The generic fiber of the universal deformation space associated to a tame Galois representation

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Abstract. We will study the generic fiber over $\mathbb{Q}_l$ of the universal deformation ring R_Q, as defined by Mazur, for deformations unramified outside a finite set of primes Q of a given Galois representation $\bar{\rho} : G_E \to \mathrm{GL}_2(k)$, E a number field, k a finite field of characteristic l. The main result will be that, if $\bar{\rho}$ is tame and absolutely irreducible, and if one assumes the Leopoldt conjecture for the splitting field E_0 of $\mathrm{ad}_{\bar{\rho}}$, then $R_Q \otimes \mathbb{Q}_l$ defines a smooth l-adic analytic variety, near the trivial lift ρ_0 of $\bar{\rho}$, whose dimension is given by cohomological constraints and as predicted by Mazur. As a corollary it follows that, in the cases considered here, R_Q is a quotient of $W(k)[[T_1, \ldots, T_n]]$ by an ideal I generated by exactly m equations, where $n = \dim_k H^1(G_E, \mathrm{ad}_{\bar{\rho}})$ and $m = \dim_k H^2(G_E, \mathrm{ad}_{\bar{\rho}})$. Under the above assumptions for $E = \mathbb{Q}$ and $\bar{\rho}$ odd, using ideas of Coleman, Gouvêa and Mazur it should now be possible to show that modular points are Zariski-dense in the component of $R_Q \otimes \mathbb{Q}_l$, that contains the trivial lift ρ_0, provided this lift satisfies the Artin conjecture and E_0 satisfies the Leopoldt conjecture.

Furthermore, in the Borel case, we show that the Krull dimension of R_Q can exceed any given number, provided Q is chosen appropriately. At the same time, we present some evidence that despite this fact, one might however expect that the dimension of the generic fiber is given by the same cohomological formula as in the tame case.

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1 Introduction

In Mazur's seminal paper on deformations of Galois representations, several questions are posed on the universal deformation rings R defined therein. A lower estimate for the Krull dimension of R is given there, which depends only on the restriction of $\mathrm{ad}_{\bar{\rho}}$, to be defined below, to the infinite places. One might wonder if this number is indeed the dimension of R, and if R is always flat over $W(k)$, i.e. l-torsion free. We shall assume from here on that $l > 2$. In many cases where $E = \mathbb{Q}$ and where $\bar{\rho} : G_E =$

$\mathrm{Gal}(\bar{Q}/E) \to \mathrm{GL}_2(k)$ is odd, i.e. a complex conjugation c satisfies $\det(\bar{\rho}(c)) = -1$, this is known by [2] building on work of Diamond and Wiles, see [8], [16].

Here we will study the ring $R \otimes \mathbb{Q}_l$ independently of even or odd, i.e. independently of the question if $\bar{\rho}$ is related to modular or Hilbert modular forms. This ring represents all the lifts, modulo strict equivalence, to $W(k)$ algebras of characteristic zero of a given residual representation $\bar{\rho}$ as above. We shall obtain the following result in the case where $\bar{\rho}$ is tame, i.e., the order of $\mathrm{Im}(\bar{\rho})$ is not divided by l - more precise definitions of the objects involved will be given later.

We shall now fix some notation. Let $\bar{\rho} : G_E \to \mathrm{GL}_2(k)$ be a Galois representation, where k is a finite field of characteristic l, and E is a number field. Let E_0 be the splitting field of the projective representation associated to $\bar{\rho}$, i.e. the fixed field in $\bar{\mathbb{Q}}$ of the inverse image under $\bar{\rho}$ of the set of homotheties in $\mathrm{GL}_2(k)$. Let ρ_0 be the trivial lift of $\bar{\rho}$ to $\mathrm{GL}_2(W(k))$, i.e. the one with $\mathrm{Im}(\rho_0) \cong \mathrm{Im}(\bar{\rho})$, which exists by the pro-finite version of the lemma of Schur Zassenhaus, as $\bar{\rho}$ is tame, and which is unique up to conjugacy. For more on this see §6 in [4]. Let $\mathrm{ad}_{\bar{\rho}}$, or simply ad if no confusion arises, be the adjoint representation, i.e. the representation of G_E on $M_2(k)$ obtained by composing the conjugation action of $\mathrm{GL}_2(k)$ with the map $\bar{\rho}$. Let Q be a set of places of E containing all the places above l and infinity and all places where $\bar{\rho}$ ramifies. Let R_Q be the universal deformation space for deformations of $\bar{\rho}$, unramified outside Q – the precise definition of R_Q will be recalled in the following section –, and let x_0 be the point of $\mathrm{Spec}(R_Q \otimes \mathbb{Q}_l)$ corresponding to ρ_0.

Theorem 1.1 *Suppose that $\bar{\rho} : G_E \to \mathrm{GL}_2(k)$ is an absolutely irreducible tame Galois representation, and that the splitting field E_0 of the projective representation associated to $\bar{\rho}$ satisfies the Leopoldt conjecture. Then the l-adic space of continuous algebra homomorphisms $Hom_{W(k)}(R_Q, \bar{\mathbb{Q}}_l)$ is a smooth l-adic manifold in a sufficiently small neighborhood of x_0 of dimension equal to*

$$n - m = 4[E : \mathbb{Q}] + \dim_k H^0(G_E, \mathrm{ad}_{\bar{\rho}}) - \sum_{\nu|\infty} \dim_k H^0(G_{E_\nu}, \mathrm{ad}_{\bar{\rho}})$$

where $n = \dim_k H^1(G_E, \mathrm{ad}_{\bar{\rho}})$, $m = \dim_k H^2(G_E, \mathrm{ad}_{\bar{\rho}})$.

We note that under our assumptions $\dim_k H^0(G_E, \mathrm{ad}_{\bar{\rho}}) = 1$. Furthermore the local expressions are easy to calculate, and, whenever $E_\nu = \mathbb{R}$, the dimension depends on $\bar{\rho}$ being even or odd at this particular place.

If R_Q is reduced, then by generic smoothness, there is always an open subset of points of $\mathrm{Spec}(R_Q[1/l])$ that are formally smooth, despite the fact that R_Q itself is usually not regular at its maximal ideal. However currently there seem to exist no criteria for R_Q being reduced. The above theorem gives us one smooth point explicitly and hence a component of R_Q that is reduced.

Inspecting the proof of the above theorem, we also find the following.

