

Finiteness conjectures for $\mathbb{F}_l[[T]]$ -analytic extensions of number fields

Gebhard Böckle

ETH-Zürich, Departement Mathematik, Rämistrasse 101, 8092 Zürich, Switzerland
E-mail: boeckle@math.ethz.ch

Received ; revised ; accepted

We formulate a finiteness conjecture on the image of the absolute Galois group of totally real fields under an even linear representation over a local field of positive characteristic. This is motivated by a recent conjecture of de Jong in the function field case. We discuss the relation to some conjectures of Boston which arise from the conjectures of Fontaine and Mazur and give group-theoretical reformulations of our conjectures. As we will explain, our conjectures have consequences for the structure of universal deformation rings of even residual representations. Finally we give some evidence for the conjectures themselves and for their consequences.

Key Words: function fields, totally real fields, $\mathbb{F}_l[[t]]$ -adic Galois representations, finiteness conjectures, universal deformation rings

1. INTRODUCTION

Notation. Throughout we fix the following notation. By κ we denote a finite field of characteristic $l > 2$ and by $W(\kappa)$ its ring of Witt vectors. K will be a global field, \bar{K} a separable closure and $G_K := \text{Gal}(\bar{K}/K)$ the absolute Galois group of K . If K is a function field, we assume that it is of residue characteristic $p \neq l$. We fix a continuous representation

$$\rho: G_K \rightarrow \text{GL}_n(\kappa[[T]]).$$

Its reduction module T is denoted $\bar{\rho}$ and is called the *residual representation* of ρ . By the *splitting* field of a representation, we mean the fixed field inside \bar{K} of the kernel of that representation. The splitting field of $\bar{\rho}$ is always denoted by L . For any global field E , we denote by E_∞ its cyclotomic \mathbb{Z}_l -extension. S will always be a finite set of places of K , and if K is a number field, then S_l will be the set of places of K above l . In any matrix group, I denotes the identity matrix.

A finiteness conjectures for function fields. In [12], de Jong states the following conjecture.

CONJECTURE 1.1 If K is a function field and ρ is ramified only at finitely many places, then $\rho(G_{K\bar{\mathbb{F}}_p})$ is finite.

Because $\rho(G_L)$ is a pro- l group, one has $\rho(G_{L\bar{\mathbb{F}}_p}) = \rho(G_{L_\infty})$. Since L/K is finite, this shows that the above conjecture is equivalent to

CONJECTURE 1.2 If K is a function field and ρ is ramified only at finitely many places, then $\rho(G_{K_\infty})$ is finite.

The case $n = 1$ follows easily from class field theory. The central result of [12] is the proof of this conjecture for $n = 2$. The proof is based on Drinfeld's reciprocity law between automorphic forms and two-dimensional Galois representations of G_K . In fact, de Jong only makes use of Drinfeld's reciprocity law in the special case of unramified representations.

Before discussing some consequences of the above finiteness conjecture, let us present what we consider an analogue for number fields.

A finiteness conjecture for number fields. There are simple examples which show that the naïve analogue of Conjecture 1.2 for number fields cannot hold. For instance the reduction modulo l of the explicit universal deformations described in [6] have an infinite image when restricted to G_{K_∞} . Thus there needs to be a restriction on the representations ρ for such an analogue.

The proof of the following simple result is left to the reader. It uses the facts that $l > 2$ and that the kernel of the reduction map $\mathrm{GL}_n(\kappa[[T]]) \rightarrow \mathrm{GL}_n(\kappa)$ is a pro- l group.

LEMMA-DEFINITION 1.3 *For a number field K , the following assertions on ρ are equivalent:*

(a) *The splitting field of the representation*

$$\rho \pmod{\{\pm I\}}: G_K \rightarrow \mathrm{GL}_n(\kappa[[T]])/\{\pm I\}$$

is totally real.

(b) *The splitting field of $\bar{\rho} \pmod{\{\pm I\}}$ is totally real.*

(c) *K is totally real and for any complex conjugation $c \in G_K$ (for any infinite place), one has $\rho(c) = \pm I$.*

If any of the above assertions hold, we call ρ (and $\bar{\rho}$) even.

The following we regard as a first analogue of de Jong's conjecture:

CONJECTURE 1.4 Suppose ρ is even and ramified only at finitely many places. Then $\rho(G_{K_\infty})$ is finite.

In Section 2, we explain the analogy between Conjectures 1.2 and 1.4 in more detail. Furthermore we present an analogue of Conjecture 1.4 for base fields that have complex multiplication. We compare Conjecture 1.4 with a conjecture of Fontaine-Mazur and a conjecture of Boston. In the end, we give some group-theoretical reformulations.

Consequences of the above finiteness conjectures for Galois representations of global fields. As observed by de Jong, [12], Thm. 3.5, Conjecture 1.1 has the following consequence for Galois representations.

THEOREM 1.5 (DE JONG) *Let K be a function field. Suppose we are given a Galois representation $\bar{\tau}: G_K \rightarrow \mathrm{GL}_n(\kappa)$ which is unramified outside a finite set S of places of K . Assume that the restriction of $\bar{\tau}$ to $G_{K\mathbb{F}_p}$ is absolutely irreducible. Let $\eta: G_K \rightarrow W(\kappa)^*$ be a continuous character such that $\eta \equiv \det \bar{\tau} \pmod{l}$. If Conjecture 1.1 holds, i.e., in particular if $n \leq 2$, and if $l \nmid n$, then the universal deformation ring for deformations of $\bar{\tau}$, unramified outside S and with determinant η , is finite flat over $W(\kappa)$.*

Following the proof of Theorem 1.8, which we give in Section 3, one can show that it suffices to assume that $\bar{\tau}$ restricted to G_{K_∞} is absolutely irreducible.

Suppose we are given a Galois representation $\bar{\tau}: G_K \rightarrow \mathrm{GL}_n(\kappa)$. By a *lift to characteristic zero*, we mean a Galois representation $\tau: G_K \rightarrow \mathrm{GL}_n(\mathcal{O})$, such that

- (a) \mathcal{O} is a finite flat local $W(\kappa)$ -algebra with residue field κ .
- (b) $\tau \equiv \bar{\tau} \pmod{l}$,
- (c) τ is ramified at most at finitely many places.

As pointed out in [12], Rem. 3.6(b), the above Theorem has the following consequence.

