Cohomological theory of crystals over function fields and applications

Advanced courses on Arithmetic Geometry in positive characteristic, CRM, Bellaterra, March 2010

Gebhard Böckle

March 7, 2012

Contents

1 Introduction **3**

2 Notation **6**

3 First basic objects **7**

1 τ-sheaves . . . 7

2 (Algebraic) Drinfeld A-modules . . . 8

3 A-motives . . . 9

4 A-crystals **11**

1 Motivation II . . . 11

2 Localization . . . 13

3 Localization for abelian categories . . . 14

4 Nilpotence . . . 15

5 A-crystals . . . 17

5 Functors on τ-sheaves and A-crystals **18**

1 Inverse Image . . . 18

2 Further functors deduced from functors on quasi-coherent sheaves . . . 20

3 Extension by zero . . . 21

6 Derived categories and derived functors **23**

7 Flatness **26**

1 Basics on flatness . . . 26

2 Flatness under functors . . . 27

3 Representability of flat crystals . . . 28

8 The L-function **29**

1 Naive L-functions . . . 29

2 Crystalline L-functions . . . 30

3 Trace formulas for L-functions . . . 31

9 Proof of Anderson's trace formula and a cohomological interpretation **33**
1 The Cartier operator . . . 33
2 Cartier sheaves . . . 34
3 Operators of trace class . . . 35
4 Anderson's trace formula . . . 36
5 Proof of Theorem 8.13 . . . 37
6 The crystalline trace formula for general (good) rings A . . . 38

10 Global L-functions for A-motives **40**
1 Exponentiation of ideals . . . 41
2 Definition and basic properties of the global L-function . . . 42
3 Global L-functions at negative integers . . . 43
4 Meromorphy and entireness . . . 44
5 The global Carlitz-Goss L-function of the affine line . . . 45

11 Relation to étale sheaves **52**
1 An equivalence of categories . . . 52
2 A result of Goss and Sinnott . . . 55

12 Drinfeld modular forms **60**
1 A moduli space for Drinfeld modules . . . 61
2 An explicit example . . . 62
3 Drinfeld modular forms via cohomology . . . 64
4 Galois representations associated to Drinfeld modular forms . . . 67
5 Ramification or Galois representations associated to Drinfeld modular forms . . . 68
6 Drinfeld modular forms and Hecke characters . . . 70
7 An extended example . . . 72

A Further results on Drinfeld modules **76**
1 Drinfeld A-modules over $\mathbb{C}_\infty$ . . . 76
2 Torsion points and isogenies of Drinfeld modules . . . 77
3 Drinfeld Hayes modules . . . 79

Bibliography **81**

Chapter 1

Introduction

This lecture series introduces in the first part a cohomological theory for varieties in positive characteristic with finitely generated rings of this characteristic as coefficients developed jointly with Richard Pink. In the second part various applications are given.

The joint work with Pink was carried out in order to give an algebraic proof of the rationality of some L-functions defined by D. Goss. Prior to our work an analytic proof using methods á la Dwork had been given by Y. Taguchi and D. Wan in [43]. Moreover by an approach dual to ours and closer in spirit to D-modules in characteristic zero, one should also be able to obtain an algebraic proof from the work [13] of M. Emerton and M. Kisin

Our expectation that such a cohomological theory should exist came from the cohomological theory of ℓ-adic étale sheaves developed by Grothendieck and his coworkers to prove the rationality of the ζ-functions introduced and studied by Weil, Hasse et al. In this case there had first been an analytic proof by Dwork. However it was only the cohomological method which in the hands of Deligne eventually led to a full proof of the Weil conjecture.

Let me be more concrete: For a variety X of finite type over $\operatorname{Spec}\mathbb{Z}$ one considers, following Weil, the ζ-function

$$\zeta_X(s) := \prod_{x\in|X|} \Big((1-T^{d_x})^{-1}\Big)|_{T=p_x^{-s}};$$

here $|X|$ is the set of closed points x of X; for such an x the residue field k_x is a finite field; p_x denotes its characteristic and d_x the degree of k_x over $\mathbb{F}_{p_x}$; s is a complex number. This infinite Euler product converges absolutely for $\Re(s) > \dim X$. It is conjectured that ζ_X has a meromorphic continuation to $\mathbb{C}$. If X is irreducible one furthermore conjectures that ζ_X has at most a simple pole at $s = \dim X$. Except for a few cases, this conjecture is wide open. If X is irreducible and if its generic points are all of characteristic zero, then $s \mapsto \zeta_X(s)/\zeta_{\operatorname{Spec}\mathbb{Z}}(s-\dim X+1)$ has a holomorphic continuation to $\Re(s) > \dim X - 1/2$ – see Exercise 1.1 and [35].

For $X = \operatorname{Spec}\mathbb{Z}$ the Euler product ζ_X is simply the Riemann ζ-function. For X the ring of integers of a number field it is the Dedekind ζ-function. For these X it is known that ζ_X has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$. Its residue is of arithmetic significance.

We can rearrange the above product as follows: For any prime p let $X_p := X \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{F}_p$ be the fiber of X above p. Then the closed points of X_p are precisely the closed points of X with $p_x = p$. Define

$$Z(X_p, T) := \prod_{x\in|X_p|} (1-T^{d_x})^{-1} \in 1 + T\mathbb{Z}[[T]]. \tag{1}$$

This is the ζ-function of A. Weil of X_p. It can also be defined in an entirely different way by counting closed points of X_p over the fields $\mathbb{F}_{p^n}$, $n \to \infty$, namely $Z(X_p, T) = \exp\Big(\sum_{r\geq 1} N_r t^r/r\Big)$ with $N_r = \#X_p(\mathbb{F}_{p^r})$.

Using $Z(X_p, T)$ it is easy to verify that

$$\zeta_X(s) = \prod_p Z(X_p, p^{-s}).$$

The Weil conjecture makes predictions about $Z(X_p, T)$.

(a) It asserts that $Z(X_p, T)$ is a rational function in T. If X_p is a smooth projective variety over $\mathbb{F}_p$ of dimension n then by Grothendieck et al. more refined assertions are true:

$$Z(X_p, T) = \prod_{j=0}^{2\dim X_p} \det\left(1 - T\,\mathrm{Frob}_p^{-1}|H^j_{\mathrm{et}}(X_p, \mathbb{Q}_\ell)\right)^{(-1)^{j+1}}; \tag{2}$$

(b) $Z(X_p, T)$ satisfies the functional equation $Z(X_p, \frac{1}{p^nT}) = \pm p^{En/2}T^E Z(X_p, T)$ where $E = \sum_{j=0}^{2n}(-1)^i B_j$ with $B_j = \dim H^j$ – or E is the self intersection number of $\Delta \subset X \times X$;

(c) the eigenvalues of Frob_p acting on $H^j_{\mathrm{et}}(X_p, \mathbb{Q}_\ell)$ are algebraic integers all of whose complex absolute values are of size $p^{j/2}$. (They are Weil numbers of weight i.)

The equality of the right hand sides of (1) and (2) is derived from a **Lefschetz trace formula**. It is a key assertion of the cohomological approach toward proving the Weil conjecture. Recommended references are [51, 36, 14, 31].

Exercise 1.1. This exercise may require additional reading on the assertions of the Well conjecture.

(a) Suppose X_p is irreducible and of dimension n (but not necessarily smooth). Then $N_r - (p^n)^r = O(p^{(n-\frac{1}{2})r})$.

(b) Suppose X is irreducible and generically of characteristic zero. Then $s \mapsto \zeta_X(s)/\zeta_{\mathrm{Spec}\,\mathbb{Z}}(s - \dim X + 1)$ has a holomorphic continuation to $\Re(s) > \dim X - 1/2$.

To explain the situation we will be interested in, we consider a slightly different setting. Let X be as above and let $\mathcal{A} \to X$ be an abelian scheme over X, e.g. an elliptic curve over X. (This means that the morphism is smooth projective and flat, it carries a section (the 0-section of the abelian scheme) and all fibers are abelian varieties.) For simplicity we assume that $X = X_p$ for some fixed prime p. For the abelian variety $\mathcal{A}_x$ it is known that $H^1_{\mathrm{et}}(\mathcal{A}_x, \mathbb{Q}_\ell)$ is the dual of the ℓ-adic Tate-module of $\mathcal{A}_x$, tensored with $\mathbb{Q}_\ell$ over $\mathbb{Z}_\ell$:

$$H^1_{\mathrm{et}}(\mathcal{A}_x, \mathbb{Q}_\ell)^\vee \cong \mathrm{Tate}_\ell(\mathcal{A}_x) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Moreover $H^j_{\mathrm{et}}(\mathcal{A}_x, \mathbb{Q}_\ell) = \Lambda^j H^1_{\mathrm{et}}(\mathcal{A}_x, \mathbb{Q}_\ell)$, the j-th exterior power of H^1. An often studied L-function in this context is

$$\prod_{x \in |X|} \det\left(1 - T^{d_x}\,\mathrm{Frob}_x^{-1}|H^1_{\mathrm{et}}(\mathcal{A}_x, \mathbb{Q}_\ell)\right)^{-1} = \prod_{x \in |X|} \det\left(1 - T^{d_x}\,\mathrm{Frob}_x^{-1}|\,\mathrm{Tate}_\ell(\mathcal{A}_x)\right)^{-1} \in 1 + T\mathbb{Z}[[T]].$$

By Grothendieck's more general formulation of the conjectures of Weil, this is also a rational function in T.

An analog of the right hand side can be defined within the framework of function field arithmetic: Consider a smooth projective curve over a finite field. Let A be the coordinate ring of the affine curve obtained from the projective curve by removing a single closed point. For such A one has the notion of Drinfeld A-module (or more generally A-motive). For every place $\mathfrak{p}$ of A, and any Drinfeld A-module φ (or more generally any A-motive M) one can associate the $\mathfrak{p}$-adic Tate-module $\mathrm{Tate}_{\mathfrak{p}}(\varphi)$. If φ is defined over a finite field k_x, then the corresponding Frobenius endomorphism acts on the Tate-module and one obtains

$$\det\left(1 - T^{d_x}\,\mathrm{Frob}_x^{-1}|\,\mathrm{Tate}_\ell(\varphi)\right) \in 1 + T^{d_x}A[[T^{d_x}]].$$

Suppose now that φ is a Drinfeld A-module over a scheme X of finite type over $\mathbb{F}_p$ – one should think of φ as a family of Drinfeld A-modules over X. For such φ Goss conjectured that

$$L(\varphi, X, T) := \prod_{x \in |X|} \det\left(1 - T^{d_x}\,\mathrm{Frob}_x^{-1}|\,\mathrm{Tate}_\ell(\varphi_x)\right)^{-1} \in 1 + TA[[T]]$$

is in fact a **rational function over** A. This was first proved by Taguchi and Wan and later by R. Pink and the author.

Goss also defined an analog of the global ζ-function considered above: For this observe that every Drinfeld A-module φ_x has a characteristic. This yields a morphism of schemes $X \to \operatorname{Spec} A$ (and the same for A-motives). As before, one considers the various fibers $X_\mathfrak{p} := X \times_{\operatorname{Spec} A} \operatorname{Spec} A/\mathfrak{p}$ and defines

$$L^{\mathrm{glob}}(\varphi, X, s) := \prod_{\mathfrak{p} \in \mathbf{Max}(A)} L(\varphi|_{X_\mathfrak{p}}, X_\mathfrak{p}, T)|_{T=\mathfrak{p}^{-s}};$$

here $s \in \mathbb{Z}_p \times \mathbb{C}_\infty$ which can be regarded as an analog of the complex plane; we skip the definition of $\mathfrak{p}^{-s}$ but note that the product only converges in an analog of a right half plane. Goss defines what it means for a function

$$\mathbb{Z}_p \times \mathbb{C}_\infty \to \mathbb{C}_\infty$$

to be **meromorphic** and **essentially algebraic**. It is shown for $A = \mathbb{F}_q[t]$ in the work [43] of Taguchi and Wan and for general A in [3] that the global L-functions of Goss possess these two properties. More will be explained in the upcoming lectures.

The aims of the present lecture series are:

- Introduce the cohomological theory of Pink and myself which is applicable to families Anderson's A-motives. (and more generally)
- Prove a Lefschetz trace formula within this theory following an argument by Anderson and obtain an algebraic proof of Goss' conjectures on L-functions of families of t-motives.
- Discuss the following topics related to the above theory:
  (a) A cohomological formula for special values of Goss global L-function at negative integers.
  (b) Goss' conjectures about the meromorphy of global L-functions as well as results and conjectures on the distribution of their zeros.
  (c) The link between the theory of Pink and myself to the étale theory of sheaves of $\mathbb{F}_p$-vector spaces on varieties X of characteristic p.
  (d) An alternative proof of a theorem of Goss and Sinnott on the relation between components on the class groups of torsion fields of Drinfeld modules and the divisibility of L-values..
  (e) The association of Galois representations (or more general A-motive like objects) to Drinfeld modular forms.

References: A detailed account of the cohomological theory treated in this course is given in the monograph [8]. The results on meromorphy of global L-functions are from [3]. The cohomological treatment of Drinfeld modular forms stems from [4]. A very important article regarding a Trace formula for Goss' L-function is Anderson's [2]. Much background on Drinfeld modules and t-motives can be found in [23]. Another rich source is [47]. Further references are given throughout the text. Some of the results we present have not yet appeared in print or preprint form.

Acknowledgments: I would like to thank the CRM at Barcelona for the invitation to present this lecture series during an advanced course on function field arithmetic from February 22 to March 5, 2010 and for the pleasant stay at CRM in the spring of 2010 during which a preliminary version of these lecture notes were written. I also thank the NCTS in Hsinchu, Taiwan, and in particular Chieh-Yu Chang and Winnie Li for inviting me in September 2010 for giving another lecture series on the above results. It much helped with the revisions of the original notes. I thank D. Thakur for his many remarks on the Goss L-function of the Carlitz module. For help with the correction of a preliminary version, I thank A. Karumbidza and I. Longhi. I acknowledge financial support by the Deutsche Forschungsgemeinschaft through the SFB/TR 45.

Chapter 2

Notation

Let p be a prime number and q a power of p. We fix a finite field k with q elements. All schemes X, Y, Z, U etc. are assumed to be noetherian and separated over k. All morphisms, fiber products, tensor products of modules and algebras are taken over k unless specified otherwise. By a (quasi)-coherent sheaf on a scheme X we will always mean a (quasi)-coherent sheaf of $\mathcal{O}_X$-modules. Any homomorphism of such sheaves is assumed to be $\mathcal{O}_X$-linear, and any tensor product of such sheaves is taken over $\mathcal{O}_X$. The Frobenius morphism on X over k, which acts on functions by $x \mapsto x^q$, is denoted $\sigma\colon X \to X$.

Throughout most of the lectures we fix a scheme C which is assumed to be a localization of a scheme of finite type over k. The notation is intended to reflect the role of C as a $\underline{C}$oefficient system. To guarantee the existence of sufficiently many functions we assume that C is affine; thus $C = \operatorname{Spec} A$, where A is a localization of a finitely generated k-algebra. Interesting special cases of such A are the coordinate ring of any smooth affine curve over k, any field which is finitely generated over k, and any finite Artin ring over k.

The assumptions on C imply that $X \times C$ is noetherian for every noetherian scheme X over k. This is useful in dealing with coherent sheaves on $X \times C$. As a general rule, sheaves on X will be distinguished from those on $X \times C$ by an index $(\_\_)_0$. Throughout we let $\mathrm{pr}_1\colon X \times C \to X$ denote the projection to the first factor. For any coherent sheaf of ideals $\mathcal{I}_0 \subset \mathcal{O}_X$ we abbreviate $\mathcal{I}_0\mathcal{F} := (\mathrm{pr}_1^{-1}\mathcal{I}_0)\mathcal{F}$.

In the special case where C is an irreducible smooth affine curve over k whose smooth compactification is obtained by adjoining precisely one closed point ∞, we define: K as the fraction field of A, K_∞ as the completion of K at ∞ and $\mathbb{C}_\infty$ as the completion of the algebraic closure of K_∞. Similarly, for any place v of K we denote by K_v the completion of K at v and by $\mathcal{O}_v$ the ring of integers of K_v and by k_v the residue field of K_v.

Chapter 3

First basic objects

In this chapter we shall introduce τ-sheaves. These are the first building blocks in the theory developed with Pink. We shall see how they arise from (families of) Drinfeld A-modules and A-motives.

1 τ-sheaves

Definition 3.1. A *τ-sheaf over A on X* is a pair $\underline{\mathcal{F}} := (\mathcal{F}, \tau_{\mathcal{F}})$ where $\mathcal{F}$ is a quasi-coherent sheaf on $X \times C$ and $\tau_{\mathcal{F}}$ is an $\mathcal{O}_{X\times C}$-linear homomorphism

$$(\sigma \times \mathrm{id})^*\mathcal{F} \xrightarrow{\tau_{\mathcal{F}}} \mathcal{F}.$$

As A remains fixed for the most part, we usually speak of *τ-sheaves on X*.

A *homomorphism* of τ-sheaves $\underline{\mathcal{F}} \to \underline{\mathcal{G}}$ on X is a homomorphism of the underlying sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ such that

$$\begin{tikzcd} (\sigma \times \mathrm{id})^*\mathcal{F} \arrow[r, "\tau_{\mathcal{F}}"] \arrow[d, "(\sigma\times\mathrm{id})^*\varphi"'] & \mathcal{F} \arrow[d, "\varphi"] \\ (\sigma \times \mathrm{id})^*\mathcal{G} \arrow[r, "\tau_{\mathcal{G}}"] & \mathcal{G}. \end{tikzcd}$$

commutes.

The sheaf underlying a τ-sheaf $\underline{\mathcal{F}}$ will always be denoted $\mathcal{F}$. We will mostly abbreviate $\tau = \tau_{\mathcal{F}}$ (if the underlying sheaf is clear from the context).

Exercise 3.2. Let $f : Y \to X$ be a morphism of schemes, let $\mathcal{F}$ and $\mathcal{G}$ be sheaves of $\mathcal{O}_X$ and $\mathcal{O}_Y$-modules, respectively. Prove that there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{O}_Y}(f^*\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, f_*\mathcal{G})$$

of abelian groups, called *adjunction*. In the case where X and Y are affine schemes, reformulate adjunction in terms of modules.