Corollary 1.2 *Under the above assumptions with the same notation, R_Q is the quotient of $W(k)[[T_1, \ldots, T_n]]$, where n is minimal, by an ideal I generated by exactly m relations, where again m is minimal.*

Remark 1.3 Our result above cannot give any information about the ring being flat over $W(k)$, or all irreducible components of the generic fiber being smooth, or about the generic fiber being equidimensional, as can be seen from the example $W(k)[[x,y]]/(px(y+p)^2, py(y+p)^2)$, which satisfies our conclusion but fails all the properties listed.

Problems 1.4 We cannot say anything about neighborhoods of other lifts to characteristic zero. It would be interesting to know if all of them are smooth, or if modular points, whenever this notion makes sense, are smooth.

Also it would be interesting to know how many irreducible components there are in $R \otimes \mathbb{Q}_l$ and what dimensions they have. One might hope that all the components are smooth and of the same dimension. For $E = \mathbb{Q}$ and $\bar{\rho}$ coming from a modular form, one might expect that the number of connected components - at least over $\bar{\mathbb{Q}}_l$ - is related to the number of certain new forms of a conductor only involving the primes in S that give rise to $\bar{\rho}$.

Remark 1.5 There is the following possible application of our result, for $E = \mathbb{Q}$, to the density of modular representations. Suppose $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(k)$ is tame, odd and absolutely irreducible. The trivial lift $\rho_0 : G_{\mathbb{Q}} \to \mathrm{GL}_2(W(k))$ has finite image, hence we may assume that in fact the image is contained in $\mathrm{GL}_2(\bar{\mathbb{Q}})$. If the Artin conjecture holds for ρ_0, then there is a modular form f of weight one corresponding to ρ_0 and hence giving rise to $\bar{\rho}$ after reduction mod l. We also assume that E_0 satisfies the Leopoldt conjecture. We note, that in principle for a given representation $\bar{\rho}$ all of our assumptions can be checked explicitly. So then what we proved is that the generic fiber near ρ_0, which is now a modular point, i.e. associated to a modular form, is smooth of dimension three.

In [10], it is shown that modular representations are dense in the universal deformation space they consider. If one could construct l-adic analytic arcs of modular curves as in [7] starting at f, i.e. a form of weight one, then arguments similar to those in [10] would show that modular representations are Zariski-dense in the component of $R_S \otimes \mathbb{Q}_l$ containing ρ_0. The argument there works near any smooth modular point in a variety of dimension three, i.e. something that looks like $\mathbb{Q}_l \otimes \mathbb{Z}_l[[X_1, X_2, X_3]]$, because one can keep all the modular arcs and all the twists inside such a given neighborhood by restricting them to small l-adic discs near the starting point.

Remark 1.6 Apart from the fact that it is interesting to consider a ring describing deformations to characteristic zero, there is another reason for considering $R_Q \otimes \mathbb{Q}_l$. Unlike in the case where $\bar{\rho}$ is absolutely irreducible, where one conjectures that R_Q is flat over $W(k)$, of relative dimension as given in our theorem for the generic fiber, and hence that all deformations lift to characteristic zero, in the case that $\mathrm{Im}(\bar{\rho})$ is of Borel type inside $\mathrm{GL}_2(k)$, and sufficiently large for R_Q to be defined, matters are different. There one has examples where the dimension of $R_Q/(l)$ exceeds any given bound, for fixed $\bar{\rho}$, provided Q is chosen appropriately. Also it seems that this phenomenon is related to deformations that do not lift to characteristic zero, i.e. to torsion in R_Q. So it might be more natural to consider $R_Q \otimes \mathbb{Q}_l$, hoping that it behaves similarly to what one would expect, e.g. that its dimension is given by cohomological expressions that come from obstruction theory, like the expression we write down in our theorem.

Further as one is often only interested in lifts to characteristic zero, $R_Q \otimes \mathbb{Q}_l$ is a reasonable space to consider. E.g. for $E = \mathbb{Q}$, this is the part containing all modular points, and one could hope that they are Zariski-dense in this space. While it seems reasonable to hope that the image of the modular points is always Zariski-dense in deformation spaces of Krull dimension three, it would seem unlikely that it could fill out spaces R_Q of large Krull dimension.

We will provide some examples of universal deformation spaces of a simplified type, R'_Q, in Section 4, where the dimension of $R'_Q/(l)$ is rather large, but after tensoring with $\mathbb{Q}_l$ their dimension is given in terms of dimensions of cohomology groups, where the cohomologies take extra constraints like ordinary, ramification, etc. into account. Those spaces R'_Q can arise as surjective images of the spaces R_Q, and so based on this evidence, one might speculate that $R_Q \otimes \mathbb{Q}_l$ behaves well, also if $\bar{\rho}$ is reducible.

This discussion fits in well with some conjectures by Tilouine, see §7 of [15], according to which the dimension of $R_Q^{n.o} \otimes \mathbb{Q}_l$ should be equal to two for $E = \mathbb{Q}$ while the dimension of $R_Q^{n.o}/(l)$ might be larger.

The paper is organized as follows. In Section 2, we give all the relevant definitions, describe the deformation problem and indicate the reduction to a deformation problem with fixed determinant. The following section contains the proof of Theorem 1.1. Here the Leopoldt conjecture will be used to obtain information about the generators and relations in a presentation of the pro-l Galois group that is relevant for our deformation problem. This will result in some properties of the equations describing the universal deformation space near the origin, which will then easily imply the above theorem. In the final section we shall discuss our observations in the Borel case.

2 The set-up

Let k be a finite field of odd characteristic l. Let E be any number field. We assume that we are given a representation $\bar{\rho} : G_E \to \mathrm{GL}_2(k)$, which is tame and absolutely irreducible. It will be called a residual representation. By $\bar{H}$ we denote the image of $\bar{\rho}$ inside $\mathrm{PGL}_2(k)$, by E_0 the corresponding Galois extension of E.

Let $\mathcal{C}$ be the category of complete noetherian local rings with residue field k and local ring homomorphisms which induce the identity on residue fields. So the objects of $\mathcal{C}$ are in particular $W(k)$-algebras, where $W(k)$ is the ring of Witt vectors of k. For R in $\mathcal{C}$ we define $N_2(R) := \ker(\mathrm{GL}_2(R) \to \mathrm{GL}_2(k))$. Two liftings $\rho, \rho' : G_{\mathbb{Q}} \to \mathrm{GL}_2(R)$ of $\bar{\rho}$ are called *strictly equivalent* if there is an $M \in N_2(R)$ such that $\rho = M\rho' M^{-1}$. A strict equivalence class of lifts of $\bar{\rho}$ to R is called a *deformation*.