COROLLARY 1.6 (DE JONG) *If Conjecture 1.1 holds, e.g., if $n \leq 2$, then any $\bar{\tau}$ as in the previous theorem has a lift to characteristic zero which ramifies precisely at the places at which $\bar{\tau}$ ramifies.*

We will show in Section 3 that this has the following implication:

COROLLARY 1.7 *Suppose K is a function field of characteristic $p > 3$ and of genus at least two. Then for infinitely many l , there exist a finite constant field extension E of K and a continuous, irreducible, nowhere ramified representation $\rho': G_E \rightarrow \mathrm{SL}_2(\mathcal{O})$ for some finite flat \mathbb{Z}_l -algebra \mathcal{O} , such that $\rho'(G_{E\mathbb{F}_p}) = \rho'(G_{E_\infty}) = \rho'(G_E)$ is infinite.*

This gives examples of infinite, unramified l -adic analytic Galois extensions of function fields E over finite fields with finite constant field. Previous examples had been constructed by Ihara, [11], and Frey, Kani, Völklein, [9]. The interest in such examples stems from a question posed by Holden, [10], who asks for an analogue of a conjecture by Fontaine and Mazur as studied in [4] over function fields.

Suppose now that K is number field and let $\bar{\tau} : G_K \rightarrow \mathrm{GL}_n(\kappa)$ be an even representation unramified outside a finite set of places S of K . Following the arguments in [12], which are used to derive Corollary 1.6 from Conjecture 1.1, we shall prove the following in Section 3:

THEOREM 1.8 *Let K , $\bar{\tau}$ and S be as in the previous paragraph and assume that $\bar{\tau}$ is absolutely irreducible when restricted to G_{K_∞} . Let η be any lift of $\det \bar{\tau}$ to $W(\kappa)$. Assume that Conjecture 1.4 holds for any κ' finite over κ and any representation $\tau : G_K \rightarrow \mathrm{GL}_n(\kappa'[[T]])$ whose residual representation is isomorphic to $\bar{\tau}$. Then the universal deformation ring for deformations of $\bar{\tau}$ unramified outside S and with determinant η is finite flat over $W(\kappa)$.*

The arguments in [12], Rem. 3.6(b), which yield Corollary 1.6 as a consequence of Theorem 1.5, carry over verbatim to show the following.

COROLLARY 1.9 *Under the assumptions of the previous theorem, any $\bar{\tau}$ has a lift to characteristic zero which ramifies at most at the set $S \cup \{\mathfrak{l} : \mathfrak{l} | l\}$.*

For $n = 1$, the assertion of the above corollary is a simple consequence of class field theory. For $n = 2$, one has the following theorem due to Ramakrishna, [18], which does not rely on Conjecture 1.4.

THEOREM 1.10 *Let $K = \mathbb{Q}$, $l \geq 7$ and χ the cyclotomic (mod l) character. Assume that the restriction of $\bar{\tau}$ to a decomposition group at l is not twist equivalent to $\begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} \chi^{-1} & * \\ 0 & 1 \end{pmatrix}$, then there exists a finite set of places $S' \supset S$ and a lift $\tau : G_{\mathbb{Q}} \rightarrow \mathrm{GL}_2(W(\kappa))$ of $\bar{\tau}$ which is unramified outside S' .*

In terms of ramification the result of Ramakrishna is weaker than the result predicted in Corollary 1.9. However it is stronger in the sense that one has a lift to $W(\kappa)$ and not just some local finite flat \mathbb{Z}_l -algebra \mathcal{O} with residue field κ .

There is hope that Ramakrishna's methods will allow similar results for two-dimensional Galois representations over arbitrary totally real ground fields K .

Evidence for Conjecture 1.4. The above result of Ramakrishna, which is close to the prediction stated in Corollary 1.9, may be viewed as

some evidence towards the validity of Conjecture 1.4. In the last section of this article, Section 4, we give further support for our conjectures. We present some results by Khare, Ramakrishna and the author that support the predictions of Theorem 1.8. The one-dimensional case is related to Leopoldt's conjecture. Reducible cases are investigated, and a positive result is obtained in the case of completely reducible cases over the base field \mathbb{Q} . Finally for those pro- l Galois groups with restricted ramification which are of a simple type, we can verify a reformulation of Conjecture 1.4.

2. VARIOUS FINITENESS CONJECTURES

Parallels between Conjectures 1.2 and 1.4. Because L/K is a finite Galois extension and the group $\rho(G_L)$ is a pro- l group, Conjecture 1.2 is equivalent to

CONJECTURE 2.1 If K is a function field, ρ is ramified at only finitely many places and $\rho(G_K)$ is a pro- l group, then $\rho(G_{K_\infty})$ is finite.

In the number field case, the situation is analogous:

LEMMA 2.2 *Conjecture 1.4 is equivalent to the following conjecture.*

CONJECTURE 2.3 If K is a totally real number field, ρ is ramified at most at finitely many places and $\rho(G_K)$ is a pro- l group, then $\rho(G_{K_\infty})$ is finite.

Proof of Lemma 2.2. As l is different from 2, any pro- l extension of K is totally real and hence it remains to show that Conjecture 2.3 implies Conjecture 1.4.

For this let $\rho: G_K \rightarrow \mathrm{GL}_n(\kappa[[T]])$ be totally real with residual representation $\bar{\rho}$. Let K' be the splitting field of $\bar{\rho} \pmod{\pm I}: G_K \rightarrow \mathrm{GL}_n(\kappa)/\{\pm I\}$. As ρ is totally real, K' is a finite totally real Galois extension of K . Let $\mathrm{GL}_n^1(\kappa[[T]])$ denote the subgroup of $\mathrm{GL}_n(\kappa[[T]])$ which consists of matrices that reduce to I modulo T . Define $G' := \rho(G_{K'})$. Because $\mathrm{GL}_n^1(\kappa[[T]])$ as well as $\{\pm I\}$ are normal subgroups of G' , the group G' is the direct product of these two. Let $\rho': G_{K'} \rightarrow \mathrm{GL}_n^1(\kappa[[T]])$ be the component of $\rho|_{G_{K'}}$ into $\mathrm{GL}_n^1(\kappa[[T]])$. Then Conjecture 2.3 applies to ρ' and predicts that $\rho'(G_{K'_\infty})$ is finite. It follows readily that $\rho(G_{K_\infty})$ is finite as well.

