppose $\dim_E V = 1$. Then $h^1(F,V) = [F:\mathbb{Q}_p]$ unless $V$ is trivial or $G_F$ acts via the (mod $p$) cyclotomic character (if $E$ is finite).*

Let $\rho\colon G_F \to \mathrm{GL}_2(E)$, $g \mapsto \left(\begin{smallmatrix} \eta & b \\ 0 & \lambda \end{smallmatrix}\right)$ be a reducible continuous representation for characters $\eta, \lambda\colon G_F \to E^*$. Then

$$b \in \mathrm{Ext}^1_{G_F}(\eta, \lambda) \cong \mathrm{Ext}^1_{G_F}(\mathbb{1}, \lambda\eta^{-1}) \cong H^1(G_F, \lambda\eta^{-1}).$$

The class $b$ is trivial if and only if the extension splits, i.e., if and only if $\rho$ is semisimple.

**Corollary 3.9.3.** *Suppose $\ell \neq p$. If $\rho$ is non-split, then $\eta\lambda^{-1}$ is trivial or the cyclotomic character (mod $p$).*

Let $F^{\mathrm{nr}}$ be the maximal unramified extension of $F$ inside a fixed algebraic closure $\overline{F}$ of $F$.

**Proposition 3.9.4.** *Suppose that* $\dim_E V = 1$ *and $V$ is unramified, i.e., $I_F$ acts trivially on $V$. Then the groups $H^i(\mathrm{Gal}(F^{\mathrm{nr}}/F), V)$, $i \in \mathbb{Z}$, are finite-dimensional over $E$. Moreover $H^1(\mathrm{Gal}(F^{\mathrm{nr}}/F), V) = 0$ unless $V$ is trivial, in which case* $h^1(\mathrm{Gal}(F^{\mathrm{nr}}/F), V) = 1$.

**Corollary 3.9.5.** *Any unramified 2-dimensional representation is either split or an extension of an unramified character by itself.*

Combining the previous two corollaries, one deduces:

**Corollary 3.9.6.** *Suppose $\ell \neq p$. If $\rho$ as above is non-split and not unramified up to twist, then $\eta\lambda^{-1}$ is the cyclotomic character (mod $p$).*

### 3.9.3 Weil–Deligne representations

Let $F$ be a finite extension of $\mathbb{Q}_\ell$ as above with $\ell \neq p$ and residue field $k$. Consider the canonical homomorphism $\pi\colon G_F \to G_k$. The arithmetic Frobenius automorphism $\sigma$ is a canonical topological generator of $G_k$. The Weil group of $W_F$ is defined as $\pi^{-1}(\sigma^{\mathbb{Z}})$, so that one has a short exact sequence $1 \to I_F \to W_F \to \sigma^{\mathbb{Z}} \to 1$. Let $q = |k|$ and define

$$||-||\colon W_F \longrightarrow \mathbb{Q},\ \sigma^n g \longmapsto q^n$$

for $g \in I_F$ and $n \in \mathbb{Z}$.

**Definition 3.9.7.** Let $L$ be a field of characteristic 0 equipped with the discrete topology. *A Weil–Deligne representation*[3] over $L$ is a triple $(V_L, \rho', N)$ such that

(a) $V_L$ is a finite-dimensional $L$-vector space,

(b) $\rho'\colon W_F \to \mathrm{Aut}_L(V_L)$ is a continuous representation with respect to the discrete topology on $V_L$, and

[3] We follow the conventions of [Tat2, §4] except that we express everything in terms of an arithmetic Frobenius.

(c) $N$ is a nilpotent endomorphism of $V_L$ such that $\rho'(w)N\rho'(w)^{-1} = ||w||N$ for all $w \in W_F$.

If $L$ is a complete discretely valued field, then $A \in \mathrm{Aut}_L(V_L)$ is called *bounded* if all its eigenvalues have valuation zero, or equivalently if the characteristic polynomials of $A$ and $A^{-1}$ have coefficients in the ring of integers of $L$. The Weil–Deligne representation $(V_L, \rho', N)$ is called *bounded* if $\rho'(\sigma)$ is bounded.

Observe that condition (b) is equivalent to the assertion that $\mathrm{Ker}(\rho'|_{I_F})$ is finite.

Let $E$ be a $p$-adic field. For any continuous representation $\rho\colon G_F \to \mathrm{Aut}(V_E)$, where $V_E$ is a finite-dimensional $E$-vector space, we may consider its restriction to $W_F$. One has the following important and elementary result due to Deligne:

**Theorem 3.9.8** (Deligne)**.** *The following assignment sets up a bijection between pairs $(V_E, \rho)$, where $V_E$ is a finite-dimensional $E$-vector space with the $p$-adic topology and $\rho\colon W_F \to \mathrm{Aut}_E(V_E)$ is a continuous representation, and Weil–Deligne representations $(V_E, \rho', N)$ over $E$ (where, as in (b) above, $V_E$ is given the discrete topology). Given $(V_E, \rho', N)$, one defines*

$$\rho(\sigma^n g) = \rho'(\sigma^n g)\exp(t_p(g)N) \quad \text{for } g \in I_F,\ n \in \mathbb{Z}.$$

The key step in the proof is Theorem 3.2.1, due to Grothendieck. The assignment $(V_E, \rho', N) \mapsto (V_E, \rho)$ is less explicit. It can be deduced from the proof of Theorem 3.2.1.

The advantage of the Weil–Deligne representation associated to a $p$-adic representation is that it can be expressed without any use of the $p$-adic topology involved —at the expense of introducing $N$. The concept is enormously important in the definition of a strictly compatible system of Galois representations to have a good description also at the ramified places! For instance, let $E/\mathbb{Q}$ be an elliptic curve with semistable but bad reduction at the prime $\ell$. For any prime $p \neq \ell$, consider the representation $V_p$ of $G_{\mathbb{Q}}$ on the $p$-adic Tate module of $A$. Then the action of $I_{\mathbb{Q}_\ell}$ on $V_p$ is unipotent and non-trivial, i.e., it is a non-trivial action via the unique quotient $\mathbb{Z}_p$ of $I_{\mathbb{Q}_\ell}$. In particular, the representation depends on $p$. However, as the reader may verify, the associated Weil–Deligne representation is independent of $p$. One has $\rho'(I_p) = 1$ and $N \sim \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.

It is possible to describe a Weil–Deligne representation also as a representation between algebraic groups. For this, one needs to define the Weil–Deligne group $\mathrm{WD}_F$. As a group one has $W_F = \varprojlim_{J \subset I_F} W_F/J$, where $J$ ranges over the open subgroups of $I_F$. For any discrete group $H$ (which may be infinite) and any ring $R$, denote by $\underline{H}_R$ the constant group scheme on $R$ with group $H$. Then the group schemes

$$\underline{W_F/J}_{\mathbb{Q}}, \ J \subset I_F \text{ an open subgroup},$$

form an inverse system. One defines

$$\underline{W_F}_{\mathbb{Q}} = \varprojlim_J \underline{W_F/J}_{\mathbb{Q}}.$$

Suppose $E$ is any field of characteristic zero. Then a homomorphism of algebraic groups $\underline{W_F}_E \to \mathrm{Aut}_E(V_E)$ will factor via $\underline{W_F/J}$ for some open $J \subset I_F$ and hence is nothing else but a continuous representation $W_F \to \mathrm{Aut}(V_E)$, where $V_E$ carries the discrete topology.

**Definition 3.9.9.** The *Weil–Deligne group* is the semidirect product

$$(\mathrm{WD}_F)_{\mathbb{Q}} = \mathbb{G}_a \rtimes \underline{W_F}_{\mathbb{Q}} = \mathbb{G}_a \rtimes \bigcup_{n\in\mathbb{Z}} \sigma^n \underline{I_F}_{\mathbb{Q}},$$

where multiplication is obtained as the inverse limit of the action of $W_F/J$ on $\mathbb{G}_a$ defined by

$$(a,w)\cdot(a',w') = (a + a'||w||, ww'), \text{ where } a, a' \in \mathbb{G}_a(R),\ w, w' \in W_F/J.$$

The following result is straightforward:

**Proposition 3.9.10.** *For any (discrete) field $E$ of characteristic zero, there is a canonical bijection between $d$-dimensional Weil–Deligne representations and algebraic representations* $(\mathrm{WD}_F)_E \to \mathrm{GL}_{d,E}$.

For further background we refer to [Tat2].

### 3.9.4 Finite flat group schemes

Let $R$ be a commutative ring (or a scheme). By a *finite flat group scheme over* $R$ one means a group scheme $\mathcal{G}$ which is finite flat over $R$. In particular $\mathcal{G}$ is affine. Let $\mathcal{A}$ denote its coordinate ring. It is a locally free (sheaf of) algebra(s) over $R$. The rank of $\mathcal{G}$ is the rank of $\mathcal{A}$ over $R$. The group scheme structure on $\mathcal{G}$ translates into a cocommutative Hopf algebra structure on $\mathcal{A}$. This means that $\mathcal{A}$ is an $R$-algebra equipped with $R$-linear maps $\mu\colon \mathcal{A} \to \mathcal{A} \otimes_R \mathcal{A}$ (comultiplication), $\varepsilon\colon \mathcal{A} \to R$ (counit), $\iota\colon \mathcal{A} \to \mathcal{A}$ (coinverse) satisfying axioms which are dual to those satisfied by a group (scheme).

**Example 3.9.11.** (a) For an abstract finite group $\Gamma$, the ring $\mathcal{A} = \mathrm{Maps}(\Gamma, R)$ is naturally an $R$-algebra. Moreover, with $\mu(f)(s,t) = f(st)$ as comultiplication, $\varepsilon(f)(s) = f(1)$ as counit and $\iota(f)(s) = f(s^{-1})$ as coinverse, it is a cocommutative Hopf algebra.

(b) Let $\mathcal{A} = R[X]/(X^m - 1)$ with $\mu(X) = X \otimes X$, $\iota(X) = X^{-1}$ and $\varepsilon(X) = 1$. This defines the multiplicative group scheme $\mu_m$. It is étale over $R$ if and only if $m$ is invertible in $R$.

For a fixed finite extension $K$ of $\mathbb{Q}_p$ we now present some basic properties on finite flat group schemes over $\mathcal{O}_K$ and flat representations of $G_K$.

A *flat representation of* $G_K$ is a continuous representation of $G_K$ on a finite abelian group $V$ such that there exists a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$ so that

$V \cong \mathcal{G}(\overline{K})$ as a $\mathbb{Z}[G_K]$-module. Such a representation on $V$ can be decomposed into its primary parts. Flatness for components of order prime to $p$ is characterized by the following result —it will not be needed in the main part of the lecture, but we include it for completeness:

**Proposition 3.9.12** ([Sha, §4, Corollary 3])**.** *Suppose $\mathcal{G}$ is a finite flat group scheme over $\mathcal{O}_K$ of order prime to $p$. Then the following three equivalent conditions hold:*

(a) *$\mathcal{G}$ is étale.*

(b) *The action of $G_K$ on $\mathcal{G}(\overline{K})$ is via $\pi_1(\operatorname{Spec}\mathcal{O}_K)$.*

(c) *The action of $G_K$ is unramified.*

*Conversely (see Exercise* 3.10.5*), any unramified continuous representation of $G_K$ on a finite abelian group is flat.*

From now on we assume that $V$ is of $p$-power order. The following descent result is presumably well known. Lacking a reference, we give a proof. Its idea was suggested to us by J.-P. Wintenberger.

**Lemma 3.9.13.** *Suppose that $V$ is a continuous linear representation of $G_K$ on a finite abelian $p$-group. If $V$ is flat over $K^{\mathrm{nr}}$, then it is flat over $K$.*

*Proof.* The Hopf algebra over $\mathcal{O}_{K^{\mathrm{nr}}}$ giving the flatness of $V$ restricted to $G_{K^{\mathrm{nr}}}$ is already defined over a finite unramified extension $L/K$ such that $G_L$ acts trivially on $V$. Let $\mathcal{A}_{\mathcal{O}_L}$ denote a Hopf algebra over $\mathcal{O}_L$ whose associated finite flat group scheme $\mathcal{G}_{\mathcal{O}_L}$ satisfies $\mathcal{G}_{\mathcal{O}_L}(\overline{K}) \cong V$ as $\mathbb{Z}[G_L]$-modules. Define $\mathcal{A}_L$ as $\mathcal{A}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} L$. By Exercise 3.10.5(c), the invariants of $\mathcal{A}_L$ under $\mathrm{Gal}(L/K)$ form a finite Hopf algebra $\mathcal{A}_K$ over $K$ such that $\mathcal{G}_K(\overline{K}) \cong V$ as $\mathbb{Z}[G_K]$-modules for the associated group scheme $\mathcal{G}_K$.

Define $\mathcal{A}_{\mathcal{O}_K} = (\mathcal{A}_{\mathcal{O}_L})^{\mathrm{Gal}(L/K)}$. We shall prove that

$$\mathcal{A}_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathcal{O}_L \cong \mathcal{A}_{\mathcal{O}_L} \tag{3.9.1}$$

is an isomorphism under the naturally given homomorphism (which regards $\mathcal{A}_{\mathcal{O}_K}$ as a subring of $\mathcal{A}_{\mathcal{O}_L}$ and $\mathcal{A}_{\mathcal{O}_L}$ as a $\mathcal{O}_L$-algebra). In other words, we shall show that $\mathcal{A}_{\mathcal{O}_L}$ satisfies Galois descent for $\mathrm{Gal}(L/K)$; see for example [Wa, §17]. By Galois descent one also shows that the Hopf algebra structure descends from $\mathcal{A}_{\mathcal{O}_L}$ to $\mathcal{A}_{\mathcal{O}_K} = \mathcal{A}_K \cap \mathcal{A}_{\mathcal{O}_L}$. For instance, to see that the comultiplication descends to $\mathcal{A}_{\mathcal{O}_K}$, one may proceed as follows. Since comultiplication on $\mathcal{A}_L$ arises by base change from $\mathcal{A}_K$, the comultiplication $\mu\colon \mathcal{A}_L \otimes_L \mathcal{A}_L \to \mathcal{A}_L$ is Galois equivariant. Its restriction to $\mathcal{A}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathcal{A}_{\mathcal{O}_L}$ maps to $\mathcal{A}_{\mathcal{O}_L}$. Hence it induces an $\mathcal{O}_K$-linear homomorphism $(\mathcal{A}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} \mathcal{A}_{\mathcal{O}_L})^{\mathrm{Gal}(L/K)} \to (\mathcal{A}_{\mathcal{O}_L})^{\mathrm{Gal}(L/K)}$. However (3.9.1) allows us to identify the left-hand side with $\mathcal{A}_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathcal{A}_{\mathcal{O}_K}$. Further details are left to the reader.