We now consider the following deformation problem. Let Q be a finite set of places containing all places above l and ∞. Let $G_{E_\mathfrak{p}} = G_\mathfrak{p}$ denote a decomposition group inside G_E for a prime $\mathfrak{p}$ of E, and $I_\mathfrak{p}$ the corresponding inertia group. Define

$$F_Q(R) = \{\text{deformations } [\rho] \text{ of } \bar{\rho} \text{ to } R \text{ unramified outside } Q\}$$

As we want to fix the determinant of our deformations, we shall assume the existence of a lift ρ_0 of $\bar{\rho}$ to a complete discrete valuation ring $\mathcal{O}$, finite over $W(k)$, whose strict equivalence class is in $F_Q(\mathcal{O})$. If $\bar{\rho}$ is tame, we shall assume that ρ_0 is the trivial lift to $W(k)$ which then exists. We define $\varepsilon = \det(\rho_0)$. By $\mathcal{C}_\mathcal{O}$ we denote the full sub-category of $\mathcal{O}$ algebras inside $\mathcal{C}$.

We define a second deformation problem that we will investigate here. For $R \in \mathcal{C}_\mathcal{O}$ we let

$$D_Q(R) = \{\text{deformations } [\rho] \in F_Q(R) \text{ such that } \det(\rho) = \varepsilon\}.$$

Since strict equivalence preserves determinants, D_Q is well-defined. Finally we define

$$\mathrm{Det}_Q(R) = \{\text{lifts of } G_E \to \{1\} \to k^* \text{ to } R^* \text{ unramified outside } Q\}$$

Let $\mathrm{ad}^0 = \mathrm{ad}^0_{\bar\rho}$ be the restriction of $\mathrm{ad} = \mathrm{ad}_{\bar\rho}$, defined in the introduction, to the set of trace zero matrices inside $M_2(k)$.

As we impose no conditions at the primes above l and the conditions at the primes $\mathfrak{p} \in Q$ are rather simple we can appeal to [13] to obtain

Proposition 2.1 *D_Q, F_Q and Det_Q are representable. If the corresponding universal objects are denoted by (R_Q, ρ_Q), (S_Q, α_Q) and $(\Lambda_Q, \varepsilon_Q)$ where $R_Q, S_Q, \Lambda_Q \in \mathcal{C}_{\mathcal{O}}$, then this means, for example for D_Q, that ρ_Q represents a class in $D_Q(R_Q)$, unique up to isomorphism, such that*

$$D_Q(R) \cong Hom(R_Q, R).$$

where the isomorphism is induced from composing the class of ρ_Q with elements of $Hom(R_Q, R)$ and Hom denotes homomorphisms in $\mathcal{C}_{\mathcal{O}}$.

Furthermore $\Lambda_Q = \mathbb{Z}_l[[\Gamma_Q]]$ where Γ_Q is the maximal abelian pro-l extension of E unramified outside Q, and $(S_Q, \alpha_Q) \cong (R_Q, \rho_Q)\hat{\otimes}(\Lambda_Q, \varepsilon_Q)$.

Proof: All but the last statement can be found in [13]. The isomorphism

$$(S_Q, \alpha_Q) \cong (R_Q, \rho_Q)\hat{\otimes}(\Lambda_Q, \varepsilon_Q)$$

is easy to see. Given a representation $\rho : G_E \to \mathrm{GL}_2(R)$, one can associate to it the pair $((\det(\rho)^{-1}\varepsilon)^{1/2}\rho, (\det(\rho)^{-1}\varepsilon)^{1/2})$ where the first element is a lift of $\bar\rho$ with determinant ε and the second a map $G_E \to R^*$, which is trivial after composition with $R^* \to k^*$. Vice versa to any such pair one can associate a lift $\rho : G_E \to \mathrm{GL}_2(R)$. This is all compatible with strict equivalence and gives rise to the above isomorphism. Here we make use of $l > 2$ to be able to take square roots inside R. ∎

Remark 2.2 If E satisfies the Leopoldt conjecture, as has been remarked in [13], then Γ_Q is isomorphic to the product of $\mathbb{Z}_l^{c_E}$ with a finite group where c_E is the number of complex places of E. It is then rather obvious to see that $\Lambda_Q \otimes \bar{\mathbb{Q}}_l$ is the direct sum of copies of $\bar{\mathbb{Q}}_l[[T_0, \ldots, T_{c_E}]]$ and thus of the form described in the main theorem. If E_0 satisfies the Leopoldt conjecture, then the same is clearly true for the subfield E. Furthermore as $\mathrm{ad} = \mathrm{ad}^0 \oplus k^{triv}$, and

$$c_E + 1 = [E : \mathbb{Q}] + \dim_k H^0(G_E, k^{triv}) - \sum_{\nu|\infty} \dim_k H^0(G_{E_\nu}, k^{triv}),$$

the dimension is as predicted. So we only need and will consider from now on the pair (D_Q, ρ_Q) and establish the main theorem for it.

From now on we shall assume that $\bar\rho$ is tame and that ρ_0 is the trivial lift. By P_Q we shall denote the Galois group of the maximal pro-l extension of L that is unramified outside Q, where L is the splitting field of $\bar\rho$. E_0 is a subfield of L. We let $H = \mathrm{Gal}(L/E)$. By the lemma of Schur-Zassenhaus, as described in [4], the sequence

$$1 \to G_L \to G_E \to H \to 1$$

is split, and uniquely up to inner automorphism. The same is true for

$$1 \to N_2(W(k)) \to \pi_{W(k)}^{-1}(H) \to H \to 1$$

where π_R is the natural map from $\mathrm{GL}_2(R) \to \mathrm{GL}_2(k)$, for $R \in \mathcal{C}$. We fix such splittings. Then by [4], §2, there is a natural isomorphism between D_Q and the following functor HE_Q.

$$HE_Q(R) = \{H\text{-equivariant homomorphisms from } P_Q \text{ to } N_2^0(R)\}$$

where $N_2^0(R)$ are the matrices of determinant one inside $N_2(R)$.