To compare Conjectures 2.1 and 2.3, we fix some notation. Let S be a finite set of places of K , let K_S be the maximal separable extension of K unramified outside S , and define $G_{K,S} := \mathrm{Gal}(K_S/K)$. For an arbitrary

group H denote by $H(l)$ its pro- l completion, and let $K_S(l)$ be the subfield of K_S corresponding to $G_{K,S}(l)$. Then

$$1 \longrightarrow G_{K_\infty,S}(l) \longrightarrow G_{K,S}(l) \longrightarrow \text{Gal}(K_\infty/K) \cong \mathbb{Z}_l \longrightarrow 1. \quad (1)$$

Suppose first that K is a function field. Then $\text{Gal}(K_\infty/K)$ is topologically generated by the arithmetic (or geometric) Frobenius automorphism. Furthermore if X denotes the smooth projective model of K , then, at least after a finite constant field extension, $G_{K_\infty,S}(l)$ is isomorphic to the pro- l completion of the geometric fundamental group of $X - S$, cf. [20], Thm 2.5. For the latter one has an explicit presentation. In particular it is a finitely generated pro- l group. If K is rational or if S is non-empty, this group is in fact free. In the remaining case it is a Demuškin group. Furthermore, the abelianization of $G_{K_\infty,S}(l)$ is a free abelian pro- l group and has a geometric description as a $\mathbb{Z}_l[[\text{Gal}(K_\infty/K)]]$ -module in terms of generalized Jacobians.

Let K now be a number field. The expected analogy between function fields and number fields led Iwasawa to study abelian pro- l extensions of K_∞ for number fields K . This greatly enlarged our knowledge about number fields and led to many deep insights into the structure of such extensions. Based on this analogy one conjectures that the μ -invariant of all number fields is zero. So let us assume this. Suppose also that S contains all places above l , in which case it is well-known that $G_{K_\infty,S}(l)$ is a free pro- l group. Because of the vanishing of μ , the group $G_{K_\infty,S}(l)$ is finitely generated if and only if K is totally real.

Taking the analogy used by Iwasawa a bit further into a non-commutative setting, one might expect that for free finitely generated $G_{K_\infty,S}(l)$ the extension $G_{K,S}(l)$ has a similar structure independently of whether K is a function or a number field. For example, one might expect similarities when considering l -adic or $\kappa[[T]]$ -adic representations. This suggests that if Conjecture 2.1 holds, then so does Conjecture 2.3.

A finiteness conjecture for Galois extensions of CM fields. We first give a further reformulation of Conjecture 1.4.

LEMMA 2.4 *Conjecture 1.4 is equivalent to the following conjecture.*

CONJECTURE 2.5 If K is a totally real number field, ρ is unramified outside S_l and $\rho(G_K)$ is a pro- l group, then $\rho(G_{K_\infty})$ is finite.

Before proving Lemma 2.4, we need the following auxiliary result.

LEMMA 2.6 Fix $1 \neq m \in 1 + l\mathbb{Z}_l$ and let G be the semi-direct product

$$\mathbb{Z}_l \rtimes \mathbb{Z}_l := \langle s, t | sts^{-1} = t^m \rangle(l).$$

Let H be the closed subgroup generated of G by t and $\rho: G \rightarrow \mathrm{GL}_n(\kappa((T)))$ a continuous representation of G . Then $\rho(H)$ is finite.

Proof. We write M, N for the images of s, t under ρ , respectively. Let us denote the eigenvalues of N by $\lambda_1, \dots, \lambda_n$. Because G is compact the λ_i lie in $\kappa[[T]]$, and because it is pro- l and κ is of characteristic l , the λ_i lie in $1 + T\kappa[[T]]$. The relation $MNM^{-1} = N^m$ implies that N and N^m have the same eigenvalues. Thus one has $\lambda_i^{m^{n!}-1} = 1$ for each i . As $m^{n!} - 1$ is in $l\mathbb{Z}_l - \{0\}$, an explicit computation shows that $\lambda_i = 1$ for all i . Therefore $N - I$ is nilpotent. Because $\kappa((T))$ is of positive characteristic, N must have finite order, whence $\rho(H)$ is finite. \blacksquare

Proof of Lemma 2.4. We show that Conjecture 2.3 is equivalent to Conjecture 2.5, which suffices due to Lemma 2.2. One direction is obvious, and so we assume Conjecture 2.5. Suppose we are given $\rho: G_K \rightarrow \mathrm{GL}_n(\kappa[[T]])$ such that ρ is unramified outside a finite set S , K is totally real and $\rho(G_K)$ is a pro- l group. We claim that there exists a finite extension L of K inside the splitting field of ρ such that the restriction of ρ to G_L is unramified outside S_l . If we then apply Conjecture 2.5 to $\rho|_{G_L}$, we find that $\rho(G_{L_\infty})$ is finite, which readily implies that $\rho(G_{K_\infty})$ is finite, as asserted.

To prove the claim, we follow the proof of [12], Lem 2.12, which in turn is based on a result of Grothendieck. Let ν be a place of S which is not above l , let D_ν be a decomposition group in G_K above ν and let q_ν be the cardinality of the residue field of K at ν . It is well-known that D_ν is isomorphic to the semi-direct product

$$\mathbb{Z}_l \rtimes \mathbb{Z}_l = \langle s_\nu, t_\nu | s_\nu t_\nu s_\nu^{-1} = t_\nu^{q_\nu} \rangle,$$

where t_ν is a generator of the inertia subgroup I_ν of D_ν . By Lemma 2.6, I_ν has finite image under $\bar{\rho}$.

Thus for each place ν of $S - S_l$, we can find a positive integer n_ν such that the $t_\nu^{n_\nu}$ have trivial image under ρ . As the group $\rho(G_K)$ is profinite, there exists a finite Galois extension L of K inside the splitting field of ρ such for each $\nu \in S - S_l$ there exists a place of L above ν whose inertia group is inside $I_\nu^{n_\nu}$. Because L is Galois over K , this implies that ρ restricted to L is unramified outside S_l , and the claim is shown.

Now let E be a CM field with totally real subfield K . Assume that all places of K above l are split in E/K . For a place \mathfrak{l} of K above l , denote by

$K_{\mathfrak{l}}(l)$ the maximal pro- l extension of the completion of K at \mathfrak{l} . Following [19], we let \tilde{E} be the union of all finite l -extension E' of E which satisfy

- (a) E'/E is Galois
- (b) For all $\mathfrak{L}'|\mathfrak{L}|\mathfrak{l}|l$, where \mathfrak{L}' is a place of E' , \mathfrak{L} of E and \mathfrak{l} of K , one has $E'_{\mathfrak{L}'} \subset K_{\mathfrak{l}}(l)E_{\mathfrak{L}}$ for the completions at \mathfrak{L}' , \mathfrak{L} and \mathfrak{l} , resp.

Following [19], we call \tilde{E} the *maximal outside l unramified extension of E which is positively ramified at all places above l* . In [20], the term *positively decomposed* is used instead of positively ramified.