We now prove (3.9.1). Let $\pi$ denote a uniformizer of $\mathcal{O}_K$. Because $L/K$ is unramified, $\pi$ is also a uniformizer of $\mathcal{O}_L$. We shall prove by induction on $n$ that for all $n$ one has a natural isomorphism

$$(\mathcal{A}_{\mathcal{O}_L}/\pi^n\mathcal{A}_{\mathcal{O}_L})^{\mathrm{Gal}(L/K)} \otimes_{\mathcal{O}_K/\pi^n\mathcal{O}_K} \mathcal{O}_L/\pi^n\mathcal{O}_L \cong \mathcal{A}_{\mathcal{O}_L}/\pi^n\mathcal{A}_{\mathcal{O}_L}. \tag{3.9.2}$$

For $n = 1$ recall that, by Hilbert 90, or rather the normal basis theorem, one has $k_L \cong k_K[\mathrm{Gal}(L/K)]$ as Galois modules, where $k_L$ and $k_K$ denote the residue fields of $L$ and $K$, respectively. Since $\mathcal{A}_{\mathcal{O}_L}$ is a free $\mathcal{O}_L$-module, say of rank $r$, it follows that

$$\mathcal{A}_{\mathcal{O}_L}/\pi\mathcal{A}_{\mathcal{O}_L} \cong k_L^r \cong k_K\big[\,\mathrm{Gal}(L/K)]^r \cong k_K^r \otimes_{k_K} k_K[\mathrm{Gal}(L/K)]$$

as $k_K[G]$-modules. One immediately deduces (3.9.2) for $n = 1$. For the induction step, consider the sequence

$$0 \longrightarrow \mathcal{A}_{\mathcal{O}_L}/\pi\mathcal{A}_{\mathcal{O}_L} \xrightarrow{\pi^n} \mathcal{A}_{\mathcal{O}_L}/\pi^{n+1}\mathcal{A}_{\mathcal{O}_L} \longrightarrow \mathcal{A}_{\mathcal{O}_L}/\pi^n\mathcal{A}_{\mathcal{O}_L} \longrightarrow 0.$$

Abbreviate $G = \mathrm{Gal}(L/K)$. Observe first that taking $G$-invariants is exact. This is so because the group $H^1(G, \mathcal{A}_{\mathcal{O}_L}/\pi\mathcal{A}_{\mathcal{O}_L})$ vanishes —again by the normal basis theorem. Tensoring the resulting short exact sequence with $\mathcal{O}_L$ over $\mathcal{O}_K$ and comparing it with the given sequence yields

$$\begin{array}{ccccc}
(\mathcal{A}_{\mathcal{O}_L}/\pi\mathcal{A}_{\mathcal{O}_L})^G \otimes_{\mathcal{O}_K} \mathcal{O}_L & \xrightarrow{\pi^n} & (\mathcal{A}_{\mathcal{O}_L}/\pi^{n+1}\mathcal{A}_{\mathcal{O}_L})^G \otimes_{\mathcal{O}_K} \mathcal{O}_L & \longrightarrow & (\mathcal{A}_{\mathcal{O}_L}/\pi^n\mathcal{A}_{\mathcal{O}_L})^G \otimes_{\mathcal{O}_K} \mathcal{O}_L \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A}_{\mathcal{O}_L}/\pi\mathcal{A}_{\mathcal{O}_L} & \xrightarrow{\pi^n} & \mathcal{A}_{\mathcal{O}_L}/\pi^{n+1}\mathcal{A}_{\mathcal{O}_L} & \longrightarrow & \mathcal{A}_{\mathcal{O}_L}/\pi^n\mathcal{A}_{\mathcal{O}_L}.
\end{array}$$

By induction hypothesis, the right and left vertical arrows are isomorphisms. By the Snake Lemma, the same follows for the middle term. This proves (3.9.2). The isomorphism (3.9.1) now follows by taking the inverse limit of (3.9.2). □

**Corollary 3.9.14.** *If $M$ is a finite continuous $\mathbb{Z}[\mathrm{Gal}(K^{\mathrm{nr}}/K)]$-module and $\mathcal{G}$ is a finite flat group scheme over $\mathcal{O}_K$, then the representation $M \otimes \mathcal{G}(\overline{K})$ arises from a finite flat group scheme. In particular, $M$ arises from a finite flat group scheme.*

*Proof.* Let $V$ be the representation of $G_K$ on $\mathcal{G}(\overline{K}) \otimes M$. Because $M$ is discrete and unramified, there is a finite unramified extension $L/K$ over which $M$ becomes trivial. Since $\mathcal{G}$, being flat over $\mathcal{O}_K$, will also be flat over $\mathcal{O}_L$, we may apply the previous result to $V$.

For the second assertion, note that for any $n \in \mathbb{N}$ the trivial $G_K$-module $\mathbb{Z}/(p^n)$ arises from a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$; cf. Exercise 3.10.5(d). □

Write $\Gamma_K = \mathrm{Gal}(K^{\mathrm{nr}}/K)$. Our next aim is to provide some background on $H^1_f(G_K, M(1))$ as defined in (3.7.1) for $M$ a discrete (possibly infinite) representation of $\mathbb{Z}_p[\mathrm{Gal}(K^{\mathrm{nr}}/K)]$. Recall that the group schemes $\mu_{p^n}$ are flat over any ring. The following result follows from [KM, Prop. 8.10.5] and its proof.

**Proposition 3.9.15** ([KM]). *The group $H^1_f(G_K, \mathbb{Z}_{p^n}(1)) \cong \mathcal{O}_K^*/\mathcal{O}_K^{*p^n}$ is naturally isomorphic to the group of flat extensions*

$$0 \longrightarrow \mu_{p^n,K} \longrightarrow V \longrightarrow \mathbb{Z}/p^n\mathbb{Z} \longrightarrow 0$$

*of $G_K$-modules such that $V$ is $p^n$-torsion within the group*

$$H^1(G_K, \mathbb{Z}_{p^n}(1)) \cong K^*/K^{*p^n}$$

*of all such extensions of $G_K$-modules.*

Note that the identification of the group of all extensions $0 \to \mu_{p^n}(\overline{K}) \to V \to \mathbb{Z}/p^n\mathbb{Z} \to 0$ of $G_K$-modules with $H^1(G_K, \mathbb{Z}_{p^n}(1))$ is via Kummer theory.

The result in [KM] is based on the construction of a universal flat extension of $\mathbb{Z}/p^n$ by $\mu_{p^n}$ over $\mathbb{Z}[x^{\pm 1}]$. To have such a flat extension defined over a ring $R$, such as $R = \mathcal{O}_K$, the image of $x$ has to be a unit in $R$.

**Proposition 3.9.16.** *Let $M$ be any finite discrete $\Gamma_K$-module which is $p^n$-torsion, so that in particular $M$ is flat. Then the group $H^1_f(G_K, M(1))$ is naturally isomorphic to the group of flat extensions*

$$0 \longrightarrow \mu_{p^n,K} \longrightarrow V \longrightarrow M \longrightarrow 0$$

*of $G_K$-modules as a subgroup of the group $H^1(G_K, M(1))$ of all such extensions of $G_K$-modules.*

If $M = \varinjlim_{i \in I} M_i$ for finite abelian $p$-groups $M_i$ with a linear action of $\Gamma_K$, the lemma gives a corresponding interpretation for $H^1_f(G_K, M(1))$.

*Proof.* By [Tat1, § 4.2], the map that associates to any flat extension $0 \longrightarrow \mu_{p^n,K} \longrightarrow V \longrightarrow M \longrightarrow 0$ the corresponding extension of $G_K$-modules on the generic fiber is injective. Hence any flat extension is described by a unique class $c$ in $H^1(G_K, M(1))$. By Lemma 3.9.13, the extension is flat if and only if it is flat over some, or any unramified extension of $K$. Passing to a suitable such extension, we may assume that $\Gamma_K$ acts trivially on $M$. So then $M$ is a finite direct sum of trivial group schemes $\mathbb{Z}/p^i\mathbb{Z}$. But then by Proposition 3.9.15, flatness of $c$ is equivalent to being a class in $\mathcal{O}_K^* \otimes_{\mathbb{Z}} M \subset K^* \otimes_{\mathbb{Z}} M$. Again by Lemma 3.9.13 we can pass to the limit over all unramified extensions of $K$, and hence $c$ is flat if and only if its image lies $\mathcal{O}_{K^{\mathrm{nr}}}^* \otimes_{\mathbb{Z}} M \subset K^{\mathrm{nr}*} \otimes_{\mathbb{Z}} M$. By the definition of $H^1_f$, the latter condition is equivalent to $c \in H^1_f(G_K.M(1))$. □

## 3.10 Exercises

*Exercise* 3.10.1. Let $\ell \neq p$ and $R = \mathbb{Z}_p[[x]]$. Construct a continuous representation $\rho\colon (\widehat{G_{\mathbb{Q}_\ell}})^p \to \mathrm{GL}_2(R)$ such that there are two closed points $x, y$ on the generic fiber $\operatorname{Spec} R[1/p]$ whose logarithmic monodromy satisfies $N_x = 0$ and $N_y \neq 0$.

*Exercise* 3.10.2. Let $\bar{\rho}\colon G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be the trivial representation. Determine, depending on the (non-zero) residue of $\ell \bmod p$, the set of all bounded Weil–Deligne representations $(V_{\overline{\mathbb{Q}}_p}, \rho', N)$ whose corresponding $p$-adic representation admits a model $G_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\overline{\mathbb{Z}}_p)$ whose mod $p$ reduction is $\bar{\rho}$.

*Exercise* 3.10.3. (a) Let $\mathfrak{Aug}_{\mathbb{F}}$ be the full subcategory of $\mathfrak{Aug}_{W(\mathbb{F})}$ whose objects are pairs $(A \to B)$ in $\mathfrak{Aug}_{W(\mathbb{F})}$ with $A = \mathbb{F}$. Prove that the restriction of the groupoid $L^{\chi,\square}_{V_{\mathbb{F}}}$ to $\mathfrak{Aug}_{\mathbb{F}}$ is either the scheme $\operatorname{Spec}\mathbb{F}$ or the scheme $\mathbb{P}^1_{\mathbb{F}}$, depending on $V_{\mathbb{F}}$.

(b) Let $\mathfrak{Aug}'_{\mathbb{F}}$ be the full subcategory of $\mathfrak{Aug}_{\mathbb{F}}$ whose objects are pairs $(\mathbb{F} \to B)$ in $\mathfrak{Aug}_{\mathbb{F}}$ such that $B$ is in $\mathfrak{Ar}_{W(\mathbb{F})}$. Prove that if $L^{\chi,\square}_{V_{\mathbb{F}}}$ over $\mathfrak{Aug}_{\mathbb{F}}$ is represented by the scheme $\mathbb{P}^1_{\mathbb{F}}$, then $L^{\chi,\square}_{V_{\mathbb{F}}}$ over $\mathfrak{Aug}'_{\mathbb{F}}$ is represented by the 0-dimensional scheme $\varinjlim_{X \subset \mathbb{P}^1_{\mathbb{F}}} X$, where $X$ runs over the zero-dimensional (not necessarily reduced) subscheme of $\mathbb{P}^1_{\mathbb{F}}$.

*Exercise* 3.10.4. Compute $h^1(K, \mathbb{Q}_p(n))$ and $h^1_{\mathrm{cris}}(K, \mathbb{Q}_p(n))$ for a finite extension $K$ of $\mathbb{Q}_p$, for all $n \in \mathbb{Z}$. *Hints:* Use without proof the following results; cf. [Nek].

(a) The dimension formulae derived from Tate's local duality theory; cf. Theorem 3.9.1.

(b) For $V$ a crystalline representation and $D_{\mathrm{cris}}(V)$ its associated filtered $\varphi$-module[4] one has

$$\begin{aligned} h^1_{\mathrm{cris}}(K, V) &= \dim_{\mathbb{Q}_p} \mathrm{Ext}^1_{\mathrm{cris}}(\mathbb{1}, V) \\ &= h^0(K, V) + [K : \mathbb{Q}_p](\dim_{\mathbb{Q}_p} V - \dim_K \mathrm{Fil}^0(D_{\mathrm{cris}}(V))). \end{aligned}$$

(c) For $K_0 \subset K$ the maximal subfield unramified over $\mathbb{Q}_p$ and $\sigma$ the Frobenius automorphism of $K_0$ one has $D_{\mathrm{cris}}(\mathbb{Q}_p(n)) = (K_0, \varphi = p^{-n}\sigma, \mathrm{Fil}^{-n} = K, \mathrm{Fil}^{-n+1} = 0)$.

*Exercise* 3.10.5. Let $K$ be a finite extension of $\mathbb{Q}_p$ and let $V$ be a finite abelian group carrying a continuous linear action of $G_K$. Let $L/K$ be a finite Galois extension where $G_L$ acts trivially on $V$. Define a flat $\mathcal{O}_L$-algebra $\mathcal{A}_{\mathcal{O}_L} = \mathrm{Maps}(V, \mathcal{O}_L)$ as in Example 3.9.11(a). Show the following:

(a) The algebra $\mathcal{A}_{\mathcal{O}_L}$ is the cocommutative Hopf algebra underlying a finite flat group scheme $\mathcal{G}_{\mathcal{O}_L}$ over $\mathcal{O}_L$.

(b) If for $f\colon V \to \mathcal{O}_L$ and $g \in \mathrm{Gal}(L/K)$ one defines $(gf)(v) = g(f(g^{-1}v))$, then this defines an action of $\mathrm{Gal}(L/K)$ on $\mathcal{A}_{\mathcal{O}_L}$ which is compatible with the Hopf algebra structure.

[4]See Section 4.2 and Appendix 4.6.2.

(c) The $L$-algebra $\mathcal{A}_L = \mathcal{A}_{\mathcal{O}_L} \otimes_{\mathcal{O}_L} L$ inherits a Hopf algebra structure from $\mathcal{A}_{\mathcal{O}_L}$. The invariants $(\mathcal{A}_L)^{\mathrm{Gal}(L/K)}$ form a cocommutative Hopf algebra over $K$ defining a finite flat group scheme $\mathcal{G}_K$ over $\operatorname{Spec} K$ such that $\mathcal{G}_K(\overline{K}) \cong V$ as a $\mathbb{Z}[G_K]$-module and $\mathcal{G}_K \times_{\operatorname{Spec} K} \operatorname{Spec} L \cong \mathcal{G}_{\mathcal{O}_L} \times_{\operatorname{Spec} \mathcal{O}_L} \operatorname{Spec} L$.

(d) Suppose $L/K$ is *unramified*. Then $\mathcal{A}_{\mathcal{O}_K} = (\mathcal{A}_{\mathcal{O}_L})^{\mathrm{Gal}(L/K)}$ is the cocommutative Hopf algebra underlying a finite flat group scheme $\mathcal{G}_{\mathcal{O}_K}$ over $\operatorname{Spec} \mathcal{O}_K$ with generic fiber $\mathcal{G}_K$ and base change to $\mathcal{O}_L$ isomorphic to $\mathcal{G}_{\mathcal{O}_L}$.

*Hint:* Part (c) is proved by étale descent: one may use the additive Hilbert 90 theorem for $L/K$, which implies that $L \cong K[\mathrm{Gal}(L/K)]$ as a Galois module. For (d) one needs to show that the canonical homomorphism $\mathcal{A}_{\mathcal{O}_K} \otimes_{\mathcal{O}_K} \mathcal{O}_L \to \mathcal{A}_{\mathcal{O}_L}$ is an isomorphism, i.e., that $\mathcal{A}_{\mathcal{O}_L}$ descends to $\mathcal{O}_K$. The argument is similar to the proof of Proposition 3.9.13.

*Exercise* 3.10.6. Adapt the proof of Theorem 2.2.1 to show the following. Let $E/\mathbb{Q}_p$ be finite, $V_E$ a continuous absolutely irreducible $G$-representation of $E$, and $B \in \mathfrak{Ar}_E$. Suppose that $V_B, V_B'$ are deformations of $V_E$ to $B$ such that $\operatorname{Tr}(\sigma|V_B) = \operatorname{Tr}(\sigma|V_{B'})$ for all $\sigma \in G$. Then $V_B$ and $V_B'$ are isomorphic deformations.

# Lecture 4

# Flat deformations

We follow mostly [Ki7]. Some parts are motivated by the course of L. Berger on $p$-adic Galois representation and discussions with K. Fujiwara and J.-P. Wintenberger during the course. As a reference for much of the $p$-adic Galois representations, the lecture notes [Ber1] by L. Berger are highly recommended.

The appendix for this chapter summarizes very briefly some basic results and definitions on $p$-divisible groups, on (weakly admissible) filtered $\varphi$-modules and on Fontaine–Laffaille modules.

## 4.1 Flat deformations

Let $K/\mathbb{Q}_p$ be a finite extension field with residue field $k$; write $W = W(k)$ and $K_0 = W[1/p]$; fix an algebraic closure $\overline{K}$ of $K$, and let $G_K = \mathrm{Gal}(\overline{K}/K)$. Denote by $V_{\mathbb{F}}$ a continuous representation of $G_K$ on a finite-dimensional $\mathbb{F}$-vector space.

A representation $V$ of $G_K$ on a finite abelian $p$-group is called *flat* if it arises from a finite flat group scheme $\mathcal{G}$ over $\mathcal{O}_K$, i.e., if $V \cong \mathcal{G}(\overline{K})$ as $\mathbb{Z}[G_K]$-modules.

The following result is essentially due to Ramakrishna [Ram].