3 The proof of the main theorem

To deduce Theorem 1.1, we first note, that one has, in general, a presentation of R_Q as a quotient of $W(k)[[T_1,\ldots,T_n]]$ by an ideal I generated by at most $m = \dim_k H^2(G_E, \mathrm{ad}_{\bar\rho}^0)$ equations, see [3]. This follows by the same method as Proposition 2 in [13], where the same is shown to hold for $R_Q/(l)$ as a quotient of $k[[T_1,\ldots,T_n]]$. As a consequence, $R_Q \otimes \mathbb{Q}_l$ describes near the origin an l-adic manifold cut out by at most m equations. The number n and m will be fixed throughout this section.

We will show below, in Proposition 3.2, using the Leopoldt conjecture, that we can find m equations near ρ_0 that satisfy the Jacobian criterion and that are inside I. So they alone cut out a smooth l-adic manifold near the origin of dimension $n-m$. The above remark then excludes the existence of possible other equations, as they would force the dimension to be smaller than $n-m$, and thus the main theorem is shown.

As a consequence, this also shows that the number of equations is indeed m and cannot be lower, and so Corollary 1.2 is shown as well. A priori, this is not clear, as it could have happened that one of the relevant relations describing P_Q is automatically satisfied for H-equivariant maps to matrix groups - relevant meaning that it is not prime to adjoint viewed inside $\mathcal{R}/[\mathcal{R},\mathcal{F}]\mathcal{R}^p$ for a presentation

$$1 \to \mathcal{R} \to \mathcal{F} \to P_Q \to 1$$

that is compatible with the H action. For more on this see [3]. The following example explains what we mean by this.

Example 3.1 Let H be the group of order two, $H = \{e, c\}$, where e is the identity. Let $\mathcal{F}$ be the free pro-l group on generators x, y. We assume that H acts trivially on x and that c sends y to its inverse. We consider the relation $r = [[[x,y],x],y^{-1}][y,[[x,y^{-1}],x]]$. Then c acts on r by sending it to its inverse. We define $\mathcal{R}$ as the closed normal subgroup generated by r, then the pro-l group $P = \mathcal{F}/\mathcal{R}$ carries an action of H. We consider the functor from $\mathcal{C}$ to sets given by

$$R \mapsto Hom_H(P, (1+\mathfrak{m}_R,\cdot)^{triv} \times (1+\mathfrak{m}_R,\cdot)^{\chi})$$

where the superscripts on the multiplicative groups $(1+\mathfrak{m}_R,\cdot)$ indicate that H acts trivially or non-trivially, respectively. As the target space is abelian, all such H-equivariant homomorphisms factor via P^{ab}, but by the same argument, one could replace P by $\mathcal{F}$ without changing the set of H-equivariant homomorphisms. So the relation r in the presentation of P is irrelevant for the functor.

In this example the module corresponding to $\mathrm{ad}_{\bar\rho}$ is the module $V = \mathbf{F}_p^{triv} \oplus \mathbf{F}_p^{\chi}$. We calculate

$$H^2(P \rtimes H, V) \cong H^2(P,V)^H \cong (\mathbf{F}_p[H] \otimes_{\mathbf{F}_p} (\mathcal{R}/[\mathcal{R},\mathcal{F}]\mathcal{R}^p)^*)^H \cong \mathbf{F}_p$$

However, the space representing the above functor is $W(k)[[T_1,T_2]]$, and hence the relation module is trivial.

What we remarked above, is that this behavior cannot occur for universal deformation rings coming from tame, absolutely irreducible Galois representations into GL_2.

So our goal is the proof of the following proposition.

Proposition 3.2 *Suppose E_0 satisfies the Leopoldt conjecture, and that $\bar\rho : G_E \to \mathrm{GL}_2(k)$ is tame and absolutely irreducible. Then there is a presentation*

$$0 \to I \to U = W(k)[[T_1,\ldots,T_n]] \to R_Q \to 0$$

such that I contains an ideal I' generated by equations of the form

$$l^{e_i}T_i - \textit{expression in } (T_1,\ldots,T_n)^2 \quad i = n-m+1,\ldots,n$$

This means that near the origin, i.e. near the trivial lift, $U/I' \otimes \mathbb{Q}_l$ is a smooth l-adic manifold of dimension $n-m$ by the Jacobian criterion.

As we remarked earlier, to obtain the universal ring R_Q, we can use the functor HE_Q assigning to a ring $R \in \mathcal{C}$ the set of H-equivariant homomorphisms from P_Q to $N_2^0(R)$. To understand those sets, the following observations, described in [4], are important. If we denote by $\bar P$ the Frattini quotient of any finitely generated pro-l group P, i.e. $\bar P \cong P^{ab} \otimes \mathbf{F}_l$, then the following hold.

(i) For any finitely generated closed subgroup P of $N_2^0(R)$, all irreducible components of $\bar P \otimes k$ as a $k[H]$-module are submodules of ad^0.

(ii) If P is a pro-l group with an H-action, then for each irreducible summand A of $\bar P$ as an $\mathbf{F}_l[H]$ module, one can find a closed H invariant subgroup P_A of P whose Frattini quotient is A. In particular if we are given such an A, a $g \in H$ and a g invariant element $\bar x \in A$, we can find a lift $x \in P_A$ of $\bar x$ that is g invariant. In fact, as was shown in [2], such lifts also exist for a decomposition of $P^{ab} \otimes \mathbb{Z}_l/(l^k)$ $k = 1,2,\ldots,\infty$ into irreducible $\mathbb{Z}_l[H]$ modules. Here it is important that the order of H is prime to l.

(iii) If we have any H-equivariant homomorphism α from a pro-l group P with an H action, and if $\bar P \cong \oplus_{i=1}^s A_i$, where the A_i are as in (ii), such that $A_1,\ldots,A_j$ are isomorphic to submodules of ad^0 and the other ones are prime to ad^0, then α factors through the quotient of P by the closed normal subgroup generated by the P_{A_i} for $i > j$.

We shall first analyze the ad^0 components of P_Q^{ab}. As the order of H is prime to l, one can write $\mathbb{Z}_l[H]$ as a direct sum of irreducible projective $\mathbb{Z}_l[H]$ modules. As one has corresponding idempotents, one can decompose P_Q^{ab} as a $\mathbb{Z}_l[H]$ module into a direct sum of irreducible ones, $P_Q^{ab} \cong \oplus A_i$, such that all the $A_i \otimes \mathbf{F}_l$ are irreducible $\mathbf{F}_l[H]$ modules. Furthermore there is a bijection between irreducible $\mathbf{F}_l[H]$ and irreducible projective $\mathbb{Z}_l[H]$ modules. Hence we can define the ad^0-part of P_Q^{ab}, $P_Q^{ab,0}$, to be the direct sum over all A_i such that $A_i \otimes k$ contains a (then unique) non-trivial subrepresentation of ad^0. We also can talk about ad^0-components.