The motivation for such a definition is the search for a pro- l Galois extension for number fields whose Galois group resembles that of the pro- l completion of the Galois group of a compact Riemann surface. It relies on fundamental work of Wingberg, [20]. The following is shown in [19], Thm. 1, Cor. 2, Thm. 6 and Cor. 7:

THEOREM 2.7 *Let E, \tilde{E} be as above and assume that the μ -invariant of E is zero. Consider*

$$1 \longrightarrow H := \text{Gal}(\tilde{E}/E_{\infty}) \longrightarrow G := \text{Gal}(\tilde{E}/E) \longrightarrow \text{Gal}(E_{\infty}/E) \cong \mathbb{Z}_l \longrightarrow 1.$$

(a) *If $\zeta_l \notin E$, then H is a free pro- l group of finite rank. Furthermore G is either isomorphic to \mathbb{Z}_l , or a duality group of dimension 2.*

(b) *If $\zeta_l \in E$, then H is trivial or a Demuškin group of finite rank such that H^{ab} is torsion free. Furthermore, either $G \cong \mathbb{Z}_l$ or G is a Poincaré group of dimension 3.*

Case (a) describes an analogue of the fundamental group of the projective line over a finite field with a finite number of points removed or of the fundamental group of a curve of genus greater than zero, with at least one point removed. Case (b) is the analogue of the fundamental group of a smooth projective curve over a finite field of genus at least one. As at the end of the previous subsection, if this is truly a good analogy, then one should expect that the $\kappa[[T]]$ -analytic linear representations of $\text{Gal}(\tilde{E}/E)$ display the same general properties as those of the arithmetic fundamental group of a curve over a finite field. Thus we are led to the following conjecture:

CONJECTURE 2.8 *Let E be a CM field with totally real subfield K and assume that all places in K above l split in E/K . Then for any continuous Galois representation $\rho : \text{Gal}(\tilde{E}/E) \rightarrow \text{GL}_n(\kappa[[T]])$, the group $\rho(\text{Gal}(\tilde{E}/E_{\infty}))$ is finite.*

PROPOSITION 2.9 *Let E, K be as above and assume that the μ -invariant of E is zero. Then Conjecture 2.8 for E implies Conjecture 2.5 for K .*

Proof. Let $\rho : G_{K, S_l} \rightarrow \mathrm{GL}_n(\kappa[[T]])$ be a continuous even representation whose image is a pro- l group. The definition of \tilde{E} implies that K_{S_l} is a subfield of \tilde{E} . Thus the restriction of ρ to G_E factors through $\mathrm{Gal}(\tilde{E}/E)$ and by Conjecture 2.8 it follows that $\rho(G_{E_\infty})$ is finite. The assertion is now obvious as $\rho(G_{E_\infty}) \subset \rho(G_{K_\infty})$ is of index at most two. \blacksquare

Relations to conjectures by Fontaine-Mazur and Boston. Let us first recall one of the conjectures of Fontaine and Mazur, [8], in a suitable form.

CONJECTURE 2.10 (FONTAINE-MAZUR) Let K be a number field. Then any finite-dimensional, finitely ramified l -adic representation of G_K which is unramified above S_l has finite image.

Let us compare Conjecture 2.5 with Conjecture 2.10. The former is an assertion on $\kappa[[T]]$ -adic representations, the latter on l -adic ones. The former allows ramification only at places of S_l , the latter only away from S_l . So the conjectures as stated are not directly related. However by working out some explicit cases, [4], Boston was led to the following strengthening of Conjecture 2.10:

CONJECTURE 2.11 (BOSTON) Let R be a complete noetherian local ring with finite residue field of characteristic l and S a finite set of places disjoint from S_l . Then every continuous homomorphism $G_{K, S} \rightarrow \mathrm{GL}_n(R)$ has finite image.

In fact it is easy to see that Boston's conjecture is equivalent to the following:

CONJECTURE 2.12 Let S be disjoint from S_l , K be any number field, and κ be any finite extension of \mathbb{F}_l . Let ρ' be either an l -adic representation or a $\kappa[[T]]$ -adic representation, and assume that ρ' is ramified only at S . Then ρ' has finite image.

The first assertion is precisely the above conjecture of Fontaine and Mazur. The second, at least in the case of totally real K , is a consequence of our conjectures:

PROPOSITION 2.13 *Let K be totally real. Then Conjecture 1.4 implies Conjecture 2.11 for $R = \kappa[[T]]$.*

We first prove a lemma.

LEMMA 2.14 *Let G be a profinite group which is given as an extension*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{\pi} \bar{G} \longrightarrow 1,$$

where \bar{G} is procyclic and H is finite. Then there exists a normal subgroup N of G of finite index which contains H and such that $N \cong H \times G_0$ for some subgroup G_0 of G .

Proof. To prove the lemma, let $\bar{\gamma}$ be a generator of \bar{G} and γ a preimage in G . Conjugation by γ induces an automorphism $\tilde{\gamma}$ of the finite group H . Thus $\tilde{\gamma}$ has finite order, say n . Choosing G_0 as the closed subgroup of G generated by γ^n , the assertion follows easily. ■

Proof of Proposition 2.13. Let $\rho : G_{K,S} \rightarrow \mathrm{GL}_n(\mathbb{F}_q[[T]])$ be a continuous Galois representation where S is a finite set of places disjoint from S_l . Conjecture 1.4 states that $\rho(G_{K_\infty})$ is finite. We need to show that $\rho(G_K)$ is finite. Since the splitting field L of $\bar{\rho}$ is finite over K , we may assume that $L = K$. Furthermore by passing to a finite extension of K inside K_∞ , we may assume that every subextension of K_∞/K is ramified at a place above l . Let L' be the splitting field of $\rho|_{G_{K_\infty}}$.

We apply the above lemma and Galois theory to

$$1 \longrightarrow \mathrm{Gal}(L'/K_\infty) \longrightarrow \mathrm{Gal}(L'/K) \longrightarrow \mathrm{Gal}(K_\infty/K) \longrightarrow 1.$$

Thus there exist finite extensions $K_0 \subset K' \subset L'$ of K such that $K_0 \subset K_\infty$,

$$\mathrm{Gal}(L'/K_0) \cong \mathrm{Gal}(L'/K_\infty) \times \mathrm{Gal}(L'/K') \text{ and } \mathrm{Gal}(L'/K') \cong \mathrm{Gal}(K_\infty/K_0).$$

But now, $\mathrm{Gal}(K_\infty/K_0)$ is totally ramified above l . Hence by our assumption on ρ , the image of $\mathrm{Gal}(L'/K')$ under ρ is trivial. In other words, the restriction of ρ to $G_{K'}$ is trivial. As K'/K is finite, the proposition follows.

