Appendix 1: On the isomorphism $R_\emptyset \to T_\emptyset$

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In this appendix, using the notation of Sections 2 and 3 of the main article, we prove the following theorem:

Theorem 1 *Suppose for some auxiliary set of primes Q one has the isomorphism $R_Q \to \mathbf{T}_Q$ of finite flat $W(k)$-algebras. Then the canonical morphism $R_\emptyset \to \mathbf{T}_\emptyset$ of minimal rings is an isomorphism as well.*

We first need some simple preparatory results. Let $\mathcal{O}$ be a discrete valuation ring which is finite flat over $W(k)$ and has residue field k. For a given residual representation $\overline{\rho}\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(k)$ as in the main part of this article, let $X = (X_l)$ be a set of deformation conditions for deformations to CNL $\mathcal{O}$-algebras as in [1], pp. 205 and 211, where l runs over all the places of $\mathbf{Q}$.

For example one could require that X_p is the condition that the deformations considered are finite at p, or Selmer but not finite at p. For all but finitely many places, X_l is the condition that the deformation is unramified at l. For a finite place $l \neq p$ at which ramification is allowed, X_l could be the condition that arbitrary ramification is allowed. If l is a prime in Q, where Q is as in Section 2.1, then one could require that X_l is the condition used in the definition of $R_Q^{Q-\mathrm{new}}$. If one fixes a lift $\eta : G_{\mathbf{Q}} \to \mathcal{O}^*$ of $\det \overline{\rho}$, then a set of deformation conditions which includes this choice of determinant is denoted by X^η.

Furthermore, for any place l of $\mathbf{Q}$, one denotes by $R_{X,l}$ the versal deformation ring which parametrizes deformations of $\overline{\rho}_{|G_l}$ subject to the condition X_l. Also R_X denotes the universal deformation ring which satisfies all the conditions of X. The existence of R_X and $R_{X,l}$ is shown in [1], §3,4. Finally define $h^i(G_l, \mathrm{Ad}(\overline{\rho})) := \dim_k H^i(G_l, \mathrm{Ad}(\overline{\rho}))$ for $i \in \mathbf{N}_0$ and l a place of $\mathbf{Q}$, and similarly for $\mathrm{Ad}^0(\overline{\rho})$ instead of $\mathrm{Ad}(\overline{\rho})$.

Define $d \in \{0,1\}$ and $\mathrm{Ad}_X(\overline{\rho}) \in \{\mathrm{Ad}(\overline{\rho}), \mathrm{Ad}^0(\overline{\rho})\}$ as $d=0$ and $\mathrm{Ad}_X(\overline{\rho}) = \mathrm{Ad}^0(\overline{\rho})$ if X contains a fixed choice of determinant and $d=1$ and $\mathrm{Ad}_X(\overline{\rho}) = \mathrm{Ad}(\overline{\rho})$ otherwise.

Proposition 1 *Suppose one is given $\overline{\rho}$ and X as above. Let S be a finite set of places of $\mathbf{Q}$ such that deformations of type X are unramfied outside S. Suppose further that*

(a) For $l \in S \setminus \{p, \infty\}$, the ring $R_{X,l}$ is a complete intersection, flat over $\mathbf{Z}_p$ and of relative dimension $h^0(G_l, \mathrm{Ad}_X(\overline{\rho})) - \Delta_l$ for some $\Delta_l \in \mathbf{Z}$.

(b) The ring $R_{X,p}$ is a complete intersection, flat over $\mathbf{Z}_p$ and of relative dimension $h^0(G_p, \mathrm{Ad}_X(\overline{\rho})) + 1 + d - \Delta_p$ for some $\Delta_p \in \mathbf{Z}$.

Let $\Delta = \sum \Delta_l$ (which is defined because $\Delta_l = 0$ at all places of $\mathbf{Q}$ where X does not allow ramification). Then $R_X \cong \mathcal{O}[[x_1, \ldots, x_{n+d}]]/(f_1, \ldots, f_{n+\Delta})$ for suitable $n \in \mathbf{N}$ and $f_i \in \mathcal{O}[[x_1, \ldots, x_{n+d}]]$.