Exercise 3.3. (i) Suppose N is an S-module and $\tilde\sigma : S \to S$ is a ring homomorphism. Denote by $N_{\tilde\sigma}$ the same underlying abelian group however with S acting via $\tilde\sigma$, i.e. $s \cdot_{\tilde\sigma} n := \tilde\sigma(s)\cdot n$. Call a morphism $\alpha\colon N \to N$ $\tilde\sigma$-linear if $\alpha(sn) = \tilde\sigma(s)\alpha(n)$. Denote by $S[\tilde\sigma]$ the not necessarily commutative polynomial ring in $\tilde\sigma$ over S with the commutation rule $\tilde\sigma s = \tilde\sigma(s)\tilde\sigma$ for $s \in S$. Then the following are equivalent for a map $\alpha\colon N \to N$:

(a) α is $\tilde\sigma$-linear

(b) $\alpha\colon N \to N_{\tilde\sigma}, n \mapsto \alpha(n)$ is linear

(c) $\alpha^{\text{lin}}: S^{\tilde\sigma} \otimes_S N \to N, s \otimes n \mapsto s\alpha(n)$ is S-linear where the map $S \to S$ used in the tensor product is $\tilde\sigma$.

(d) N is a left $S[\tilde\sigma]$-module via $\sum s_i \tilde\sigma^i \cdot n = \sum_i s_i \alpha^i(n)$, i.e. via the unique action extending that of S on N so that $\alpha(n) = \tilde\sigma \cdot n$.

(ii) On any affine chart $\operatorname{Spec} R \subset X$ a τ-sheaf over A corresponds to an $R \otimes A$-module M together with a $\sigma \otimes \text{id}$-linear homomorphism $\tau: M \to M$. In other words, it corresponds to a left module over the non-commutative polynomial ring $(R \otimes A)[\tau]$, defined by the commutation rule $\tau(u \otimes a) := (u^q \otimes a)\tau$ for all $u \in R$ and $a \in A$.

Definition 3.4. The category formed by all τ-sheaves over A on X and with the above homomorphisms is denoted $\mathbf{QCoh}_\tau(X, A)$. The full subcategory of all *coherent* τ-sheaves (those $\underline{\mathcal{F}}$ for which $\mathcal{F}$ is coherent) is denoted $\mathbf{Coh}_\tau(X, A)$.

Exercise 3.5. Find an example of a non-zero τ-sheaf which contains no coherent τ-subsheaf except for 0. Show that any quasi-coherent sheaf (without τ) is the direct limit of its coherent subsheaves.

Because of the above example and for various technical reasons, in [8] we also introduce the category of ind-coherent τ-sheaves, i.e. τ-sheaves which are the filtered direct limit of their coherent τ-subsheaves.

The above two categories are abelian A-linear categories, and all constructions like kernel, cokernel, etc. are the usual ones on the underlying quasi-coherent sheaves, with the respective τ added by functoriality. In particular, the formation of kernel, cokernel, image and coimage is preserved under the inclusions $\mathbf{Coh}_\tau(X, A) \subset \mathbf{QCoh}_\tau(X, A)$.

Proposition 3.6. $\mathbf{Coh}_\tau(X, A) \subset \mathbf{QCoh}_\tau(X, A)$ *is a Serre subcategory – cf. Definition 4.9.*

$\mathbf{QCoh}_\tau(X, A)$ *is a* Grothendieck category*, i.e., it is closed under exact filtered direct limits and it posses a generator: an element* $\underline{\mathcal{U}}$ *such that any element of* $\mathbf{QCoh}_\tau(X, A)$ *is a quotient of* $\oplus_I \underline{\mathcal{U}}$ *for some index set* I.

The proof of the first part is obvious, that of the second can be found in [8, Thm 3.2.7]

2 (Algebraic) Drinfeld A-modules

Throughout this section, we assume that $C = \operatorname{Spec} A$ is an irreducible smooth curve over k whose smooth compactification $\bar{C}$ is obtained by adjoining precisely one closed point called ∞. Our prime example will be $A = k[t]$. For any non-zero element $a \in A$, we set $\deg(a) := \log_q \#(A/Aa)$.

By a line bundle L on X we mean a group scheme over X which is Zariski locally isomorphic to the additive group scheme $\mathbb{G}_a \times X$. Its endomorphism ring $\operatorname{End}_k(L)$ consists of all k-linear endomorphisms as a group scheme over X. If $X = \operatorname{Spec} R$ is affine and L is trivial (and thus $L = \operatorname{Spec} R[x]$), one can identify $\operatorname{End}_k(L)$ with the non-commutative polynomial ring $R[\tau]$ defined by the commutation rule $\tau u := u^q \tau$ for all $u \in R$. Here τ acts on a polynomial $f = \sum r_i x^i \in R[x]$ as $\tau f = \sum a_i x^{qi}$ and thus on $r \in R = \mathbb{G}_a(\operatorname{Spec} R)$ as $\tau(r) = r^q$. This means that τ is simply the Frobenius endomorphism on the sections $\mathbb{G}_a(\operatorname{Spec} R) = R$.

For arbitrary X, let $\mathcal{L}$ denote the invertible sheaf of sections of L over X. Since on sections, Frobenius is exponentiation to the power q, it defines a homomorphism $\mathcal{L} \to \mathcal{L}^{\otimes q}$, and thus only if composed with a linear homomorphism $\mathcal{L}^{\otimes q} \to \mathcal{L}$ one obtains a q-linear homomorphism. From this one deduces that $\operatorname{End}_k(L)$ is isomorphic to the module of global sections of $\oplus_{n \geq 0} \mathcal{L}^{\otimes(q^n - 1)}$.

Definition 3.7. A *Drinfeld A-module of rank $r > 0$ on X* consists of a line bundle L on X and a ring homomorphism $\varphi: A \to \operatorname{End}_k(L)$, $a \mapsto \varphi_a$, such that for all points $x \in X$ with residue field k_x the induced map

$$\varphi_x: \ A \to \operatorname{End}_k(L|x) \cong k_x[\tau], \ a \mapsto \sum_{i=0}^{\infty} u_i(a) \tau^i$$

has coefficients $u_i(a) = 0$ for $i > r \deg(a)$ and $u_{r \deg(a)}(a) \in k_x^*$.

A *homomorphism* $(L,\varphi) \to (L',\varphi')$ *of Drinfeld A-modules over* X is a homomorphism of line bundles $L \to L'$ that is equivariant with respect to the actions φ and φ'. It is called an *isogeny* if it is non-zero on any connected component of X. The latter implies that its kernel is a finite subgroup scheme of L.

The *characteristic of* (L,φ) is the morphism of schemes $\mathrm{Char}_\varphi\colon X \to C$ corresponding to the ring homomorphism $d\varphi:\ A \to \mathrm{End}_{\mathcal{O}_X}(\mathrm{Lie}(L)) \cong \Gamma(X,\mathcal{O}_X)$. An element $a \in A$ is prime to the characteristic of φ if $d\varphi_a$ is non-zero. A Drinfeld-module over a field is of generic characteristic if all $a \in A \smallsetminus \{0\}$ are prime to its characteristic. Else it is of special characteristic $\mathrm{Ker}(d\varphi)$, which is a non-zero prime ideal of A.

Exercise 3.8. Verify that the above definitions agree with the "usual ones" in the case $X = \operatorname{Spec} F$ for any field F of characteristic p.

Example 3.9. *In the special case where $X = \operatorname{Spec} R$ is affine and L is trivial over X, any Drinfeld A-module is isomorphic to one in the standard form*

$$\varphi\colon A \to R[\tau],\ a \mapsto \sum_{i=0}^{r\deg(a)} u_i(a)\tau^i,$$

where $u_{r\deg(a)}(a)$ is a unit in R for all $a \in A \smallsetminus \{0\}$. The characteristic of φ is the morphism corresponding to the ring homomorphism $A \to R$, $a \mapsto u_0(a)$.

An isogeny $\varphi \to \varphi'$ between Drinfeld modules in standard form over $\operatorname{Spec} R$ *is given by some $\psi \in R[\tau]$ with leading coefficient a unit in R such that $\varphi'_a\psi = \psi\varphi_a$ for all $a \in A$. Such a ψ defines an isomorphism if its degree is zero.*

Further results on Drinfeld modules such as their analytic definition via lattices, a discussion of their torsion points and the existence of isogenies and on Drinfeld Hayes modules can be found in Appendix A.

3 A-motives

The following construction due to Drinfeld attaches a coherent τ-sheaf to any Drinfeld A-module (L,φ) of rank r on X: The functor

$$U \mapsto \mathrm{Hom}_k\big(L|U,\ \mathbb{G}_a \times U\big)$$

defines a quasi-coherent sheaf of $\mathcal{O}_X$-modules on X. Letting each $a \in A$ act via right composition with φ_a defines on it the structure of a sheaf of $\mathcal{O}_X \otimes A$-modules. Let $\mathcal{M}(\varphi)$ be the corresponding quasi-coherent sheaf of $\mathcal{O}_{X\times C}$-modules on $X \times C$.

Example 3.10. *In Example 3.9 the module underlying $\mathcal{M}(\varphi)$ is $M(\varphi) := R[\tau]$. Here R and τ act by left multiplication, and $a \in A$ by right multiplication with φ_a. It is easy to see that $M(\varphi)$ is finitely generated over $R \otimes A$. In the special case $A = k[t]$ it is free over $R \otimes A \cong R[t]$ with basis $\{1, \tau, \tau^2, \dots, \tau^{r-1}\}$.* If $\varphi_t = \theta + \alpha_1 + \ldots + \alpha_r\tau^r$ with $\alpha_r \in R^*$, then the matrix representing τ is given by

$$\tau = \begin{vmatrix} 0 & 0 & \dots & 0 & \frac{t-\theta}{\alpha_r} \\ 1 & 0 & \dots & 0 & \frac{-\alpha_1}{\alpha_r} \\ 0 & 1 & \dots & 0 & \frac{-\alpha_2}{\alpha_r} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & 1 & \frac{-\alpha_{r-1}}{\alpha_r} \end{vmatrix} (\sigma_R \times \mathrm{id}_t)$$

Exercise 3.11. For any Drinfeld A-module (L,φ) of rank r on X, the sheaf $\mathcal{M}(\varphi)$ is a locally free sheaf on $X \times C$ of rank r. (Hint: Reduce to X affine, φ of standard form; treat first $A = \mathbb{F}_q[t]$; reduce the general case to this.)

Let $\sigma \in \mathrm{End}_k(\mathbb{G}_a \times X)$ denote the Frobenius endomorphism relative to X. Left composition with σ defines an $\mathcal{O}_{X\times C}$-linear homomorphism $\mathcal{M}(\varphi) \to (\sigma \times \mathrm{id})_*\mathcal{M}(\varphi)$, and thus via adjunction an $\mathcal{O}_{X\times C}$-linear homomorphism

$$\tau\colon\ (\sigma \times \mathrm{id})^*\mathcal{M}(\varphi) \longrightarrow \mathcal{M}(\varphi).$$

The resulting τ-sheaf is denoted $\underline{\mathcal{M}}(\varphi)$. This construction is functorial in (L, φ). Moreover $\mathrm{Coker}(\tau)$ is supported on the graph of Char_φ and locally free of rank 1 over X.

The following definition is essentially due to Anderson: We fix a morphism $\mathrm{Char}\colon X \to C$.

Definition 3.12. A *family of A-motives over X*, or short an *A-motive on X, of rank r and of characteristic* Char is a coherent τ-sheaf $\underline{\mathcal{M}}$ on X such that

(a) the underlying sheaf $\mathcal{M}$ is locally free of rank r over $\mathcal{O}_{X\times C}$, and

(b) as subsets of $X \times C$ the support of $\mathrm{Coker}(\tau)$ is a subset of the support of the graph of Char.

When X is the spectrum of a field F, $A = \mathbb{F}_q[t]$ and the module corresponding to $\underline{\mathcal{M}}$ is finitely generated over $F[\tau]$, then $\underline{\mathcal{M}}$ is a t-motive in the sense of Anderson [1].

Exercise 3.13. (a) Denote by $\mathcal{O}_X[\tau]$ the sheaf of rings on X defined on affine charts $\mathrm{Spec}\, R \subset X$ by $R[\tau]$ and with the obvious gluing morphisms. Given a Drinfeld-module (L, φ) on X we defined $\underline{\mathcal{M}}(\varphi)$ as $\mathcal{H}om_{k/X}(L, \mathbb{G}_{a,X})$ with the induced τ and $\mathcal{O}_{X\times C}$-actions. Show that to recover (L, φ) one may proceed as follows:
Define a functor on X-schemes $\pi\colon Y \to X$ by assigning to π the value $\mathrm{Hom}_{\mathcal{O}_Y[\tau]}(\pi^*\mathcal{M}(\varphi), \mathcal{O}_Y)$ where $\mathcal{O}_Y$ is considered as a sheaf on Y with the obvious action by $\mathcal{O}_Y[\tau]$. Show that this functor is represented by L on X and that one can recover φ from the A-action on $\underline{\mathcal{M}}(\varphi)$. (*Hints:* Exercise 3.3; us a affine covers.)

(b) The assignment $(L, \varphi) \to \underline{\mathcal{M}}(\varphi)$ defines a contravariant functor which is fully faithful. For $X = \mathrm{Spec}\, F$, the image is the set of those A-motives which are free over $F[\tau]$ of rank 1.

(c) Call a map $\underline{\mathcal{M}} \to \underline{\mathcal{N}}$ between A-motives on X an isogeny if its kernel and cokernel are finite over X. Show that any isogeny has trivial kernel and that $(L, \varphi) \to \underline{\mathcal{M}}(\varphi)$ maps isogenies of Drinfeld A-modules to isogenies of A-motives.

(d) Show that for $X = \mathrm{Spec}\, F$ any non-zero subobject of $\underline{\mathcal{M}}(\varphi)$ is isogenous to $\underline{\mathcal{M}}(\varphi)$, i.e. that it is an object which is irreducible up to isogeny. (Hint: use that $\mathrm{End}(\varphi)$ is an order in a division algebra.)

The category of Drinfeld A-modules does not permit the formation of direct sums or tensor products or related operations from linear algebra.

The passage to Anderson's t-motives, and more generally to A-motives, adds this missing flexibility.

We will see that A-crystals are even more flexible in that they form an abelian category with tensor product, which possesses a cohomology theory with compact support with many of the usual properties.

Chapter 4

A-crystals

In the previous chapter we introduced the first basic objects. Their definition was natural in light of the definition of families of Drinfeld modules and A-motives. In Section 1 we shall revisit the motivation given at the beginning of this lecture series. This will indicate that τ-sheaves are not in all respects suitable for the sought-for cohomological theory. Namely it suggest that we should find a new category built out of τ-sheaves in which those homomorphisms of τ-sheaves whose kernels and cokernels have nilpotent τ into isomorphisms. The formal procedure to obtain this category is localization. We briefly recall this in Sections 2 and 3 and refer to [8, §1.2] for further details and further references. In Section 4 we introduce the important notions of nilpotent τ-sheaf and of nil-isomorphism. Their understanding is a prerequisite to Section 5 where we introduce the category of A-crystals. This is the category for which we shall in the following chapters investigate the cohomological formalism introduced in [8].

1 Motivation II

The objects of $\mathbf{Coh}_\tau(X, A)$ are pairs of a coherent $\mathcal{O}_{X\times C}$-module and an endomorphism. For such pairs, the definition of inverse image, $\otimes$ and direct image can be defined in an obvious way, and we will do this later. One problem with direct image is that coherence is not preserved. But this does not come unexpectedly: already direct image between categories of quasi-coherent sheaves does not preserve coherence.

What is lacking at this point?

- For a trace formula, we need $Rf_!$, direct image with proper support, for any morphism f of finite type. The standard construction in the setting of schemes is $Rf_! = R\bar{f}_* \circ j_!$ where $f = \bar{f} \circ j$ with $\bar{f}$ proper and j an open immersion. (Such a factorization is called a (relative) compactification.) It remains the question of how to define $j_!$ for j an open immersion.

  Note that $j_!$ from quasi-coherent sheaves is not a useful functor here, since $j_!$ of a coherent sheaf is not necessarily quasi-coherent. On the other hand, in the present setting we would like $j_!$ to preserve coherence.
- If we are mainly interested in L-functions, we should regard pairs $(\mathcal{F}, 0)$ with the zero morphism as zero. More generally we should regard pairs $(\mathcal{F}, \tau)$ with τ nilpotent as zero!
- Expanding on the previous example we might like to regard a homomorphism $\underline{\mathcal{F}} \to \underline{\mathcal{G}}$ as an isomorphism if its kernel and cokernel have nilpotent τ.
- We would like to have a simple categorical characterization of objects $\underline{\mathcal{F}}$ to which we can attach an L-function, i.e., to which we can attach a pointwise L-factor at all closed points.

For this further motivation, suppose that X is of finite type over $\mathbb{F}_q$ and that $\mathcal{F}$ is the pullback of a coherent sheaf $\mathcal{F}_0$ on X. To any such $\underline{\mathcal{F}}$ one can assign an L-function as a product of pointwise L-factors as follows:

For any $x \in |X|$ let k_x denote its residue field and d_x its degree over k. Then the pullback $\mathcal{F}_x$ of $\mathcal{F}$ to $x \times C$ is equal to the pullback of $\mathcal{F}_0$ from X to x, pulled back under $x \times C \to x$; hence it corresponds to a free $k_x \otimes A$-module of finite rank M_x. The induced homomorphism $\tau_x \colon (\sigma_x \times \mathrm{id})^* \mathcal{F}_x \to \mathcal{F}_x$ corresponds to a $\sigma_x \otimes \mathrm{id}_A$-linear endomorphism $\tau_x \colon M_x \to M_x$. The iterate $\tau_x^{d_x}$ of the latter is $k_x \otimes A$-linear, and one can prove that

$$\det_{k_x \otimes A}\big(\mathrm{id} - t^{d_x}\tau_x^{d_x} \bigm| M_x\big) = \det_A\big(\mathrm{id} - t\tau_x \bigm| M_x\big).$$

This is therefore a polynomial in $1 + t^{d_x}A[t^{d_x}]$. Since there are at most finitely many $x \in |X|$ with fixed d_x, the following product makes sense:

Definition 4.1. The *naive L-function* of $\underline{\mathcal{F}}$ is

$$L^{\mathrm{naive}}(X, \underline{\mathcal{F}}, T) \;:=\; \prod_{x \in |X|} \det_A\big(\mathrm{id} - T\tau_x \bigm| M_x\big)^{-1} \;\in\; 1 + TA[[T]],$$

For the trace formula suppose first that X is proper over k. Then for every integer i the coherent cohomology group $H^i(X, \mathcal{F}_0)$ is a finite dimensional vector space over k. Moreover, the equality $\mathcal{F} = \mathrm{pr}_1^* \mathcal{F}_0$ yields a natural isomorphism $H^i(X \times C, \mathcal{F}) \cong H^i(X, \mathcal{F}_0) \otimes A$. This is therefore a free A-module of finite rank. It also carries a natural endomorphism induced by τ; hence we can consider it as a coherent τ-sheaf on $\operatorname{Spec} k$, denoted by $H^i(X, \underline{\mathcal{F}})$. The first instance of the trace formula for L-functions then states:

Theorem 4.2. $L^{\mathrm{naive}}(X, \underline{\mathcal{F}}, T) \;=\; \prod_{i \in \mathbb{Z}} L^{\mathrm{naive}}\big(\operatorname{Spec} k, H^i(X, \underline{\mathcal{F}}), T\big)^{(-1)^i}$.