The following is left as a simple exercise:

PROPOSITION 2.15 *Let K be totally real and ρ a representation which is unramified outside S_l and finitely ramified at any place above l . Then Conjecture 2.11 implies Conjecture 1.4 for ρ .*

Group-theoretical reformulations. For K a number field and S disjoint from S_l , the structure of $G_{K,S}(l)$ remains rather mysterious. For S sufficiently large it is always infinite. But if Conjecture 2.11 is valid, this cannot be detected by looking at linear representations. In recent work, cf. [5], Boston conjectures that such groups can have infinite representations

on the pro- l completion of the automorphism group of a rooted tree. Yet, no examples seem to be known.

For K a number field and $S \supset S_l$, one has the short exact sequence (1). While this still does not give a complete description of $G_{K,S}$, at least it shows that it is an extension of $\text{Gal}(K_\infty/K) \cong \mathbb{Z}_l$ by the free, possibly infinitely generated pro- l group $G_{K_\infty,S}(l)$. Say we impose the further condition that K is totally real and assume that the μ -invariant is zero. Then $G_{K_\infty,S}(l)$ is finitely generated. In this situation, our conjecture predicts that $G_{K,S}(l)$ admits non-solvable infinite l -adic representations, cf. Theorem 1.8, but no non-solvable infinite $\kappa[[T]]$ -adic representations. This leads naturally to the following purely group-theoretical question:

QUESTION 2.16 Let G be a pro- l group which is the extension of \mathbb{Z}_l by a free finitely generated pro- l group F . Give group-theoretical conditions on G such that for any representation $\varrho: G \rightarrow \text{GL}_n(\kappa((T)))$ the group $\varrho(F)$ is finite.

Via a non-canonical lift of a generator of \mathbb{Z}_l , we may regard G as the semi-direct product of F with \mathbb{Z}_l . If γ denotes a topological generator of \mathbb{Z}_l , then it may be regarded as a continuous automorphism of F . So one may reformulate the above question as follows:

QUESTION 2.17 Can one classify those continuous automorphisms of F which generate a group isomorphic to \mathbb{Z}_l and have the property that the resulting semi-direct product $G := F \rtimes \mathbb{Z}_l$ satisfies $\#\varrho(F) < \infty$ for any $\varrho: G \rightarrow \text{GL}_n(\kappa((T)))$?

As explained on page 6, the expected situation in the case of function fields is very similar. Unless K is of strictly positive genus and no ramification is allowed, one is led to precisely the same group-theoretical questions as above. In the remaining case, one is led to:

QUESTION 2.18 Can one classify those continuous automorphisms of a given Demuškin group D which generate a group isomorphic to \mathbb{Z}_l and have the property that the resulting semi-direct product $G := D \rtimes \mathbb{Z}_l$ satisfies $\#\varrho(D) < \infty$ for any $\varrho: G \rightarrow \text{GL}_n(\kappa((T)))$?

3. CONSEQUENCES OF THE STATED CONJECTURES

Infinite, everywhere unramified l -adic extensions of function fields with finite constant fields.

Proof of Corollary 1.7. As we assume that $p > 3$, there exist infinitely many primes l such that p does not divide $l^3 - l$. We carry out the construction for any such l which also satisfies $l > 5$.

By the pro- p' completion of a profinite group G , we mean the filtered inverse limit $\varprojlim G/H$ over all (closed) normal subgroups H such that G/H has order prime to p . It is well-known that the pro- p' completion of the geometric fundamental group of the smooth proper model of $K\bar{\mathbb{F}}_p$ is isomorphic to the pro- p' completion G' of the discrete group with the presentation

$$\langle a_1, b_1, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} \rangle,$$

where g is the genus of K . As we assumed that $g \geq 2$, G' has a quotient which is isomorphic to the free pro- p' group F_2' on two generators.

Our assumption $p \nmid l^3 - l$ implies that $\mathrm{SL}_2(\mathbb{F}_l)$ occurs as a quotient \bar{G}' of F_2' . Let L' be the corresponding extension of $K\bar{\mathbb{F}}_p$ with Galois group $\mathrm{SL}_2(\mathbb{F}_l)$. Consider the short exact sequence

$$1 \longrightarrow \mathrm{Gal}(L'/K\bar{\mathbb{F}}_p) \longrightarrow \mathrm{Gal}(L'/K) \longrightarrow \mathrm{Gal}(K\bar{\mathbb{F}}_p/K) \cong \hat{\mathbb{Z}} \longrightarrow 1.$$

By Lemma 2.14 and Galois theory, there exist finite extensions $E \subset E'$ of K inside L' with $E \subset K\bar{\mathbb{F}}_p$ such that $\mathrm{Gal}(L'/E) = \mathrm{Gal}(E'/E) \times \mathrm{Gal}(K\bar{\mathbb{F}}_p/E)$. It follows that there is a continuous unramified surjective morphism $\bar{\tau} : G_E \rightarrow \mathrm{SL}_2(\mathbb{F}_l)$ such that $\bar{\tau}(G_{E\bar{\mathbb{F}}_p}) \cong \mathrm{SL}_2(\mathbb{F}_l)$ is absolutely irreducible.

By Theorem 1.5 the universal deformation ring R' for unramified deformations of $\bar{\tau}$ with trivial determinant is finite flat over \mathbb{Z}_l . Therefore, the ring $R'[1/l]$ is a finite \mathbb{Q}_l vector space whose dimension equals the rank of R' over \mathbb{Z}_l . So if \mathcal{O} denotes the quotient of R' by some minimal prime ideal, it is an integral domain whose fraction field is a finite extension of \mathbb{Q}_l . By the universality of R' , we obtain a lift $\rho' : G_E \rightarrow \mathrm{SL}_2(\mathcal{O})$ of $\bar{\tau}$.

We first show that the image of ρ' must be infinite. If this was not the case, then choose any inclusion of \mathcal{O} into the complex numbers \mathbb{C} and consider the resulting representation ρ'' into $\mathrm{PGL}_2(\mathbb{C})$. As the map $\mathrm{GL}_2(\mathbb{C}) \rightarrow \mathrm{PGL}_2(\mathbb{C})$ has abelian kernel and as the image of ρ' has a quotient isomorphic to $\mathrm{SL}_2(\mathbb{F}_l)$, the image of ρ'' must surject onto $\mathrm{PSL}_2(\mathbb{F}_l)$. But this is absurd, as this group does not occur as a quotient of any of the possible finite subgroups contained in $\mathrm{PGL}_2(\mathbb{C})$, cf. [7], Sections 255, 260.