**Proposition 4.1.1.** *Let $A$ be in $\mathfrak{A}_{\mathfrak{r}W(\mathbb{F})}$ and $V_A$ in $D_{V_{\mathbb{F}}}(A)$. There exists a quotient $A^{\mathrm{flat}}$ of $A$ such that, for any morphism $A \to A'$ in $\mathfrak{A}_{\mathfrak{r}W(\mathbb{F})}$, $V_{A'} = V_A \otimes_A A'$ is flat if and only if $A \to A'$ factors through $A^{\mathrm{flat}}$.*

*Proof.* Let $V$ denote any flat representation of $G_K$ on a finite abelian $p$-group, say $V \cong \mathcal{G}(\overline{K})$, and let $V'$ be any subrepresentation. Define $\mathcal{G}' \subset \mathcal{G}$ as the scheme theoretic closure of $V' \subset \mathcal{G}(\overline{K})$; cf. [Ray, §2.1]. Then $\mathcal{G}'$ is a finite flat group scheme over $\mathcal{O}_K$ and $V' = \mathcal{G}(\overline{K})$. Moreover the functor $\mathcal{G}/\mathcal{G}'$ is representable by a finite flat group scheme over $\mathcal{O}_K$ with generic fiber $V/V'$. Let us give some details. Let $\mathcal{A}$ be the affine coordinate ring of $\mathcal{G}$. It is a free $\mathcal{O}_K$-module of finite rank and carries the structure of a cocommutative Hopf algebra. The $\overline{K}$-points of $V'$ correspond to $\mathcal{O}_K$-homomorphisms $\mathcal{A} \to \overline{K}$. The intersection of the kernels of these homomorphisms is an ideal $\mathcal{I}$ of $\mathcal{A}$. Since it is the same as the intersection of the corresponding ideal $\mathcal{I}_K$ of the generic Hopf algebra $\mathcal{A}_K = \mathcal{A} \otimes_{\mathcal{O}_K} K$ with $\mathcal{A}$, the ideal $\mathcal{I}$ is saturated as an $\mathcal{O}_K$-submodule of $\mathcal{A}$. Over $\operatorname{Spec} K$ the subgroup $V'$ is represented by the finite flat subscheme $\operatorname{Spec} \mathcal{A}/\mathcal{I}$ of $\mathcal{G}_K$. This shows that $\mathcal{I}_K \subset \mathcal{A}_K$ is a Hopf ideal. The latter property is inherited by $\mathcal{I}$. Hence $\mathcal{A}' = \mathcal{A}/\mathcal{I}$ is a Hopf algebra which is finite flat over $\mathcal{O}_K$. One verifies that $\mathcal{G}' = \operatorname{Spec} \mathcal{A}'$ is the desired subgroup scheme of $\mathcal{G}$.

The above shows that, if $\theta\colon A \to A'$ is a morphism in $\mathfrak{Ar}_{W(\mathbb{F})}$, then $V_{A'}$ is flat if and only if $V_{\theta(A)}$ is flat. (For one direction use that, if $\theta(A)^r \to A'$ is a $\theta(A)$-module epimorphism, then $V_{A'}$ is a quotient of $V_{\theta(A)}^r$.) Similarly, if $I, J \subset A$ are ideals such that $V_{A/I}$ and $V_{A/J}$ are flat, then $V_{A/(I\cap J)} \subset V_{A/I} \oplus V_{A/J}$ is flat. The second assertion implies the existence of a largest quotient $A_0$ of $A$ such that $V_{A_0}$ is flat. By the first assertion, this $A_0$ is the desired $A^{\mathrm{flat}}$. □

**Corollary 4.1.2.** *Let $D_{V_{\mathbb{F}}}^{\mathrm{flat}} \subset D_{V_{\mathbb{F}}}$ denote the subfunctor corresponding to flat deformations. Then $D_{V_{\mathbb{F}}}^{\mathrm{flat}} \subset D_{V_{\mathbb{F}}}$ is relatively representable.*

*Proof.* Relative representability for groupoids over categories was defined in Definition 2.4.4. It simply means that for all $\xi$ in $D_{V_{\mathbb{F}}}$ the functor $(D^{\mathrm{flat}})_\xi$ is representable. The latter is the functor of flat representations arising from $\xi \in D_{V_{\mathbb{F}}}(A)$ via a homomorphism $A \to A'$. The corollary follows from Lemma 4.1.1. □

## 4.2 Weakly admissible modules and smoothness of the generic fiber

**Proposition 4.2.1.** *Let $E/W(\mathbb{F})[1/p]$ be a finite extension and $\xi \in D_{V_{\mathbb{F}}}^{\mathrm{flat}}(\mathcal{O}_E)$ with corresponding representation $V_\xi$ over $E$. Then there is a natural isomorphism of groupoids over $\mathfrak{Ar}_E$,*

$$D_{V_{\mathbb{F}},(\xi)}^{\mathrm{flat}} \longrightarrow D_{V_\xi}^{\mathrm{flat}},$$

*where $D_{V_\xi}^{\mathrm{flat}}$ is the subgroupoid of $D_{V_\xi}$ of representations which are crystalline with Hodge–Tate weights in $\{0,1\}$. In particular, one has*

$$D_{V_{\mathbb{F}},(\xi)}^{\mathrm{flat}}(E[\varepsilon]) \cong \operatorname{Ext}^1_{\mathrm{cris}}\big(V_\xi, V_\xi\big).$$

*Moreover, for any $(A \stackrel{\alpha}{\to} \mathcal{O}_E)$ in $\widehat{\mathfrak{Ar}}_{W(\mathbb{F})}$ such that $A$ is flat over $\mathcal{O}_E$ and $\alpha$ becomes $B \to E$ in $\mathfrak{Ar}_E$ after inverting $p$, and for any $V_A \in D_{V_{\mathbb{F}}}(A)$ mapping to $V_\xi$ under $\alpha$, one has*

$$V_A \in D_{V_{\mathbb{F}}}^{\mathrm{flat}}(A) \iff V_A \cong \operatorname{Tate}_p \mathcal{G} \text{ for } \mathcal{G}/\mathcal{O}_K \text{ a } p\text{-divisible group} \tag{4.2.1}$$

$$\iff V_A \otimes_A B \text{ is crystalline with weights in } \{0,1\}. \tag{4.2.2}$$

*Proof.* The equivalence in (4.2.1) is a result of Raynaud: $V_A$ lies in $D_{V_{\mathbb{F}}}^{\mathrm{flat}}(A)$ if and only if for all $n \in \mathbb{N}$ the representation $V_A \otimes_A A/\mathfrak{m}_{\mathcal{O}_E}^n A$ is finite flat. By [Ray, 2.3.1], the latter is equivalent to $V_A$ being isomorphic to the Tate module of a $p$-divisible group.

The equivalence in (4.2.2) uses Breuil's result that a crystalline representation with all Hodge–Tate weights equal to 0 or 1 arises from a $p$-divisible group [Bre, Thm. 5.3.2], [Ki2, 2.2.6].

From (4.2.1) and (4.2.2) and the definition of $D^{\mathrm{flat}}_{V_{\mathbb{F}},(\xi)}$, the equivalence of functors is immediate, as is then the identification of the tangent space. $\square$

Suppose that $D_{V_{\mathbb{F}}}$ is pro-represented by $R_{V_{\mathbb{F}}}$ and let $R^{\mathrm{flat}}_{V_{\mathbb{F}}}$ be the quotient of $R_{V_{\mathbb{F}}}$ which pro-represents $D^{\mathrm{flat}}_{V_{\mathbb{F}}}$. For $\xi$ as above, denote by $\widehat{R}^{\mathrm{flat}}_{\xi}$ the completion (after $-\otimes_{W(\mathbb{F})} E$) along the kernel of $\xi$. The equivalence in the above proposition allows one to use Fontaine theory to show that $\widehat{R}^{\mathrm{flat}}_{\xi}$ is formally smooth over $E$ and to compute its relative dimension; see Corollary 4.2.4. If $K/\mathbb{Q}_p$ is ramified, the difficulty of $R^{\mathrm{flat}}_{V_{\mathbb{F}}}$ lies in its special fiber. As shown in [Ki4], its analysis may require delicate arguments.

To compute $\mathrm{Ext}^1_{\mathrm{cris}}(V_\xi, V_\xi)$, we recall some facts on weakly admissible filtered $\varphi$-modules from the lectures of L. Berger; cf. [Ber2] —see also Appendix 4.6.2. Consider the fully faithful functor

$$\begin{aligned} D_{\mathrm{cris}}\colon \big\{\text{crystalline representations of } G_K\big\} &=: \mathbf{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}\big(G_K\big) \\ &\longrightarrow \mathrm{Fil}^{\varphi}_K := \big\{\text{filtered } \varphi\text{-modules on } K\big\}. \end{aligned}$$

It is elementary to extend this equivalence to an equivalence with $E$-coefficients for any finite extension $E/\mathbb{Q}_p$

$$D_{\mathrm{cris}}\colon\ \mathbf{Rep}^{\mathrm{cris}}_{E}\big(G_K\big) \longrightarrow \mathrm{Fil}^{\varphi}_{K,E}.$$

Denote by $D_\xi$ the image of $V_\xi$ under $D_{\mathrm{cris}}$. From the definitions and properties of $D_{\mathrm{cris}}$ one deduces that

$$H^1_f\big(G_K, \mathrm{ad}V_\xi\big) = \mathrm{Ext}^1_{\mathrm{cris}}\big(V_\xi, V_\xi\big) \cong \mathrm{Ext}^1_{\mathrm{Fil}^{\varphi}_{K,E}}\big(D_\xi, D_\xi\big).$$

For the definition of $H^1_f$, see [Ber2]. Using the period rings $B_{\mathrm{cris}}$ and $B_{\mathrm{deR}}$ in [Ber2], the following formula is derived:

$$\dim_E H^1_f\big(G_K, \mathrm{ad}V_\xi\big) = \dim_E H^0_f\big(G_K, \mathrm{ad}V_\xi\big) + d^2 - \dim_E \mathrm{Fil}^0 \mathrm{ad} D_\xi. \tag{4.2.3}$$

Let us rederive the latter dimension formula by an elementary approach given, for instance, in [Ki7]. For any weakly admissible filtered $\varphi$-module $D$ over $K$, denote by $C^\bullet(D)$ the complex

$$D \xrightarrow{(1-\varphi,\,\mathrm{id})} D \oplus D_K/\mathrm{Fil}^0 D_K \tag{4.2.4}$$

concentrated in degrees $0, 1$.

**Lemma 4.2.2.** *There is a canonical isomorphism*

$$\mathrm{Ext}^1_{\mathrm{w.a.}}\big(\mathbb{1}, D\big) \xrightarrow{\cong} H^1\big(C^\bullet(D)\big)$$

*where $\mathbb{1} = K_0$ denotes the unit object in the category of weakly admissible modules.*

*Proof.* Consider an extension

$$0 \longrightarrow D \longrightarrow \widetilde{D} \longrightarrow \mathbb{1} \longrightarrow 0 \tag{4.2.5}$$

of weakly admissible filtered modules. Let $\tilde{d} \in \widetilde{D}$ be a lift of $1 \in \mathbb{1} = K_0$. Since (4.2.5) is short exact, so is the sequence of $\mathrm{Fil}^0$-terms of the induced sequence obtained by base change from $K_0$ to $K$ (by the definition of exactness for filtered modules). This shows that

$$D_K/\mathrm{Fil}^0 D_K \xrightarrow{\cong} \widetilde{D}_K/\mathrm{Fil}^0 \widetilde{D}_K$$

is an isomorphism, and so we may regard $\tilde{d}$ as an element of $D_K/\mathrm{Fil}^0 D_K$. Moreover $(1-\varphi)(\tilde{d}) \in D$ (because $1 = \varphi_{\mathbb{1}}(1)$). We thus associate the class

$$((1-\varphi)\tilde{d}, \tilde{d}) \in H^1(C^\bullet(D))$$

to the given extension.

Suppose now that $(d_0, d_1) \in D \oplus D_K/\mathrm{Fil}^0 D_K$. To construct a corresponding extension of $\mathbb{1}$ by $D$, set $\widetilde{D} = D \oplus K_0$ on underlying $K_0$-vector spaces, define $\varphi$ on $\widetilde{D}$ by $\varphi((d,1)) = (\varphi_D(d) + d_0, 1)$, and a filtration by

$$\mathrm{Fil}^i \widetilde{D}_K = \mathrm{Fil}^i D_K + K \cdot d_1 \text{ for all } i \le 0$$

and $\mathrm{Fil}^i \widetilde{D}_K = \mathrm{Fil}^i D_K$ for $i > 0$. The extensions which arise from elements in the image of $(1-\varphi, \mathrm{id})$ in (4.2.4) are split extensions. It is the content of Exercise 4.7.2 to show that these two constructions induce the asserted isomorphism and its inverse. (*Note:* The proof uses that the category of weakly admissible filtered $\varphi$-modules is closed under extensions within the category filtered of $\varphi$-modules. Hence any extension of weakly admissible modules is again weakly admissible.) ☐

Let the notation be as in Proposition 4.2.1 and let $D_\xi = D_{\mathrm{cris}}(V_\xi)$ be in $\mathrm{Fil}^\varphi_{K,E}$. For $B \in \mathfrak{A}_{\mathfrak{r}E}$, following Kisin one defines the category $\mathrm{Fil}^\varphi_{K,B}$ of filtered $\varphi$-modules on $K$ over $B$: the objects are free and finitely generated $K_0 \otimes_{\mathbb{Q}_p} B$-modules $D_B$ with a $\sigma_{K_0} \otimes \mathrm{id}_B$-linear automorphism $\varphi$ together with a filtration on $D_B \otimes_{K_0} K$ such that the associated graded pieces are free over $B$ (but not necessarily over $K \otimes B$). An object is weakly admissible if and only it is so if considered in $\mathrm{Fil}^\varphi_K$. One now defines the groupoid $D^{\mathrm{cris}}_{V_\xi}$ over $\mathfrak{A}_{\mathfrak{r}E}$ by defining $D^{\mathrm{cris}}_{V_E}(B)$ as the category of crystalline deformations of $V_\xi$ to $B$, and similarly $D^{\mathrm{w.a.}}_{D_\xi}$ over $\mathfrak{A}_{\mathfrak{r}E}$ by defining $D^{\mathrm{w.a.}}_{D_\xi}(B)$ as the category of all weakly admissible deformations of $D_\xi$ to $B$.

**Lemma 4.2.3.** *The functor* $D_{\mathrm{cris}}$ *induces an equivalence of groupoids* $D^{\mathrm{cris}}_{V_\xi} \to D^{\mathrm{w.a.}}_{D_\xi}$ *over* $\mathfrak{A}_{\mathfrak{r}E}$. *Moreover, each of these groupoids is formally smooth.*

*Proof.* For the proof of the first statement, see Exercise 4.7.5 below. The proof of the second statement for $D^{\mathrm{w.a.}}_{D_E}$ is rather straightforward. Indeed, one lifts the free $K \otimes_{\mathbb{Q}_p} B/I$-module $D_{B/I}$ to a free $K \otimes_{\mathbb{Q}_p} B$-module $D_B$. The isomorphism $\varphi_{B/I} \colon \varphi^* D_{B/I} \to D_{B/I}$ lifts (non-uniquely) to a $K \otimes_{\mathbb{Q}_p} B$-linear isomorphism $\varphi_B \colon \varphi^* D_B \to D_B$. To lift the filtration, one can use a complete set of idempotents for $K \otimes_{\mathbb{Q}_p} E$; via the canonical $E$-module structure of $B$, these idempotents lift uniquely to $K \otimes_{\mathbb{Q}_p} B$. □

**Corollary 4.2.4.** *Let the notation and hypotheses be as in Proposition* 4.2.1 *and let* $D_\xi = D_{\mathrm{cris}}(V_\xi)$. *Then the* $E$*-algebra* $\widehat{R}^{\mathrm{flat}}_\xi$ *is formally smooth of dimension* $1 + \dim_E \mathrm{ad} D_{\xi,K}/\mathrm{Fil}^0 \mathrm{ad} D_{\xi,K}$.

The corollary assumes that $D_{V_{\mathbb{F}}}$ is representable. One could instead work with $D^{\square}_{V_{\mathbb{F}}}$ and $D^{\mathrm{flat},\square}_{V_{\mathbb{F}}}$. The functor $D^{\mathrm{flat},\square}_{V_{\mathbb{F}},(\xi)}$ is then formally smooth of dimension $d^2 + \dim_E \mathrm{ad} D_{\xi,K}/\mathrm{Fil}^0 \mathrm{ad} D_{\xi,K}$.