Lemma 3.3 *If E_0 satisfies the Leopoldt conjecture, then the number of torsion free ad^0-components of P_Q is $n - m$.*

Proof: We shall compare the ad^0-components of P_Q^{ab} and P_{Q,E_0}^{ab}, where P_{Q,E_0} is the Galois group of the maximal pro-l extension of E_0 unramified outside the places above Q. We shall see that they are isomorphic. Then the claim will follow from the Leopoldt conjecture for E_0.

By class field theory P_Q^{ab} is the cokernel of the map

$$U_L \otimes \mathbb{Z}_l \to \prod_{\mathfrak{L}|l} U_{L_\mathfrak{L}}$$

This map is H-equivariant. We are interested in the ad^0-part of the torsion free part of the quotient. We have the diagram

$$\begin{array}{ccc} U_{E_0} \otimes \mathbb{Z}_l & \longrightarrow & \prod_{\mathfrak{l}|l} U_{E_{0\mathfrak{l}}} \\ \downarrow & & \downarrow \\ U_L \otimes \mathbb{Z}_l & \longrightarrow & \prod_{\mathfrak{L}|l} U_{L_\mathfrak{L}} \end{array}$$

Here all the $\mathbb{Z}_l$ modules in the diagram have an action of $H = \mathrm{Gal}(L/E)$. In the first row the action of the subgroup $\mathrm{Gal}(L/E_0)$ is trivial. By the remarks above about $\mathbb{Z}_l[H]$, one has the same diagram if one considers ad^0-parts everywhere, as all maps between modules corresponding to non-isomorphic irreducible projective $\mathbb{Z}_l[H]$ modules are zero. As $\mathrm{Gal}(L/E_0)$ acts trivial on ad^0, and as the invariants of this group taken on the second row gives an isomorphism with the first row, the vertical arrows in the diagram for the ad^0-parts are isomorphisms. Hence the number of torsion free ad^0-components is the same for both cokernels.

Assuming the Leopoldt conjecture for E_0 means that the top arrow is an injection. So we only have to count the number of ad^0-components inside $U_{E_0} \otimes \mathbb{Q}_l$ and $\prod_{\mathfrak{l}|l} U_{E_{0\mathfrak{l}}} \otimes \mathbb{Q}_l$. By a modification of the argument in [6], §2.2, one obtains for the rank of the first expression $\sum_{\nu|\infty} H^1(G_{E_\nu}, \mathrm{ad}^0)$ and for that of the second $3[E : \mathbb{Q}]$. One way to obtain this is to enlarge E_0 to its normal closure, calculate everything there and descend to E_0 by using restriction of induced characters. Using the global Euler Poincaré characteristic formula as in [13], §1.10, one finds $n - m$ for this difference. ∎

Proof of Proposition 3.2: By the remarks above the previous lemma, we can decompose P_Q^{ab} into a direct sum of irreducible $\mathbb{Z}_l[H]$ modules, and we can assume that the ad^0-components are the submodules A_i, $i = 1, \ldots, n$. That n is the number of such components can be found in [1], Proposition 2.8. Strictly speaking, in [1] it was shown that the number of such components in $P_Q^{ab} \otimes \mathbb{Z}/(l)$ is n. But this can easily be lifted to P^{ab} as the number of components is invariant under tensoring with $\mathbf{F}_l$.

We pick P_{A_i} as in part (ii) above. By checking all possible modules ad^0 than can occur, i.e. all adjoint representations on $M_2(k)$ of a non-abelian subgroup of $\mathrm{GL}_2(k)$ of order prime to l, one can find an element $\bar{x}_i \in A_i$ for $i = 1, \ldots, n$, and a non-trivial element $g_i \in \bar{H}$, the image of H in $\mathrm{PGL}_2(k)$ such that $\bar{x}_i$ is g_i invariant. The list of possible subgroups of $\mathrm{PGL}_2(k)$ is given for example in [9], Sections 255, 260. By x_i we denote lifts of the elements $\bar{x}_i$ to P_{A_i}. By the Burnside basis theorem, it is clear that the H-orbit of x_i generates P_{A_i}. In particular, any H-equivariant map α from P_Q to $N_2^0(R)$ for $R \in \mathcal{C}$ is uniquely determined by the images of the elements x_i, by (iii) above.

Using the above lemma, we shall assume that $A_1, \ldots, A_{n-m}$ is the complete list of torsion free ad^0-components of P_Q^{ab}. The images of the x_i we shall call $B_i \in N_2^0(R)$. At this point we want k to be large enough, so that all elements g_i can be diagonalized. This can always be assumed without loss of generality by base changing to a possibly larger field (the unique quadratic extension of k is always sufficient). This means that with respect to an appropriate base change of on $\mathrm{GL}_2(W(k))$, depending on g_i, the image of B_i is of the form $\begin{pmatrix} 1+T_i & 0 \\ 0 & (1+T_i)^{-1} \end{pmatrix}$. It is then clear that R_Q is a quotient of $W(k)[[T_1, \ldots, T_n]]$ modulo the ideal I that is generated by all the equations that have to be satisfied among the B_i in order to satisfy all the relations that hold among the x_i inside P_Q. The universal representation is given by sending the x_i to the above matrices. As $n = \dim_k H^1(G_E, \mathrm{ad}^0)$, none of the T_i is superfluous.