It remains to prove the assertion that $\rho'(G_E) = \rho'(G_{E_\infty}) = \rho'(G_{E\bar{\mathbb{F}}_p})$. We use arguments similar to those of [3]. Let D be the subgroup of $\mathrm{SL}_2(\mathbb{F}_l)$ generated by the matrices

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \quad x \in \mathbb{F}_l^*.$$

Let E' be the finite extension of E corresponding to the subgroup D of $\bar{G}' \cong \mathrm{SL}_2(\mathbb{F}_l)$ and, as usual, let L be the splitting field of $\bar{\tau}$. We claim that

$\rho'(G_{E'}) = \rho'(G_{E'_\infty})$. This will finish the proof, since (a), it implies that $\rho'(G_L) = \rho'(G_{L_\infty}) = \rho'(G_{L_{\mathbb{F}_p}})$, because $\rho(G_L)$ is a pro- l group, and (b), we have by construction that $\bar{\tau}(G_E) = \bar{\tau}(G_{E_{\mathbb{F}_p}})$.

To prove the claim, let $L_\emptyset(l)$ denote the maximal unramified pro- l extension of L . This extension is clearly Galois over E' . We observe that D is of order prime to l . Therefore by a profinite version of the Lemma of Schur-Zassenhaus, the quotient D of $\text{Gal}(L_\emptyset(l)/E')$ can be realized as a subgroup, so that

$$\text{Gal}(L_\emptyset(l)/E') \cong G_{L,\emptyset}(l) \rtimes D.$$

Let γ' denote an element of $G_{L,\emptyset}(l)$ whose image in $\text{Gal}(L_\infty/L) \cong \mathbb{Z}_l$ is a generator. By the arguments given in [3], §2, we may choose a γ' on which D acts trivially. Thereby we may further identify

$$\text{Gal}(L_\emptyset(l)/E') \cong G_{L_\infty,\emptyset}(l) \rtimes (\text{Gal}(E'_\infty/E') \times D).$$

Since the kernel of $\text{GL}_2(\mathcal{O}) \rightarrow \text{GL}_2(\mathbb{F}_l)$ is a pro- l group, ρ' restricted to $G_{E'}$ must factor through $\text{Gal}(L_\emptyset(l)/E')$. Also ρ' yields a two-dimensional irreducible representation of D . Because D acts trivially on γ' , the image of $\rho'(D)$ must commute with $\rho'(\gamma')$. By irreducibility, $\rho'(\gamma)$ is in the center of $\text{SL}_2(\mathcal{O})$, i.e. in the subgroup $\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Hence $\rho'(\gamma)$ is trivial, as $l \neq 2$, and we have shown $\rho'(G_{E'}) = \rho'(G_{E'_\infty})$. ■

On universal deformation rings of even Galois representations.

For the remainder of this section, we assume that K is a totally real number field. Let us fix an even residual representation $\bar{\tau} : G_{K,S} \rightarrow \text{GL}_n(\kappa)$ and some lift $\eta : G_K \rightarrow W(\kappa)^*$ of $\det(\bar{\tau})$. By $R_{K,S}^\eta$ we denote the universal deformation ring in the sense of Mazur, [14], of deformations of $\bar{\tau}$ that are unramified outside S and whose determinant is η .

THEOREM 3.1 *Suppose that $\bar{\tau}$ is absolutely irreducible when restricted to G_{K_∞} . Then Conjecture 1.4 for K implies that $R_{K,S}^\eta$ is a complete intersection, finite flat over $W(\kappa)$.*

Conversely, let l^v denote the maximal l -power dividing n and \tilde{K} the unique subextension of K_∞ of order l^v over K . If $\tilde{R}_{\tilde{K},S}^\eta := R_{\tilde{K},S}^\eta/(l)$ is finite and if the Leopoldt conjecture holds for K , then Conjecture 1.4 holds for all $\tau : G_{K,S} \rightarrow \text{GL}_n(\kappa[[T]])$ whose reduction modulo T is $\bar{\tau}$.

A similar formulation in the case $n = 2$ was given in [1], Theorem 4.14.

Note that the above theorem implies Theorem 1.8!

Proof. (This is a variation of 3.14 in [12].) By Theorem 2.4 of [2] the ring $R_{K,S}^\eta$ is a quotient of a power series ring $W(\kappa)[[x_1, \dots, x_m]]$ modulo

at most $r \leq m$ equations f_1, \dots, f_r . Our assertion follows if we show that $\bar{R}_{K,S}^\eta/(l)$ is finite, because then the height of the ideal (f_1, \dots, f_r, l) is $m+1 \leq r+1$, and hence $m=r$ and f_1, \dots, f_r, l is a regular sequence.

So we assume that $\dim \bar{R}_{K,S}^\eta > 0$. The ring $\bar{R}_{K,S}^\eta$ parameterizes mod l deformations of $\bar{\tau}$ with determinant $\det(\bar{\tau})$. Let A be any quotient of $\bar{R}_{K,S}^\eta$ by a prime of coheight one, and let $\bar{\tau}_A$ be the corresponding representation. If A' denotes the normalization of A , then $A' \cong \kappa'[[T]]$ for some finite extension κ' of κ .

We apply Conjecture 1.4 to $\bar{\tau}_A$ as a representation into $\mathrm{GL}_n(\kappa'[[T]])$. Therefore $\bar{\tau}_A(G_{K_\infty})$ is finite. By Lemma 2.14, there exists a finite extension E of K inside K_∞ and a finite extension E' of E inside the splitting field of τ_A , such that $\bar{\tau}_A(G_E) = \bar{\tau}_A(G_{K_\infty}) \times \bar{\tau}_A(G_{E'})$.

Because $\bar{\tau}_A(G_{K_\infty})$ is absolutely irreducible, the group $\bar{\tau}_A(G_{E'})$ maps to the center of $\mathrm{GL}_n(A)$. But the determinant map $\mathrm{GL}_n(\kappa'[[T]]) \rightarrow \kappa'[[T]]^*$ has finite kernel when restricted to the center. Because $\det \bar{\tau}_A = \det \bar{\tau}$ it follows that $\bar{\tau}_A(G_{E'})$ is finite, and hence so is $\bar{\tau}_A(G_K)$.

By [12], Lem. 3.15, the representation $\bar{\tau}_A$, viewed as a representation into $\mathrm{GL}_n(\kappa'[[T]])$, must factor via $\mathrm{GL}_n(\kappa')$. Thus we may factor $\bar{R}_{K,S}^\eta \rightarrow A \rightarrow \kappa'[[T]]$ as the composite $\bar{R}_{K,S}^\eta \rightarrow \kappa' \rightarrow \kappa'[[T]]$. Let $\bar{\mathfrak{m}}$ be the maximal ideal of $\bar{R}_{K,S}^\eta$. It must map to the maximal ideal of κ' , hence to zero. But this implies that $\bar{\mathfrak{m}}$ is also in the kernel of the surjection $\bar{R}_{K,S}^\eta \rightarrow A$, whence $A \cong \kappa$, a contradiction.