Proof: By [1], Cor. 6.4, we can find a (not necessarily optimal) set of auxiliary primes S_{aux}, in the sense of op. cit. For $l \in S' := S \cup S_{\mathrm{aux}} \cup \{p\}$, let $R_{X,l} \cong \mathcal{O}[[X_{l,1}, \ldots, X_{l,n_l}]]/J_l$ be a presentation such that $n_l = h^1(G_l, \mathrm{Ad}_X(\overline{\rho}))$ for $l \in S_{\mathrm{aux}}$ and $n_l = \dim_k \mathfrak{m}_{R_{X,l}}/(p, \mathfrak{m}^2_{R_{X,l}})$ for $l \in S \cup \{p\}$, where $\mathfrak{m}_{R_{X,l}}$ is the maximal ideal of $R_{X,l}$. Let j_l denote the minimal number of generators of J_l. By our assumptions on $l \in S \cup \{p\}$ and because $R_{X,l}$ is smooth over $W(k)$ of relative dimension $h^0(G_l, \mathrm{Ad}_X(\overline{\rho}))$ for $l \in S_{\mathrm{aux}}$, we have

$$n_l = j_l + h^0(G_l, \mathrm{Ad}_X(\overline{\rho})) - \Delta_l + \begin{cases} 0 & \text{for } l \in S' \setminus \{p\}, \\ 1+d & \text{for } l = p. \end{cases}$$

Let X' be the deformation problem with the same local constraints as X at primes l not in S_{aux} and no local constraints at $l \in S_{\mathrm{aux}}$, and $\mathfrak{m}_{R_{X'}}$ the maximal ideal of $R_{X'}$. Then by [1], Thm. 5.6, there exists a presentation $R_X \cong \mathcal{O}[[x_1, \ldots, x_{n+d}]]/J$, where $n + d = \dim_k \mathfrak{m}_{R_{X'}}/(p, \mathfrak{m}^2_{R_{X'}})$ and J is generated by at most $j := \sum_{l \in S'} j_l$ elements (all coming from the J_l). Because $\overline{\rho}$ is odd, the formula in [1], Lem. 5.5(ii), yields

$$n + d = d + \sum_{l \in S'} (n_l - h^0(G_l, \mathrm{Ad}_X(\overline{\rho}))) = d - \Delta + \sum_{l \in S'} j_l,$$

so that $n + \Delta = j$, as asserted. ∎

Suppose now that a global choice of determinant η is fixed in the deformation problem X^η. If X^η_p is the condition that the deformations are finite at p or Selmer, then $\Delta_p = 0$, cf. [5], Sect. 2, except in the case when the restriction of $\overline{\rho}$ to a decomposition group at p is decomposable and flat. This case is treated by the following lemma, which essentially follows from Ramakrishna's thesis [7]:

Lemma 1 *Suppose the restriction $\overline{\rho}|_{G_p}$ is isomorphic to $\begin{pmatrix} \bar\varepsilon & 0 \\ 0 & 1 \end{pmatrix}$, where $\bar\varepsilon$ is the mod p reduction of the cyclotomic character ε. Let X^η_p be the condition that the lifts are flat and of determinant ε. Then the versal deformation ring $R_{X^\eta,p}$ is isomorphic to $W(k)[[X_1, X_2]]$.*

Proof: The functor of flat lifts of determinant ε and of the form

$$\rho\colon G_p \to \mathrm{GL}_2(R), g \mapsto \begin{pmatrix} \varepsilon\chi(g) & b(g) \\ 0 & \chi^{-1}(g) \end{pmatrix}$$

is representable and a versal hull of the functor in the lemma (note that by [4], Thm. 1.8(ii), any flat lift is reducible!). Therefore, it suffices to show that the universal ring for the latter be isomorphic to $W(k)[[X_1, X_2]]$. By [4], Lem. 3.4., the mod p tangent space of the latter is of dimension 2, i.e., one has a presentation $R_{X^\eta,p} \cong W(k)[[X_1, X_2]]/I$ for some ideal I.