Let $n - m < i \leq n$. Then the image of x_i in P_Q^{ab} is torsion. Thus there exists a positive integer e_i such that $x^{l^{e_i}}$ is in the commutator subgroup of P_Q. The image of the commutator subgroup under α is generated by arbitrary commutators of the H orbits of the B_i as a closed subgroup. It is easy to check that all such commutators are congruent to the identity modulo $(T_1, \ldots, T_n)^2$. It is also easy to check that, in the appropriate basis for B_i, the element $B_i^{l^{e_i}}$ is given by $\begin{pmatrix} 1+l^{e_i}T_i & 0 \\ 0 & 1-l^{e_i}T_i \end{pmatrix}$ modulo $(T_1, \ldots, T_n)^2$. Hence considering the $(1,1)$ entries, one obtains equations

$$0 = l^{e_i}T_i - f_i(T_1, \ldots, T_n) \quad i = n-m+1, \ldots, n \text{ where } f_i \in (T_1, \ldots, T_n)^2$$

We now take for I' the ideal generated by those m functions. If we evaluate the Jacobian with respect to the variables $T_{n-m+1}, \ldots, T_n$ at the origin, we obtain the diagonal matrix with entries $(l^{e_{n-m+1}}, \ldots, l^{e_n})$. Applying the inverse function theorem, shows that U/I' describes near zero a smooth l-adic manifold of dimension $n - m$. ∎

Remark 3.4 Without having a bound on the number of equations, it is not clear at this point that the above equations describe the space that we are interested in. There could well be more equations necessary. Also, we only used one relation for every generator of P_Q that becomes torsion in P_Q^{ab}. But there may well be relations among the other generators, too, as long as they are trivial inside the abelianization, see [17].

4 Digression to the Borel case

Here we shall briefly expose some examples related to Remark 1.6 of a restricted deformation problem in the Borel case with $\mathbb{Q}$ as the base field. The first examples in this direction were described in [4]. A rather detailed account of such examples is given in [14]. Our examples are basically the ones treated in [4]. For the background we refer to the above sources.

Let S be a finite set of places of $\mathbb{Q}$. Let L over $\mathbb{Q}$ be an abelian extension of order prime to l, let L_∞ be the maximal cyclotomic $\mathbb{Z}_l$-extension of L and M_S be the maximal abelian pro-l extension of L_∞ unramified outside S, $C_S = \mathrm{Gal}(M_S/L)$, $A_S = \mathrm{Gal}(M_S/L_\infty)$. We shall from the start assume the following amenable situation. As a $\mathbb{Z}_l[[T]]$ Iwasawa module, A_S is isomorphic – and not just pseudo-isomorphic – to

$$\coprod_{\chi \text{ odd}} \mathbb{Z}_l[[T]]^\chi \oplus \coprod_{\chi} \left(\coprod_{(i,\chi)} \mathbb{Z}_l[[T]]^\chi/(f_{(i,\chi)}(T)) \right)$$

where χ denotes any character of $H = \mathrm{Gal}(L/\mathbb{Q})$, odd refers to complex conjugation, and so the first coproduct is zero if L is totally real, and all the $f_{(i,\chi)}$ are distinguished polynomials. We also assume that the functions $f_{(i,\chi)}$ for fixed χ are relatively prime over $\mathbb{Q}_l[[T]]$. The superscript refers to the χ action of H. Finally $\mathrm{Gal}(M_S/L) \cong A_S \rtimes \mathbb{Z}_l$ where the right term refers to $\mathrm{Gal}(L_\infty/L)$ and H acts trivially on it. Let G be the resulting semi-direct product. By our assumption, G is a quotient of $G_{\mathbb{Q},S}$.

We now start with a map from H into the diagonal matrices $\mathrm{GL}_2(k)$ with image in $\mathrm{PGL}_2(k)$ of order strictly larger than two. The action of H on the $(1,2)$ entry of $M_2(k)$ via conjugation composed with this map will be the character denoted by ψ. We assume that the ψ-component of A_S is non-trivial. We then extend the map constructed so far to a representation $\bar\rho$ into the Borel matrices by selecting one of the ψ-components and sending its generator $\bar{1}$ to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. There are essentially three cases to consider.

(i) The resulting representation is even. Then $\mathbb{Z}_l[[T]]^\psi = 0$.

(ii) The representation is odd and the chosen ψ-component is $\mathbb{Z}_l[[T]]^\psi$.

(iii) The representation is odd and the chosen ψ-component is $\mathbb{Z}_l[[T]]^\psi/(f_{(1,\psi)})$.

Even and odd refers to the image of complex conjugation being trivial or non-trivial in $\mathrm{PGL}_2(k)$.

By [13], there exists a universal deformation R_G for the deformations of $\bar\rho$ that factor via G. We will now briefly derive its explicit shape as in [4]. As described there, or in [2], Proposition 2.3, R_G is the universal ring representing H-equivariant maps from $A_S \rtimes \mathbb{Z}_l$ to $\mathrm{GL}_2(R)$ for local rings R, such that the generator $\bar{1}$ from above gets sent to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the other elements get send to the kernel of $\mathrm{GL}_2(R) \to \mathrm{GL}_2(k)$ for $R \in \mathcal{C}$. Thus we only need to consider the parts of A_S where H acts trivially or via ψ, as it is not hard to see that there cannot be any images with other characters – for

ψ^{-1} one uses, that the images of elements in A_S have to commute with the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

By x_i we denote the generator $\bar{1}$ of $\mathbb{Z}_l[[T]]^{triv}/(f_{(i,triv)})$. By X_i its image. So

$$X_i = (1+a_i)\begin{pmatrix} \sqrt{1+d_i} & 0 \\ 0 & \sqrt{1+d_i}^{-1} \end{pmatrix}.$$

Similarly the image of a generator x of $\mathbb{Z}_l = \mathrm{Gal}(L_\infty/L)$ is mapped to

$$X = (1+a)\begin{pmatrix} \sqrt{1+d} & 0 \\ 0 & \sqrt{1+d}^{-1} \end{pmatrix}.$$

By y_j we denote the generator $\bar{1}$ of $\mathbb{Z}_l[[T]]^{\psi}/(f_{(j,\psi)})$, by y that of $\mathbb{Z}_l[[T]]^{\psi}$, by Y_j its image $\begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix}$ and by Y the image $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, resp.

As explained in [4], from the comparison of the action of $\mathrm{Gal}(L_\infty/L)$ on A_S and on the images – one has to make sure that the images of $f_{(j,\psi)}$ and $f_{(i,triv)}$ get mapped to the identity –, one finds the following conditions for the above cases. If c_i is the zeroth coefficient of the polynomial $f_{(i,triv)}$ and m_j the degree of $f_{(j,\psi)}$, then

(i) $(1+a_i)^{c_i} = 1$, $d_i = 0$, $f_{(1,\psi)}(d) = 0$, $b_j f_{(j,\psi)}(d) = 0$ for $j \geq 2$, $b_1 = 1$. (There is no b in this case.)

(ii) $(1+a_i)^{c_i} = 1$, $d_i = 0$, $b_j f_{(j,\psi)}(d) = 0$, for all j, $b = 1$.