We now turn to the second assertion, and so we assume that $\bar{R}_{K,S}^\eta$ is finite. We denote by $\tau: G_{K,S} \rightarrow \mathrm{GL}_n(\kappa[[T]])$ any lift of $\bar{\tau}$ and let η' be its determinant. Then the character $\psi := \eta'^{-1} \det \bar{\tau}$ of $G_{K,S}$ takes its image in $(1 + T\kappa[[T]], \cdot)$. We would like to twist τ by the n -th root of ψ . But this may not exist. So we proceed as follows.

First we note that $(1 + T\kappa[[T]], \cdot)$ is a torsion free pro- l group. As we assume the Leopoldt conjecture for K , class field theory shows that ψ must factor via $\mathrm{Gal}(K_\infty/K) \cong \mathbb{Z}_l$. Let γ be a topological generator of $\mathrm{Gal}(K_\infty/K)$ and let w be the unique n/l^v -th root of $\psi(\gamma)$ in $1 + T\kappa[[T]]$. The following character of $G_{\bar{K}}$ provides us with an ' n -th root of $\psi|_{G_{\bar{K}}}$ ':

$$\psi': G_{\bar{K}} \rightarrow \mathrm{Gal}(K_\infty/K') \rightarrow (1 + T\kappa[[T]], \cdot) : \gamma^{l^v} \rightarrow w.$$

Therefore $\tau' := \tau|_{G_{\bar{K}}} \otimes \psi'$ is a deformation of $\bar{\tau}|_{G_{\bar{K}}}$. Hence it arises from the universal such deformation via a local ring homomorphism $\bar{R}_{\bar{K},S}^\eta \rightarrow \kappa[[T]]$. This shows that $\tau'(G_{\bar{K}})$ and hence also $\tau'(G_{\bar{K}_\infty}) = \tau'(G_{K_\infty})$ are finite. As clearly $\psi'(G_{K_\infty})$ is trivial, it follows that $\tau(G_{K_\infty})$ is finite. ■

4. EVIDENCE FOR CONJECTURE 1.4

Throughout this section let K be a totally real number field.

Results which support the implications of Conjecture 1.4. In Theorem 3.1, we showed, under suitable hypothesis, that Conjecture 1.4 is equivalent to the finite flatness of $R_{K,S}^\eta$ over $W(\kappa)$. In fact all the rings $R_{K,S}^\eta$ which are explicitly known and arise from residual representations $\bar{\tau}$ as in Theorem 3.1 are finite flat over $W(\kappa)$. However there are very few examples for which the corresponding deformation ring is obstructed and different from $W(\kappa)$.

The only such examples are based on [15] and [16], and shown to be finite flat over $W(\kappa)$ in Corollary 7.9 of [2]. There is numerical and theoretical evidence that there is an abundance of such examples, cf. the probabilistic results in [17].

As stated in Theorem 1.10, in the case $n = 2$ and $K = \mathbb{Q}$ there is ample evidence that the conclusion of Corollary 1.9 holds in general. Based on the methods that led to Theorem 1.10, it is observed in [13] that for given $\bar{\tau}$ and S sufficiently large, any $R_{\mathbb{Q},S}^\eta$ contains a component which is finite flat over $W(\kappa)$. Both can be viewed as further evidence for Conjecture 1.4.

Reducible representations. Let us first look at the simplest case of a representation ρ .

PROPOSITION 4.1 *Suppose K is totally real. For one-dimensional representations, Conjecture 1.4 for K is equivalent to Leopoldt's conjecture for K .*

Proof. For one-dimensional representations the conjecture is a conjecture on abelian extensions of number fields. If Leopoldt's conjecture holds, then the maximal abelian quotient of $G_{K,S}$ is a finite extension of $\mathbb{Z}_l \cong \text{Gal}(K_\infty/K)$. The conjecture is immediate. If Leopoldt's conjecture is wrong, then $G_{K,S}$ has a quotient isomorphic to $\mathbb{Z}_l \times \text{Gal}(K_\infty/K)$. Obviously this implies the existence of representations that violate Conjecture 1.4. ■

By $\tilde{\rho}$ we will always mean a Galois representation $G_K \rightarrow \text{GL}_n(\kappa((T)))$. Given ρ as in the introduction, one can canonically attach a $\tilde{\rho}$ as the composite

$$G_K \xrightarrow{\rho} \text{GL}_n(\kappa[[T]]) \longrightarrow \text{GL}_n(\kappa((T))).$$

Conversely if one is given $\tilde{\rho}$, by compactness of G_K one can choose a suitable basis of $\kappa((T))^n$ such that the image of $\tilde{\rho}$ lies in $\text{GL}_n(\kappa[[T]])$. Hence Conjecture 1.4 is equivalent to:

CONJECTURE 4.2 Suppose $\tilde{\rho}: G_K \rightarrow \mathrm{GL}_n(\kappa((T)))$ is even and ramified only at finitely many places. Then $\tilde{\rho}(G_{K_\infty})$ is finite.

The reason for introducing $\tilde{\rho}$ is that over fields it is simpler to talk about reducibility of representations.

PROPOSITION 4.3 *Suppose that the μ -invariant of all totally real number fields is zero. If $\tilde{\rho}$ is even, continuous and reducible, and if Conjecture 4.2 holds for all irreducible subquotients of $\tilde{\rho}$, then the conjecture holds for $\tilde{\rho}$.*

Proof. By passing to a finite totally real extension E of K , we may assume that all irreducible subquotients have the property that when the representation is restricted to $\mathrm{Gal}(E_S/E_\infty)$, it is trivial. It follows that $\mathrm{Gal}(E_S/E_\infty)$ maps to a unipotent subgroup of $\mathrm{GL}_n(\kappa((T)))$. Any unipotent subgroup of $\mathrm{GL}_n(\kappa((T)))$ is of nilpotency degree at most $l^{\lfloor \log_l n \rfloor + 1}$. Thus $\rho(\mathrm{Gal}(E_S/E_\infty))$ is of finite nilpotency degree. Moreover it is topologically finitely generated, because we assumed that the μ -invariant of L is zero. Hence $\rho(\mathrm{Gal}(E_S/E_\infty))$ is finite, and the same must therefore hold for $\rho(G_{K_\infty})$. ■

COROLLARY 4.4 *If $\tilde{\rho}: G_{\mathbb{Q},S} \rightarrow \mathrm{GL}_n(\kappa((T)))$ is completely reducible with one-dimensional composition factors, then Conjecture 1.4 holds unconditionally.*

Proof. Clearly Leopoldt's conjecture holds for \mathbb{Q} , and so by Proposition 4.1 there exists a finite abelian extension L of \mathbb{Q} , such that the image of G_L under ρ is a unipotent pro- l group. Because L is abelian, it is known that its μ -invariant is zero. The result follows from the argument given in the previous proof. ■

Galois groups of simple structure with restricted ramification.