A standard procedure to extend this formula to non-proper X is via cohomology with compact support. For this we fix a dense open embedding $j \colon X \hookrightarrow \overline{X}$ into a proper scheme of finite type over k. (The existence of such a compactification is a result due to Nagata; [37, 38] or [34].) We want to extend the given $\underline{\mathcal{F}}$ on X to a coherent τ-sheaf $\underline{\widetilde{\mathcal{F}}}$ on $\bar{X}$ **without changing the L-function**.) Any extension whose τ_z on $\widetilde{\mathcal{F}}_z$ is zero for all $z \in |\bar{X} \smallsetminus X|$ has that property, and it is not hard to construct such an extension: In fact, any coherent sheaf on $\bar{X}$ extending $\mathcal{F}$, multiplied by a sufficiently high power of the ideal sheaf of $\bar{X} \smallsetminus X$, does the job. However, there are many choices for this $\underline{\widetilde{\mathcal{F}}}$, and none is functorial. Thus there is none that we can consider a natural extension by zero "$j_! \underline{\mathcal{F}}$" in the sense of τ-sheaves. For the purpose of L-functions however, any such is a reasonable choice since it satisfies

$$L^{\mathrm{naive}}(X, \underline{\mathcal{F}}, T) \;= L^{\mathrm{naive}}(\bar{X}, \underline{\widetilde{\mathcal{F}}}, T).$$

One can in fact show the following: Given any two extension $\underline{\widetilde{\mathcal{F}}}_1$, $\underline{\widetilde{\mathcal{F}}}_2$ of $\underline{\mathcal{F}}$ to X, there exists a third one $\underline{\widetilde{\mathcal{F}}}_3$ and injective homomorphisms of τ-sheaves $\varphi_i : \underline{\widetilde{\mathcal{F}}}_3 \to \underline{\widetilde{\mathcal{F}}}_i$, $i = 1, 2$, whose cokernels have nilpotent τ. In particular one would like to regard all such $\underline{\mathcal{F}}_i$ as isomorphic.

Ignoring the ambiguity in the definition of $j_! \underline{\mathcal{F}}$ for the moment, let us nevertheless provisionally regard $H^i(\bar{X}, \underline{\widetilde{\mathcal{F}}})$ as the cohomology with compact support $H^i_c(X, \underline{\mathcal{F}})$. Then from Theorem 4.2 we obtain the more general trace formula

Theorem 4.3. $L^{\mathrm{naive}}(X, \underline{\mathcal{F}}, T) \;=\; \prod_{i \in \mathbb{Z}} L^{\mathrm{naive}}\big(\operatorname{Spec} k, H^i_c(X, \underline{\mathcal{F}}), T\big)^{(-1)^i}$.

Since the factors on the right hand side are polynomials in $1 + tA[t]$ or inverses of such polynomials, the *rationality* of $L^{\mathrm{naive}}(X, \underline{\mathcal{F}}, T)$ is an immediate consequence.

The order of presentation of the above theorems is for expository purposes only. In [8], Theorem 4.3 is proved first when X is regular and affine over k and then generalized to arbitrary X by devissage. The proof in the affine case is based on a trace formula by Anderson from [2]. While Anderson formulated it only for $A = k$, Taguchi and Wan [44] already noted that it holds whenever A is a field, and [8] extends it further. Also, the formula in [2] is interpreted in [8] as the Serre dual of the one in Theorem 4.3. This explains the absence of cohomology in the trace formula given in [2].

2 Localization

Let $\mathfrak{C}$ be a category and let $\mathcal{S}$ denote a collection of morphisms in $\mathfrak{C}$. Morphisms in $\mathcal{S}$ are drawn as double arrows $\Longrightarrow$ to distinguish them from arbitrary morphisms $\longrightarrow$ in $\mathfrak{C}$.

Definition 4.4. The collection $\mathcal{S}$ is a **multiplicative system** if it satisfies the following three axioms:

(a) $\mathcal{S}$ is closed under composition and contains the identity morphism for every object of $\mathfrak{C}$.

(b) For any $t\colon N' \Rightarrow N$ in $\mathcal{S}$ and any $f: M \to N$ in $\mathfrak{C}$, there exist $s\colon M' \Rightarrow M$ in $\mathcal{S}$ and $f': M' \to N'$ in $\mathfrak{C}$ such that the following diagram **commutes**:

$$\begin{tikzcd} M' \arrow[r, "f'"] \arrow[d, Rightarrow, "s"'] & N' \arrow[d, Rightarrow, "t"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

The same statement with all arrows reversed is also required.

(c) For any pair of morphisms $f, g\colon M \to N$ the following are equivalent:

  (i) There exists $s \in \mathcal{S}$ such that $sf = sg$.

  (ii) There exists $t \in \mathcal{S}$ such that $ft = gt$.

Suppose $\mathcal{S}$ is a multiplicative system. Then one constructs a new category $\mathcal{S}^{-1}\mathfrak{C}$ as follows:

The **objects** are those of $\mathfrak{C}$, i.e., $\mathrm{Ob}(\mathcal{S}^{-1}\mathfrak{C}) := \mathrm{Ob}(\mathfrak{C})$.

Morphisms will be equivalence classes of certain diagrams: A *right fraction from M to N* is a diagram

$$M \xrightarrow{f} L \overset{s}{\Longleftarrow} N$$

in $\mathfrak{C}$ with $s \in \mathcal{S}$. Two right fractions $M \xrightarrow{f_i} L_i \overset{s_i}{\Longleftarrow} N$ are called *equivalent* if there is a commutative diagram

$$\begin{tikzcd} & L_1 \arrow[d] & \\ M \arrow[ur, "f_1"] \arrow[r, "f"] \arrow[dr, "f_2"'] & L & N \arrow[ul, Rightarrow, "s_1"'] \arrow[l, Rightarrow, "s"'] \arrow[dl, Rightarrow, "s_2"] \\ & L_2 \arrow[u] & \end{tikzcd}$$

in $\mathfrak{C}$ with $s \in \mathcal{S}$.

Exercise 4.5. Using the axioms 4.4, show that this defines an equivalence relation on the class of all right fractions from M to N.

One defines

$$\mathrm{Hom}_{\mathcal{S}^{-1}\mathfrak{C}}(M, N)$$

as the set of equivalence classes of right fraction, **provided** the following set theoretic condition is satisfied:

Definition 4.6. $\mathcal{S}$ is called *essentially locally small* if for all M and N the class of right fractions from M to N possesses a set of representatives.

Due to the symmetry in the definition of multiplicative systems one can also work with left fractions $M \Leftarrow L \to N$. The axioms 4.4 imply that every equivalence class of left fractions corresponds to a unique equivalence class of right fractions, and vice versa, so that one obtains the same result. Using axiom 4.4 (b) one defines the composition of right fractions. Thus under the condition 4.6 one obtains a well-defined category $\mathcal{S}^{-1}\mathfrak{C}$, called the *localization of* $\mathfrak{C}$ *by* $\mathcal{S}$.

There is a natural *localization functor* $q\colon \mathfrak{C} \to \mathcal{S}^{-1}\mathfrak{C}$ mapping any object to itself and any morphism $M \xrightarrow{f} N$ to the equivalence class of the right fraction $M \xrightarrow{f} N \xLeftarrow{\mathrm{id}} N$. To distinguish morphisms in $\mathcal{S}^{-1}\mathfrak{C}$ from those in $\mathfrak{C}$ we often denote them by dotted arrows $\dashrightarrow$. A morphism of the form $q(f)$ is, by abuse of notation, also denoted by a solid arrow. We will often abbreviate $\overline{\mathfrak{C}} := \mathcal{S}^{-1}\mathfrak{C}$ when $\mathcal{S}$ is clear from the context.

Exercise 4.7. Let $\mathcal{S}$ be an essentially locally small multiplicative system $\mathcal{S}$ in a category $\mathfrak{C}$.

(a) For every $s \in \mathcal{S}$ the morphism $q(s)$ is an isomorphism in $\overline{\mathfrak{C}}$.

(b) For any category $\mathfrak{D}$ and any functor $F\colon \mathfrak{C} \to \mathfrak{D}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exists a unique functor $\bar{F}\colon \overline{\mathfrak{C}} \to \mathfrak{D}$ such that $F = \bar{F}\,q$.

(c) If $\mathfrak{C}$ is an additive category, then so is $\overline{\mathfrak{C}}$ and q is an additive functor. In this case, if the functor F in (b) is additive, then so is $\bar{F}$.

(d) Suppose $\mathcal{S}$ and $\mathcal{S}'$ are multiplicative systems of categories $\mathfrak{C}$ and $\mathfrak{C}'$, respectively. If a functor $F\colon \mathfrak{C}' \to \mathfrak{C}$ satisfies $F(\mathcal{S}) \subset \mathcal{S}'$, then there is an induced functor $\bar{F}\colon \overline{\mathfrak{C}}' \to \overline{\mathfrak{C}}$.

Remark: The properties 4.7 (a)–(b) characterize $\overline{\mathfrak{C}}$ and q up to equivalence of categories.

Definition 4.8. A multiplicative system $\mathcal{S}$ is *saturated* if, in addition to 4.4 (a)–(c), it also satisfies the condition

(d) For any morphism $f\colon M \to N$ in $\mathfrak{C}$, if there exist $g\colon N \to N'$ and $h\colon M' \to M$ such that gf and fh are in $\mathcal{S}$, then f is in $\mathcal{S}$.

If $\mathcal{S}$ if saturated, one easily shows that for any morphisms $L \xrightarrow{f} M \xrightarrow{g} N$, if two of f, g, and gf are in $\mathcal{S}$, then so is the third. This property is useful in simplifying arguments.

3 Localization for abelian categories

Definition 4.9. A full subcategory $\mathfrak{B}$ of an abelian category $\mathfrak{A}$ which is closed under taking subobjects, quotients, extensions, and isomorphisms, is called a *Serre subcategory.*

Exercise 4.10. ([50, Exer. 10.3.2]) For any abelian category $\mathfrak{A}$ there is a bijection between the class of saturated multiplicative systems $\mathcal{S}$ and the class of Serre subcategories $\mathfrak{B}$. Explicitly, given $\mathfrak{B}$ one defines $\mathcal{S}$ as the class of those morphisms whose kernel and cokernel are in $\mathfrak{B}$. Conversely, given $\mathcal{S}$ one defines $\mathfrak{B}$ as the full subcategory consisting of those objects M of $\mathfrak{A}$ such that $0 \to M$ is in $\mathcal{S}$.

Definition 4.11. An abelian category is called *locally small* if for every object the equivalence classes of subobjects form a set.

Exercise 4.12. Suppose $\mathfrak{A}$ is locally small and let $\mathcal{S}$ be the multiplicative system associated to any Serre subcategory $\mathfrak{B}$. Let $f\colon M \to N$ denote a morphism in $\mathfrak{A}$. Then

(a) (see [50, Ex. 10.3.2]) $\mathcal{S}$ is essentially locally small, the localized category $\bar{\mathfrak{A}} := \mathcal{S}^{-1}\mathfrak{A}$ is abelian and the functor $q\colon \mathfrak{A} \to \overline{\mathfrak{A}}$ is exact.

(b) (i) The object $q(M) \in \overline{\mathfrak{A}}$ is zero if and only if $M \in \mathfrak{B}$.

(ii) The morphism $q(f)$ is zero if and only if $\operatorname{Im} f \in \mathfrak{B}$.

(iii) The morphism $q(f)$ is a monomorphism if and only if $\operatorname{Ker} f \in \mathfrak{B}$.

(iv) The morphism $q(f)$ is an epimorphism if and only if $\operatorname{Coker} f \in \mathfrak{B}$.

(v) The morphism $q(f)$ is an isomorphism if and only if both $\operatorname{Ker} f$, $\operatorname{Coker} f \in \mathfrak{B}$.

(c) If $q(f)$ is an isomorphism, then f can be factored as $f = gh$ where h is an epimorphism with kernel in $\mathfrak{B}$ and h is a monomorphism with cokernel in $\mathfrak{B}$.

(d) Every short exact sequence in $\overline{\mathfrak{A}}$ is isomorphic to the image of a short exact sequence in $\mathfrak{A}$.

(e) Every complex in $\overline{\mathfrak{A}}$ is isomorphic to the image of a complex in $\mathfrak{A}$.

Next recall that an object $M \in \mathfrak{A}$ is *noetherian* if every increasing sequence of subobjects becomes stationary.

Exercise 4.13. If $M \in \mathfrak{A}$ is noetherian, then $q(M) \in \overline{\mathfrak{A}}$ is noetherian.

Exercise 4.14. Is the category of A-motives on a scheme X abelian?

Show that the category which is the localization of the category of A-motives at the set of isogenies is an F-linear abelian tensor category and that any morphism is given by a diagram

$$\underline{\mathcal{M}} \Leftarrow \underline{\mathcal{H}} \to \underline{\mathcal{N}}.$$

4 Nilpotence

For a τ-sheaf $\underline{\mathcal{F}}$ one defines the iterates $\tau_{\mathcal{F}}^n$ of $\tau_{\mathcal{F}}$ by setting inductively

$$\tau_{\mathcal{F}}^0 := \mathrm{id} \quad \text{and} \quad \tau_{\mathcal{F}}^{n+1} := \tau_{\mathcal{F}} \circ (\sigma \times \mathrm{id})^* \tau_{\mathcal{F}}^n.$$

Thus

- $\tau^n \colon (\sigma^n \times \mathrm{id})^* \mathcal{F} \longrightarrow \mathcal{F}$ is an $\mathcal{O}_{X \times C}$-linear homomorphisms.
- Each $(\sigma^n \times \mathrm{id})^* \underline{\mathcal{F}} := \big((\sigma^n \times \mathrm{id})^* \mathcal{F}, (\sigma^n \times \mathrm{id})^* \tau_{\mathcal{F}}\big)$ is a τ-sheaf.
- $\tau_{\mathcal{F}}^n \colon (\sigma^n \times \mathrm{id})^* \underline{\mathcal{F}} \to \underline{\mathcal{F}}$ is a homomorphism of τ-sheaves.

Definition 4.15. (a) A τ-sheaf $\underline{\mathcal{F}}$ is called *nilpotent* if $\tau_{\mathcal{F}}^n$ vanishes for some, or equivalently all, $n \gg 0$.

(b) A τ-sheaf $\underline{\mathcal{F}}$ is called *locally nilpotent* if it is a union of nilpotent τ-subsheaves.

The full subcategories of $\mathbf{QCoh}_\tau(X, A)$ formed by all nilpotent **and** coherent, respectively locally nilpotent τ-sheaves are denoted $\mathbf{Nil}_\tau(X, A) \subset \mathbf{LNil}_\tau(X, A)$.

We have the following inclusions of categories

$$\begin{tikzcd} \mathbf{Nil}_\tau(X,A) \arrow[r, hook] \arrow[d, hook] & \mathbf{LNil}_\tau(X,A) \arrow[d, hook] \\ \mathbf{Coh}_\tau(X,A) \arrow[r, hook] & \mathbf{QCoh}_\tau(X,A), \end{tikzcd} \tag{1}$$

where furthermore $\mathbf{Nil}_\tau(X, A) = \mathbf{Coh}_\tau(X, A) \cap \mathbf{LNil}_\tau(X, A)$ (essentially by definition).

Remark 4.16. We observe the following obvious fact: Suppose X is a scheme of finite type over k and $\underline{\mathcal{F}}$ is a locally free τ-sheaf. Then if $\underline{\mathcal{F}}$ is nilpotent, its L-function is trivial, i.e., $L(X, \underline{\mathcal{F}}, T) = 1$.

Proposition 4.17. *The categories in (1) are Serre subcategories of* $\mathbf{QCoh}_\tau(X, A)$.

Proof. The non-trivial part is to prove the invariance under extensions for $\mathbf{LNil}_\tau(X,A)$. Consider a short exact sequence $0 \to \underline{\mathcal{F}}' \xrightarrow{\alpha} \underline{\mathcal{F}} \xrightarrow{\beta} \underline{\mathcal{F}}'' \to 0$ in $\mathbf{QCoh}_\tau(X,A)$ with $\underline{\mathcal{F}}'$ and $\underline{\mathcal{F}}''$ in $\mathbf{LNil}_\tau(X,A)$. We claim that every coherent subsheaf $\mathcal{G}$ is contained in a nilpotent coherent τ-subsheaf of $\underline{\mathcal{F}}$. By hypothesis on $\underline{\mathcal{F}}''$, the image $\beta(\mathcal{G})$ is contained in a sheaf underlying a nilpotent coherent τ-subsheaf $\underline{\mathcal{G}}''$ of $\underline{\mathcal{F}}''$. In particular there exists n, such that $\tau^n_{\mathcal{G}''} = 0$.

Hence $\tau^n((\sigma^n \times \mathrm{id})^*\mathcal{G}) \subset \mathcal{F}'$. Now we apply our hypothesis on $\underline{\mathcal{F}}'$. It yields a nilpotent coherent τ-subsheaf containing $\tau^n((\sigma^n \times \mathrm{id})^*\mathcal{G})$. In particular there exists n' such that $\tau^{n'}(\tau^n(\underline{\mathcal{G}})) = 0$. One easily deduces that $\sum_{i=0}^{n+n'} \tau^i((\sigma^i \times \mathrm{id})^*\mathcal{G})$ is a nilpotent coherent τ-subsheaf of $\underline{\mathcal{F}}$ which contains $\mathcal{G}$. □

By Proposition 4.10, the Serre subcategory $\mathbf{LNil}_\tau(X,A)$ defines a corresponding multiplicative system:

Definition 4.18. A homomorphism of τ-sheaves is called a *nil-isomorphism* if both its kernel and cokernel are locally nilpotent.

Note that by diagram (1) a homomorphism of *coherent* τ-sheaves is a nil-isomorphism if and only if its kernel and cokernel are *nilpotent*.

The following characterization of nil-isomorphisms will be useful. Note that inverse image by $\sigma^n \times \mathrm{id}$ always preserves coherence.