*Proof.* By Proposition 4.2.1 and the previous lemma, formal smoothness is clear. The complex (4.2.4) shows that

$$h^1_{\mathrm{w.a.}}(G_K, \mathrm{ad} D_\xi) - h^0_{\mathrm{w.a.}}(G_K, \mathrm{ad} D_\xi) = \dim_E \, \mathrm{ad} D_{\xi,K}/\mathrm{Fil}^0 \mathrm{ad} D_{\xi,K}.$$

As we assume the representability of the groupoid $D_{V_{\mathbb{F}}}$, it has no extra automorphisms and so $\mathrm{End}_{\mathbb{F}[T]}(V_{\mathbb{F}}) \cong V_{\mathbb{F}}$, which implies that $h^0_{\mathrm{w.a.}}(G_K, \mathrm{ad} D_\xi) = 1$. Now use that

$$\begin{aligned} \dim_E \mathrm{Ext}^1_{\mathrm{cris}}\big(V_\xi, V_\xi\big) &= \dim_E \mathrm{Ext}^1_{\mathrm{w.a.}}\big(D_\xi, D_\xi\big) \\ &= \dim_E \mathrm{Ext}^1_{\mathrm{w.a.}}\big(\mathbb{1}, \mathrm{ad} D_\xi\big) = \dim_E H^1\big(C^\bullet\big(\mathrm{ad} D_\xi\big)\big) \end{aligned}$$

to obtain the assertion on the dimension from Lemma 4.2.2. Alternatively one can simply use (4.2.3). □

## 4.3 The Fontaine–Laffaille functor and smoothness when $e = 1$

So far we have seen that the generic fiber of $D^{\mathrm{flat}}_{V_{\mathbb{F}}}$ is smooth. In general its special fiber may have a complicated structure. However, in the case where $K/\mathbb{Q}_p$ is unramified the groupoid is smooth over $W(\mathbb{F})$. The principal tool to prove this is Fontaine–Laffaille theory, which we now recall.

The Fontaine–Laffaille category $\mathrm{MF}^1_{\mathrm{tor}}$ is defined as follows. Its objects are finite, torsion $W$-modules $M$ together with a submodule $M^1 \subset M$ and Frobenius semilinear maps

$$\varphi \colon M \longrightarrow M \ \text{ and } \ \varphi^1 \colon M^1 \longrightarrow M$$

such that

(a) $\varphi|_{M^1} = p\varphi^1$;

(b) $\varphi(M) + \varphi^1(M^1) = M$.

The category $\mathrm{MF}^1_{\mathrm{tor}}$ is an abelian subcategory of the category of filtered $W$-modules of finite length [FL, 9.1.10]. In particular, any morphism on $\mathrm{MF}^1_{\mathrm{tor}}$ is strict for filtrations.

Note that, if $p \cdot M = 0$, then $\varphi(M^1) = 0$, and so comparing the lengths of the two sides of (b) above shows that $\varphi^1$ is injective and

$$\varphi(M) \oplus \varphi^1(M^1) \xrightarrow{\cong} M. \tag{4.3.1}$$

**Theorem 4.3.1** (Fontaine–Laffaille, Raynaud)**.** *Suppose that $K = K_0$ and $p > 2$. Then there exist (covariant) equivalences of abelian categories*

$$\mathrm{MF}^1_{\mathrm{tor}} \xrightarrow[\mathrm{FL}]{\cong} \{\textit{finite flat group schemes}/W\} \xrightarrow[\mathrm{Raynaud}]{\cong} \{\textit{flat reps. of } G_K\}.$$

*Proof.* The first equivalence is obtained by composing the anti-equivalence [FL, 9.11] with Cartier duality. The second follows from Raynaud's result [Ray, 3.3.6] that, when $e(K/K_0) < p - 1$, the functor $\mathcal{G} \mapsto \mathcal{G}(\overline{K})$ is fully faithful and the category of finite flat group schemes over $\mathcal{O}_K$ is abelian. □

*Remark* 4.3.2. For $A \in \mathfrak{A}\mathfrak{r}_{W(\mathbb{F})}$, one defines a category $\mathrm{MF}^1_A$ as follows: its objects are quadruples $(M, M^1, \varphi, \varphi^1)$, where $M$ is a finitely generated $W \otimes_{\mathbb{Z}_p} A$-module, $M^1 \subset M$ is a $W \otimes_{\mathbb{Z}_p} A$-submodule, and $\varphi\colon M \to M$ and $\varphi^1\colon M^1 \to M$ are $\sigma_W \otimes \mathrm{id}_A$-linear homomorphisms such that (a) and (b) hold. This is an $A$-linear abelian category; see Exercise 4.7.5.

**Theorem 4.3.3.** *Suppose $K = K_0$ and $p > 2$. Then $D^{\mathrm{flat}}_{V_{\mathbb{F}}}$ is formally smooth.*

Independently of the condition $\mathrm{End}_{\mathbb{F}[G_K]}(V_{\mathbb{F}}) = \mathbb{F}$, the proof below will also show the formal smoothness of $D^{\mathrm{flat},\square}_{V_{\mathbb{F}}}$. Without using frames it is considerably more difficult to study $D^{\mathrm{flat}}_{V_{\mathbb{F}}}$ and its properties if $\mathrm{End}_{\mathbb{F}[G_K]}(V_{\mathbb{F}}) \supsetneq \mathbb{F}$. This problem had been considered by K. Fujiwara.

*Proof.* Let $M_{\mathbb{F}} \in \mathrm{MF}^1_{\mathrm{tor}}$ denote the object corresponding to $V_{\mathbb{F}}$. Then $M_{\mathbb{F}}$ lies naturally in $\mathrm{MF}^1_{\mathbb{F}}$ by the full faithfulness of Theorem 4.3.1. Its underlying module is free and finite over $W \otimes_{\mathbb{Z}_p} \mathbb{F}$, and the submodule $\mathrm{MF}^1_{\mathbb{F}}$ is a $W \otimes_{\mathbb{Z}_p} \mathbb{F}$-direct summand by (4.3.1). Let $D_{M_{\mathbb{F}}}$ denote the groupoid over $\mathfrak{A}\mathfrak{r}_{W(\mathbb{F})}$ such that $D_{M_{\mathbb{F}}}(A)$ is the category of tuples $(M_A, M^1_A, \varphi_A, \varphi^1_A, \iota_A)$ such that $(M_A, M^1_A, \varphi_A, \varphi^1_A)$ lies in $\mathrm{MF}^1_A$ and $M_A$ is a finite free $W \otimes_{\mathbb{Z}_p} A$-module, $M^1_A$ is a $W \otimes_{\mathbb{Z}_p} A$-direct summand, and conditions (a) and (b) hold, and moreover $\iota_A$ is an isomorphism

$$(M_A, M^1_A, \varphi_A, \varphi^1_A) \otimes_A \mathbb{F} \xrightarrow{\iota_A} (M_{\mathbb{F}}, M_{\mathbb{F}}{}^1, \varphi_{\mathbb{F}}, \varphi_{\mathbb{F}}{}^1).$$

Note that $W \otimes_{\mathbb{Z}_p} \mathbb{F}$ will in general not be a local ring —because in fact $k \otimes_{\mathbb{F}_p} \mathbb{F}$ will not be a field whenever the fields $k$ and $\mathbb{F}$ over $\mathbb{F}_p$ are not linearly disjoint inside $\overline{\mathbb{F}}_p$. However, observe that, since $W(\mathbb{F})$ is a ring of Witt vectors, the complete set of indecomposable idempotents $e_1, \ldots, e_n$ for $k \otimes_{\mathbb{F}_p} \mathbb{F}$ will lift to the unique such set over $W \otimes_{\mathbb{Z}_p} W(\mathbb{F})$ and in turn induce the unique such set on $W \otimes_{\mathbb{Z}_p} A$ for any $A \in \mathfrak{Ar}_{W(\mathbb{F})}$.

The following result is immediate from Exercise 4.7.5.

**Lemma 4.3.4.** *The Fontaine–Laffaille functor of Theorem* 4.3.1 *induces an equivalence of categories*

$$\mathrm{FL}\colon D_{M_{\mathbb{F}}} \xrightarrow{\cong} D_{V_{\mathbb{F}}}^{\text{flat}}.$$

Having the lemma at our disposal, to prove the formal smoothness of $D_{V_{\mathbb{F}}}^{\text{flat}}$ it suffices to prove the formal smoothness of $D_{M_{\mathbb{F}}}$. Let $A$ be in $\mathfrak{Ar}_{W(\mathbb{F})}$, $I \subset A$ an ideal and $M_{A/I}$ in $D_{M_{\mathbb{F}}}(A/I)$. We have to show that $M_{A/I}$ lifts to an object of $D_{M_{\mathbb{F}}}(A)$. Consider first the given data displayed in the following diagram:

$$\begin{array}{ccc} M_{A/I} & \xrightarrow{\varphi_{A/I}} & M_{A/I} \\ \big\uparrow & & \big\uparrow {\scriptstyle p\cdot\_} \\ M_{A/I}^1 & \xrightarrow{\varphi_{A/I}^1} & M_{A/I}. \end{array}$$

The module $M_{A/I}^1$ is a direct summand of $M_{A/I}$ as a $W \otimes_{\mathbb{Z}_p} A$-module, by definition of $D_{M_{\mathbb{F}}}$. The homomorphism $\varphi_{A/I}^1$ is injective, because this holds true for $\varphi_{\mathbb{F}}^1$. This in turn implies that $L_{A/I} := \varphi_{A/I}^1(M_{A/I}^1)$ is a projective $W \otimes_{\mathbb{Z}_p} A/I$-module of finite rank. Using the idempotents mentioned above and the fact that any free submodule of a local Artin ring is a direct summand, one can see that $L_{A/I}$ is a direct summand of $M_{A/I}$.

We can thus choose a free $W \otimes_{\mathbb{Z}_p} A$-module $M_A$, and projective $W \otimes_{\mathbb{Z}_p} A$-modules $M_A^1$ and $L_A$ lifting $M_{A/I}$, $M_{A/I}^1$, $L_{A/I}$, respectively, and in such a way that $M_A^1$ and $L_A$ are direct summands of $M_A$. By the projectivity of $M_A^1$, one can lift the $\sigma_W \otimes \mathrm{id}_{A/I}$-linear homomorphism $\varphi_{A/I}^1$ to an isomorphism $\varphi_A^1\colon M_A^1 \xrightarrow{\cong} L_A$. Using a complement to $M_A^1$ inside $M_A$ it is also straightforward to show that $\varphi_{A/I}$ can be lifted. □

## 4.4 The dimension of $D_{V_{\mathbb{F}}}^{\text{flat}}$

We wish to compute the dimension of the mod $p$ tangent space of $D_{V_{\mathbb{F}}}^{\text{flat}}$ in the case $K = K_0$. A direct way using Fontaine–Laffaille theory is described in [Ki7, 5.3.3]. We take a different route by working over the generic fiber. There it amounts to finding a more explicit form of the formula in Corollary 4.2.4. The computation here is valid for all $K$ and is taken from [Ki4]. For $K = K_0$ one can relate the final result via Fontaine–Laffaille theory to the filtered torsion $\varphi$-module $M_{V_{\mathbb{F}}}$ associated to $V_{\mathbb{F}}$. Moreover, in that case $D_{V_{\mathbb{F}}}^{\mathrm{flat}}$ is smooth and so the result also yields its dimension.

Let $\xi$ be a closed point on the generic fiber of $R_{V_{\mathbb{F}}}^{\mathrm{flat}}$, say with values in the finite extension $E$ of $\mathbb{Q}_p$ and associated $E$-representation $V_\xi$. Let $D_\xi$ be the corresponding filtered $\varphi$-module and $\mathcal{G}$ a $p$-divisible group over $\mathcal{O}_K$ whose Tate module $\mathrm{Tate}\,\mathcal{G} = \mathcal{G}(\overline{K})$ satisfies $V_\xi \cong V_{\mathcal{G}} := \mathrm{Tate}\,\mathcal{G} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (see the proof of Proposition 4.2.1).

Denote by $\mathfrak{t}_{\mathcal{G}}$ the tangent space of $\mathcal{G}$ and by $\mathcal{G}^\vee$ its Cartier dual. Clearly, $\mathfrak{t}_{\mathcal{G}}$ is the tangent space of the connected component $\mathcal{G}^0$ of $\mathcal{G}$. By [Tat1, Prop. 1], the $p$-divisible group $\mathcal{G}^0$ arises as the $p$-power torsion from a unique $p$-divisible smooth formal Lie group $\Lambda$ over $W$. The dimension $\dim \mathcal{G}$ of $\mathcal{G}$ is defined to be the dimension of $\Lambda$ or, equivalently, the dimension of $\mathfrak{t}_{\mathcal{G}}$.

Let $\mathbb{C}_p$ denote the completion of $\overline{K}$. The following isomorphism of continuous $G_K$-modules is taken from [Tat1, p. 180, Corollary 2]:

$$V_{\mathcal{G}} \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \mathfrak{t}_{\mathcal{G}}(\mathbb{C}_p)(1) \oplus \mathfrak{t}_{\mathcal{G}^\vee}(\mathbb{C}_p)^*. \tag{4.4.1}$$

Here, for any complete field $L \subset \mathbb{C}_p$ containing $K$, one has $\mathfrak{t}_{\mathcal{G}}(L) \cong L^{\dim \mathcal{G}}$ as $L[G_K]$-modules and similarly for $\mathfrak{t}_{\mathcal{G}^\vee}$. Hence $V_{\mathcal{G}}$ has Hodge–Tate weights $-1$ and $0$ with multiplicities $\dim \mathcal{G}$ and $\dim \mathcal{G}^\vee$.

To relate this to the functor $\mathrm{MF}^1_{\mathrm{tor}} \xrightarrow{\cong} \{\text{finite flat group schemes}/W\}$ from Theorem 4.3.1, we observe that the inverse of this functor, extended to the isogeny category of $p$-divisible groups, takes the form

$$\mathcal{G} \longmapsto D_{\mathcal{G}} := D_{\mathrm{cris}}(V_{\mathcal{G}}(-1)) \xrightarrow{\cong} \mathrm{Hom}_{G_K}(V_p\mathcal{G}^\vee, B_{\mathrm{cris}}).$$

In particular, as $K$-vector spaces, we have

$$\mathrm{Fil}^1 D_{\mathcal{G},K} \cong \mathfrak{t}_{\mathcal{G}^\vee}(K)^*. \tag{4.4.2}$$

However this is not quite sufficient for the desired dimension calculation! The point is that, so far, on the side of the $p$-divisible group we have ignored the action of $E$ (or its ring of integers). The action of $E$ on $V_\xi$ induces an action on $\mathcal{G}$ and hence on $\mathfrak{t}_{\mathcal{G}}$ as well, as its Cartier dual. This makes $\mathfrak{t}_{\mathcal{G}^\vee}(K)^*$ into a $K \otimes E$-module and the isomorphism (4.4.2) one of $K \otimes E$-modules.