(iii) $(1+a_i)^{c_i} = 1$, $d_i = 0$, $f_{(1,\psi)}(d) = 0$, $b_j f_{(j,\psi)}(d) = 0$ for $j \geq 2$, $b_1 = 1$, and b is arbitrary.

Thus R_G is explicitly given by

(i) $W(k)[[a, d, a_i, b_j]]/((1+a_i)^{c_i} - 1, f_{(1,\psi)}(d), b_j f_{(j,\psi)}(d)$ for $j \geq 2, b_1 - 1)$,

(ii) $W(k)[[a, b, d, a_i, b_j]]/((1+a_i)^{c_i} - 1, b_j f_{(j,\psi)}(d), b - 1)$,

(iii) $W(k)[[a, b, d, a_i, b_j]]/((1+a_i)^{c_i} - 1, f_{(1,\psi)}(d), b_j f_{(j,\psi)}(d)$ for $j \geq 2, b_1 - 1)$.

The images of the generators of the universal deformation are as above.

For $R_G/(l)$ we obtain.

(i) $k[[a, d, a_i, b_j]]/(a_i^{c_i}, d^{m_1}, b_j d^{m_i}$ for $j \geq 2, b_1 - 1)$.

(ii) $k[[a, b, d, a_i, b_j]]/(a_i^{c_i}, b_j d^{m_j}, b - 1)$.

(iii) $k[[a, b, d, a_i, b_j]]/(a_i^{c_i}, d^{m_1}, b_j d^{m_j}$ for $j \geq 2, b_1 - 1)$.

And so after modding out d we obtain a ring of dimension equal to zero or one, plus the number of summands in $\coprod_j \mathbb{Z}_l[[T]]^{\psi}/(f_{(j,\psi)})$ The latter can be made arbitrarily large by suitably enlarging S, and thus the universal deformation ring for the ramification set S, of which the above is a quotient, can have arbitrarily large dimension in the Borel case.

After tensoring with $\mathbb{Q}_l$ on the other hand, where now the functions $f_{(j,\psi)}$ become units modulo $f_{(1,\psi)}$, due to our assumption that the $f_{(j,\psi)}$ are relatively prime, we find for $R_G \otimes \mathbb{Q}_l$.

(i) $(W(k)[[d]]/(f_{(1,\psi)(d)})) \otimes \mathbb{Q}_l[[a, a_i]]/((1+a_i)^{c_i} - 1)$.

(ii) To understand $R_G \otimes \mathbb{Q}_l$, we first consider the map

$$\begin{aligned} &W(k)[[d, b_j]]/(b_j f_{(j,\psi)}(d)) \to \\ &\qquad W(k)[[d, b_j]]/(b_j) \oplus \coprod_{j'} W(k)[[d, b_j]]/(f_{j'}(d), b_j f_{(j,\psi)}(d)) \end{aligned}$$

coming from the primary decomposition of 0 of this ring. We note that by our assumption there is a power l^n in all ideals generated by any two of the b_j. For the rings appearing here, tensoring with $\mathbb{Q}_l$ was described in the other cases - or is obvious in the case of $W(k)[[d, b_j]]/(b_j)$. As the primary decomposition is interchangeable with tensoring, the ring $\mathbb{Q}_l \otimes W(k)[[d, b_j]]/(b_j f_{(j,\psi)}(d))$ has dimension one, and hence $R_G \otimes \mathbb{Q}_l$ has dimension two.

(iii) $(W(k)[[d]]/(f_{(1,\psi)(d)})) \otimes \mathbb{Q}_l[[a, b, a_i]]/((1+a_i)^{c_i} - 1)$.

Hence the universal ring tensored with $\mathbb{Q}_l$ has dimension one in the even case and two in the odd case.

At the same time it is not difficult to calculate the difference

$$\dim_k H^1(G, \tau) - \dim_k H^2(G, \tau)$$

explicitly, where τ is the adjoint action of G on the upper triangular matrices in $M_2(k)$. It is one in the even and two in the odd case. We shall sketch the necessary steps. First one uses the Hochschild Serre spectral sequence twice - to pull out H and then $\mathbb{Z}_l$.

$$H^i(\mathbb{Z}_l, H^j(A_S, \tau))^H \Rightarrow H^{i+j}(G, \tau).$$

The H action is easy to analyze, and also $\tau = k^{triv} \oplus V_2$ where V_2 is indecomposable with composition series

$$0 \to k^{\psi} \to V_2 \to k^{triv} \to 0. \tag{1}$$

The dimension of H^1 we already calculated above as the minimal number of topological generators of R_G. For H^2 one finds

$$(H^1(A_S, \tau)_{\mathbb{Z}_l})^H = H^1(\mathbb{Z}_l, H^1(A_S, \tau))^H = H^2(G, \tau).$$

The decomposition of A_S into the single summand with image $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and all the others gives a decomposition of the cohomology - for the second summand the coefficients change. The second term is calculated as a set of homomorphisms. Using the H action, one easily can see that its k dimension is one less than the number of summands of A_S, which are $\mathbb{Z}_l[[T]]$ torsion and on which H acts trivially or by ψ. For the first summand one can use the long exact homology sequence associated to (1) to see that its k dimension is one. Subtracting respective numbers gives our claim on $\dim_k H^1(G, \tau) - \dim_k H^2(G, \tau)$, and we showed the following.

Proposition 4.1 *The Krull dimension of $R_G \otimes \mathbb{Q}_l$ for the above deformation problem agrees with the cohomologically computed one and the resulting ring is smooth over $\mathbb{Q}_l$. On the other hand R_G is not flat over $W(k)$, it has components of different dimensions and can have arbitrarily large Krull dimension, depending on S.*

There is an easier way to see that the dimension of the universal deformation ring R_Q, as defined in Section 2, can be arbitrarily. First, it is clearly enough to see this for $R_Q/(l)$. Further, if we replace the original deformation problem by one that corresponds to a quotient group property in the sense of Mazur [13], §2.1, it is enough to show it for that one.

Let $\bar{\rho} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbf{F}_l)$ be given, where the image in $\mathrm{PGL}_2(\mathbf{F}_l)$ is inside a Borel subgroup and contains elements of order l and of order prime to l and greater than two. Let U be the unipotent part of $\mathrm{Im}(\bar{\rho})$. By L we denote the field corresponding to the inverse image of U under $\bar{\rho}$. Let $H = \mathrm{Gal}(L/\mathbb{Q})$, so H is abelian, and let ψ be the character by which H acts, via the adjoint representation, on the $(1,2)$ entry of $M_2(\mathbf{F}_l)$ - respectively its Teichmüller lift.