The other evidence we have for our conjectures comes about by considering pairs K, S such that $G_{K,S}(l)$ has a particularly simple structure and by investigating Conjecture 2.3. For every pair of a totally real number field K and a finite set of places S , we denote by $C_{K,S}$ the following assertion:

$C_{K,S}$ Any $\kappa[[T]]$ -linear representation ρ of $G_{K,S}$ satisfies $\#\rho(G_{K_\infty}) < \infty$.

Clearly Conjecture 2.3 is equivalent to the assertion $C_{K,S}$ for all pairs (K, S) .

The simplest structure of $G_{K,S}(l)$ to occur is that it is isomorphic to \mathbb{Z}_l . Here Conjecture 2.3 trivially holds. The case we treat in the following proposition might be considered as the simplest non-trivial case.

PROPOSITION 4.5 *If $G_{K,S}(l)$ is a non-abelian Demuškin group of rank 2, then Conjecture 2.3 holds for the pair (K, S) .*

Proof. Our assumption means that there is an isomorphism

$$G_{K,S}(l) \cong \mathbb{Z}_l \ltimes \mathbb{Z}_l \cong \langle s, t | sts^{-1} = t^m \rangle$$

for some $1 \neq m \in 1 + l\mathbb{Z}_l$. Furthermore, if H denotes the closed subgroup of $G_{K,S}$ generated by t , then the fixed field of H must be K_∞ . As we assume that the images of the representations we consider are pro- l groups, they factor via $G_{K,S}(l)$. Lemma 2.6 completes the proof. \blacksquare

REMARK 4.6 If one assumes Leopoldt's conjecture for K and if $G_{K,S}(l)$ is a Demuškin group, then it cannot be abelian.

In [21] explicit conditions in terms of invariants attached to K and S are given (for $l \neq 2$) such that $G_{K,S}(l)$ is a Demuškin group of rank 2. We quote the following examples: (a) $l = 3$, $K = \mathbb{Q}$, $S = \{3, 7, \infty\}$, (b) $l = 3$, $K = \mathbb{Q}(\sqrt{6})$, $S = \{3, \infty\}$, (c) $l = 37$, $K = \mathbb{Q}(\zeta_{37} + \zeta_{37}^{-1})$, $S = \{3, \infty\}$.

ACKNOWLEDGMENTS

I gratefully acknowledge the financial support of the Deutsche Forschungsgemeinschaft through a Habilitation grant as well as the hospitality of the University of Illinois at Urbana-Champaign that I received while working on this project. For some interesting discussions related to this work during my stay at Urbana, I would like to thank Nigel Boston. Many thanks also to the referee for his extremely careful reading of the original manuscript and a great number of useful suggestions.

REFERENCES

1. G. Böckle, *Explicit universal deformations of even Galois representations*, Math. Nachr. **206** (1999), 85–110.
2. G. Böckle, *A local-to-global principle for deformations of Galois representations*, J. Reine Angew. Math. **509** (1999), 199–236.
3. N. Boston, *Explicit Deformations of Galois Representations*, Invent. Math. **103** (1990), p. 181–196.
4. N. Boston, *Some cases of the Fontaine-Mazur conjecture. II*, J. Number Theory **75** (1999), no. 2, 161–169.
5. N. Boston, *Tree representations of Galois groups*, preprint 2000.
6. N. Boston and B. Mazur, *Explicit Universal Deformations of Galois Representations*, Adv. Stud. Pure Math. **17** (1989), 1–21.
7. L.E. Dickson, *Linear Groups with an Exposition of the Galois Field Theory*, Teubner, Leipzig, 1901.
8. J.-M. Fontaine, B. Mazur, *Geometric Galois representations*, in 'Elliptic curves, modular forms and Fermat's last theorem', J. H. Coates, S. T. Yao, eds., International Press, Boston-Hong Kong, 1993.
9. G. Frey, E. Kani, H. Völklein, *Curves with infinite K -rational geometric fundamental group*, in 'Aspects of Galois theory' (Gainesville, FL, 1996), 85–118, LMS Lecture Note Ser. 256, Cambridge Univ. Press, Cambridge, 1999.

10. J. B. Holden, *On the Fontaine-Mazur Conjecture for Number Fields and an Analogue for Function Fields*, J. Number Theory **81** (2000), 16–47.
11. Y. Ihara, *On unramified extensions of function fields over finite fields*, in ‘Galois groups and their representations’ (Nagoya, 1981), 89–97, Adv. Stud. Pure Math. 2, North-Holland, Amsterdam-New York, 1983.
12. A. J. de Jong, *A conjecture on arithmetic fundamental groups*, preprint.
13. C. Khare, R. Ramakrishna, *Finiteness of Selmer groups and deformation rings*, preprint 2000.
14. B. Mazur, *Deforming Galois representations*, in: Galois groups over \mathbb{Q} , Y. Ihara, K. Ribet, J.-P. Serre eds., MSRI Publ. **16**, Springer-Verlag, New-York, Berlin, Heidelberg, 1987, 385–437.
15. R. Ramakrishna, *Deforming an even representation*, Inventiones Math. **132** (1998), 563–580.
16. R. Ramakrishna, *Deforming an even representation. II. Raising the level*, J. Number Theory **72** (1998), 92–109.
17. R. Ramakrishna, *Lifting Galois representations*, Invent. Math. **138** (1999), no. 3, 537–562.
18. R. Ramakrishna, *Deforming Galois representations and the conjectures of Serre and Fontaine-Mazur*, preprint 1999.
19. A. Schmidt, *Positively ramified extensions of algebraic number fields*, J. reine angew. Math. **458** (1995), 93–126.
20. K. Wingberg, *Ein Analogon zur Fundamentalgruppe einer Riemann’schen Fläche im Zahlkörperfall*, Invent. Math. **77** (1984), 557–584.
21. K. Wingberg, *Galois Groups of Local and Global Type*, J. Reine Angew. Math. **517** (1999), 223–239.