Proposition 4.19. *A homomorphism of τ-sheaves $\varphi\colon \underline{\mathcal{F}} \to \underline{\mathcal{G}}$ is a nil-isomorphism if there exist $n \geq 0$ and a homomorphism of τ-sheaves α making the following diagram commute:*

$$\begin{tikzcd}
(\sigma^n \times \mathrm{id})^*\underline{\mathcal{F}} \arrow[r, "\tau^n"] \arrow[d, "(\sigma^n\times\mathrm{id})^*\varphi"'] & \underline{\mathcal{F}} \arrow[d, "\varphi"] \\
(\sigma^n \times \mathrm{id})^*\underline{\mathcal{G}} \arrow[r, "\tau^n"] \arrow[ur, "\alpha"] & \underline{\mathcal{G}}.
\end{tikzcd} \tag{2}$$

If $\underline{\mathcal{F}}$ and $\underline{\mathcal{G}}$ are coherent, the converse is also true.

Proof. For this proof we abbreviate $\underline{\mathcal{H}}_{(n)} := (\sigma^n \times \mathrm{id})^*\underline{\mathcal{H}}$ for any τ-sheaf $\underline{\mathcal{H}}$. We only give the proof of the first assertion. The reader is advised to try to prove the converse by herself. Let us suppose that α exists, so that we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\mathrm{Ker}\,\varphi)_{(n)} \arrow[r] \arrow[d, "\tau^n"] & \underline{\mathcal{F}}_{(n)} \arrow[r] \arrow[d, "\tau^n"] & \underline{\mathcal{G}}_{(n)} \arrow[r] \arrow[d, "\tau^n"] \arrow[dl, "\alpha"'] & (\mathrm{Coker}\,\varphi)_{(n)} \arrow[r] \arrow[d, "\tau^n"] & 0 \\
0 \arrow[r] & \mathrm{Ker}\,\varphi \arrow[r] & \underline{\mathcal{F}} \arrow[r, "\varphi"] & \underline{\mathcal{G}} \arrow[r] & \mathrm{Coker}\,\varphi \arrow[r] & 0.
\end{tikzcd}$$

Here the bottom row is exact and, since inverse image on coherent sheaves is a right exact functor, the top row is a complex whose right half is exact. It is straightforward to deduce that the outer vertical homomorphisms vanish. This shows that $\mathrm{Ker}\,\varphi$ and $\mathrm{Coker}\,\varphi$ are nilpotent; hence φ is a nil-isomorphism, as desired. □

Applying Proposition 4.19 to $\varphi = \tau^n_{\mathcal{F}}$ and $\alpha = \mathrm{id}_{\mathcal{F}}$ yields:

Corollary 4.20. *For every τ-sheaf and every $n \geq 0$ the homomorphism $\tau^n_{\mathcal{F}} : (\sigma^n \times \mathrm{id})^*\underline{\mathcal{F}} \longrightarrow \underline{\mathcal{F}}$ is a nil-isomorphism.*

Exercise 4.21. Suppose $\varphi\colon \underline{\mathcal{M}} \to \underline{\mathcal{M}}'$ is a nil-isomorphism of A-motives. Show that it is an isomorphism.

Remark 4.22. From 4.16 we deduce the following rather trivial observation: Suppose X is a scheme of finite type over k and $\underline{\mathcal{F}} \to \underline{\mathcal{G}}$ a homomorphism of locally free τ-sheaves whose kernels and cokernels are locally free as well. Then $L(X, \underline{\mathcal{F}}, T) = L(X, \underline{\mathcal{G}}, T)$.

5 A-crystals

By Proposition 4.17 the categories in the upper row of (1) are Serre subcategories of the categories in the lower row. Proposition 4.10 identifies the corresponding saturated multiplicative systems with the respective classes of nil-isomorphisms. In this section we will study basic properties of the associated localized categories.

Regarding existence, it is shown in [8, Prop. 2.6.1, Thm. 3.2.7] that $\mathbf{QCoh}_\tau(X, A)$ is a Grothendieck category and so in particular it is locally small. Thus by Exercise 4.12 for any Serre subcategory the localization at the corresponding multiplicative system exists.

Definition 4.23. In the following commutative diagram the lower row is obtained from the upper row by localization with respect to nil-isomorphisms, the vertical arrows are the respective localization functors, and the lower horizontal arrows are obtained from the upper horizontal arrows by the universal property of localization:

$$\begin{tikzcd} \mathbf{Coh}_\tau(X,A) \arrow[r, hook] \arrow[d, "q"'] & \mathbf{QCoh}_\tau(X,A) \arrow[d, "q"'] \\ \mathbf{Crys}(X,A) \arrow[r] & \mathbf{QCrys}(X,A). \end{tikzcd}$$

We refer to the objects of $\mathbf{Crys}(X, A)$ as *A-crystals on X*, and to the objects of $\mathbf{QCrys}(X, A)$ as *A-quasi-crystals on X*. As A usually remains fixed, we mostly speak only of *(quasi)-crystals on X*.

Both $\mathbf{Crys}(X, A)$ and $\mathbf{QCrys}(X, A)$ are A-linear abelian categories and the horizontal functor in the bottom row of diagram 4.23 is fully faithful.

Remark 4.24. As before we use solid arrows $\longrightarrow$ to denote homomorphisms in $\mathbf{QCoh}_\tau(X, A)$, double arrows $\Longrightarrow$ to denote nil-isomorphisms, and dotted arrows $\dashrightarrow$ to emphasize homomorphisms in $\mathbf{QCrys}(X, A)$. We retain solid arrows for functors and natural transformations, even if their target is a category of (quasi)-crystals. In any case, the rule regarding dotted arrows will be relaxed to some extent in the later chapters.

There is a standard way to represent homomorphisms of crystals which is derived from Proposition 4.19 for nil-isomorphisms between coherent τ-sheaves.

Proposition 4.25. *Any homomorphism $\varphi\colon \underline{\mathcal{F}} \dashrightarrow \underline{\mathcal{G}}$ in $\mathbf{Crys}(X, A)$ can be represented for suitable n by a diagram*

$$\underline{\mathcal{F}} \overset{\tau^n}{\Longleftarrow} (\sigma^n \times \mathrm{id})^*\underline{\mathcal{F}} \longrightarrow \underline{\mathcal{G}}.$$

Proof. The rather straightforward proof, building on Proposition 4.19, is left to the reader. $\square$

Based on this one can give the following alternative description of crystals: The category $\mathbf{Crys}(X, A)$ has the same *objects* as the category $\mathbf{Coh}_\tau(X, A)$. Given coherent τ-sheaves $\underline{\mathcal{F}}$ and $\underline{\mathcal{G}}$, the set of *morphisms* from $\underline{\mathcal{F}}$ to $\underline{\mathcal{G}}$ in $\mathbf{Crys}(X, A)$ is defined as

$$\mathrm{Hom}_{\mathrm{crys}}(\underline{\mathcal{F}}, \underline{\mathcal{G}}) := \Big(\bigcup_{n\in\mathbb{N}} \mathrm{Hom}_\tau((\sigma^n \times \mathrm{id})^*\underline{\mathcal{F}}, \underline{\mathcal{G}}) \Big) / \sim,$$

where the equivalence relation $\sim$ is defined as follows: Morphisms $\varphi\colon (\sigma^n \times \mathrm{id})^*\underline{\mathcal{F}} \to \underline{\mathcal{G}}$ and $\psi\colon (\sigma^m \times \mathrm{id})^*\underline{\mathcal{F}} \to \underline{\mathcal{G}}$ are equivalent, if there exists $\ell \geq \max\{m, n\}$ such that

$$\varphi \circ (\sigma^n \times \mathrm{id})^*(\tau^{\ell-n}) = \psi \circ (\sigma^m \times \mathrm{id})^*(\tau^{\ell-m}).$$

Composition of morphisms in $\mathbf{Crys}$ is defined in the obvious way, i.e., the composite of $\varphi\colon (\sigma^n \times \mathrm{id})^*\underline{\mathcal{F}} \to \underline{\mathcal{G}}$ and $\psi\colon (\sigma^m \times \mathrm{id})^*\underline{\mathcal{G}} \to \underline{\mathcal{H}}$ is defined as

$$\psi \circ (\sigma^m \times \mathrm{id})^*\varphi\colon (\sigma^{m+n} \times \mathrm{id})^*\underline{\mathcal{F}} \longrightarrow \underline{\mathcal{H}}.$$

Chapter 5

Functors on τ-sheaves and A-crystals

We indicate the basic construction of all functors on τ-sheaves and A-crystals from [8]. For τ-sheaves these are inverse and direct image, tensor product and change of coefficients. For crystals we have in addition an exact functor extension by zero. Since our approach follows closely the well-known constructions for coherent sheaves we mostly omit details. In Section 1 or 3, respectively, inverse image and extension by zero on crystals are discussed in greater detail. The inverse image functor has properties different from those known for coherent sheaves. The extension by zero is not derived from a functor on coherent sheaves.

1 Inverse Image

We fix a morphism $f\colon Y \to X$.

Definition 5.1. For any τ-sheaf $\underline{\mathcal{F}}$ on X we let $f^*\underline{\mathcal{F}}$ denote the τ-sheaf on Y consisting of $(f \times \mathrm{id})^*\mathcal{F}$ and the composite homomorphism

$$\begin{tikzcd}
(\sigma \times \mathrm{id})^*(f \times \mathrm{id})^*\mathcal{F} \arrow[r, "\tau_{f^*\mathcal{F}}"] \arrow[d, equal] & (f \times \mathrm{id})^*\mathcal{F}. \\
(f \times \mathrm{id})^*(\sigma \times \mathrm{id})^*\mathcal{F} \arrow[ur, "(f\times\mathrm{id})^*\tau_{\mathcal{F}}"'] &
\end{tikzcd}$$

For any homomorphism $\varphi\colon \underline{\mathcal{F}} \to \underline{\mathcal{F}}'$ we abbreviate $f^*\varphi := (f \times \mathrm{id})^*\varphi$.

This defines an A-linear functor

$$f^*\colon\ \mathbf{QCoh}_\tau(X, A) \longrightarrow \mathbf{QCoh}_\tau(Y, A)$$

which is clearly right exact. When f is flat, it is exact. In general, its exactness properties are governed by associated Tor-objects.

Proposition 5.2. *(a) If φ is a nil-isomorphism, then so is $f^*\varphi$.*

(b) The functor f^* induces a functor

$$f^*\colon \mathbf{QCrys}(X, A) \longrightarrow \mathbf{QCrys}(Y, A)$$

which preserves coherence, i.e., $f^*(\mathbf{Crys}(X, A)) \subset \mathbf{Crys}(Y, A)$.

Proof. Note that f^* is in general not exact. Thus (a) is not entirely trivial. However it can be easily reduced to the case of nil-isomorphisms where either kernel or cokernel are zero. These cases are easier to treat. For coherent τ-sheaves, a direct proof is obtained by applying Proposition 4.19. Once (a) is proved (b) is immediate. □

The following result, whose proof we omit, shows that quasi-crystals behave like sheaves:

Proposition 5.3. *Let $X = \bigcup_i U_i$ be an open covering with embeddings $j_i : U_i \hookrightarrow X$.*

(a) A quasi-crystal $\underline{\mathcal{F}}$ on X is zero if and only if $j_i^*\underline{\mathcal{F}}$ is zero in $\mathbf{QCrys}(U_i, A)$ for all i.

(b) A homomorphism φ in $\mathbf{QCrys}(X, A)$ is a monomorphism, an epimorphism, an isomorphism, respectively zero, if and only if its inverse image $j_i^*\varphi$ has that property for all i.

Next consider any point $x \in X$. Let k_x denote its residue field and $i_x : x \cong \operatorname{Spec} k_x \hookrightarrow X$ its natural embedding. The object $i_x^*\underline{\mathcal{F}}$ can be viewed as the *stalk of $\underline{\mathcal{F}}$ at x* in the category of crystals! This is justified by the following result.

Theorem 5.4. *The following assertions hold in* **Crys***:*

(a) A crystal $\underline{\mathcal{F}} \in \mathbf{Crys}(X, A)$ is zero if and only if the crystals $i_x^*\underline{\mathcal{F}}$ are zero for all $x \in X$.

(b) The functors i_x^* are exact on $\mathbf{Crys}(X, A)$ for all $x \in X$.

Proof. The 'only if' part of (a) follows from the well-definedness of i_x^* on **Crys**. For the 'if' part denote by $\underline{\mathcal{F}}$ also a τ-sheaf representing the crystal $\underline{\mathcal{F}}$. The images of $\tau_{\mathcal{F}}^n$ form a decreasing sequence of coherent subsheaves of $\mathcal{F}$. Thus their supports form a decreasing sequence of closed subschemes $Z_n \subset X \times C$. As $X \times C$ is noetherian, we deduce that $Z_\infty := Z_n$ is independent of n whenever $n \gg 0$. By replacing $\underline{\mathcal{F}}$ by the image of $\tau_{\mathcal{F}}^n$ for $n \gg 0$ we may assume $\operatorname{Supp}(\operatorname{Im}(\tau_{\mathcal{F}}^n)) = Z_\infty$ for all $n \geq 0$. If $Z_\infty = \varnothing$ we are done. Otherwise let η be a generic point of Z_∞ and set $x := \operatorname{pr}_1(\eta)$. We shall deduce a contradiction.

For this we replace X by its localization at x, after which $X = \operatorname{Spec} R$ for a noetherian local ring R, and x corresponds to the maximal ideal $\mathfrak{m} \subset R$. Let M be the $(R \otimes A)$-module corresponding to $\mathcal{F}$. By construction the support of M lies in $x \times C$. Since M is of finite type over the noetherian ring $R \otimes A$ it follows that $\mathfrak{m}^r M = 0$ for any $r \gg 0$, say for $r \geq r_0$.

If $i_x^*\underline{\mathcal{F}}$ is nilpotent, we have $\tau^s M \subset \mathfrak{m}M$ for some $s \geq 1$. For every $i \geq 0$ this implies

$$\tau^{i+s}M \subset \tau^i(\mathfrak{m}M) \subset \mathfrak{m}^{q^i}M,$$

and for $i \gg 0$, so that $q^i \geq r_0$ it follows that $\tau^{i+s}M = 0$. This contradicts our assumption on Z_∞ and thus proves (a).

To prove (b) we may again replace X by its localization at x, so that $X = \operatorname{Spec} R$ for a noetherian local ring with maximal ideal $\mathfrak{m} \subset R$. We consider an arbitrary short exact sequence of $(R \otimes A)[\tau]$-modules which are finitely generated over $R \otimes A$

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0.$$

The long exact Tor-sequence induces the exact sequence

$$\ldots \longrightarrow \operatorname{Tor}_1^{R\otimes A}\big(M'', (R/\mathfrak{m}) \otimes A\big) \longrightarrow M'/\mathfrak{m}M' \longrightarrow M/\mathfrak{m}M \longrightarrow M''/\mathfrak{m}M'' \longrightarrow 0.$$

Thus it suffices to show that the left hand term is nilpotent for any M''. Using Lemma 5.6 below, we may write M'' as the quotient of an $(R \otimes A)[\tau]$-module P which is free of finite type over $R \otimes A$. From the resulting short exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow M'' \longrightarrow 0$$

and the long exact Tor-sequence one now obtains the exact sequence

$$0 \longrightarrow \operatorname{Tor}_1^{R\otimes A}\big(M, (R/\mathfrak{m}) \otimes A\big) \longrightarrow K/\mathfrak{m}K \longrightarrow P/\mathfrak{m}P \longrightarrow M''/\mathfrak{m}M'' \longrightarrow 0.$$

It yields $\operatorname{Tor}_1^{R\otimes A}\big(M, (R/\mathfrak{m}) \otimes A\big) \cong (K \cap \mathfrak{m}P)/\mathfrak{m}K$. By the Artin-Rees lemma there exists j_0 such that for all $j \geq j_0$ we have

$$K \cap \mathfrak{m}^j P = \mathfrak{m}^{j-j_0}(K \cap \mathfrak{m}^{j_0}P).$$

For any ℓ with $q^\ell > j_0$ we then have

$$\tau^\ell(K \cap \mathfrak{m}P) \subset K \cap \mathfrak{m}^{q^\ell} P = \mathfrak{m}^{q^\ell - j_0}(K \cap \mathfrak{m}^{j_0} P) \subset \mathfrak{m}K.$$

Thus the endomorphism of $(K \cap \mathfrak{m}P)/\mathfrak{m}K$ induced by τ is nilpotent, as desired. $\square$

Remark 5.5. The analogous statement for quasi-crystals is false. An example is given in [8, Rem. 4.1.8]

Lemma 5.6. *If X is affine, every coherent τ-sheaf on X is the quotient of a coherent τ-sheaf whose underlying coherent sheaf is free.*

Proof. Suppose that $X = \operatorname{Spec} R$ and let M be the $(R \otimes A)[\tau]$-module corresponding to a coherent τ-sheaf on X. As M is of finite type over $R \otimes A$, we may write it as a quotient of a free module of finite type $N := (R \otimes A)^r$. Since N is free, the semi-linear endomorphism τ of M can be lifted to a semi-linear endomorphism of N. $\square$

Using Theorem 5.4, the following assertion can be reduced to the case of fields where it is rather obvious:

Theorem 5.7. *The functor $f^* \colon \mathbf{Crys}(X, A) \longrightarrow \mathbf{Crys}(Y, A)$ is exact.*

One also has the following more difficult result regarding stalks:

Theorem 5.8. *Suppose that X is of finite type over k. Then the following hold:*

(a) For any $\underline{\mathcal{F}} \in \mathbf{Crys}(X, A)$, its crystalline support $\text{Crys-Supp}(\underline{\mathcal{F}}) := \{x \in X \mid i_x^* \underline{\mathcal{F}} \neq 0$ *in* $\mathbf{Crys}\}$ *is a constructible subset of X.*

(b) A crystal $\underline{\mathcal{F}} \in \mathbf{Crys}(X, A)$ is zero if and only if the crystals $i_x^* \underline{\mathcal{F}}$ are zero for all $x \in X$.

2 Further functors deduced from functors on quasi-coherent sheaves

Tensor product: The assignment $(\underline{\mathcal{F}}, \underline{\mathcal{G}}) \mapsto (\mathcal{F} \otimes_{\mathcal{O}_{X \otimes C}} \mathcal{G}, \tau_{\mathcal{F}} \otimes \tau_{\mathcal{G}})$ with the usual tensor product of homomorphisms defines an A-bilinear bi-functor

$$\otimes : \ \mathbf{QCoh}_\tau(X, A) \times \mathbf{QCoh}_\tau(X, A) \longrightarrow \mathbf{QCoh}_\tau(X, A)$$

which is right exact in both variables. Its exactness properties are governed by associated Tor-objects.