We denote by $\tilde{L}$ the maximal outside Q unramified l-extension of L containing L_∞, for which the corresponding Galois group is isomorphic to $\mathbb{Z}_l^{triv} \times \coprod_i V_i$ as a $\mathbb{Z}_l[H]$ module, where all V_i are isomorphic to $\mathbf{F}_l^{\psi}$. We shall now consider the deformations factoring through $\tilde{L}$. Let N_2 be as in Section 2. The following is an easy, but crucial, observation. For $R \in \mathcal{C}$,

$$Hom_H(\mathbf{F}_l^{\psi}, N_2(R/(l))) = Hom_{\mathbf{F}_l-\text{algebras}}(\mathbf{F}_l[[T]], R/(l))$$

and $\mathbf{F}_l[[T]]$ represents the corresponding functor.

By the equivalence of functors as in [2], Proposition 2.3, now for $\mathbf{F}_l$-algebras instead of all rings in $\mathcal{C}$, the universal $\mathbf{F}_l$-algebra that we want to find is the universal ring representing elements in $Hom_H(\mathrm{Gal}(\tilde{L}/L), N_2(R))$ that commute with $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We shall denote the elements by $Hom'_H(\mathrm{Gal}(\tilde{L}/L), N_2(R))$. For $(R, \mathfrak{m})$ a local ring of characteristic l, we calculate

$$\begin{aligned} Hom'_H(\mathrm{Gal}(\tilde{L}/L), N_2(R)) &= Hom'_H(\mathbb{Z}_l^{triv}, N_2(R)) \oplus \coprod_i Hom_H(\mathbf{F}_l^{\psi}, N_2(R)) \\ &= Hom(\mathbb{Z}_l^{triv}, \{aI : a \in 1 + \mathfrak{m}\}) \oplus \coprod_i Hom(\mathbf{F}_l[[T]], R) \\ &= Hom(\mathbf{F}_l[[T]], R) \oplus \coprod_i Hom(\mathbf{F}_l[[T]], R) \end{aligned}$$

So the universal ring is $\mathbf{F}_l[[T_1, \ldots, T_m]]$, where m is the number of summands in $\coprod_i V_i$. The number of such summands can be easily calculated using class field theory, provided Q is large enough so that $V_Q = E^{*l}$ in the notation of [12], §11. If one enlarges Q in this situation by a prime p, the number of additional components V_i is given by

$$\dim_{\mathbf{F}_l} Hom(\mathbf{F}_l^{\psi}, \mathrm{Ind}_{H_p}^{H} \mu_l(L_p)) = \dim_{\mathbf{F}_l} Hom(\mathbf{F}_l^{\psi}|_{H_p}, \mu_l(L_p)).$$

In particular this number is one for every completely split p such that $p \equiv 1 (\mathrm{mod}\ l)$. Hence m can become arbitrarily large depending on Q.

Remark 4.2 The same reasoning can be made for representations $\bar{\rho} : G_E \to \mathrm{GL}_2(k)$ where k is any finite field of characteristic l. Furthermore the argument, slightly modified, is still valid if one imposes the condition that the deformations are ordinary, or nearly ordinary at the primes above l. For the definitions of those terms, see [11], for the modifications necessary at those primes above l, see §7 in [2].

So for all those deformation functors, given a number $n > 0$, there exists a finite set of places Q, containing all places above p and infinity, such that the Krull dimension of $R_Q/(p)$, or $R_Q^{ord}/(p)$ or $R_Q^{n.o.}/(p)$ is larger than this given bound. At the same time as mentioned in the introduction, in §7 of [15], it is conjectured that the dimension of $R_Q^{n.o.} \otimes \mathbb{Q}_l$ is equal to a certain cohomological expression. In particular, conjecturally there is a large number of torsion classes.

We would like to end with the remark that despite our evidence, it seems still an open problem to give an example of any particular universal deformation ring R_Q associated to a residual representation of Borel type, such that the Krull dimension of $R_Q \otimes \mathbb{Q}_l$ equals the cohomological expression given in Theorem 1.1, while that of $R_Q/(p)$ is larger than this expression, i.e. that there exist cases where one has torsion classes.

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References

[1] G. Böckle, Explicit Universal Deformations of Even Galois Representations, to appear in Math. Nachr.

[2] G. Böckle, Demuškin groups with group actions and and applications to deformations of local Galois representations, preprint.

[3] G. Böckle, A local to global principle for deformations of Galois representations, preprint.

[4] N. Boston, Explicit Deformations of Galois Representations, Invent. Math. 103 (1990), p. 181–196.

[5] N. Boston, Families of Galois Representations - Increasing the Ramification, Duke Math. Journ. 66 vol. 3 (1992), p. 357–367.

[6] N. Boston, B. Mazur Explicit universal deformations of Galois representations, Adv. Stud. Pure Math. **17** (1989), 1–21.

[7] R. Coleman, p-adic Banach spaces and families of modular forms, to appear in Invent. Math.

[8] F. Diamond, The Taylor-Wiles construction and multiplicity one, preprint.

[9] L.E. Dickson, Linear Groups with an Exposition of the Galois Field Theory, Teubner, Leipzig, 1901.

[10] F. Gouvêa and B. Mazur, On the density of modular representations, preprint 1996.

[11] H. Hida, Adjoint modular Selmer groups of several variables over totally real fields, Notes of lectures given at Université de Paris-Nord, June 17, 1997.

[12] H. Koch, Galoissche Theorie der p-Erweiterungen, Mathematische Monographien, Band 10, VEB Deutscher Verlag der Wissenschaften, Berlin 1970.

[13] B. Mazur, Deforming Galois Representations, in Galois groups over $\mathbb{Q}$, ed. Ihara et al., Springer-Verlag 1987.

[14] A. Mézard, Computation of a universal deformation ring in the Borel case, preprint.

[15] J. Tilouine, Deformations of Galois Representations and Hecke Algebras, Mehta Research Institute, Narosa Publishing House 1996.

[16] A. Wiles, Modular Elliptic Curves and Fermat's Last Theorem, Ann. of Math. 141 No. 3 (1995), p. 443–551.

[17] M. Yamagishi, A note on free pro-p extensions of algebraic number fields, Journ. Th. de Nomb. Bordeaux **5** (1993), 165–178.