To unify the arguments, we assume that $E$ contains the Galois closure of $K/\mathbb{Q}_p$. Then $K \otimes E \cong \prod_{\psi\colon K \hookrightarrow E} E$, where $\psi$ ranges over the embeddings of $K$ into $\overline{K}$ —these factor via $E$. Let $e_\psi$ be the corresponding idempotents. Write $d_\psi$ for $\dim_E e_\psi \mathfrak{t}_{\mathcal{G}}(K)$. From equation (4.4.1) one deduces that

$$d - d_\psi = \dim_E e_\psi \mathfrak{t}_{\mathcal{G}}^\vee(K) = \dim_E e_\psi \mathfrak{t}_{\mathcal{G}}^\vee(K)^*.$$

**Theorem 4.4.1.**

$$\dim_E D_{V_\xi}^{\mathrm{flat}}(E[\varepsilon]) = 1 + \sum_\psi d_\psi(d - d_\psi).$$

*Proof.* By Proposition 4.2.4, we need to compute $\dim_E \mathrm{ad}D_{\mathcal{G},K}/\mathrm{Fil}^0\mathrm{ad}D_{\mathcal{G},K}$. Now $\mathrm{ad}D_{\mathcal{G},K} = D_{\mathcal{G},K} \otimes_{K\otimes E} D^*_{\mathcal{G},K}$. The filtration of $D_{\mathcal{G},K}$ satisfies $\mathrm{Fil}^0 = D_{\mathcal{G},K} \supset \mathrm{Fil}^1 \supset \mathrm{Fil}^2 = 0$. The filtration on $D^*_{\mathcal{G},K}$ is given by $\mathrm{Fil}^i(D^*_{\mathcal{G},K}) = (\mathrm{Fil}^{1-i}D_{\mathcal{G},K})^\perp$ where $D^\perp \subset D^*_{\mathcal{G},K}$ denotes the annihilator of $D \subset D_{\mathcal{G},K}$ under the duality pairing from linear algebra. Thus

$$\mathrm{Fil}^{-1}D^*_{\mathcal{G},K} = D^*_{\mathcal{G},K} \supset \mathrm{Fil}^0 D^*_{\mathcal{G},K} = (\mathrm{Fil}^1 D_{\mathcal{G},K})^\perp \supset \mathrm{Fil}^{-1}D^*_{\mathcal{G},K} = 0,$$

and it follows that $\mathrm{Fil}^0\mathrm{ad}D_{\mathcal{G},K} = \mathrm{Fil}^1 D_{\mathcal{G},K} \otimes D^*_{\mathcal{G},K} + D_{\mathcal{G},K} \otimes (\mathrm{Fil}^1 D_{\mathcal{G},K})^\perp$. We deduce that

$$\begin{aligned}
&\mathrm{ad}D_{\mathcal{G},K}/\mathrm{Fil}^0\mathrm{ad}D_{\mathcal{G},K}\\
&\qquad \cong (D_{\mathcal{G},K}/\mathrm{Fil}^1 D_{\mathcal{G},K} \otimes D^*_{\mathcal{G},K})/(D_{\mathcal{G},K}/\mathrm{Fil}^1 D_{\mathcal{G},K} \otimes (\mathrm{Fil}^1 D_{\mathcal{G},K})^\perp)\\
&\qquad \cong (D_{\mathcal{G},K}/\mathrm{Fil}^1 D_{\mathcal{G},K}) \otimes (D^*_{\mathcal{G},K}/(\mathrm{Fil}^1 D_{\mathcal{G},K})^\perp)\\
&\qquad \cong (D_{\mathcal{G},K}/\mathrm{Fil}^1 D_{\mathcal{G},K}) \otimes \mathrm{Fil}^1 D^*_{\mathcal{G},K}.
\end{aligned}$$

Using the idempotents introduced above and the isomorphism in (4.4.2), the asserted dimension for $\dim_{\mathbb{F}} D^{\mathrm{flat}}_{V_\xi}(E[\varepsilon])$ can easily be verified. □

Suppose now that $K = K_0$. Then $D^{\mathrm{flat}}_{V_{\mathbb{F}}}$ is smooth. In particular there is a unique finite flat group scheme $\mathcal{G}_1$ mod $p$ which gives rise to $V_{\mathbb{F}}$. Moreover we can assume that $\mathcal{G}$ has coefficients $W(\mathbb{F})[1/p]$. One has $\mathcal{G}_1 = \mathcal{G}[p]$ and the dimension of the tangent space $\mathfrak{t}_{\mathcal{G}}$ (as well as its decomposition into $\psi$-equivariant parts) only depend on $\mathcal{G}_1$. Moreover, by the theory of Fontaine–Laffaille modules, $M_{\mathbb{F}}{}^1$ agrees with $\mathfrak{t}_{\mathcal{G}}(\mathbb{F})$ as an $\mathbb{F}$-module. We introduce integers $d_\psi$ as above for the automorphisms $\psi$ of $K_0 = K$. The following is an immediate corollary of Theorems 4.3.3 and 4.4.1.

**Corollary 4.4.2.** *If $K = K_0$ and $p > 2$, so that $D^{\mathrm{flat}}_{V_{\mathbb{F}}}$ is formally smooth, then*

$$\dim_{\mathbb{F}} D^{\mathrm{flat}}_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) = 1 + \sum_{\psi} d_\psi(d - d_\psi).$$

For arbitrary $K/\mathbb{Q}_p$, an important result of Kisin [Ki4, Cor. 2.1.13] constructs a projective $\mathbb{F}$-scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ such that, for any finite extension $\mathbb{F}'$ of $\mathbb{F}$, the finite flat group scheme models of $V_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$ are in bijection with the $\mathbb{F}'$-valued points of this scheme. The connected components of the scheme $\mathcal{GR}_{V_{\mathbb{F}},0}$ are in bijection with the connected components of the generic fiber $\mathrm{Spec}\, R^{\mathrm{flat}}[1/p]$, by [Ki4, Cor. 2.4.10]. The latter components are smooth and their dimension is given by Theorem 4.4.1. Since the tangent space of a $p$-divisible group $\mathcal{G}$ depends on $\mathcal{G}[p]$ only, the dimension can be computed from any model from $\mathcal{GR}_{V_{\mathbb{F}},0}$ in the corresponding component. Different components for the same $V_{\mathbb{F}}$ can have different dimensions. The tuple $(d_\psi)$ is called a *p-adic Hodge type* in [Ki4]. If $K$ is unramified over $\mathbb{Q}_p$, then $\mathcal{GR}_{V_{\mathbb{F}},0} \cong \mathrm{Spec}\,\mathbb{F}$ —which follows already from Raynaud's results.

Suppose again that $K/\mathbb{Q}_p$ is an arbitrary finite extension. Assume now that $\mathcal{G}$ is isogenous to $\mathcal{G}^\vee$, at least after restriction to a finite extension of $K$, i.e., that $\mathcal{G}$ is potentially (Cartier) self-dual. This happens in the following situations relevant to deformations of Galois representations associated to weight 2 Hilbert modular forms:

(a) $\mathcal{G}$ is the $p$-divisible group associated with an abelian variety over $\mathcal{O}_K$ (i.e., with good reduction): a polarization exists over $\mathcal{O}_{K'}$ for a finite extension $K'$ of $K$. It induces an isomorphism $\mathcal{G}^\vee \cong \mathcal{G}$ over $\mathcal{O}'_K$.

(b) $\mathcal{G}$ is the $p$-divisible group associated with a parallel weight 2 Hilbert modular Hecke eigenform $f$ whose level is prime to $p$. At least if $f$ arises from a Shimura curve $C$ over a totally real field $F$, then the $p$-adic Galois representation of $f$ arises from a subfactor of the Jacobian $J_C$ of $C$ over $F$, which has good reduction at $p$. Essentially by part (a) the associated $p$-divisible group is potentially self-dual.

From the isogeny over $K'$ it follows that $\mathfrak{t}_{\mathcal{G}^\vee}(K') \cong \mathfrak{t}_{\mathcal{G}}(K')$ and that this isomorphism is compatible with extra endomorphisms such as those coming from $E$. In particular, $d$ is even and $d_\psi = d/2$ for all $\psi$. Thus

$$\dim_E D^{\mathrm{flat}}_{V_\xi}(E[\varepsilon]) = 1 + [K:\mathbb{Q}_p](d/2)^2.$$

For $d = 2$ one recovers the expected result $\dim_E D^{\mathrm{flat}}_{V_\xi}(E[\varepsilon]) = 1 + [K:\mathbb{Q}_p]$.

Note that the argument basically rests on the fact that the Hodge–Tate weight is invariant under finite extensions of the base field.

## 4.5 Complements

Suppose that $V_{\mathbb{F}}$ is an irreducible 2-dimensional representation of $G_{\mathbb{Q}_p}$ of any Serre weight $2 \le k(V_{\mathbb{F}}) \le p$. Then the methods of the present lecture on flat (hence weight 2) deformations can be generalized to study (low weight) crystalline deformations. The reason is simply that in this range of weight ($2 \le k \le p$), the theory of Fontaine–Laffaille is still applicable. On the Fontaine–Laffaille side, one considers 2-dimensional filtered torsion modules of weight at most $k$. An $A$-representation ($A \in \mathfrak{Ar}_{W(\mathbb{F})}$) is then said to be crystalline of weight $k$ if it arises via the (inverse) Fontaine–Laffaille functor from an FL-module of weight $k$. In this perspective, the analogue of Lemma 4.3.4 is no longer an assertion but a definition. We simply state the results from [KW2], in particular [KW2, 3.2.3]. (Analogous results hold whenever $K/\mathbb{Q}_p$ is unramified.)

**Theorem 4.5.1** ([KW2, §3.2])**.** *Suppose that $2 \le k \le p$, that $V_{\mathbb{F}}$ is irreducible of Serre weight $k$, and that $p > 2$. Then the deformation functor for framed weight $k$ crystalline deformations of $V_{\mathbb{F}}$ of determinant $\chi^{k-1}$ is formally smooth over $W(\mathbb{F})$ of relative dimension 4.*

There is one further deformation condition considered in [KW2, §3.2] for $V_{\mathbb{F}}$ irreducible and of weight 2: semistable deformations with associated Weil–Deligne parameter (in the sense of Fontaine and for $p$-adic lifts) given by the pair $((\chi \bmod p)\oplus 1, N)$ and with $N$ non-trivial. The result in this case is due to Savitt [Sav, Thm. 6.2.2(3)].

**Theorem 4.5.2** ([KW2, §3.2])**.** *Suppose that $V_{\mathbb{F}}$ is irreducible of Serre weight 2 and $p > 2$. Let $\mathcal{O}$ be the ring of integers of a totally ramified extension of $K_0$. Then the deformation functor for framed weight 2 semistable deformations of $V_{\mathbb{F}}$ of determinant $\chi$ on $\mathfrak{Ar}_{\mathcal{O}}$ is representable. Provided that $\mathcal{O}$ is sufficiently large, it is isomorphic to $\mathcal{O}[[X_1,\ldots,X_5]]/(X_4X_5-p)$.*

## 4.6 Appendix

### 4.6.1 $p$-divisible groups

We only recall the most basic notions on $p$-divisible groups. As a reference we recommend Tate's seminal article [Tat1] and his notes [Tat3].

**Definition 4.6.1.** Let $h \geq 0$ be an integer and let $S$ be a scheme. A *$p$-divisible group $\mathcal{G}$ of height $h$ over a scheme $S$* is an inductive system

$$\mathcal{G} = \left(\mathcal{G}_n, \iota_n\right)_{n\geq 0}$$

where, for each $n$,

(a) $\mathcal{G}_n$ is a finite flat commutative group scheme over $S$ of order $p^{nh}$, and

(b) the sequence
$$0 \longrightarrow \mathcal{G}_n \xrightarrow{\iota_n} \mathcal{G}_{n+1} \xrightarrow{p^n} \mathcal{G}_{n+1}$$
is exact (i.e., $(\mathcal{G}_n, \iota_n)$ can be identified with the kernel of the homomorphism multiplication by $p^n$ on $\mathcal{G}_{n+1}$).

A homomorphism $f\colon \mathcal{G}\to\mathcal{H}$ of $p$-divisible groups $\mathcal{G}=(\mathcal{G}_n,\iota_n)$, $\mathcal{H}=(\mathcal{H}_n,\iota'_n)$ is a compatible system $f=(f_n)_{n\geq 0}$ of $S$-group homomorphisms $f_n\colon \mathcal{G}_n\to\mathcal{H}_n$ such that $\iota'_n f_n = f_{n+1}\iota_n$ for all $n\geq 0$.

If $\mathcal{G}=(\mathcal{G}_n)$ is a $p$-divisible group, we shall often use the perhaps more intuitive notation $\mathcal{G}[p^n]$ for $\mathcal{G}_n$ (see Examples 4.6.2 below).

Note that, if $p$ is invertible in $S$, then the $\mathcal{G}_n$ will be étale over $S$.

**Examples 4.6.2.** Let $A\to S$ be an abelian scheme over $S$. Then multiplication by $p^n$ is a finite flat homomorphism $A \xrightarrow{p^n} A$ of group schemes. Thus the kernel, denoted by $A[p^n]$, is a finite flat commutative group scheme over $S$. Denote by $\iota_n\colon A[p^n]\hookrightarrow A[p^{n+1}]$ the canonical inclusion. If $g$ denotes the dimension of $A$, then $A[p^\infty] := (A[p^n],\iota_n)_{n\geq 0}$ is a $p$-divisible group of height $2g$.

Consider the particular case where $A = E$ is an elliptic curve over a finite extension $K$ of $\mathbb{Q}_p$. Then $E[p^\infty]$ is a $p$-divisible group over $K$. It is completely determined by the Tate module of $E$ at $p$. Suppose that $E$ has good reduction and denote by $E$ a model over the ring of integers $\mathcal{O}$ of $K$. Then $E[p^\infty]$ is a $p$-divisible group over $\operatorname{Spec}\mathcal{O}$ of height 2.

### 4.6.2 Weakly admissible filtered $\varphi$-modules

Much of the material of this and the following section goes back to Fontaine and his coauthors. We suggest [Ber1] and [BC3, Ch. 8, §12.4] as references. They contain many further references.

Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and let $K$, $E$ be finite extensions of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$. The field $K$ will take the role of the *base* and the field $E$ that of a *coefficient ring*. Suppose $k$ is the residue field of $K$, so that $K$ contains $K_0 := W(k)[1/p]$. Let v be the valuation on $\overline{\mathbb{Q}}_p$ such that $\mathrm{v}(p) = 1$. Let $\sigma\colon K_0 \to K_0$ be the Frobenius automorphism induced on $k$, e.g. via the Witt vector construction.

**Definition 4.6.3.** A *filtered $\varphi$-module of rank $n$ on $K$ over $E$* is a tuple

$$\underline{D} = \big(D, \varphi, \{\mathrm{Fil}^i D_K\}_{i\in\mathbb{Z}}\big)$$

consisting of

(a) a free $K_0 \otimes_{\mathbb{Q}_p} E$-module $D$ of rank $r$,

(b) an isomorphism $\varphi\colon (\sigma\otimes\mathrm{id}_E)^* D \to D$, i.e., a $K_0$-semilinear automorphism $\varphi$, and

(c) an exhaustive separating decreasing filtration

$$\big(\mathrm{Fil}^i D_K\big)_{i\in\mathbb{Z}}$$

of $D_K := D \otimes_{K_0} K$ by $K \otimes_{\mathbb{Q}_p} E$-submodules.

A *morphism* $\psi\colon \underline{D} \to \underline{D}'$ between filtered $\varphi$-modules $\underline{D} = (D, \varphi_D, \{\mathrm{Fil}^i D_K\}_{i\in\mathbb{Z}})$ and $\underline{D}' = (D', \varphi_{D'}, \{\mathrm{Fil}^i D'_K\}_{i\in\mathbb{Z}})$ is a $K_0 \otimes_{\mathbb{Q}_p} E$-linear homomorphism $\psi\colon D \to D'$ which is compatible with the action of $\varphi$ and preserves the filtration.

The category of all filtered $\varphi$-modules on $K$ over $E$ is denoted by $\mathrm{MF}^{\varphi}_{K,E}$.

Note that the filtration in (c) need not satisfy any compatibilities with the previous data. However, the filtration datum imposes a strong restriction on the morphisms in the category $\mathrm{MF}^{\varphi}_{K,E}$. In particular it limits the set of subobjects of a given filtered $\varphi$-module. Note also that the $\mathrm{Fil}^i D_K$ need not be free over $K \otimes_{\mathbb{Q}_p} E$.

**Definition 4.6.4.** Suppose $\alpha\colon \underline{D}' \to \underline{D}$ and $\beta\colon \underline{D} \to \underline{D}''$ are morphisms in $\mathrm{MF}^{\varphi}_{K,E}$. Then $\underline{D}' \overset{\alpha}{\to} \underline{D} \overset{\beta}{\to} \underline{D}''$ is a *short exact sequence*, written

$$0 \longrightarrow \underline{D}' \longrightarrow \underline{D} \longrightarrow \underline{D}'' \longrightarrow 0,$$

if $0 \to D' \overset{\alpha}{\to} D \overset{\beta}{\to} D'' \to 0$ is an exact sequence of $K_0$-vector spaces and for all $i \in \mathbb{Z}$ the induced sequences $0 \to \mathrm{Fil}^i D'_K \to \mathrm{Fil}^i D_K \to \mathrm{Fil}^i D''_K \to 0$ are exact as sequences of $K$-vector spaces.

One says that $\underline{D}$ is an *extension of* $\underline{D}''$ *by* $\underline{D}'$ if there exists a short exact sequence $0 \to \underline{D}' \to \underline{D} \to \underline{D}'' \to 0$.

For $\underline{D} \in \mathrm{MF}^{\varphi}_{K,E}$ one can define exterior and symmetric powers as well as duals, where one takes the induced endomorphisms and filtrations.

If $\underline{D}$ is in $\mathrm{MF}^{\varphi}_{K,E}$, it is clearly also in $\mathrm{MF}^{\varphi}_{K,\mathbb{Q}_p}$. Under this forgetful functor, the dimension will increase by a factor of $\dim_{\mathbb{Q}_p} E$. By $\det_{K_0} \underline{D}$ we denote the element

$$\bigwedge\nolimits_{K_0}^{\dim_{K_0} D} \underline{D}$$

in $\mathrm{MF}^{\varphi}_{K,\mathbb{Q}_p}$ of rank one. By the previous remark,

$$\dim_{K_0} D = \dim_{K_0 \otimes_{\mathbb{Q}_p} E} D \cdot \dim_{\mathbb{Q}_p} E.$$

**Definition 4.6.5.** The *Hodge slope* of $\underline{D} \in \mathrm{MF}^{\varphi}_{K,E}$ is defined as

$$t_H(\underline{D}) = \max\{i \in \mathbb{Z} \,:\, \mathrm{Fil}^i(\det_{K_0} \underline{D})_K \neq 0\},$$

and its *Newton slope* is defined as

$$t_N(\underline{D}) = \mathrm{v}(\det_{K_0} \varphi(x)/x), \text{ for any } x \in \det_{K_0} D \smallsetminus \{0\}.$$

The Newton slope is well-defined, since $\det_{K_0} \underline{D}$ is of rank one over $K_0$ and since for any $x \in K_0 \smallsetminus \{0\}$ one has $\mathrm{v}(\sigma(x)) = \mathrm{v}(x)$.