Coefficient change: For any homomorphism $h \colon A \to A'$, the assignment $\underline{\mathcal{F}} \mapsto (\mathcal{F} \otimes_A A', \tau_{\mathcal{F}} \otimes_A \mathrm{id}_{A'})$ with the usual change of coefficients of homomorphisms defines an A-bilinear functor

$$\otimes_A A' : \ \mathbf{QCoh}_\tau(X, A) \longrightarrow \mathbf{QCoh}_\tau(X, A')$$

which is right exact. Its exactness properties are again governed by associated Tor-objects.

Direct image: Consider a morphism $f : Y \to X$ and $\underline{\mathcal{F}} \in \mathbf{QCoh}_\tau(Y, A)$. Using $\sigma^* f_* \to f_* \sigma^*$ deduced from adjunction of inverse and direct image, one obtains a functorial assignment

$$\underline{\mathcal{F}} \mapsto ((f \times \mathrm{id})_* \mathcal{F}, \tau \text{ induced from} (f \times \mathrm{id})_* \tau_{\mathcal{F}}).$$

With the usual direct image of homomorphisms this defines an A-linear functor

$$f_* : \mathbf{QCoh}_\tau(Y, A) \longrightarrow \mathbf{QCoh}_\tau(X, A')$$

which is left exact. Its exactness properties are governed by associated higher derived images.

The above three functors all preserve nil-isomorphisms and thus pass to functors on crystals.

One has the following remarkable property which also explains the term *crystal*, describing something which does not deform (note that the canonical morphism $X_{\text{red}} \to X$ is finite radicial and surjective):

Theorem 5.9. *If f is finite radicial and surjective, the adjunction homomorphism* $\mathrm{id} \to f_* f^*$ *is an isomorphism in* $\mathbf{QCrys}(X, A)$ *and the functors*

$$\mathbf{QCrys}(X,A) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathbf{QCrys}(Y,A)$$

are mutually quasi-inverse equivalences of categories.

3 Extension by zero

Let $j\colon U \hookrightarrow X$ be an open immersion and $i\colon Z \hookrightarrow X$ be a closed complement with ideal sheaf $\mathcal{I}_0$.

The following is the main result:

Theorem 5.10. *(a) For any crystal* $\underline{\mathcal{F}}$ *on* U *there exists a crystal* $\underline{\widetilde{\mathcal{F}}}$ *on* X*, such that* $j^*\underline{\widetilde{\mathcal{F}}} \cong \underline{\mathcal{F}}$*, and* $i^*\underline{\widetilde{\mathcal{F}}} = 0$ *in* $\mathbf{Crys}(Z, A)$*.*

(b) The pair in (a) consisting of $\underline{\widetilde{\mathcal{F}}}$ *and the isomorphism* $j^*\underline{\widetilde{\mathcal{F}}} \cong \underline{\mathcal{F}}$ *is unique up to unique isomorphism; it depends functorially on* $\underline{\mathcal{F}}$*.*

(c) For any $\underline{\mathcal{F}}$ *and* $\underline{\widetilde{\mathcal{F}}}$ *as in (a) and any quasi-crystal* $\underline{\widetilde{\mathcal{G}}}$ *on* X*, inverse image under* j *induces a bijection*

$$j^* : \ \mathrm{Hom}_{\mathbf{QCrys}}(\underline{\widetilde{\mathcal{F}}}, \underline{\widetilde{\mathcal{G}}}) \longrightarrow \mathrm{Hom}_{\mathbf{QCrys}}(\underline{\mathcal{F}}, j^*\underline{\widetilde{\mathcal{G}}}).$$

(d) The assignment $\underline{\mathcal{F}} \mapsto \underline{\widetilde{\mathcal{F}}}$ *with* $\underline{\widetilde{\mathcal{F}}}$ *from (a) defines an A-linear functor* extension by zero

$$j_!\colon \mathbf{Crys}(U, A) \longrightarrow \mathbf{Crys}(X, A),$$

One should be aware that $j_!$ is *not* induced from a functor of coherent τ-sheaves, because in general a homomorphism $j_!\underline{\mathcal{F}} \dashrightarrow j_!\underline{\mathcal{G}}$ in $\mathbf{Crys}(X, A)$ lifts to a homomorphism in $\mathbf{Coh}_\tau(X, A)$ only after $\underline{\mathcal{F}}$ or $\underline{\mathcal{G}}$ is replaced by a nil-isomorphic τ-sheaf.

Remarks 5.11. Property (c) is the expected universal property of extension by zero.

Property (a) is technically a very simple characterization of $j_!\underline{\mathcal{F}}$.

From (a) one easily deduces $L(U, \underline{\mathcal{F}}, T) = L(X, j_!\underline{\mathcal{F}}, T)$ whenever the left hand side is defined.

Proof of Theorem 5.10 (a). Let $\underline{\mathcal{F}}$ be a τ-sheaf representing the same-named crystal. The first step is to construct a coherent extension of $\mathcal{F}$ to $X \times C$. This is standard, e.g. [10, n°1 Cor. 2] or [27, Ch. II Ex. 5.15]. For the convenience of the reader, we repeat the short argument.

Observe that $(j \times \mathrm{id})_*\mathcal{F}$ is a quasi-coherent extension of $\mathcal{F}$ to $X \times C$. Thus we can write $(j \times \mathrm{id})_*\mathcal{F}$ as a filtered direct limit $\varinjlim_{i\in I} \mathcal{F}_i$ over its coherent subsheaves $\mathcal{F}_i$ (with no τ). It follows that $\mathcal{F} \cong \varinjlim_{i\in I} j^*\mathcal{F}_i$. As $\mathcal{F}$ is coherent and the $j^*\mathcal{F}_i$ are still filtered, there exists an i such that $\mathcal{F} = j^*\mathcal{F}_i$. Thus $\widetilde{\mathcal{F}} := \mathcal{F}_i$ is a coherent extension of $\mathcal{F}$.

Next we wish to extend τ. As $\widetilde{\mathcal{F}} \subset (j \times \mathrm{id})_*\mathcal{F}$, the homomorphism $\tau_{\mathcal{F}}$ yields a homomorphism

$$\tau\colon (\sigma \times \mathrm{id})^*\widetilde{\mathcal{F}} \longrightarrow (j \times \mathrm{id})_*\mathcal{F}.$$

We would like the morphism to factor via $\widetilde{\mathcal{F}}$. Consider the image of $(\sigma \times \mathrm{id})^*\widetilde{\mathcal{F}}$ in the (quasi-coherent) quotient sheaf $(j \times \mathrm{id})_*\mathcal{F}/\widetilde{\mathcal{F}}$. Being the image of a coherent sheaf, it is itself coherent. Since it also vanishes on $U \times C$, it is annihilated by $\mathcal{I}_0^n$ for some integer $n \geq 0$. In other words, we have

$$\mathcal{I}_0^n \tau\big((\sigma \times \mathrm{id})^*\widetilde{\mathcal{F}}\big) \subset \widetilde{\mathcal{F}}.$$

Select an integer m with $(q-1)m > n$. Since $\sigma^*\mathcal{I}_0 \subset \mathcal{I}_0^q$, we can calculate

$$\begin{array}{rcl} \tau\big((\sigma \times \mathrm{id})^*(\mathcal{I}_0^m \widetilde{\mathcal{F}})\big) & \subset & \mathcal{I}_0^{qm}\tau\big((\sigma \times \mathrm{id})^*\widetilde{\mathcal{F}}\big) \\ & \subset & \mathcal{I}_0^{qm-n}\widetilde{\mathcal{F}} \\ & \subset & \mathcal{I}_0\big(\mathcal{I}_0^m\widetilde{\mathcal{F}}\big). \end{array}$$

Thus after replacing $\widetilde{\mathcal{F}}$ by $\mathcal{I}_0^m\widetilde{\mathcal{F}}$ the homomorphism $\tau_{\mathcal{F}}$ extends to a homomorphism $(\sigma \times \mathrm{id})^*\widetilde{\mathcal{F}} \longrightarrow \mathcal{I}_0\widetilde{\mathcal{F}}$. Let $\underline{\widetilde{\mathcal{F}}}$ be the corresponding crystal on X. Then the first condition of (a) holds by construction, and the second follows from the fact that $\tau_{i^*\widetilde{\mathcal{F}}}$ vanishes.

For the proof of the remaining assertions we refer to [8]. $\square$

Example 5.12. Let $X = \mathbb{A}^1 = \operatorname{Spec} k[\theta]$ and $C = \mathbb{A}^1 = \operatorname{Spec} k[t]$. Let $\underline{\mathcal{C}}$ denote the Carlitz τ-sheaf on X over A. Its underlying module is $M = \mathbb{F}_q[\theta, t]$, the endomorphism $\tau\colon M \to M$ is given by $(t-\theta)(\sigma \times \mathrm{id})$. For $j\colon \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$ we wish to determine $j_!\underline{\mathcal{C}}^{\otimes n}$ for $n \in \mathbb{N}$.

For any $m \in \mathbb{Z}$ define $\mathcal{F}_m := \mathcal{O}_{\mathcal{P}^1}(m\infty) \otimes A$ on $X \times C$. If $\mathcal{O} := \mathcal{O}_{\mathcal{P}^1}(-1)$ denotes the ideal sheaf of ∞, then $\mathcal{F}_m = \mathcal{I}_0^{-m}\mathcal{F}_0$.

We consider $\mathcal{F}_m$ near ∞, more concretely on $\mathbb{P}^1 \smallsetminus \{0\}$. Here

$$\Gamma\big(\operatorname{Spec} \mathbb{F}_q\big[\frac{1}{\theta}, t\big], \mathcal{F}_m\big) = \theta^m \mathbb{F}_q\big[\frac{1}{\theta}, t\big].$$

(It would suffice to consider a formal neighborhood of ∞. But notationally it is actually simpler to consider $\mathbb{P}^1 \smallsetminus \{0\}$.) On $\mathbb{P}^1 \smallsetminus \{0, \infty\}$, we have

$$\theta^m f\big(\frac{1}{\theta}, t\big) \xmapsto{\tau^{\otimes n}} (t-\theta)^n \theta^{qm} f\big(\frac{1}{\theta^q}, t\big) = \theta^{(q-1)m+n}\theta^m\Big(\frac{t}{\theta} - 1\Big)^n f\big(\frac{1}{\theta^q}, t\big).$$

For the right hand expression to lie in $\theta^m\mathbb{F}_q\big[\frac{1}{\theta}, t\big]$, one needs $(q-1)m + n \leq 0$. For it also to be zero at ∞ one requires $(q-1)m + n < 0$. This leads to

$$m < \frac{-n}{q-1}.$$

Chapter 6

Derived categories and derived functors

In [8] we carefully develop derived categories of τ-sheaves and (quasi-)crystals and derived functors between such categories. For lack of time, this cannot be exposed in the present lecture series. Sot I shall confine myself to point to some issues which led in [8] to work out the derived setting in great detail:

Derived categories are the appropriate setting for derived functors. This is not particular to the case at hand. It allows one to apply standard techniques of homological algebra to deduce consequences for derived functors. For instance, spectral sequences often arise from the universal properties of the involved derived functors.

Derived functors are the main reason for introducing large ambient categories such as $\mathbf{QCrys} \supset \mathbf{Crys}$. Only in this setting functors like Rf_* can be properly defined. Therefore only in this generality theorems like proper base change and the projection formula can be established.

To be more concrete let $f: Y \to X$ be a morphism Then one way of defining Rf_* is via Čech resolutions. Even if the initial object resolved is coherent, the objects of the resolution are only quasi-coherent. – The same holds if one works with injective resolutions. – Thus a priori one only obtains a functor

$$Rf_*\colon \mathbf{D}^*(\mathbf{QCrys}(Y, A)) \longrightarrow \mathbf{D}^*(\mathbf{QCrys}(X, A)).$$

For proper f it maps $\mathbf{D}^*(\mathbf{Crys}(Y, A))$ as well as $\mathbf{D}^*_{\mathrm{crys}}(\mathbf{QCrys}(Y, A))$ to $\mathbf{D}^*_{\mathrm{crys}}(\mathbf{QCrys}(X, A))$. Thus an important theorem will be the equivalence of derived categories

$$\mathbf{D}^*(\mathbf{Crys}(X, A)) \longrightarrow \mathbf{D}^*_{\mathrm{crys}}(\mathbf{QCrys}(X, A)) \tag{1}$$

which holds for $* \in \{b, -\}$. In the proper case it allows one to deduce, again for $* \in \{b, -\}$, a functor

$$Rf_*\colon \mathbf{D}^*(\mathbf{Crys}(Y, A)) \longrightarrow \mathbf{D}^*(\mathbf{Crys}(X, A)).$$

Note that in the present situation the equivalence (1) is not a simple formal result as it is in the case of quasi-coherent sheaves without an endomorphism, see [26]. The reason is that not all quasi-crystals are direct limits of crystals.

We defined crystals from τ-sheaves by a localization procedure. But one also defines derived categories from homotopy categories of complexes by a localization procedure. Thus for the theory of derived categories of quasi-crystals it is important to know that the following functor is an equivalence

$$\mathcal{S}_{\mathrm{nilqi}}^{-1}\,\mathbf{D}^*(\mathbf{QCoh}_\tau) = \mathcal{S}_{\mathrm{nilqi}}^{-1}\,\mathbf{K}^*(\mathbf{QCoh}_\tau) \to \mathcal{S}_{\mathrm{qi}}^{-1}\,\mathbf{K}^*(\mathcal{S}_{\mathrm{nil}}^{-1}\,\mathbf{QCoh}_\tau) = \mathcal{S}_{\mathrm{qi}}^{-1}\,\mathbf{K}^*(\mathbf{QCrys}).$$

Another important issue concerns the computation of the derived functors Lf^* (on quasi-crystals) and Rf_* (e.g. for f proper). We define these derived functor on derived categories of an abelian category whose objects are pairs of a sheaf and an endomorphism. However we do prove that one can compute these objects by computing the derived functors on the underlying categories of sheaves and then adding the induced endomorphism which can also be obtained from the derived functors on sheaves without endomorphisms. For further results we refer to [8].

We conclude this section by collecting the main results on the existence of derived functors: Let $f\colon Y \to X$ be any morphism and denote by $j\colon U \hookrightarrow X$ an open immersion.

Theorem 6.1. *The exact functor f^* on* **Crys** *induces for any $* \in \{b, +, -, \varnothing\}$ an exact functor*

$$f^*\colon \mathbf{D}^*(\mathbf{Crys}(X, A)) \longrightarrow \mathbf{D}^*(\mathbf{Crys}(Y, A)).$$

The functor f^* on **QCrys** *induces a left derived functor*

$$Lf^*\colon\ \mathbf{D}^-(\mathbf{QCrys}(X, A)) \longrightarrow \mathbf{D}^-(\mathbf{QCrys}(Y, A))$$

via resolutions inside $\mathbf{D}^-(\mathbf{QCrys}(X, A))$. The second functor if restricted to **Crys** *agrees with the first.*

In the following section on flatness we shall define flat crystals and very flat quasi-crystals. In Corollary 7.6 we shall see that the category $\mathbf{D}^-(\mathbf{Crys}(X, A))$ has enough flat objects. They are not flat within $\mathbf{D}^-(\mathbf{QCrys}(X, A))$. Nevertheless we have:

Theorem 6.2. *Constructed via flat resolutions within $\mathbf{D}^-(\mathbf{Crys}(X, A))$, the bi-functor $\otimes$ on crystals possesses a left bi-derived functor*

$$\overset{L}{\otimes}\colon\ \mathbf{D}^-(\mathbf{Crys}(X, A)) \times \mathbf{D}^-(\mathbf{Crys}(X, A)) \longrightarrow \mathbf{D}^-(\mathbf{Crys}(X, A)).$$

Constructed via very flat resolutions within $\mathbf{D}^-(\mathbf{QCrys}(X, A))$, the bi-functor $\otimes$ on quasi-crystals possesses a left bi-derived functor

$$\overset{L}{\otimes}\colon\ \mathbf{D}^-(\mathbf{QCrys}(X, A)) \times \mathbf{D}^-(\mathbf{QCrys}(X, A)) \longrightarrow \mathbf{D}^-(\mathbf{QCrys}(X, A)).$$

The second functor if restricted to crystals agrees with the first one.

Theorem 6.3. *Let $A \to A'$ denote a homomorphism of coefficient rings. Constructed via flat resolutions within $\mathbf{D}^-(\mathbf{Crys}(X, A))$, the functor $\_\_ \otimes_A A'$ on A-crystals possesses a left derived functor*

$$\_\_ \overset{L}{\underset{A}{\otimes}} A'\colon \mathbf{D}^-(\mathbf{Crys}(X, A)) \longrightarrow \mathbf{D}^-(\mathbf{Crys}(X, A')).$$

Constructed via very flat resolutions within $\mathbf{D}^-(\mathbf{QCrys}(X, A))$, the functor $\_\_ \otimes_A A'$ on A-quasi-crystals possesses a left derived functor

$$\_\_ \overset{L}{\underset{A}{\otimes}} A'\colon \mathbf{D}^-(\mathbf{QCrys}(X, A)) \longrightarrow \mathbf{D}^-(\mathbf{QCrys}(X, A')).$$

The second functor if restricted to crystals agrees with the first one.

Theorem 6.4. *For any $* \in \{b, +, -, \varnothing\}$ the functor f_* possesses a right derived functor*

$$Rf_*\colon \mathbf{D}^*(\mathbf{QCrys}(Y, A)) \longrightarrow \mathbf{D}^*(\mathbf{QCrys}(X, A)).$$

It can be defined via Čech resolutions. When f is proper, the subcategory $\mathbf{D}^*_b(\mathbf{QCrys}(\_\_, A))$ is preserved under f_* and thus by (1) it induces a functor

$$Rf_*\colon \mathbf{D}^*(\mathbf{Crys}(Y, A)) \longrightarrow \mathbf{D}^*(\mathbf{Crys}(X, A)).$$

for $* \in \{b, -\}$.

Since $j_!$ is exact on crystals it clearly induces an exact functor

$$j_! \colon \mathbf{D}^*(\mathbf{Crys}(U,A)) \longrightarrow \mathbf{D}^*(\mathbf{Crys}(X,A)).$$

Combined with Rf_* and using Nagata's theorem that any morphism f of finite type has a relative compactification, i.e., that it lies in a commutative diagram

$$\begin{tikzcd} Y \arrow[r, hook, "j"] \arrow[d, "f"'] & \bar{Y} \arrow[dl, "\bar{f}"] \\ X & \end{tikzcd} \tag{2}$$

where j is an open embedding and $\bar{f}$ is proper, one obtains the following.