Let us assume $I \neq 0$. Write $\mathfrak{m} := (p)$ for the maximal ideal of $W(k)$ and consider the subspace

$$V^{\text{flat}} := \mathrm{Hom}_{W(k)}(R_{X^\eta,p}, W(k)) \cong \{(\alpha, \beta) \in \mathfrak{m}^2 | f(\alpha, \beta) = 0 \ \ \forall f \in I\}$$

of $\mathfrak{m}^2$. Let σ_p be a Frobenius element in G_p and denote by $\rho_{\alpha,\beta}$ the lift corresponding to the pair $(\alpha, \beta) \in V^{\text{flat}}$. The same convention is used for χ and b. Since χ can be expressed as a power series in X_1, X_2, the map

$$\phi\colon V^{\text{flat}} \to \mathfrak{m} : (\alpha, \beta) \to \chi_{\alpha,\beta}(\sigma_p) - 1$$

can be represented by $h \in W(k)[[\alpha, \beta]]$. Let $|\,| : W(k) \to U := \{p^{-n} : n \in \mathbf{N}_0\} \cup \{0\}$ be the normalized valuation on $W(k)$ and I_p the ramification subgroup of G_p. Define

$$\psi\colon V^{\text{flat}} \to U : (\alpha, \beta) \mapsto |b(I_p)|,$$

where $|b(I_p)| = \max\{|b(g)| : g \in I_p\}$. Using the results in [4], § 4,5, or in [7], one can show that $(\phi, \psi) : V^{\text{flat}} \to \mathfrak{m} \times U$ is surjective, i.e., given any character χ and any $n \in \mathbf{N} \cup \{\infty\}$, there exists a flat deformation to $W(k)$ with the given character and $|b(I_p)| = p^{-n}$.

Let now $0 \neq f$ be an element of I. Then

$$X_f := \{(\alpha, \beta) \in \mathbb{C}_p^2 : |\alpha|, |\beta| \leq \frac{1}{p}, f(\alpha, \beta) = 0\}$$

defines a possibly reducible one-dimensional affinoid variety. On each component, the analytic function h is either constant or has finite fibers. Therefore almost all the fibers of h, considered as a function on X_f, are finite. Because U is infinite and (ϕ, ψ) is surjective, all the fibers of h restricted to V^{flat} are infinite. Since $V^{\text{flat}} \subset X_f$, the set V^{flat} must be the union of finitely many fibers of h. This contradicts the surjectivity of h onto the infinite set $\mathfrak{m}$. Hence we must have $I = 0$ as asserted. ∎

We claim that for $l \in S \setminus \{p, \infty\}$ one has $\Delta_l = 0$ in the following cases:

(i) X_l^η allows no ramification at l, or

(ii) X_l^η imposes no local conditions at l, or

(iii) ρ is ramified at l as in Section 2 and X_l is the condition that the deformation is minimal at l, or

(iv) l is a prime in the set Q of Section 2, and X_l^η is the condition imposed for $R_Q^{Q-\text{new}}$ at l.

Case (i) follows from [1], Thm. 2.4, case (ii) from [1], Thm. 3.8, and case (iii) from [1], Lem.3.10 and Rem. 3.11. In case (iv), it is implicitly shown in [8], that $R_{X^\eta,l} \cong \mathcal{O}[[T]]$ for some parameter T, since clearly one has $h^0(G_l, \text{Ad}^0(\overline{\rho})) = 1$ and it is shown in loc. cit. that any deformation to $W(k)/(p^n)$ of type X_l^η can be lifted to a deformation to $W(k)/(p^{n+1})$ of the same type. This yields the following corollary to Proposition 1:

Corollary 1 *Each of the rings $R_\emptyset$ and $R_Q^{\alpha-\text{new}}$ has a presentation*

$$\mathcal{O}[[x_1, \ldots, x_n]]/(f_1, \ldots, f_m)$$

for suitable $m \leq n$ and $f_i \in \mathcal{O}[[x_1, \ldots, x_n]]$.