Hodge and Newton slopes are used to define a semistability condition on filtered $\varphi$-modules:

**Definition 4.6.6.** A filtered $(\varphi, K, E)$-module is called *(weakly) admissible* if

$$t_H(\underline{D}) = t_N(\underline{D})$$

and for all subobjects $\underline{D}' \subset \underline{D}$ in the category $\mathrm{MF}^{\varphi}_{K,\mathbb{Q}_p}$ one has $t_H(\underline{D}') \leq t_N(\underline{D}')$.

This is a priori a rather tricky definition, since the subobjects to be considered for weak admissibility are subobjects in $\mathrm{MF}^{\varphi,N}_{K,\mathbb{Q}_p}$. However, one has the following [BM, Prop. 3.1.1.5]:

**Proposition 4.6.7.** *A filtered $(\varphi, K, E)$-module is admissible iff $t_H(\underline{D}) = t_N(\underline{D})$ and for all $\varphi$-stable sub-$K_0 \otimes_{\mathbb{Q}_p} E$-modules $D' \subset D$ one has $t_H(\underline{D}') \leq t_N(\underline{D}')$, where $D'$ carries the induced $\varphi$ and filtration.*

Note that the test objects $D'$ need not be free over $K_0 \otimes_{\mathbb{Q}_p} E$, and so they may not lie in $\mathrm{MF}^{\varphi,N}_{F,E}$.

Categorically, the introduction of the semistability concept has the following remarkable consequence (for a proof, see [BC3, Thm. 8.2.11]):

**Theorem 4.6.8** (Fontaine)**.** *The full subcategory of $\mathrm{MF}^{\varphi}_{K,E}$ of weakly admissible objects is abelian and closed under extensions.*

### 4.6.3 Fontaine–Laffaille modules

We assume that $K = K_0$ and so we drop the subscript $K$ at $D_K$. Let

$$\underline{D} = (D, \varphi, \{\mathrm{Fil}^i D_F\}_{i\in\mathbb{Z}})$$

be an *admissible* filtered $\varphi$-module with coefficients in $E$. Let $W = \mathcal{O}_{K_0} = W(k)$. Suppose that $\underline{D}$ is *effective*, i.e., that $\mathrm{Fil}^0 D = D$, and moreover $\mathrm{Fil}^p D = 0$.

**Definition 4.6.9.** A *strongly divisible* $\mathcal{O}_E$*-lattice in* $D$ is a free $W \otimes_{\mathbb{Z}_p} \mathcal{O}_E$-submodule $\Lambda \subset D$ such that

(a) $\Lambda[1/p] = D$,

(b) $\Lambda$ is stable under $\varphi$,

(c) $\varphi(\mathrm{Fil}^i\Lambda) \subset p^i\Lambda$ for all $i \geq 0$, where $\mathrm{Fil}^i\Lambda = \Lambda \cap \mathrm{Fil}^i D$, and

(d) $\sum_{i\geq 0} p^{-i}\varphi(\mathrm{Fil}^i\Lambda) = \Lambda$.

A strongly divisible lattice $\Lambda$ is called *connected* if $\varphi_\Lambda$ is topologically nilpotent for the $p$-adic topology on $\Lambda$.

For the following, see [FL] or [BC3, Thm. 12.4.8]:

**Theorem 4.6.10** (Fontaine–Laffaille)**.** *There are exact quasi-inverse anti-equivalences between the category of strongly divisible lattices* $\Lambda$ *with* $\mathrm{Fil}^p\Lambda = 0$ *and the category of* $\mathcal{O}_E[G_K]$*-lattices in crystalline* $G_K$*-representations with Hodge–Tate weights in the set* $\{0, \ldots, p-1\}$.

**Definition 4.6.11.** A *Fontaine–Laffaille module* $\underline{M} = (M, \varphi_M, (\mathrm{Fil}^i M)_{i\in\mathbb{Z}})$ over $W$ is a finite length $W$-module $M$ equipped with a finite and separated decreasing filtration $(\mathrm{Fil}^i M)$ and $\sigma$-semilinear endomorphisms $\varphi^i_M \colon \mathrm{Fil}^i M \longrightarrow M$ such that

(a) for all $i \geq 0$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Fil}^i M & \xrightarrow{\varphi^i} & M \\ \uparrow & & \uparrow{\scriptstyle p\cdot(-)} \\ \mathrm{Fil}^{i+1} M & \xrightarrow{\varphi^{i+1}} & M, \end{array}$$

(b) $\sum_i \mathrm{Im}(\varphi^i_M) = M$, and

(c) $\mathrm{Fil}^0_M = M$.

The category of such is denoted by $\mathrm{MF}_{\mathrm{tor}}$. If the filtration step 1 is non-zero, but 2 is zero, then we write $\mathrm{MF}^1_{\mathrm{tor}}$. One says that $\underline{M}$ is connected if $\varphi^0_M$ is nilpotent.

**Example 4.6.12.** If $\Lambda$ is a strongly divisible lattice, then for each $n > 0$ we obtain a Fontaine–Laffaille module $\underline{M}$ by setting $M = \Lambda/p^n\Lambda$, taking $\mathrm{Fil}^i M$ to be the image of $\mathrm{Fil}^i\Lambda$ under the natural quotient map, and letting $\varphi_M^i$ be the reduction of $p^{-i}\varphi_\Lambda$ restricted to $\mathrm{Fil}_M^i$.

This Fontaine–Laffaille module is connected if and only if $\Lambda$ is connected.

The following result is stated in [BC3, Thm. 12.4.12] —unfortunately without proof.

**Theorem 4.6.13.** *Consider the contravariant functor*

$$M \longrightarrow \mathrm{Hom}_{\mathrm{Fil},\varphi}(M, A_{\mathrm{cris}} \otimes \mathbb{Q}_p/\mathbb{Z}_p)$$

*from the category of Fontaine–Laffaille modules $\underline{M}$ with one-step filtration that satisfies $\mathrm{Fil}^0 M = M$ and $\mathrm{Fil}^p M = 0$ to the category of $p$-power torsion discrete $G_K$-modules. If $p > 2$, this is an exact and fully faithful functor into the category $\mathbf{Rep}_{\mathrm{tor}}\, G_K$, i.e., continuous $p$-torsion $G_K$-modules. If $p = 2$, the same statement holds if one restricts the functor to connected Fontaine–Laffaille modules.*

## 4.7 Exercises

*Exercise* 4.7.1. Formulate and prove Proposition 4.2.1 for framed deformations and verify the assertion made after Corollary 4.2.4 .

*Exercise* 4.7.2. Check that the two constructions in the proof of Proposition 4.2.2 of the isomorphism

$$\mathrm{Ext}^1_{\mathrm{w.a.}}\left(\mathbb{1}, D\right) \xrightarrow{\cong} H^1\left(C^\bullet(D)\right)$$

are well-defined and inverse.

*Exercise* 4.7.3. Give an explicit description of the isomorphism

$$\mathrm{Ext}^1_{\mathrm{w.a.}}\left(D_\xi, D_\xi\right) \cong \mathrm{Ext}^1_{\mathrm{w.a.}}\left(\mathbb{1}, \mathrm{ad} D_\xi\right)$$

used in the proof of Corollary 4.2.4.

*Exercise* 4.7.4. Prove that the functor $D^{\mathrm{w.a.}}_{D_E}$ in Corollary 4.2.3 is formally smooth.

*Exercise* 4.7.5. Let $C$ be a ring (commutative with 1). Recall that an additive category $\mathfrak{C}$ is $C$-linear if for all $M \in \mathfrak{C}$ one has a homomorphism $\varphi_M\colon C \to \mathrm{End}_{\mathfrak{C}}(M)$ such that for all $M, N \in \mathfrak{C}$ and all $\psi \in \mathrm{Hom}_{\mathfrak{C}}(M, N)$ diagram (4.7.1) commutes (this also makes $\mathrm{Hom}_{\mathfrak{C}}(M, N)$ into a $C$-module). This exercise provides a categorical approach to equipping suitable subcategories of a $C$-linear category with a larger endomorphism ring than $C$. It will be applied to several of the categories in this lecture.

Let now $\mathfrak{C}$ be a $C$-linear abelian category in which all objects have finite length over $C$.

(a) For $A \in \mathfrak{A}\mathfrak{r}_C$, define a category $\mathfrak{C}_A$ as follows. Objects of $\mathfrak{C}_A$ are pairs $(M, \varphi)$ with $M \in \mathfrak{C}$ and $\varphi\colon A \to \mathrm{End}_{\mathfrak{C}}(M)$ a $C$-linear homomorphism. Morphisms $\psi$ from $(M, \varphi_M)$ to $(N, \varphi_N)$ in $\mathfrak{C}_A$ are morphisms $\psi\colon M \to N$ such that for all $a \in A$ the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\varphi_M(a)} & M \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ N & \xrightarrow{\varphi_N(a)} & N. \end{array} \tag{4.7.1}$$

Show that $\mathfrak{C}_A$ is abelian and $A$-linear. Show also that for any finitely generated $A$-module $N$ the tensor product $- \otimes_A N$ is well-defined.

*Hint:* If $N$ is free over $A$, this is obvious. Else use a 2-step resolution of $N$ by free finitely generated $A$-modules.

(b) For $C = \mathbb{Z}_p$ and $\mathfrak{C} = \mathrm{MF}^1_{\mathrm{tor}}$, describe $\mathfrak{C}_A$ for $A \in \mathfrak{A}\mathfrak{r}_C$. An object of $\mathfrak{C}_A$ contains in particular an inclusion $M^1 \subset M$ of $A$-modules. Show that $M^1$ must be a direct summand.

*Hint:* Use the notion of pure submodule from [Mat, Appendix to §7] *and* the abelianness of $\mathfrak{C}$.

(c) Suppose that $\mathfrak{D}$ is a second $C$-linear abelian category in which all objects are of finite $C$-length and that $F\colon \mathfrak{C} \to \mathfrak{D}$ is an exact $C$-linear functor. Show that for all $A \in \mathfrak{A}\mathfrak{r}_C$ it induces via the construction in (a) an exact $C$-linear functor $F_A\colon \mathfrak{C}_A \to \mathfrak{D}_A$.

(d) The functor $F_A$ from (c) is compatible with the operation $- \otimes_A N$ for any finitely generated $A$-module $N$ and it restricts to an exact subfunctor on objects which are $A$-flat.

(e) For $C$ a finite extension of $\mathbb{Q}_p$ and $\mathfrak{C}$ the category of weakly admissible $\varphi$-modules on $K$ over $C$, describe $\mathfrak{C}_B$ for $B \in \mathfrak{A}\mathfrak{r}_C$. An object of $\mathfrak{C}_B$ is equipped with a filtration (over $K \otimes_{\mathbb{Q}_p} B$). Show that the subobjects of this filtration are direct summands as $K \otimes_{\mathbb{Q}_p} B$-modules.

# Lecture 5

# Presenting global over local deformation rings

A $p$-adic Galois representation of the absolute Galois group of a number field is called *geometric* if it is unramified outside finitely many places and at all places above $p$ it is de Rham in the sense of Fontaine. When the number field $F$ is totally real and the representation is into $\mathrm{GL}_2$, one typically also requires the representation to be totally odd. Conjecturally, the latter should automatically be satisfied if not all Hodge–Tate weights are equal —but no proof is known. By the Fontaine–Mazur conjecture, geometric 2-dimensional odd Galois representations over totally real fields should (up to twisting by powers of the cyclotomic character) arise from Hilbert modular forms. Then they are called *modular*. This is proven in some instances. But even for $\mathbb{Q}$ the proof of the Fontaine–Mazur conjecture is not complete.

In practice, it is important to construct geometric Galois representations, even in situations when it is not known that they are modular. An important method is to combine the proof of the potential modularity theorem by Taylor [Tay2, Tay3], i.e., an $R = T$ theorem over an enlarged (totally real) base field, with a technique from deformation theory. The method has proved useful in many instances beyond $\mathrm{GL}_2$, such as Galois representations of unitary or symplectic type, e.g. [BGGT, BGHT].

In the present lecture, the focus will be on the deformation theoretic part: we shall construct and analyze universal deformations rings $\overline{R}_S^{\psi}$ for representations of Galois groups of global fields which locally satisfy conditions that ensure that the deformations are geometric in the above sense. Concretely, one requires that the deformations are unramified outside finitely many places and odd. At those primes $\ell \neq p$ where ramification is allowed, one fixes a finite set of inertial WD-types and imposes these on the deformations. Finally, at places above $p$ one chooses deformation conditions that lead to $p$-adic Galois representations which are crystalline or ordinary of low weight, semistable of weight 2, or of potential Barsotti–Tate type (for instance).

The main result of this lecture —see Theorem 5.4.1 for a precise statement— is the following dimension bound: if $\overline{R}_S^\psi$ is non-zero, then

$$\dim_{\mathrm{Krull}} \overline{R}_S^\psi \geq 1. \tag{5.0.1}$$

Let us indicate how to derive from Taylor's results on potential modularity, e.g. [Tay2, Tay3], the existence of lifts of mod $p$ Galois representations satisfying the conditions in the definition of $\overline{R}_S^\psi$. If Taylor's result is applicable, then it implies that $\overline{R}_S^\psi/(p)$ is finite. Since the length of $\overline{R}_S^\psi/(p^n)$ is at most $n$ times the length of $\overline{R}_S^\psi/(p)$, and since $\overline{R}_S^\psi$ is $p$-adically complete, one deduces that $\overline{R}_S^\psi$ is a finitely generated $\mathbb{Z}_p$-module. The lower bound (5.0.1) thus implies that

$$\left(\overline{R}_S^\psi\left[\frac{1}{p}\right]\right)_{\mathrm{red}} \cong E_1 \times \cdots \times E_r$$

for suitable $p$-adic fields $E_i$ (i.e., finite extensions of $\mathbb{Q}_p$). The defining properties of $\overline{R}_S^\psi$ yield geometric Galois representations $G_F \to \mathrm{GL}_2(E_i)$ satisfying the conditions prescribed by the corresponding functor at all places above $p$ and $\infty$ and possibly at some further places.

Lower bounds as in (5.0.1) were first obtained in [Bö1]; cf. also [Bö2] —and in fact the results therein were sufficient for the proof of Serre's conjecture in the level one case [Kh]. However, the results in [Bö1] required the local deformation rings to be complete intersections. In recent work [Ki1], Kisin gave a different approach to obtain such bounds. This greatly enlarged the range where a lower bound as in (5.0.1) can be proved. Moreover it simplified the arguments considerably. So here we follow Kisin's approach. As a further reference we recommend [KW2, Ch. 1–4].

In this lecture we fix the following notation pertaining to number fields:

- $F$ will be a number field and $S$ will denote a finite set of places of $F$ containing all places $v \mid p$ and $v \mid \infty$.
- By $G_{F,S}$, or simply $G_S$, we denote the Galois group of the maximal outside $S$ unramified extension of $F$ inside a fixed algebraic closure $\overline{F}$ of $F$.
- For any place $v$ of $F$, we denote by $G_v$ the absolute Galois group of the completion $F_v$ of $F$ at $v$. We fix for each $v$ a homomorphism $\overline{F} \hookrightarrow \overline{F}_v$ extending $F \hookrightarrow F_v$. This yields a group homomorphism $G_v \to G_S$.
- By $V_{\mathbb{F}}$ we denote a continuous $\mathbb{F}[G_{F,S}]$-module of finite dimension $d$ over $\mathbb{F}$. We write $\mathrm{ad}^0 \subset \mathrm{ad} = \mathrm{ad}V_{\mathbb{F}}$ for the subrepresentation on trace zero matrices.
- All deformation functors (or categories of groupoids) considered will be functors on either the category $\mathfrak{Ar}_{\mathcal{O}}$ or $\widehat{\mathfrak{Ar}}_{\mathcal{O}}$, where $\mathcal{O}$ is the ring of integers of a totally ramified extension field of $W[1/p]$ and thus with residue field $\mathbb{F}$.