Theorem 6.5. *For any such diagram and any $* \in \{b, -\}$ one defines an exact functor* direct image with proper support

$$Rf_! := R\bar{f}_* \circ j_! \colon \mathbf{D}^*(\mathbf{Crys}(Y,A)) \longrightarrow \mathbf{D}^*(\mathbf{Crys}(X,A)), \tag{3}$$

which, by a standard procedure, is independent of the chosen compactification. It is compatible with the composition of morphisms and it satisfies the proper base change theorem and the projection formula, cf. [8, S 6.7].

We end this section by stating two main theorems on the above derived functors: For the first we consider a cartesian diagram

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"'] & X. \end{tikzcd} \tag{4}$$

Adjunction between direct image and inverse image yields a natural transformation

$$Lg^* Rf_* \longrightarrow Rf'_* Lg'^* \tag{5}$$

called the *base change* homomorphism.

Theorem 6.6 (Proper Base Change)**.** *In the cartesian diagram 4 assume that f is compactifiable. Then f' is compactifiable and there is a natural isomorphism of functors $g^* Rf_! \cong Rf'_! g'^*$.*

Theorem 6.7 (Projection Formula)**.** *For compactifiable $f : Y \to X$ there is a natural isomorphism of functors $Rf_! \_\_ \overset{L}{\otimes} \_\_ \cong Rf_! \big(\_\_ \overset{L}{\otimes} f^* \_\_\big)$.*

Chapter 7

Flatness

The present chapter is mainly preparatory for the following one. There we shall define (naive and crystalline) L-functions. Their definition requires some hypotheses on the underlying crystals. This is to be expected since given an arbitrary A-module with an endomorphism there need not be a well-defined characteristic polynomial of such an endomorphism taking values in the polynomial ring over A. A sufficient condition is that the underlying module is free over A. In the context of crystals it turns out that the proper setting to define L-functions (at least over good coefficient rings, cf. 8.7) is that of flat crystals. Flat crystals are the theme of the present chapter. In Section 1 we give their definition and discuss some basic results, in Section 2 their behavior under all (derived) functors defined so far is studied and in Section 3 we give a partial answer to the question to what extent a flat crystal is representable by a locally free τ-sheaf.

1 Basics on flatness

Proposition 7.1. *The following properties for $\underline{\mathcal{F}} \in \mathbf{Crys}(X, A)$ are equivalent:*

(a) The functor $\underline{\mathcal{F}} \otimes \_\_ \colon \mathbf{Crys}(X, A) \to \mathbf{Crys}(X, A)$ is exact.

(b) $\mathrm{Tor}_i(\underline{\mathcal{F}}, \underline{\mathcal{G}}) = 0$ for all $i \geq 1$ and all $\underline{\mathcal{G}} \in \mathbf{Crys}(X, A)$.

(b) $\mathrm{Tor}_1(\underline{\mathcal{F}}, \underline{\mathcal{G}}) = 0$ for all $\underline{\mathcal{G}} \in \mathbf{Crys}(X, A)$.

Definition 7.2. If any of the above conditions are satisfied, then $\underline{\mathcal{F}}$ is called a *flat A-crystal*.

A quasi-crystal $\underline{\mathcal{G}}$ is called *very flat* if the functor $\underline{\mathcal{G}} \otimes \_\_ \colon \mathbf{QCrys}(X, A) \to \mathbf{QCrys}(X, A)$ is exact.

As remarked earlier, flat crystals need not be very flat. In the sequel we shall exclusively consider flat crystals. We introduced very flat ones only for completeness sake. Their main use is to provide $\otimes$-acyclic resolutions within **QCrys**.

Flatness of crystals is a pointwise property:

Proposition 7.3. *A crystal $\underline{\mathcal{F}}$ on X is flat if and only if $i_x^* \underline{\mathcal{F}}$ is flat for every $x \in X$.*

Proof. For any $x \in X$ and any $\underline{\mathcal{G}} \in \mathbf{Crys}(X, A)$, one has

$$i_x^* \, \mathrm{Tor}_1\big(\underline{\mathcal{F}}, \underline{\mathcal{G}}\big) \cong \mathrm{Tor}_1\big(i_x^* \underline{\mathcal{F}}, i_x^* \underline{\mathcal{G}}\big).$$

The assertion follows easily. □

Definition 7.4. A τ-sheaf is called of *pullback type*, if its underlying sheaf is a pullback from the first factor. A crystal is called of *pullback type*, if it has a representing τ-sheaf with this property.

Example 7.5. We can now give some examples:

- Any crystal represented by a locally free τ-sheaf is flat.
- Any crystal of pullback type $\underline{\mathcal{F}}$ is flat: This follows from Proposition 7.3 since for any $x \in X$ the τ-sheaf $i_x^*\underline{\mathcal{F}}$ is then the pullback of a vector space on the residue field k_x of x and thus free.
- If $\underline{\mathcal{F}}$ is a flat crystal on U, then $j_!\underline{\mathcal{F}}$ is a flat crystal on X.

Corollary 7.6. *The category* $\mathbf{Crys}(X, A)$ *possesses enough flat objects.*

Proof. Let $\mathfrak{U} = \{U_i\}$ denote a finite affine cover of X and let $j_i : U_i \to X$ the open embedding for i. Then for any $\underline{\mathcal{F}} \in \mathbf{Crys}(X, A)$, the natural homomorphism $\oplus_i j_{i!}\underline{\mathcal{F}}|_{U_i} \to \underline{\mathcal{F}}$ is surjective. Thus it suffices to prove that each $j_{i!}\underline{\mathcal{F}}|_{U_i}$ is the image of a flat crystal. Because $j_{i!}$ preserves flatness and is exact, in turn it suffices that each $\underline{\mathcal{F}}|_{U_i}$ is the image of a flat crystal. But this is immediate from Lemma 5.6. □

Without proof we state the following result:

Theorem 7.7. *A crystal* $\underline{\mathcal{F}}$ *is flat if and only if for all* $c \in C$ *and all* $i \geq 1$ *one has* $\mathrm{Tor}_i(\underline{\mathcal{F}}, k_c) = 0$.

An immediate corollary is the following:

Corollary 7.8. *If A is a field, then any A-crystal is flat.*

2 Flatness under functors

Proposition 7.9. *Let* $j : U \hookrightarrow X$ *be an open immersion. If* $\underline{\mathcal{F}}$ *and* $\underline{\mathcal{G}}$ *are flat crystals, then so are*

(a) $f^*\underline{\mathcal{F}}$,

(b) $\underline{\mathcal{F}} \otimes \underline{\mathcal{G}}$,

(c) $\underline{\mathcal{F}} \otimes_A A'$,

(d) $j_!\underline{\mathcal{F}}$.

Except for (a) all parts are rather straightforward. For (a) one may use Theorem 7.7.

Theorem 7.10. *Let* $f : Y \to X$ *be a proper morphism. Suppose* $\underline{\mathcal{F}}^\bullet \in \mathbf{C}^b(\mathbf{Crys}(Y, A))$ *is quasi-isomorphic to a bounded complex of flat crystals. Then so is* $Rf_*\underline{\mathcal{F}}^\bullet$ *within* $\mathbf{C}^b(\mathbf{Crys}(X, A))$.

The following result shows that for regular coefficient rings all objects possess flat resolutions:

Theorem 7.11. *Suppose A is regular. Then any* $\underline{\mathcal{F}}^\bullet \in \mathbf{C}^b(\mathbf{Crys}(Y, A))$ *is quasi-isomorphic to a bounded complex of flat crystals.*

Remark 7.12. Crystals of pullback type are particular examples of flat crystals. For them it is particularly easy to show that they are preserved under all of our functors, including $R^i f_*$. It is however not clear whether the complex $Rf_*\underline{\mathcal{F}}$ of a crystal $\underline{\mathcal{F}}$ of pullback type is representable by a complex in $\mathbf{D}^b(\mathbf{Crys}(X, A))$ all of whose objects are of pullback type.

3 Representability of flat crystals

It is a basic question to what extent flat crystals are representable by coherent τ-sheaves whose underlying sheaves are flat over $\mathcal{O}_X \otimes A$. The following results given without proof provide partial answers:

Proposition 7.13. *Suppose that $X = \operatorname{Spec} F$ for a field F and that C is regular of dimension ≤ 1. Then every flat A-crystal on X can be represented by a coherent τ-sheaf whose underlying sheaf is free.*

Theorem 7.14. *Suppose that X is reduced of dimension ≤ 1 and that A is artinian. Then for every flat A-crystal $\underline{\mathcal{F}}$ on X there exists an open dense embedding $j\colon U \hookrightarrow X$ such that $j^*\underline{\mathcal{F}}$ can be represented by a coherent τ-sheaf whose underlying sheaf is free.*

A particularly important result in relations to L-functions is Proposition 7.16 which provides for $x = \operatorname{Spec} k_x$ with k_x a finite field and A artinian a canonical locally free representative of a flat crystal:

Definition 7.15. A τ-sheaf $\underline{\mathcal{F}}$ is called *semisimple* if $\tau_{\mathcal{F}}\colon (\sigma \times \mathrm{id})^*\mathcal{F} \to \mathcal{F}$ is an isomorphism.

Proposition 7.16. *Let $x = \operatorname{Spec} k_x$ with k_x a finite field, let A be artinian and consider $\underline{\mathcal{F}}$, $\underline{\mathcal{G}} \in \mathbf{Coh}_\tau(x, A)$.*

(a) There exists a unique direct sum decomposition $\underline{\mathcal{F}} = \underline{\mathcal{F}}_{\mathrm{ss}} \oplus \underline{\mathcal{F}}_{\mathrm{nil}}$ such that $\underline{\mathcal{F}}_{\mathrm{ss}}$ is semisimple and $\underline{\mathcal{F}}_{\mathrm{nil}}$ is nilpotent. The summands are called the semisimple part *and the* nilpotent part *of $\underline{\mathcal{F}}$, respectively.*

(b) The decomposition in (a) is functorial in $\underline{\mathcal{F}}$.

(c) Any nil-isomorphism $\underline{\mathcal{F}} \to \underline{\mathcal{G}}$ induces an isomorphism $\underline{\mathcal{F}}_{\mathrm{ss}} \to \underline{\mathcal{G}}_{\mathrm{ss}}$.

(d) The construction induces a functor $\mathbf{Crys}(x, A) \to \mathbf{Coh}_\tau(x, A) : \underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\mathrm{ss}}$.

Proof. Note first that $\tau_{\mathcal{F}}^{d_x}$ is an endomorphism $\mathcal{F} \to \mathcal{F}$, because σ^{d_x} is the identity on k_x. Moreover $\mathcal{F}$ has finite length, because $k_x \otimes A$ is artinian. Thus for $n \gg 0$ the subsheaves $\mathcal{F}_{\mathrm{ss}} := \operatorname{Im} \tau_{\mathcal{F}}^{nd_x}$ and $\mathcal{F}_{\mathrm{nil}} := \operatorname{Ker} \tau_{\mathcal{F}}^{nd_x}$ are independent of n. Clearly $\tau_{\mathcal{F}}$ maps them to themselves, so they define τ-subsheaves $\underline{\mathcal{F}}_{\mathrm{ss}}$ and $\underline{\mathcal{F}}_{\mathrm{nil}}$. By construction $\tau_{\mathcal{F}_{\mathrm{ss}}}$ is surjective and $\tau_{\mathcal{F}_{\mathrm{nil}}}$ nilpotent. Since $\mathcal{F}$ and hence $\mathcal{F}_{\mathrm{ss}}$ has finite length, $\tau_{\mathcal{F}_{\mathrm{ss}}}$ is then also injective. Thus $\underline{\mathcal{F}}_{\mathrm{ss}}$ is semisimple and $\underline{\mathcal{F}}_{\mathrm{nil}}$ nilpotent. Furthermore, the construction yields a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}_{\mathrm{nil}} \arrow[r] & \mathcal{F} \arrow[r, "\tau_{\mathcal{F}}^{nd_x}"] & \mathcal{F}_{\mathrm{ss}} \arrow[r] & 0. \\ & & \mathcal{F}_{\mathrm{ss}} \arrow[u, phantom, "\cup"] \arrow[ur, "\cong"'] & & \end{tikzcd}$$

This shows that $\underline{\mathcal{F}} = \underline{\mathcal{F}}_{\mathrm{ss}} \oplus \underline{\mathcal{F}}_{\mathrm{nil}}$, proving the existence part of (a). The uniqueness follows from the fact that any semisimple τ-subsheaf of $\underline{\mathcal{F}}$ is contained in $\underline{\mathcal{F}}_{\mathrm{ss}}$ and any nilpotent τ-subsheaf of $\underline{\mathcal{F}}$ is contained in $\underline{\mathcal{F}}_{\mathrm{nil}}$.

Part (b) follows directly from the above construction of $\mathcal{F}_{\mathrm{ss}}$ and $\mathcal{F}_{\mathrm{nil}}$. Also (b) implies that the kernel and cokernel of any homomorphism $\underline{\mathcal{F}}_{\mathrm{ss}} \to \underline{\mathcal{G}}_{\mathrm{ss}}$ have trivial nilpotent part. This implies (c) and hence (d). $\square$

Proposition 7.17. *Let $x = \operatorname{Spec} k_x$ with k_x a finite field and let A be artinian and consider $\underline{\mathcal{F}} \in \mathbf{Crys}(x, A)$.*

(a) $\underline{\mathcal{F}}_{\mathrm{ss}}$ is the unique semisimple τ-sheaf representing $\underline{\mathcal{F}}$.

(b) The functor $\mathbf{Crys}(x, A) \to \mathbf{Coh}_\tau(x, A) : \underline{\mathcal{F}} \mapsto \underline{\mathcal{F}}_{\mathrm{ss}}$ *is exact.*

(c) $\underline{\mathcal{F}}$ is flat if and only if the sheaf $\mathcal{F}_{\mathrm{ss}}$ underlying $\underline{\mathcal{F}}_{\mathrm{ss}}$ is locally free.

Proof. (a) follows from Proposition 7.16, especially from 7.16 (c). Part (b) is a consequence of 7.16 (b) and the fact that any exact sequence in $\mathbf{Crys}(x, A)$ is the image of an exact sequence in $\mathbf{Coh}_\tau(x, A)$.

For (c) note that one direction, the "if", is obvious. For the other direction, let $\underline{\mathcal{G}}$ be a representative of the crystal $\underline{\mathcal{F}}$ whose underlying sheaf is free. cf. Proposition 7.14. By Proposition 7.17(a) we have $\underline{\mathcal{G}} = \underline{\mathcal{G}}_{\mathrm{nil}} \oplus \underline{\mathcal{G}}_{\mathrm{ss}}$. Considering the underlying sheaves it follows that $\underline{\mathcal{G}}_{\mathrm{ss}}$ is locally free on $\mathcal{O}_{x \times C}$. By part (d) of the same proposition, we have $\underline{\mathcal{G}}_{\mathrm{ss}} = \underline{\mathcal{F}}_{\mathrm{ss}}$. $\square$

Chapter 8

The L-function

In this chapter we introduce two L-functions. A naive one for locally free τ-sheaves or τ-sheaves which are of pullback type, and a crystalline one for flat A-crystals, provided the ring A satisfies some mild hypotheses. The word naive simply refers to the fact that the naive L-function is not necessarily invariant under nil-isomorphism and thus it does not, in general, induce an L-function for crystals. In the last section we shall state a trace formula for naive L-functions and indicate some consequences for crystalline L-functions. The proof of the trace formula will be postponed to Chapter 9.

1 Naive L-functions

As a preparation we briefly recall without proof some basic properties of the dual characteristic polynomial for endomorphisms of projective modules.

Lemma–Definition 8.1. *Let A be a commutative ring, M a finitely generated projective A-module, and $\varphi\colon M \to M$ an A-linear endomorphism.*

(a) Let M' be any finitely generated projective A-module such that $M \oplus M'$ is free over A. Let $\varphi' : M' \to M'$ be the zero endomorphism. Then the following expression is independent of the choice of M':

$$\det_A\bigl(\mathrm{id} - t(\varphi \oplus \varphi') \bigm| M \oplus M'\bigr) \in 1 + tA[t]$$

From now on we simply write $\det_A(\mathrm{id} - t\varphi \,|\, M)$ for it and call it the dual characteristic polynomial of (M, φ).

(b) The assignment $(M, \varphi) \mapsto \det_A(\mathrm{id} - t\varphi \,|\, M)$ is multiplicative in short exact sequences.

Lemma 8.2. *Let A be an algebra over a field k. Let k' be a finite cyclic Galois extension of k of degree d, and σ a generator of $\mathrm{Gal}(k'/k)$. Let M be a finitely generated projective module over $k' \otimes_k A$ and $\varphi\colon M \to M$ an A-linear endomorphism satisfying $\varphi(xm) = {}^{\sigma}x \cdot \varphi(m)$ for all $x \in k'$ and $m \in M$. Then φ^d is $k' \otimes A$-linear and*

$$\det_A\bigl(\mathrm{id} - t\varphi \bigm| M\bigr) = \det_{k' \otimes A}\bigl(\mathrm{id} - t^d\varphi^d \bigm| M\bigr).$$

In particular, both sides lie in $1 + t^d A[t^d]$.

Now we return to τ-sheaves. Let X be a variety of finite type over k and let $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X, A)$ be either of pullback type, or locally free. For any $x \in |X|$, writing $\underline{\mathcal{F}}_x := i_x^*\underline{\mathcal{F}}$, Lemma 8.2 shows that

$$\det_A\bigl(\mathrm{id} - T\tau \bigm| \underline{\mathcal{F}}_x\bigr) \;=\; \det_{k_x \otimes A}\bigl(\mathrm{id} - T^{d_x}\tau^{d_x} \bigm| \underline{\mathcal{F}}_x\bigr) \;\in\; 1 + T^{d_x}A[T^{d_x}].$$

Definition 8.3. The *naive L-function of* $\underline{\mathcal{F}}$ over x is

$$L^{\text{naive}}(x, \underline{\mathcal{F}}, T) \;:=\; \det_A\bigl(\mathrm{id} - T\tau \bigm| \underline{\mathcal{F}}_x\bigr)^{-1} \;\in\; 1 + T^{d_x} A[[T^{d_x}]].$$

As the number of points in $|X|$ of any given degree d_x is finite, we can form the product over all x within $1 + tA[[T]]$:

Definition 8.4. The *naive L-function of* $\underline{\mathcal{F}}$ over X is

$$L^{\text{naive}}(X, \underline{\mathcal{F}}, T) \;:=\; \prod_{x \in |X|} L^{\text{naive}}(x, i_x^* \underline{\mathcal{F}}, T) \;\in\; 1 + TA[[T]].$$

In the special case where A is reduced, L^{naive} is invariant under nil-isomorphisms. For more general A, this property may fail.

2 Crystalline L-functions

We first assume that A is artinian.