To further exploit the above corollary, we need the following result:

Lemma 2 *Suppose R is a CNL $\mathcal{O}$-algebra which is a finitely generated module over $\mathcal{O}$. If R has a presentation $R \cong \mathcal{O}[[x_1, \ldots, x_n]]/(f_1, \ldots, f_m)$ where $m \leq n$, then R is a complete intersection, finite flat over $\mathcal{O}$.*

Proof: Let π be a uniformizing parameter of $\mathcal{O}$. Because $\mathcal{O}$ is a discrete valuation ring, the ring $A := \mathcal{O}[[x_1, \ldots, x_n]]$ is a regular local ring of dimension $n+1$. The given presentation of R shows that $\bar{R} := R/(\pi)$ is a quotient of A by the ideal $I := (\pi, f_1, \ldots, f_m)$. By the finiteness of R over $\mathcal{O}$, the ring $\bar{R}$ is a finite local ring over k. Hence I is of height $n+1$ and the Krull intersection theorem implies $n = m$. Because A is regular local, by [6], Thm. 17.4, the elements $\pi, f_1, \ldots, f_n$ form a regular sequence in A. On the one hand this shows that $f_1, \ldots, f_n$ is a regular sequence in A, and therefore $R = A/(f_1, \ldots, f_n)$ is a complete intersection, on the other it implies that π is a regular sequence of R, i.e. that multiplication by π is injective. Since $\mathcal{O}$ is a discrete valuation ring, R must be flat over $\mathcal{O}$. ∎

Combining Corollary 1 with Lemma 2 yields:

Corollary 2 *If R_Q is finite over $\mathcal{O}$, then $R_\emptyset$ is a complete intersection, finite flat over $\mathcal{O}$.*

For any $\alpha \subset Q$, if $R_Q^{\alpha-\mathrm{new}}$ is finite over $\mathcal{O}$, then it is a complete intersection, finite flat over $\mathcal{O}$.

Note that in the proof of the first assertion, one uses that $R_Q \to R_\emptyset$ is surjective.

Proof of Theorem 1: The following commutative diagram will be basic in our proof:

$$\begin{array}{ccc} R_Q & \xrightarrow{\cong} & \mathbf{T}_Q \\ \downarrow & & \downarrow \\ R_\emptyset & \longrightarrow & \mathbf{T}_\emptyset. \end{array} \tag{1}$$

Due to the previous corollary and the standard fact that $\mathbf{T}_Q$ and $\mathbf{T}_\emptyset$ are finite flat over $W(k)$, all the rings in the above diagram are finite flat $W(k)$-algebras. To show that the bottom vertical map is an isomorphism, it therefore suffices to prove this for a geometric point over the generic fiber of $\operatorname{Spec} W(k)$. Let $\bar{K}$ be an algebraic closure of the fraction field K of $W(k)$.

We introduce some notation. Let M_Q denote the set of normalized eigenforms for $\Gamma_0(l^\delta N(\rho)Q)$ such that their associated l-adic Galois representation ρ_f has mod l reduction isomorphic to $\overline{\rho}$. Analogously, one defines $M_\emptyset$ (with $Q = 1$). Finally, let $\rho_\emptyset\colon G_{\mathbf{Q}} \to \mathrm{GL}_2(R_\emptyset)$ denote a representation that represents the universal one.

Upon tensoring the above diagram over $W(k)$ with $\overline{K}$, one obtains

$$\begin{tikzcd} R_Q \otimes_{W(k)} \overline{K} \arrow[r, "\cong"] \arrow[d, two heads] & \mathbf{T}_Q \otimes_{W(k)} \overline{K} \arrow[r, "\cong"] \arrow[d, two heads] & \prod_{f \in M_Q} \overline{K} \arrow[d, two heads] \\ R_\emptyset \otimes_{W(k)} \overline{K} \arrow[r] & \mathbf{T}_\emptyset \otimes_{W(k)} \overline{K} \arrow[r, "\cong"] & \prod_{f \in M_\emptyset} \overline{K}. \end{tikzcd} \tag{2}$$

The isomorphisms on the right are a direct consequence of the definitions of $\mathbf{T}_Q$ and $\mathbf{T}_\emptyset$, and the fact that these rings are reduced (in the former case this is by choice of Q).

Suppose the map on the bottom left is not an isomorphism. Then there exists a modular form $f \in M_Q - M_\emptyset$ and a non-trivial morphism $\alpha\colon R_\emptyset \to \overline{K}$ such that the deformation class of ρ_f is obtained from that of $\rho_\emptyset$ via the morphism α. The definition of $R_\emptyset$ shows that ρ_f must be unramified at all places dividing Q. But this forces the conductor of f to be prime to Q by [3] and contradicts the fact that $f \in M_Q - M_\emptyset$. ∎

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