- We fix a lift $\psi\colon G_S \to \mathcal{O}^*$ of $\det V_{\mathbb{F}}$. This defines subfunctors $\mathrm{D}^{\psi}$ and $\mathrm{D}^{\psi,\square}$ of D and $\mathrm{D}^{\square}$ by requiring $\det V_A = \psi$ for lifts.

## 5.1 Tangent spaces

We provide some complements to Section 1.4. Here $G$ stands either for $G_{F,S}$ or $G_v$, and $V_{\mathbb{F}}$ for $V_{\mathbb{F}|G}$. As in Lecture 1, one proves:

**Proposition 5.1.1.** (a) *The functor $\mathrm{D}^{\psi,\square} \to \mathrm{D}^{\psi}$ of groupoids over $\widehat{\mathfrak{Ar}}_{\mathcal{O}}$ is formally smooth. The functor $\mathrm{D}^{\psi,\square}$ is always representable and the functor $\mathrm{D}^{\psi}$ is representable when $h^0(G,\mathrm{ad}) = 1$.*

(b) *The tangent space $\mathrm{D}^{\psi}(\mathbb{F}[\varepsilon])$ is isomorphic to*

$$H^1\big(G,\mathrm{ad}^0\big)' := \mathrm{Im}\left(H^1\big(G,\mathrm{ad}^0\big) \longrightarrow H^1\big(G,\mathrm{ad}\big)\right).$$

(c) *There is a short exact sequence*

$$0 \longrightarrow \mathrm{ad}^0/H^0\big(G,\mathrm{ad}^0\big) \longrightarrow \mathrm{D}^{\psi,\square}\left(\mathbb{F}[\varepsilon]\right) \longrightarrow \mathrm{D}^{\psi}\left(\mathbb{F}[\varepsilon]\right) \longrightarrow 0.$$

*Remark* 5.1.2. If $p$ does not divide the degree $d$ of $V_{\mathbb{F}}$, then $\mathrm{ad} \cong \mathrm{ad}^0 \oplus \mathbb{F}$ as a $G$-representation, and in this case $H^1(\ldots)' \cong H^1(\ldots)$. However, for $d = 2$ and $p = 2$ (for instance) one needs $H^1(\ldots)'$.

Applying Proposition 5.1.1(b) and (c) to the first five terms in the long exact cohomology sequence obtained from $0 \to \mathrm{ad}^0 \to \mathrm{ad} \to \mathbb{F} \to 0$, one finds

**Corollary 5.1.3.** *One has*

$$\dim_{\mathbb{F}} \mathrm{D}^{\psi,\square}(\mathbb{F}[\varepsilon]) = d^2 - 1 + h^1(G,\mathrm{ad}^0) - h^0(G,\mathrm{ad}^0).$$

*If in the deformation problem one fixes $m$ bases of $V_A$ instead of just one, then one has to add $(m-1)d^2$ to the right-hand side of the above formula.*

## 5.2 Relative presentations

We now turn to a situation which is closer to our final aim. Thus, from now on,

- the representation $\bar{\rho}\colon G_S \longrightarrow \mathrm{Aut}_{\mathbb{F}}(V_{\mathbb{F}})$ is absolutely irreducible, and
- we fix a subset $\Sigma$ of $S$ which is assumed to contain all places $v$ of $F$ dividing $p$ or $\infty$.

Corresponding to the above set-up, we introduce the following deformation functors and associated universal deformation rings:

| | deformation functor | $\longleftrightarrow$ | universal ring |
|---|---|---|---|
| $\forall v \in \Sigma:$ | $\mathrm{D}_v = \mathrm{D}_v^{\psi,\square} = \mathrm{D}_{V_{\mathbb{F}\vert G_v}}^{\psi_{\vert G_v},\square}$ | $\longleftrightarrow$ | $R_v^{\psi,\square}$ |
| | $\mathrm{D}_S^{\psi} = \mathrm{D}_{V_{\mathbb{F}\vert G_S}}^{\psi}$ | $\longleftrightarrow$ | $R_S^{\psi}$ |
| | $\mathrm{D}_{\Sigma,S}^{\psi,\square}$ | $\longleftrightarrow$ | $R_{\Sigma,S}^{\psi,\square}$, |

where the functor $\mathrm{D}_{\Sigma,S}^{\psi,\square}\colon \widehat{\mathfrak{Ar}_{\mathcal{O}}} \to \mathbf{Sets}$ is defined by the assignment

$$A \mapsto \left\{ (V_A, \iota_A, (\beta_v)_{v\in\Sigma}) \;\middle|\; \begin{matrix} (V_A,\iota_A) \in D_S^{\psi}(A), (\beta_v)_{v\in\Sigma} \text{ are bases} \\ \text{of } V_A \text{ with } \iota_A(\beta_v) = \beta_{\mathbb{F}} \;\; \forall v \in \Sigma \end{matrix} \right\} / \cong .$$

The functor $\mathrm{D}_{\Sigma,S}^{\psi,\square}$ provides the crucial link between the global and local situation:

$$\begin{array}{ccc} \mathrm{D}_{\Sigma,S}^{\psi,\square} & \xrightarrow{(V_A,\iota_A,(\beta_v)_{v\in\Sigma}) \mapsto ((V_A)_{|G_v},\iota_A,\beta_v)_{v\in\Sigma}} & \prod_{v\in\Sigma} \mathrm{D}_v \\ {\scriptstyle \text{smooth, rel. dim } 4|\Sigma|-1} \Big\downarrow {\scriptstyle (V_A,\iota_A,(\beta_v)_{v\in\Sigma}) \mapsto (V_A,\iota_A)} & & \\ \mathrm{D}_S^{\psi}\,. & & \end{array}$$

The formal smoothness of $\mathrm{D}_{\Sigma,S}^{\psi,\square}$ over $\mathrm{D}_S^{\psi}$ follows from Proposition 5.1.1(a). The formula for the relative dimension is proved in the same way as Proposition 5.1.1(c).

**Corollary 5.2.1.** (a) $R_{\Sigma,S}^{\psi,\square} \cong R_S^{\psi}[[x_1,\ldots,x_{4|\Sigma|-1}]]$.

(b) *There is a natural homomorphism* $R_{\mathrm{loc}} := \hat{\bigotimes}_{v\in\Sigma} R_v^{\psi,\square} \to R_{\Sigma,S}^{\psi,\square}$.

The ring $R_S^{\psi}$ was first studied by Mazur in [Maz]. It is an interesting object, since, for $F$ totally real and $\bar\rho$ odd, it can be naturally compared with a big Hecke algebra of Hilbert modular forms. On the other hand, it can be recovered from the universal representation $\rho_S\colon G_S \to \mathrm{GL}_d(R_{\Sigma,S}^{\psi,\square})$ as the ring generated by the traces of $\rho_S$ over $\mathcal{O}$. Ultimately it is a quotient of $R_S^{\psi}$ which will be of interest to us. The local rings at the places in $\Sigma$ will be useful in order to pass from $R_{\Sigma,S}^{\psi,\square}$ to this quotient. The use of framed deformations is a clean way to deal with non-representability issues of the functors $D_{V_{\mathbb{F}|G_v}}^{\psi_{|G_v}}$.

**Key Lemma 5.2.2.** *Consider the canonical homomorphisms*

$$\mathrm{D}_{\Sigma,S}^{\psi,\square}(\mathbb{F}[\varepsilon]) \xrightarrow{\theta^{\square,1}} \bigoplus_{v\in\Sigma} \mathrm{D}_v^{\psi,\square}(\mathbb{F}[\varepsilon]),$$

$$H^2(G_S,\mathrm{ad}^0) \xrightarrow{\theta^2} \bigoplus_{v\in\Sigma} H^2(G_v,\mathrm{ad}^0).$$

*Set* $r = \dim_{\mathbb{F}} \mathrm{Ker}\,\theta^{\square,1}$ *and* $t = \dim_{\mathbb{F}} \mathrm{Ker}\,\theta^2 + \dim_{\mathbb{F}} \mathrm{Coker}\,\theta^{\square,1}$. *Then* $R_{\Sigma,S}^{\psi,\square}$ *has a presentation*

$$R_{\mathrm{loc}}[[x_1,\ldots,x_r]]/(f_1,\ldots,f_t) \cong R_{\Sigma,S}^{\psi,\square}.$$

*Remarks* 5.2.3. (a) The proof will be given in Section 5.6.

(b) The lemma makes no assumption about the shape of the $f_i$. Some of the $f_i$ could be zero. Therefore $t$ is only an upper bound for the minimal number of relations.

(c) The value of $r$ is optimal, since $\theta^{\square,1}$ is the homomorphism of mod $\mathfrak{m}_{\mathcal{O}}$ tangent spaces induced from $\operatorname{Spec} R^{\psi,\square}_{\Sigma,S} \longrightarrow \operatorname{Spec} R_{\mathrm{loc}}$.

(d) Before we compute $r - t$ in the following section, let us determine the dimension of $\operatorname{Coker}\theta^2$. The diagram defining $\theta^2$ is part of the terms 7–9 of the 9-term Poitou–Tate sequence

$$H^2(G_S, \mathrm{ad}^0) \xrightarrow{\theta^2 \oplus \cdots} \bigoplus_{v \in \Sigma \dot{\cup} (S \smallsetminus \Sigma)} H^2(G_v, \mathrm{ad}^0) \longrightarrow H^0(G_S, (\mathrm{ad}^0)^\vee)^* \longrightarrow 0.$$

Using local Tate duality $H^2(G_v, \mathrm{ad}^0) \cong H^0(G_v, (\mathrm{ad}^0)^\vee)^*$ and some elementary linear algebra, we find (indeed!)

$$\begin{aligned}\delta &:= \dim_{\mathbb{F}} \operatorname{Coker}\theta^2 \\ &= \dim_{\mathbb{F}} \operatorname{Ker}\Big(H^0\big(G_S, (\mathrm{ad}^0)^\vee\big) \longrightarrow \bigoplus_{v \in S \smallsetminus \Sigma} H^0\big(G_v, (\mathrm{ad}^0)^\vee\big)\Big).\end{aligned}$$

If $H^0(G_S, (\mathrm{ad}^0)^\vee) = 0$, which is for instance the case whenever the image of $\bar{\rho}$ is non-solvable, or if $S \smallsetminus \Sigma \neq \varnothing$, and thus by our hypothesis on $\Sigma$ the difference contains a finite prime, then $\delta = 0$.

(e) The analogous computation for $\operatorname{Spec} R^{\psi}_{\Sigma,S}$ requires actually more bookkeeping due to the infinite places. For $\operatorname{Spec} R^{\psi}_{\Sigma,S}$, the set $\Sigma$ is supposed to only contain places at which $\mathrm{ad}^0 V_{\mathbb{F}}^{G_v} = 0$; however, the infinite places do not satisfy this requirement.

## 5.3 Numerology

**Lemma 5.3.1.** *If $\Sigma$ contains all places above $p$ and $\infty$, then $r - t + \delta = |\Sigma| - 1$.*

*Proof.* Tate's duality theory for global (and local) fields gives us the following formulas for the Euler–Poincaré characteristic of Galois cohomology (which is defined to be the alternating sum of the dimension of the zeroth, first and second term of Galois cohomology):

$$\chi(G_S, \mathrm{ad}^0) = -[F:\mathbb{Q}] \dim(\mathrm{ad}^0) + \sum_{v \mid \infty} h^0(G_v, \mathrm{ad}^0), \tag{5.3.1}$$

$$\chi(G_v, \mathrm{ad}^0) = \begin{cases} -\dim(\mathrm{ad}^0)[F_v : \mathbb{Q}_p] & \text{if } v \mid p, \\ h^0(G_v, \mathrm{ad}^0) & \text{if } v \mid \infty, \\ 0 & \text{otherwise.} \end{cases} \tag{5.3.2}$$

We deduce that

$$\begin{aligned} r - t + \delta &= \dim_{\mathbb{F}} \operatorname{Ker} \theta^{\square,1} - \dim_{\mathbb{F}} \operatorname{Coker} \theta^{\square,1} - \dim_{\mathbb{F}} \operatorname{Ker} \theta^2 + \dim_{\mathbb{F}} \operatorname{Coker} \theta^2 \\ &= \dim_{\mathbb{F}} \mathrm{D}^{\psi,\square}_{\Sigma,S}(\mathbb{F}[\varepsilon]) - \sum_{v\in\Sigma} \dim_{\mathbb{F}} \mathrm{D}_v(\mathbb{F}[\varepsilon]) - h^2(G_S, \mathrm{ad}^0) + \sum_{v\in\Sigma} h^2(G_v, \mathrm{ad}^0) \\ &\overset{\text{Cor. 5.1.3}}{=} |\Sigma| d^2 - 1 + h^1(G_S, \mathrm{ad}^0) - h^0(G_S, \mathrm{ad}^0) - h^2(G_S, \mathrm{ad}^0) \\ &\qquad - \sum_{v\in\Sigma} \big(d^2 - 1 + h^1(G_v, \mathrm{ad}^0) - h^0(G_v, \mathrm{ad}^0) - h^2(G_v, \mathrm{ad}^0)\big) \\ &= -\chi(G_S, \mathrm{ad}^0) + \sum_{v\in\Sigma} \chi(G_v, \mathrm{ad}^0) + |\Sigma| - 1 \\ &\overset{(5.3.1),\,(5.3.2)}{=} \dim(\mathrm{ad}^0)[F : \mathbb{Q}] - \sum_{v|\infty} h^0(G_v, \mathrm{ad}^0) \\ &\qquad - \sum_{v|p} \dim(\mathrm{ad}^0)[F_v : \mathbb{Q}_p] + \sum_{v|\infty} h^0(G_v, \mathrm{ad}^0) + 0 + |\Sigma| - 1 \;=\; |\Sigma| - 1, \end{aligned}$$

since $[F : \mathbb{Q}] = \sum_{v|p}[F_v : \mathbb{Q}_p]$. Note that after the third and fifth "=" the first line contains the global and the second the local contribution. □

## 5.4 Geometric deformation rings

In this and in the following section we assume the following:

- $F$ is totally real.
- $\bar\rho$ is odd and of degree 2 over $\mathbb{F}$ (and still absolutely irreducible).
- $\Sigma$ contains all places above $p$ and $\infty$ (as before).

For each place $v$ in $\Sigma$, choose a relatively representable subfunctor $\overline{\mathrm{D}}^{\psi,\square}_v \subset \mathrm{D}_v$ such that the corresponding universal ring $\overline{R}^{\psi,\square}_v$ (a quotient of $R^{\psi,\square}_v$) satisfies:

- $\overline{R}^{\psi,\square}_v$ is $\mathcal{O}$-flat,
- $\overline{R}^{\psi,\square}_v\left[\frac{1}{p}\right]$ is regular of dimension $\begin{cases} 3 & \text{if } v \nmid p, \infty, \\ 3 + [F_v : \mathbb{Q}_p] & \text{if } v \mid p, \\ 2 & \text{if } v \mid \infty. \end{cases}$

Suitable deformation conditions for $v \nmid p, \infty$ were described in Lecture 3. The natural choice is to fix a set of inertial WD-types for lifts to the generic fiber.

Rings of the above type for $v \mid p$ were constructed at the end of Lecture 3 and in Lecture 4. Possible deformation conditions are: low weight crystalline at $v$ if $\bar{\rho}$ is absolutely irreducible, low weight ordinary at $v$ for ordinary $\bar{\rho}$, and potentially Barsotti–Tate. In some cases, $F_v = \mathbb{Q}_p$ is required; in others, that $F_v$ is unramified over $\mathbb{Q}_p$, etc. Here we shall simply assume that we do have (framed) deformation functors at places above $p$ which satisfy the above requirements.

For $v \mid \infty$ we shall shortly describe the deformations and the corresponding rings. They describe odd deformations.

The above hypotheses on $\overline{R}_v^{\psi,\square}$ have the following consequences:

(a) The ring $\overline{R}_{\mathrm{loc}} := \hat{\bigotimes}_{v\in\Sigma} \overline{R}_v^{\psi,\square}$ is $\mathcal{O}$-flat. Its generic fiber is regular of dimension $3|\Sigma|$ (this uses the fact that $\sum_{v|p}[F_v : \mathbb{Q}_p] = \sum_{v|\infty} 1 = [F : \mathbb{Q}]$). Hence $\dim_{\mathrm{Krull}} \overline{R}_{\mathrm{loc}} \geq 3|\Sigma| + 1$.