Definition 8.5. The *(crystalline) L-function* of a flat crystal $\underline{\mathcal{F}}$ on x is

$$L^{\text{crys}}(x, \underline{\mathcal{F}}, T) \;:=\; L^{\text{naive}}(x, \underline{\mathcal{F}}_{\text{ss}}, T) \;\in\; 1 + T^{d_x} A[[T^{d_x}]].$$

Let now $\underline{\mathcal{F}}$ be a flat crystal on a scheme X of finite type over k. Since $L^{\text{crys}}(x, \underline{\mathcal{F}}, T) \in 1 + T^{d_x} A[[T^{d_x}]]$ for any $x \in |X|$, and the number of points of any given degree d_x is finite, we can again form the product over all x within $1 + TA[[T]]$:

Definition 8.6. The *(crystalline) L-function* of a flat crystal $\underline{\mathcal{F}}$ on X is

$$L^{\text{crys}}(X, \underline{\mathcal{F}}, T) \;:=\; \prod_{x \in |X|} L^{\text{crys}}(x, i_x^* \underline{\mathcal{F}}, T) \;\in\; 1 + TA[[T]].$$

If $\underline{\mathcal{F}}^\bullet$ is a bounded complex of flat crystals, one defines $L^{\text{crys}}(X, \underline{\mathcal{F}}^\bullet, T) \;:=\; \prod_{i \in \mathbb{Z}} L^{\text{crys}}(X, \underline{\mathcal{F}}^i, T)^{(-1)^i}$. If $\underline{\mathcal{F}}^\bullet$ is quasi-isomorphic to such a complex, we also use the notation $L^{\text{crys}}(X, \underline{\mathcal{F}}^\bullet, T)$ and mean by this the L-function of that bounded complex of flat crystals. Using the cone construction it is not difficult to show that this is well-defined, i.e., that quasi-isomorphic bounded complexes of flat crystals have the same L-function.

We now relax our condition on A and only require that A be a *good coefficient ring*. In Remark 8.8 we give several examples of classes of rings A that are good coefficient rings. The reason for introducing this notion is that for such A one can define an L-function as follows: Changing coefficients from A to its quotient ring Q_A one obtains from a flat crystal on A a flat crystal over the artinian ring Q_A. For Q_A we know how to define a crystalline L-function. The property of being a good coefficient ring is then used to prove that the pointwise L-factors defined over Q_A have in fact coefficients in A and not just in Q_A.

We let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ denote the minimal primes of A and call

$$Q_A := A_{\mathfrak{p}_1} \oplus \ldots \oplus A_{\mathfrak{p}_n}$$

the *quotient ring* of A.

Definition 8.7. We call A a *good coefficient ring* if

(a) the natural homomorphism $A \to Q_A$ is injective, and

(b) A is closed under any taking finite ring extensions $A \hookrightarrow A' \hookrightarrow Q_A$ which induce bijections $\operatorname{Spec} A' \to \operatorname{Spec} A$ and isomorphisms on all residue fields.

Remark 8.8. Artin rings, regular rings and normal domains are examples of good coefficient rings. In particular, any finite ring and any Dedekind domain is a good coefficient ring.

The following result is a main result on good coefficient rings and the reason for their definition: Consider any flat $\underline{\mathcal{F}} \in \mathbf{Crys}(x, A)$ for $x = \operatorname{Spec} k_x$ for a finite extension k_x of k.

Lemma 8.9. *If A is a good coefficient ring, then*

$$L^{\mathrm{crys}}\big(x, \underline{\mathcal{F}} \underset{A}{\otimes} Q_A, T\big)^{-1} \in 1 + TQ_A[T]$$

has in fact coefficients in A.

The proof uses properties of flat A-crystals not developed in this lecture series therefore we refer to [8, §9.7].

Putting things together, we obtain.

Theorem 8.10. *If A is a good coefficient ring, then for any complex $\underline{\mathcal{F}}^\bullet \in \mathbf{C}^b(\mathbf{Crys}^{\mathrm{flat}}(X, A))$ the L-function $L^{\mathrm{crys}}(X, \underline{\mathcal{F}}^\bullet \overset{L}{\otimes}_A Q_A, T)$ has coefficients in A.*

Definition 8.11. If A is a good coefficient ring, the *(crystalline) L-function* of $\underline{\mathcal{F}}^\bullet \in \mathbf{C}^b(\mathbf{Crys}^{\mathrm{flat}}(X, A))_{\mathrm{ftd}}$ is

$$L^{\mathrm{crys}}(X, \underline{\mathcal{F}}^\bullet, T) \ := \ L^{\mathrm{crys}}\big(X, \underline{\mathcal{F}}^\bullet \underset{A}{\overset{L}{\otimes}} Q_A, T\big) \ \in \ 1 + TA[[T]].$$

One has the following elementary comparison theorem between naive and crystalline L-functions. (The asserted equality is true for all pointwise L-factors and can be proved there by passing from A to Q_A.):

Proposition 8.12. *Suppose A is a good coefficient ring. Suppose $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X, A)$ is a τ-sheaf of pullback type. Then both $L^{\mathrm{crys}}(X, \underline{\mathcal{F}}, T)$ and $L^{\mathrm{naive}}(X, \underline{\mathcal{F}}, T)$ are defined and elements of $1 + TA[[T]]$. If A is reduced then they agree.*

3 Trace formulas for L-functions

We now have the necessary definitions at our disposal to formulate the main results regarding trace formulas of τ-sheaves and crystals.

Theorem 8.13. *Let $X = \operatorname{Spec} R$ be affine and smooth of equidimension n over k and with structure morphism $s_X \colon X \to \operatorname{Spec} k$. Let A be artinian. Suppose the underlying sheaf of $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X, A)$ is free. Let $j \colon X \hookrightarrow \overline{X}$ be a compactification of X. Then there exists $\underline{\widetilde{\mathcal{F}}} \in \mathbf{Coh}_\tau(\overline{X}, A)$ of pullback type, such that*

(a) $\underline{\widetilde{\mathcal{F}}} = j_! \underline{\mathcal{F}}$ *in* $\mathbf{Crys}(\overline{X}, A)$.

(a) $H^i(\overline{X}, \underline{\widetilde{\mathcal{F}}})$ *is nilpotent for all* $i \neq n$.

(b) $L^{\mathrm{naive}}(X, \underline{\mathcal{F}}, T) = L^{\mathrm{naive}}(\operatorname{Spec} k, H^n(\overline{X}, \underline{\widetilde{\mathcal{F}}}), T)^{(-1)^n}$.

The result strongly relies on Anderson's trace formula which we state below in Theorem 9.13. It has some rather restrictive hypotheses on the base scheme and the sheaf underlying the given τ-sheaf. But formally it has the correct shape. The following result for crystals has hypotheses as general as can be expected. However the trace formula will, in general, not be an exact equality.

We write $\mathfrak{n}_A$ for the nilradical of A. For $f, g \in 1+TA[[T]]$ we define $f \sim g$ to mean that there exists $h \in 1+T\mathfrak{n}_A[T]$ such that $g = fh$. This defines an equivalence relation on $1 + TA[[T]]$.

Theorem 8.14. *For any morphism $f: Y \to X$ of schemes of finite type over k and any bounded complex $\underline{\mathcal{F}}^\bullet$ of flat A-crystals on Y we have*

$$L^{\mathrm{crys}}(Y, \underline{\mathcal{F}}^\bullet, T) \sim L^{\mathrm{crys}}(X, Rf_! \underline{\mathcal{F}}^\bullet, T).$$

Note that A is reduced if and only if $\mathfrak{n}_A = 0$. I.e., in this case $\sim$ becomes $=$.

As we shall explain later, cf. Chapter 11, an example of Deligne from the mid 1970's showed that one cannot expect a stronger result.

It is worthwhile to state explicitly the following corollary:

Corollary 8.15. *Let X be a scheme of finite type over k with structure morphism $s_X: X \to \operatorname{Spec} k$. Then for any bounded complex $\underline{\mathcal{F}}^\bullet$ of flat A-crystals on X we have*

$$L^{\mathrm{crys}}(X, \underline{\mathcal{F}}^\bullet, T) \sim L^{\mathrm{crys}}(\operatorname{Spec} k, Rf_! \underline{\mathcal{F}}^\bullet, T).$$

Since the complex $Rf_! \underline{\mathcal{F}}^\bullet \otimes_A Q_A$ can be represented by a bounded complex of free Q_A modules carrying some endomorphism, the right hand side is a rational function over A, and thus so is the left hand side.

For completeness, we also mention an important result on change of coefficients:

Theorem 8.16. *If both A and A' are good coefficient rings, then for any $\underline{\mathcal{F}}^\bullet \in \mathbf{C}^b(\mathbf{Crys}^{\mathrm{flat}}(X, A))$ we have*

$$L^{\mathrm{crys}}\big(X, \underline{\mathcal{F}}^\bullet \overset{L}{\underset{A}{\otimes}} A', T\big) \sim \lambda\big(L^{\mathrm{crys}}(X, \underline{\mathcal{F}}^\bullet, T)\big),$$

where $\lambda: 1 + TA[[T]] \to 1 + TA'[[T]]$ is induced from $A \to A'$. If moreover A is artinian, then equality holds.

Chapter 9

Proof of Anderson's trace formula and a cohomological interpretation

The aim of this section is to give a proof of Theorem 8.13 – at least under some simplifying hypotheses. In the end we shall briefly indicate how from this result one can hope to deduce Theorem 8.14, and that for general non-reduced A the proof of the latter result shall indeed require quite some more effort than we indicate here. The proof of the important Theorem 9.13 essentially goes back to the article [2] by G. Anderson.

Throughout this section, we let $X = \operatorname{Spec} R$ be an affine scheme which is smooth and of finite type over k. The ring A is an arbitrary fixed k-algebra.

1 The Cartier operator

Let $\Omega := \Omega_{R/k}$ be the module of Kähler differentials of R. Because R/k is smooth of equidimension n, it is a finitely generated projective R-module of rank n. Let $d\colon R \to \Omega$ denote the universal derivation as well as its extension to the de Rham complex $\bigwedge^\bullet \Omega$. The following result is due to Cartier, see [29, p.199-203].

Theorem 9.1 (Cartier)**.** *There exists an isomorphism of complexes with zero differential*

$$C^{-1}\colon (\textstyle\bigwedge^\bullet\Omega, 0) \xrightarrow{\cong} (H^\bullet(\textstyle\bigwedge^\bullet\Omega, d), 0)$$

such that for all $r \in R$ and $\xi, \eta \in \bigwedge^\bullet \Omega$ one has

$$\begin{aligned} C^{-1}(r\xi) &= r^p C^{-1}(\xi), \\ C^{-1}(dr) &= r^{p-1} dr + dR, \\ C^{-1}(\xi \wedge \eta) &= C^{-1}(\xi) \wedge C^{-1}(\eta). \end{aligned}$$

The inverse C of C^{-1} on the highest non-vanishing exterior power $\omega := \bigwedge^n \Omega$ is called the Cartier operator*.*

For $m := \log_p(\#k) = \log_p q$ we call the m-fold iterate

$$C_q = \underbrace{C \circ C \ldots \circ C}_{m}\colon \omega \longrightarrow \omega$$

the q-Cartier operator ($q = p^m$). It satisfies $C_q(r^q\xi) = rC(\xi)$, i.e., it is q^{-1} linear.

2 Cartier sheaves

Definition 9.2. A *Cartier linear endomorphism* of a coherent sheaf $\mathcal{V}$ on $X\times C$ is an $\mathcal{O}_{X\times C}$-linear homomorphism $\kappa_{\mathcal{V}}\colon (\sigma\times \mathrm{id})_*\mathcal{V}\longrightarrow \mathcal{V}$. The pair $\underline{\mathcal{V}} := (\mathcal{V},\kappa_{\mathcal{V}})$ is then called a *Cartier sheaf on X over A*. A homomorphism of Cartier sheaves $\underline{\mathcal{V}}\to\underline{\mathcal{W}}$ is a homomorphism of the underlying sheaves $\varphi\colon\mathcal{V}\to\mathcal{W}$ compatible with the extra endomorphism κ.

We denote the category of locally free Cartier sheaves on X (over A) by $\mathbf{Cart}^{\mathrm{locfree}}(X,A)$

As we assume that $X=\operatorname{Spec} R$ is affine, the above notions are expressed on modules as follows. A *Cartier linear map* on a finitely generated $R\otimes A$-module V is an A-linear homomorphism $\kappa_V : V\to V$ such that $\kappa_V((x^q\otimes a)v) = (x\otimes a)\cdot\kappa_V(v)$ for all $x\in R$, $a\in A$, and $v\in V$. For simplicity such a pair $\underline{V} := (V,\kappa_V)$ is called a *Cartier module.*

Definition 9.3. For any A define $\omega_{X,A}$ as the sheaf on $X\times C$ which is the pullback along $\mathrm{pr}_1\colon X\times C\to X$ of the invertible sheaf on X associated to the module ω. Denote by $\kappa_{X,A}\colon(\sigma\times\mathrm{id})_*\omega_{X,A}\to\omega_{X,A}$ the endomorphism induced from C_q under this pullback.

Example 9.4. The pair $(\omega_{X,A},\kappa_{X,A})$ is a Cartier sheaf on X over A.

If $\mathcal{F}$ is a coherent sheaf on $X\times C$, then $\mathcal{H}om_{\mathcal{O}_{X\times C}}(\mathcal{F},\omega_{X,A})$ is again a coherent sheaf on $X\times C$. Suppose now that $\mathcal{F}$ is the underlying sheaf of a locally free τ-sheaf $\underline{\mathcal{F}}$ and let $\tilde{\tau}_{\mathcal{F}}\colon\mathcal{F}\to(\sigma\times\mathrm{id})_*\mathcal{F}$ denote the homomorphism adjoint to $\tau_{\mathcal{F}}$. The dualizing sheaf ω together with C_q allows us to assign a locally free Cartier sheaf $D(\underline{\mathcal{F}})$ on X to $\underline{\mathcal{F}}$ as follows: Its underlying sheaf is

$$D(\mathcal{F}) := \mathcal{H}om_{\mathcal{O}_{X\times C}}(\mathcal{F},\omega_{X,A}).$$

For a section $(\sigma\times\mathrm{id})_*\varphi\in(\sigma\times\mathrm{id})_*\,\mathcal{H}om_{\mathcal{O}_{X\times C}}(\mathcal{F},\omega_{X,A})$, one defines

$$\kappa_{D(\mathcal{F})}((\sigma\times\mathrm{id})_*\varphi) := \kappa_{X,A}\circ(\sigma\times\mathrm{id})_*\varphi\circ\tilde{\tau}_{\mathcal{F}}.$$

Proposition 9.5. *The functor $\underline{\mathcal{F}}\mapsto D(\underline{\mathcal{F}})$ induces an anti-equivalence of categories*

$$\mathbf{Coh}_\tau^{\mathrm{locfree}}(X,A)\longrightarrow\mathbf{Cart}^{\mathrm{locfree}}(X,A).$$

For arbitrary smooth schemes of finite type over k an analog of Proposition 9.5 holds. The proof is a simple patching argument by which one is reduced to the affine case.

Proof. Well-definedness and injectivity are easily verified. The main point is to proof essential surjectivity. For this one needs the adjunction between σ_* and $\sigma^!$, cf. [27, Exercise III.6.10], for the finite flat morphism $\sigma\colon X\to X$ – the ring R is finitely generated and smooth over k. It yields

$$\mathcal{H}om_{\mathcal{O}_{X\times C}}((\sigma\times\mathrm{id})_*\mathcal{F},\mathcal{G})\cong(\sigma\times\mathrm{id})_*\,\mathcal{H}om_{\mathcal{O}_{X\times C}}(\mathcal{F},(\sigma\times\mathrm{id})^!\mathcal{G}).$$

Moreover the adjoint of the homomorphism $C_q : \sigma_*\omega\to\omega$ is an isomorphism $\omega\to\sigma^!\omega$. Details are left to the reader; cf. also [8, §7.2]. ☐

Example 9.6. Let $X=\operatorname{Spec} k[\theta]\cong\mathbb{A}^1_k$ and $C=\operatorname{Spec} k[t]\cong\mathbb{A}^1_k$ and consider the Carlitz τ-sheaf $\underline{\mathcal{C}}$ corresponding to $(k[\theta,t],(t-\theta)(\sigma\times\mathrm{id}))$.

In the case at hand $\omega=\Omega_{k[\theta]/k}=k[\theta]\mathrm{d}\theta$ and the Cartier operator C_q, which can be remembered via $C_q(\mathrm{d}\theta/\theta)=\mathrm{d}\theta/\theta$, has the following description:

$$C_q(\theta^\ell\,\mathrm{d}\theta)=C_q(\theta^{\ell+1}\,\mathrm{d}\theta/\theta)=\begin{cases}\theta^{(\ell+1-q)/q}\,\mathrm{d}\theta & \text{if } q|(\ell+1),\\ 0 & \text{else.}\end{cases}$$

To simplify notation, we define the expression θ^α for $\alpha\in\mathbb{Q}$ to mean 0 whenever $\alpha\in\mathbb{Q}\smallsetminus\mathbb{Z}$ and to mean the respective power for $\alpha\in\mathbb{Z}$.

We now determine $D(\underline{\mathcal{C}}^{\otimes n})$: The underlying module is $k[\theta,t]\,\mathrm{d}\theta = \mathrm{Hom}(k[\theta,t],\omega)$. Here an element $f(\theta,t)\,\mathrm{d}\theta$ represents the homomorphism which maps $1 \in k[\theta,t]$ to $f(\theta,t)\,\mathrm{d}\theta$ in ω. Based on this, one computes the image of $\theta^n\,\mathrm{d}\theta \in k[\theta,t]\,\mathrm{d}\theta$ as follows:

$$\kappa := D(\tau_{\mathcal{C}^{\otimes n}}) : \theta^\ell\,\mathrm{d}\theta \longmapsto C_q((t-\theta)^n\theta^\ell\,\mathrm{d}\theta/\theta) = (-1)^n \sum_{\substack{i=0 \\ i\,\equiv\,\ell+n+1\,(\mathrm{mod}\ q)}}^{n} \binom{n}{i}(-t)^i\theta^{(\ell+1+n-i-q)/q}\,\mathrm{d}\theta.$$

Let us take $\{\theta^\ell\,\mathrm{d}\theta\}_{\ell\in\mathbb{N}_0}$ as a basis of the module underlying $D(\mathcal{C}^{\otimes n})$ over $k[t]$. Then κ is $k[t]$-linear. Above we computed the image of the basis element $\theta^\ell\,\mathrm{d}\theta$. Considering the sum on the right for $\kappa(\theta^\ell\,\mathrm{d}\theta)$ we observe that the exponent ℓ essentially is divided by q, except for the added constant $\frac{n+1-q-i}{q}$. This means that the image of $\theta^\ell\,\mathrm{d}\theta$ only involves the basis elements $\{\theta^j\,\mathrm{d}\theta\}_{j=0,1,\ldots,c+\ell/q}$ for a constant c independent of ℓ.