(b) The corresponding functors $\overline{\mathrm{D}}_S^{\psi}$ and $\overline{\mathrm{D}}_{\Sigma,S}^{\psi,\square}$ (where the latter again includes a choice of $|\Sigma|$ bases of $V_A$) are representable, where (for instance) $\overline{\mathrm{D}}_{\Sigma,S}^{\psi,\square}$ is defined as the pullback in

$$\begin{array}{ccc} \mathrm{D}_{\Sigma,S}^{\psi,\square} & \longrightarrow & \prod_{v\in\Sigma} \mathrm{D}_v \\ & & \uparrow \\ & & \prod_{v\in\Sigma} \overline{\mathrm{D}}_v. \end{array}$$

(c) The global universal ring $\overline{R}_{\Sigma,S}^{\psi,\square}$ is isomorphic to $R_{\Sigma,S}^{\psi,\square} \hat{\otimes}_{R_{\mathrm{loc}}} \overline{R}_{\mathrm{loc}}$, and therefore Lemma 5.2.2 yields

$$\overline{R}_{\Sigma,S}^{\psi,\square} \cong \overline{R}_{\mathrm{loc}}[[x_1, \ldots, x_r]]/(f_1, \ldots, f_t)$$

with $r$, $t$ as in that lemma. Since $r - t = |\Sigma| - 1 - \delta$ by Lemma 5.3.1, part (a) yields $\dim_{\mathrm{Krull}} \overline{R}_{\Sigma,S}^{\psi,\square} \geq 4|\Sigma| - \delta$.

By Remark 5.1.2(d), the map $\overline{\mathrm{D}}_{\Sigma,S}^{\psi,\square} \longrightarrow \overline{\mathrm{D}}_S^{\psi}$ is formally smooth of relative dimension $4|\Sigma| - 1$. We deduce the following from part (c):

**Theorem 5.4.1.** *If $\delta = 0$, then* $\dim_{\mathrm{Krull}} \overline{R}_S^{\psi} \geq 1$.

## 5.5 Odd deformations at real places

At a real place, any two-dimensional odd residual representation is of the form

$$\bar{\rho}_\infty \colon \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \longrightarrow \mathrm{GL}_2(\mathbb{F}),$$

with $\det \bar{\rho}_\infty(c) = -1$ in $\mathbb{F}$, for $c$ the complex conjugation in $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Up to conjugation, one of the following three cases occurs:

(i) $p > 2$, $\bar{\rho}_\infty(c) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; (ii) $p = 2$, $\bar{\rho}_\infty(c) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$;

(iii) $p = 2$, $\bar{\rho}_\infty(c) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Any framed representation of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is determined by the image $M$ of $c$ and the latter is subject to the condition that $M^2 = \mathrm{id}$. If we further want to ensure that $M$ has eigenvalues 1 and $-1$, we need to fix its characteristic polynomial to be $X^2 - 1$. Let $\mathcal{M}(X^2-1)$ denote the moduli space of $2 \times 2$ matrices of characteristic polynomial $X^2 - 1$. Its completion at the matrix $\bar{\rho}_\infty(c)$ is the wanted universal ring (as may be checked easily). This is precisely the construction that was used in the proof of existence of $R^{\square}_{V_{\mathbb{F}}}$ in Proposition 1.3.1. Let us carry out this procedure explicitly for case (iii) (the other ones being similar but simpler):

If we start with $M = \begin{pmatrix} 1+a & b \\ c & 1+d \end{pmatrix}$, then the conditions $\mathrm{Tr} = 0$ and $\det = -1$ lead to

$$\begin{aligned} \mathcal{M}(X^2-1) &= \operatorname{Spec} \mathcal{O}[a,b,c,d]/((1+a)+(1+d), (1+a)(1+d)+1-bc) \\ &= \operatorname{Spec} \mathcal{O}[a,b,c]/(-(1+a)^2+1-bc) \\ &= \operatorname{Spec} \mathcal{O}[a,b,c]/(-2a-a^2-bc) \end{aligned}$$

and hence $\overline{R}^{\psi,\square}_{\infty,\mathrm{odd}} \cong \mathcal{O}[[a,b,c]]/(2a+a^2+bc)$. The latter is a domain with generic regular fiber of dimension 2.

In cases (i) and (ii), similar calculations lead to $\overline{R}^{\psi,\square}_{\infty,\mathrm{odd}} \cong \mathcal{O}[[x_1,x_2]]$.

## 5.6 Proof of Key Lemma 5.2.2

Note that we now work again with representations of general degree $d$. To simplify notation, we set $R_{\mathrm{gl}} = R^{\psi,\square}_{\Sigma,S}$. For $r$ as in Lemma 5.2.2, we choose a surjective ring homomorphism

$$\varphi\colon \widetilde{R} := R_{\mathrm{loc}}[[x_1,\ldots,x_r]] \longrightarrow R_{\mathrm{gl}}.$$

We set $J = \operatorname{Ker}\varphi$ and denote the maximal ideals of $R_{\mathrm{gl}}$, $R_{\mathrm{loc}}$ and $\widetilde{R}$ by $\mathfrak{m}_{\mathrm{gl}}$, $\mathfrak{m}_{\mathrm{loc}}$, $\widetilde{\mathfrak{m}}$, respectively. By Nakayama's lemma, we need to show that $\dim_{\mathbb{F}} J/\widetilde{\mathfrak{m}}J \le t$. The module $J/\widetilde{\mathfrak{m}}J$ appears as the kernel in the sequence

$$0 \longrightarrow J/\widetilde{\mathfrak{m}}J \longrightarrow \widetilde{R}/\widetilde{\mathfrak{m}}J \longrightarrow \widetilde{R}/J \cong R_{\mathrm{gl}} \longrightarrow 0. \tag{5.6.1}$$

The argument to bound the dimension of $J/\widetilde{\mathfrak{m}}J$ is similar to the one given by Mazur in [Maz] to bound the number of relations in presentations of universal deformation rings as quotients of power series rings over $\mathcal{O}$. The idea is to consider the lifting problem associated to the above sequence for the universal lift $\rho_{\mathrm{gl}}\colon G_S \longrightarrow \mathrm{GL}_d(R_{\mathrm{gl}})$. The difference with Mazur's argument is that some lifting problems do have a solution and one needs to properly interpret this.

Formally, we shall construct a homomorphism $\alpha\colon \operatorname{Hom}(J/\widetilde{\mathfrak{m}}J,\mathbb{F}) \to \operatorname{Ker}\theta^2$ and show that the kernel of $\alpha$ can be interpreted as a subspace of $\operatorname{Coker}\theta^{\square,1}$. This will imply the lemma, since then

$$\dim_{\mathbb{F}} J/\widetilde{\mathfrak{m}}J = \dim_{\mathbb{F}} \operatorname{Ker}(\alpha) + \dim_{\mathbb{F}} \operatorname{Im}(\alpha) \le \dim_{\mathbb{F}} \operatorname{Coker}\theta^{\square,1} + \dim_{\mathbb{F}} \operatorname{Ker}\theta^2 = t.$$

Fix $u \in \operatorname{Hom}_{\mathbb{F}}(J/\widetilde{\mathfrak{m}}J,\mathbb{F})$. Then pushout under $u$ of the sequence (5.6.1) defines an exact sequence

$$0 \longrightarrow I_u \longrightarrow R_u \xrightarrow{\varphi_u} R_{\mathrm{gl}} \longrightarrow 0,$$

where $I_u \cong \mathbb{F}$. It is not hard to construct a set-theoretic lift $\widetilde{\rho}_u$ so that the diagram

$$\begin{array}{ccc} G_S & \xrightarrow{\widetilde{\rho}_u} & \mathrm{GL}_d(R_u) \\ & \searrow^{\rho_{\mathrm{gl}}} & \downarrow{\scriptstyle \mathrm{GL}_d(\varphi_u)} \\ & & \mathrm{GL}_d(R_{\mathrm{gl}}) \end{array}$$

commutes, and so that $\det \widetilde{\rho}_u = \psi$. (Regarding $\mathrm{GL}_d(R_u)$ as the set-theoretic product of diagonal matrices with diagonal entries $(r_u, 1, 1, \ldots, 1)$ with $\mathrm{SL}_d(R_u)$, it suffices to construct a continuous splitting of $\mathrm{SL}_d(R_u) \to \mathrm{SL}_d(R_{\mathrm{gl}})$. This can be done using the smoothness of $\mathrm{SL}_d$.)

The kernel of $\mathrm{GL}_d(\varphi_u)$ is $(1 + M_d(I_u), \cdot)$ and can thus be identified with $\mathrm{ad} \otimes_{\mathbb{F}} I_u \cong \mathrm{ad}$. Via these identifications, the set-theoretic lift yields a continuous 2-cocycle

$$c_u \in Z^2(G_S, \mathrm{ad}^0)$$

given by $1 + c_u(g_1, g_2) = \widetilde{\rho}_u(g_1, g_2)\widetilde{\rho}_u(g_2)^{-1}\widetilde{\rho}_u(g_1)^{-1}$. Its image $[c_u] \in H^2(G_S, \mathrm{ad}^0)$ is independent of the choice of the set-theoretic lifting. The representation $\rho_{\mathrm{gl}}$ can be lifted to a homomorphism $G_S \to \mathrm{GL}_d(R_u)$ precisely if $[c_u] = 0$. The existence of homomorphisms $R_{\mathrm{loc}} \to R_u \to R_{\mathrm{gl}}$ together with the universality of $R_{\mathrm{loc}}$ imply that the restrictions $[c_{u|G_v}] \in H^2(G_v, \mathrm{ad}^0)$ are zero for all $v \in \Sigma$. Thus we have constructed the desired homomorphism

$$\alpha\colon \operatorname{Hom}(J/\widetilde{\mathfrak{m}}J,\mathbb{F}) \longrightarrow \operatorname{Ker}\theta^2,\ u \longmapsto [c_u].$$

It remains to analyze the kernel of $\alpha$. Let $u$ be in the kernel, so that $[c_u] = 0$ and $\rho_{\mathrm{gl}}$ can be lifted. By the universality of $R_{\mathrm{gl}}$ we obtain a splitting $s$ of $R_u \twoheadrightarrow R_{\mathrm{gl}}$. Consider the surjective map of mod $\mathfrak{m}_{\mathcal{O}}$ cotangent spaces

$$\mathfrak{ct}_{\varphi_u} : \mathfrak{m}_{R_u}/(\mathfrak{m}_{R_u}^2 + \mathfrak{m}_{\mathcal{O}}) \longrightarrow \mathfrak{m}_{\mathrm{gl}}/(\mathfrak{m}_{\mathrm{gl}}^2 + \mathfrak{m}_{\mathcal{O}}).$$

Any surjective homomorphism $A \to B$ in $\widehat{\mathfrak{Ar}}_{\mathcal{O}}$ which induces an isomorphism on mod $\mathfrak{m}_{\mathcal{O}}$ cotangent spaces and which has a splitting is an isomorphism (exercise!). In our situation, this implies that $I_u$ can be identified with the kernel of $\mathfrak{ct}_{\varphi_u}$.

The map $\mathfrak{ct}_{\varphi_u}$ itself is induced from the homomorphism $\widetilde{R}/(J\widetilde{\mathfrak{m}}) \to R_{\mathrm{gl}}$ by pushout and from the analogous map

$$\widetilde{\mathfrak{ct}}_\varphi \colon \widetilde{\mathfrak{m}}/(\widetilde{\mathfrak{m}}^2 + \mathfrak{m}_{\mathcal{O}}) \longrightarrow \mathfrak{m}_{\mathrm{gl}}/(\mathfrak{m}_{\mathrm{gl}}^2 + \mathfrak{m}_{\mathcal{O}}).$$

Because $\widetilde{\mathfrak{m}}/(\widetilde{\mathfrak{m}}^2 + \mathfrak{m}_{\mathcal{O}}) \to \mathfrak{m}_{R_u}/(\mathfrak{m}_{R_u}^2 + \mathfrak{m}_{\mathcal{O}})$ is surjective, the induced homomorphism $\gamma_u \colon \mathrm{Ker}(\widetilde{\mathfrak{ct}}_\varphi) \to I_u$ of $\mathbb{F}$-vector spaces is non-zero. Remembering that $I_u$ is really just another name for $\mathbb{F}$ to indicate that it is an ideal in $R_u$, we have thus constructed an injective $\mathbb{F}$-linear monomorphism

$$\mathrm{Ker}(\alpha) \hookrightarrow \mathrm{Hom}_{\mathbb{F}}\left(\mathrm{Ker}(\widetilde{\mathfrak{ct}}_\varphi), \mathbb{F}\right). \tag{5.6.2}$$

By the choice of $r$ and its minimality, it follows that we have $\mathrm{Ker}(\widetilde{\mathfrak{ct}}_\varphi) = \mathrm{Ker}(\mathfrak{ct}_\varphi)$ for the canonical homomorphism

$$\mathfrak{ct}_\varphi \colon \mathfrak{m}_{\mathrm{loc}}/(\mathfrak{m}_{\mathrm{loc}}^2 + \mathfrak{m}_{\mathcal{O}}) \longrightarrow \mathfrak{m}_{\mathrm{gl}}/(\mathfrak{m}_{\mathrm{gl}}^2 + \mathfrak{m}_{\mathcal{O}}).$$

Since $\widetilde{\mathfrak{ct}}_\varphi = (\theta^{\square,1})^*$, the map (5.6.2) is the desired homomorphism $\mathrm{Ker}(\alpha) \hookrightarrow \mathrm{Coker}\,\theta^{\square,1}$. The proof of Lemma 5.2.2 is thus complete. □

*Remark* 5.6.1. For each $v \in S$, define a subspace $L_v$ of $H^1(G_v, \mathrm{ad}^0)$ by $L_v = H^1(G_v, \mathrm{ad}^0)$ for $v \in S \smallsetminus \Sigma$ and by $L_v = \mathrm{Ker}(H^1(G_v, \mathrm{ad}^0) \to H^1(G_v, \mathrm{ad}^0)')$ for $v \in \Sigma$, and denote by $H^1_{\mathcal{L}^\perp}(G_S, (\mathrm{ad}^0)^\vee)$ the corresponding dual Selmer group (cf. [KW2, Ch. 4] for a precise definition). It naturally sits in a short exact sequence

$$0 \longrightarrow \mathrm{Coker}\,\theta^{\square,1} \longrightarrow H^1_{\mathcal{L}^\perp}(G_S, (\mathrm{ad}^0)^\vee)^* \longrightarrow \mathrm{Ker}\,\theta^2 \longrightarrow 0.$$

In [KW2, proof of Prop. 4.4], it is proved directly that there is an injective homomorphism

$$\mathrm{Hom}(J/\widetilde{\mathfrak{m}}J, \mathbb{F}) \hookrightarrow H^1_{\mathcal{L}^\perp}(G_S, (\mathrm{ad}^0)^\vee)^*.$$

This gives an alternative, more conceptual method to derive the desired bound $\dim J/\widetilde{\mathfrak{m}}J \leq t$.

## 5.7 Exercises

*Exercise* 5.7.1. Verify all unproven assertions in Section 5.1.

*Exercise* 5.7.2. Check the assertions made about the cocycle $c_u$ in the proof of Lemma 5.2.2: that $[c_u]$ does not depend on the set-theoretic lifting $\widetilde{\rho}_u$ and that the class is trivial if and only if $\widetilde{\rho}_u$ can be chosen to be a homomorphism.

*Exercise* 5.7.3. Prove that any surjective homomorphism $A \to B$ in $\widehat{\mathfrak{Ar}_{\mathcal{O}}}$ which has a splitting (as $\mathcal{O}$-algebras) and induces an isomorphism $\mathfrak{ct}_A \to \mathfrak{ct}_B$ on mod $\mathfrak{m}_{\mathcal{O}}$ cotangent spaces is an isomorphism.

*Exercise* 5.7.4. Let $\mathcal{O}$ be the ring of integers of a finite totally ramified extension of $W(\mathbb{F})[1/p]$ and let $R$ be an $\mathcal{O}$-algebra which is finite over $\mathcal{O}$. Show that $R$ is flat over $\mathcal{O}$ if and only if $R$ is $p$-torsion free. *Hint:* Deduce from $\mathrm{Tor}_1^{\mathcal{O}}(R, \mathcal{O}/p) = 0$ that $\mathrm{Tor}_1^{\mathcal{O}}(R, \mathbb{F}) = 0$ and hence the assertion.

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