Thus the matrix representing κ with respect to our chosen basis has the following shape: If we draw a straight line starting at row c in the first column and with slope $-\frac{1}{q}$, then all entries below that line are zero!

3 Operators of trace class

Let V_0 be a k-vector space, typically of countably infinite dimension. Set $V := V_0 \otimes A$ and consider an A-linear operator $\kappa_V : V \to V$.

Definition 9.7. A k-subspace $W_0 \subset V_0$ is called a *nucleus for κ_V* if it is finite-dimensional and there exists an exhaustive increasing filtration of V_0 by finite dimensional k-vector spaces

$$W_0 \subset W_1 \subset W_2 \subset \ldots \subset V_0$$

such that $\kappa_V(W_{i+1} \otimes A) \subset W_i \otimes A$ for all $i \geq 1$. If (V,κ_V) possesses a nucleus we call it *nuclear*.

The following proposition collects some basic results which are easy consequences of the definition:

Proposition 9.8. *(a) If W_0 is a nucleus for κ_V, then for any $j \geq 0$ the exterior power $\bigwedge_k^j W_0 \subset \bigwedge_k^j V_0$ is a nucleus for $\bigwedge^j \kappa_V : \bigwedge_A^j V \to \bigwedge_A^j V$.*

(b) If (V,κ_V) is nuclear, the values of the following expressions are independent of the chosen nucleus W_0

$$\begin{aligned} \mathrm{Tr}(\kappa_V) &:= \mathrm{Tr}_A(\kappa_V|W_0 \otimes A) \ \in\ A, \\ \Delta(1-T\kappa_V) &:= \det{}_A(1-T\kappa_V|W_0\otimes A) \\ &= \sum_j (-1)^j\,\mathrm{Tr}\big(\textstyle\bigwedge^j \kappa_V \big| \bigwedge_k^j W_0 \otimes A\big)\,T^j \ \in\ A[T]. \end{aligned}$$

They are called the trace *and the* dual characteristic polynomial of κ_V, *respectively.*

(c) Suppose $0 \to V_0' \to V_0 \to V_0'' \to 0$ is a short exact sequence of k vector spaces and that $V' := V_0' \otimes A$ is preserved by κ_V. Define $V'' := V_0'' \otimes A$ and write $\kappa_{V'}$ and $\kappa_{V''}$ for the endomorphisms induced from κ_V. If (V,κ_V) is nuclear, then so are $(V',\kappa_{V'})$ and $(V'',\kappa_{V''})$, and

$$\begin{aligned} \mathrm{Tr}(\kappa_V) &= \mathrm{Tr}(\kappa_{V'}) + \mathrm{Tr}(\kappa_{V''}), \\ \Delta(1-T\kappa_V) &= \Delta(1-T\kappa_{V'})\cdot\Delta(1-T\kappa_{V''}). \end{aligned}$$

Note that if W_0 is a nucleus for κ_V, then $W_0 \cap V_0'$ is a nucleus for $\kappa_{V'}$.

Remark 9.9. It is unclear whether there is a reasonable theory of nuclei, trace and characteristic polynomial for pairs (V,κ_V) if the underlying module of V is not of the form $V_0 \otimes A$.

We will now see that Cartier modules provide natural examples of nuclear endomorphisms. Thus in the following we assume that V_0 is the k-vector space underlying a finitely generated R-module. (Here R is as before smooth and finitely generated over k.) Following Anderson, one introduces the following notions:

Definition 9.10. Let $r_1, \ldots, r_s$ be generators of R as a k-algebra, and let $v_1, \ldots, v_t$ be generators of V_0 as an R-module. For every integer n let $V_{0,n} \subset V_0$ denote the k-linear span of all elements admitting a representation

$$\sum_{j=1}^{t} f_j(r_1, \ldots, r_s)\, v_j$$

with polynomials $f_{ij} \in k[X_1, \ldots, X_s]$ of total degree at most n and $V_{0,-\infty} := \{0\}$. These subspaces form an exhaustive sequence of finite-dimensional k vector spaces of V_0. The function

$$\gamma \colon V \to \mathbb{Z}^{\geq 0} \cup \{-\infty\},\ v \mapsto \inf \{n \mid v \in V_{0,n} \otimes A\}$$

is called a *gauge on V*.

Definition 9.11. An A-linear operator $\kappa_V : V \to V$ is called *of trace class* if for every gauge γ on V there exist constants $0 \leq K_1 < 1$ and $0 \leq K_2$ such that $\gamma(\kappa_V(v)) \leq K_1 \cdot \gamma(v) + K_2$ for all $v \in V$.

Proposition 9.12. *([2, Props. 3, 6], [8, §8.3]) Let (V, κ_V) be as above.*

(a) Any Cartier linear operator on V is of trace class with $K_1 = 1/q$.

(b) If κ_V is of trace class with constants K_1, K_2 for some gauge γ, then $V_{0,n}$ from 9.10 is a nucleus for κ_V for any $n \geq 1 + K_2/(1 - K_1)$.

(c) If κ_V is of trace class, then so is the composite of κ_V with any $R \otimes A$-linear endomorphism φ.

(d) If φ in (c) is of the form $\varphi_0 \otimes \mathrm{id}_A$ for some R-linear endomorphism φ_0 of V_0, then $\mathrm{Tr}(\kappa_V \varphi) = \mathrm{Tr}(\varphi \kappa_V)$.

4 Anderson's trace formula

The following is the generalization of the main result [2, Thm. 1] of Anderson's article [2] to arbitrary artinian A. As explained in Remark 9.9, the Cartier module V requires a k-structure V_0 in order for the dual characteristic polynomial to be well-defined.

Theorem 9.13 (Anderson)**.** *Let $X = \operatorname{Spec} R$ be affine and smooth of equidimension n over k. Consider $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X, A)$ such that $\mathcal{F} = \mathrm{pr}_1^* \mathcal{F}_0$ for a locally free coherent sheaf $\mathcal{F}_0$ on X. Set $V_0 := \mathrm{Hom}(\mathcal{F}_0, \omega_X)$ and $V = V_0 \otimes A$, so that the Cartier module corresponding to $D(\underline{\mathcal{F}})$ has the form (V, κ_V). Then κ_V is of trace class and*

$$L^{\mathrm{naive}}(X, \underline{\mathcal{F}}, T) = \Delta(1 - T\kappa_V)^{(-1)^{n-1}}. \tag{1}$$

In particular $L^{\mathrm{naive}}(X, \underline{\mathcal{F}}, T)$ is a rational function.

Proof. We give a sketch of the proof following Anderson. Full details can be found in [2]. The proof goes by proving equation (1) modulo T^m for all $m \in \mathbb{N}$, $m \geq 2$.

Step 1: Formula (1) holds modulo T^m if $m_0 := \min\{\deg(x) \mid x \in |X|\} \geq m$. In this case the left hand side is clearly congruent to 1 modulo T^m by definition since $m_0 \geq m$. To prove the same for the right hand side, consider first the case $m = 2$. (The case $m = 1$ is trivial.)

Let I be the ideal of R generated by the set $\{r^q - r \mid r \in R\}$. Since $m_0 \geq 2$, the ideal I must be the unit ideal. Else let $\mathfrak{m} \supset I$ be a proper maximal ideal. Then in the field $R/\mathfrak{m}$ every element would satisfy the equation $\bar{r}^q = \bar{r}$. It follows $R/\mathfrak{m} \cong k$, contradicting $m_0 \geq m \geq 2$. Since $I = R$, we can find $r_1, \ldots, r_s, f_1, \ldots, f_s \in R$, such that

$$1 = \sum_{i=1}^{s} (r_i - r_i^q) f_i.$$

We deduce

$$\mathrm{Tr}(\kappa_V) = \mathrm{Tr}(\kappa_V \cdot 1) = \mathrm{Tr}(\sum_{i=1}^{s} \kappa_V(r_i - r_i^q)f_i) = \sum_{i=1}^{s} \Big(\mathrm{Tr}(\kappa_V r_i f_i) - \mathrm{Tr}(r_i \kappa_V f_i)\Big) \overset{9.12(c)}{=} 0.$$

Hence the right hand side of (1) is congruent to 1 modulo T^2.

The case $m \geq 3$ is reduced to the case $m = 2$ by regarding $\bigwedge^j V$ as a module over the ring $(R^{\otimes j})^{\Sigma_j}$ of invariants of $R^{\otimes j}$ under the natural action of the symmetric group Σ_j on j elements.

Step 2: Reduction to the case where X contains at most one point of degree less than a given m: For this one chooses an affine covering by sets U_i of the form $\operatorname{Spec} R_{f_i}$ for suitable $f_i \in R$ such that each U_i contains at most one such point. Then one proves an inclusion exclusion principle for both sides. I.e., in the simplest case where $X = U_1 \cup U_2$ one proves that both sides are functions f on open subsets on X such that $f(X) = f(U_1)f(U_1)/f(U_1 \cap U_2)$.

Step 3: Induction on the dimension of R: Suppose equation (1) is known for all smooth affine varieties of dimension less than n. Here Anderson shows that for any $f \in R$ defining a disjoint decomposition $X = \operatorname{Spec} R/f \cup \operatorname{Spec} R_f$ there is a corresponding decomposition of (V, κ_V) into the restriction to $\operatorname{Spec} R_f$ already used in step 2 and a suitably defined residue $\mathrm{Res}_f(V, \kappa_V)$ of (V, κ_V) along $\operatorname{Spec} R/f$. If this is established and we are in the situation of step 2, then we choose f such that the single point of small degree lies in $\operatorname{Spec} R/f$. Then by induction hypothesis one is done.

Step 4: Initial step of the induction: One needs to prove equation (1) explicitly in the case where $X = \operatorname{Spec} k_x$ is the spectrum of a finite field extension k_x of k. Explicitly one has to show that

$$\Delta(1 - T\mathrm{Res}_{f_1} \ldots \mathrm{Res}_{f_n}(V, \kappa_V))$$

for $f_1, \ldots, f_n$ a regular sequence defining a maximal ideal $\mathfrak{m}$ of R with $x = \operatorname{Spec} R/\mathfrak{m}$ agrees with the L-factor of $\underline{\mathcal{F}}_x$. □

5 Proof of Theorem 8.13

Here we only give a proof under the following simplifying hypotheses: The compactification $\bar{X}$ is smooth and the ideal sheaf $\mathcal{I}_0$ of a complement $i\colon Z \hookrightarrow \bar{X}$ to $j\colon X \hookrightarrow \bar{X}$ is the inverse of an ample line bundle $\mathcal{O}_{\bar{X}}(1)$. We also assume that $\mathcal{F}$ is free, say of rank r. – The latter hypothesis can be easily achieved as follows: Since X is affine, the sheaf $\mathcal{F}$ corresponds to a finitely generated projective $R \otimes A$-module. Choose a finitely generated projective complement Q, define τ to be zero on it, and replace $\underline{\mathcal{F}}$ by its direct sum with the nilpotent τ-sheaf defined by $(Q, 0)$.

For $m \geq 0$ we define $\mathcal{F}_{0,m} := \big(\mathcal{O}_{\bar{X}}(-m)\big)^{\oplus r}$ and $\mathcal{F}_m := \mathcal{F}_{0,m} \otimes_k A$. From our construction of $j_!\underline{\mathcal{F}}$ we see that for $m \gg 0$ the endomorphism τ extends (in the present case uniquely) to $\mathcal{F}_m$ and in such a way that its inverse image along i is zero. In other words, we have

$$\tau_m\colon (\sigma \times \mathrm{id})^*\mathcal{F}_m \longrightarrow \mathcal{F}_{m+1} \subset \mathcal{F}_m. \tag{2}$$

Let m_0 be the smallest m such that the above factorization exists. Then for any $m \geq m_0$ the resulting τ-sheaf $\underline{\mathcal{F}}_m := (\mathcal{F}_m, \tau_m)$ on $\bar{X}$ is a representative of the crystal $j_!\underline{\mathcal{F}}$.

We can now invoke Serre duality: It provides us with a canonical isomorphism

$$D(H^n(\bar{X}, \mathcal{F}_m)) \cong H^0(\bar{X}, D(\mathcal{F}_m));$$

observe that $D(\mathcal{F}_m)$ is simply $\mathcal{F}_m^\vee \otimes \Omega_{\bar{X}/k}$. But more is true. A careful analysis yields that the above isomorphism is compatible with the endomorphism τ_m: On the left hand side, τ_m induces a *linear* endomorphism on $H^n(\bar{X}, \mathcal{F}_m)$, and $D(\_\_)$ dualizes this to a linear endomorphism on $D(H^n(\bar{X}, \mathcal{F}_m))$. On the right hand side, $D(\_\_)$ provides us with a Cartier linear endomorphism $D(\tau_m)$ on $D(\mathcal{F}_m)$ induced from τ_m. Taking cohomology, we obtain an induced *linear* endomorphism on $H^0(\bar{X}, D(\mathcal{F}_m))$. One can show, cf. [8, §7.4], that Serre duality identifies the two obtained linear endomorphisms.

For $m \gg 0$ define $W_{0,m} := H^0(\bar{X}, \mathcal{H}om(\mathcal{F}_{0,m}, \Omega_{\bar{X}/k}))$, so that $W_{m,0} \otimes_k A = H^0(\bar{X}, D(\mathcal{F}_m))$. One easily verifies that the $W_{0,m}$ form an increasing exhaustive filtration of $V_0 := H^0(X, \mathcal{H}om(\mathcal{F}, \Omega_{X/k}))$. Dualizing (2) one deduces that $D(\tau_m) = D(\tau)$ maps $W_{0,m+1} \otimes A$ into $W_{0,m} \otimes A$. This shows that

Lemma 9.14. *For any $m \geq m_0$, the k-vector space $(W_{0,m})$ is a nucleus for $D(\underline{\mathcal{F}})$.*

Thus Anderson's trace formula tells us

$$L^{\text{naive}}(X, \underline{\mathcal{F}}, T) = \Delta(1 - TD(\tau))^{(-1)^{n-1}} = \det(1 - TD(\tau_m) \mid H^0(\bar{X}, D(\mathcal{F}_m)))^{(-1)^{n-1}}.$$

Serre duality, as explained above, yields

$$\det(1 - TD(\tau_m) \mid H^0(\bar{X}, D(\mathcal{F}_m))) = \det(1 - T\tau_{H^n(\bar{X}, \mathcal{F}_m)} \mid H^n(\bar{X}, \mathcal{F}_m)) = L(\operatorname{Spec} k, H^n(\bar{X}, \mathcal{F}_m), T).$$

To complete the proof of Theorem 8.13, it simply remains to observe that the cohomology groups $H^i(\bar{X}, \mathcal{F}_m), T)$, $i \neq n$, all vanish for m sufficiently large. This follows from Serre duality and the fact that $H^i(\bar{X}, \mathcal{O}_{\bar{X}}(m) \otimes \Omega^{-1}_{\bar{X}/k}) = 0$ for $i \neq 0$ and all $m \gg 0$, since the sheaf $\mathcal{O}_{\bar{X}}(1)$ is ample on $\bar{X}$.

6 The crystalline trace formula for general (good) rings A

Suppose first that the ring A is a good coefficient ring and reduced and that $\underline{\mathcal{F}}$ is a flat A-crystal on some scheme X of finite type over k. Under these hypotheses the L-function $L(X, \underline{\mathcal{F}}, T)$ is well-defined. Since the nil radical of A is zero, all trace formulas are exact equalities, and thus to prove them, we may pass to the quotient ring Q_A of A and therefore assume that A is artinian and reduced.

One now has several standard techniques to prove the desired trace formula in this context:

(a) If $f\colon Y \to X$ is a finite morphism, then the trace formula in Theorem 8.14 holds. Here one can in fact directly prove that for any $x \in |X|$ one has

$$\prod_{y \in f^{-1}(x)} L(y, \underline{\mathcal{F}}_y, T) = L(x, (f_*\underline{\mathcal{F}})_x, T).$$

Note that the product on the left is finite.

(b) To prove an absolute trace formula, one can decompose X into a finite disjoint union of locally closed subschemes: Suppose that X is a disjoint union $X = U \cup Z$ with $j\colon U \hookrightarrow X$ and open immersion and $i\colon Z \hookrightarrow X$ a closed complement. Then one has the short exact sequence

$$0 \longrightarrow j_! j^* \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{F}} \longrightarrow i_* i^* \underline{\mathcal{F}} \longrightarrow 0.$$

Since $j_!$ and $i_* = i_!$ are exact, the spectral sequence for direct image with proper support yields, upon applying $Rf_!$, the exact triangle

$$Rs_U! j^* \underline{\mathcal{F}} \longrightarrow Rf_! \underline{\mathcal{F}} \longrightarrow Rs_{Z!} i^* \underline{\mathcal{F}} \longrightarrow Rs_U! j^* \underline{\mathcal{F}}[1].$$

Since for the L-functions one has

$$L(X, \underline{\mathcal{F}}, T) = L(U, j^*\underline{\mathcal{F}}, T) \cdot L(Z, i^*\underline{\mathcal{F}}, T)$$

it suffices to prove an absolute trace formula for U and Z.

(c) If $f : Y = \mathbb{A}^n \to X = \mathbb{A}^{n-1}$ is the projection onto the $n-1$ first coordinates and if the trace formula is proved for $\mathbb{A}^1 \to \operatorname{Spec} k$, it follows for f: One proceeds as in (a) except that now the fibers are no longer finite. One needs to use the proper base change formula in order to deduce

$$i_x^* \, Rf_! = Rf_{|Y_x!} \, i_{Y_x!}$$

where f_{Y_x} is the restriction of f to the fiber Y_x above x and and $i_{Y_x} : Y_x \hookrightarrow Y$ is the closed immersion of the fiber obtained as the pullback along $i_x : x \hookrightarrow X$.

(d) The trace formula for the structure morphism $\mathbb{A}^1 \to \operatorname{Spec} k$ can be deduced as follows. Near the generic point, i.e., on $\mathbb{A}^1$ minus finitely many closed points, one has a locally free representative due to Proposition 7.14. Using (b) it suffices to prove the formula for this open subset. But here one can simply apply Theorem 8.13 for the naive L-function.

(e) From (a)–(d) and the Noether normalization lemma (after reduction to an affine situation), one can readily deduce the absolute trace formula for any X of finite type over k.

(f) Having the absolute trace formula, one can deduce the relative trace formula for a morphism $f : Y \to X$ of finite type from proper base change as explained in a special case in (c). The key point is that one obtains the relative version as the product over the closed points $x \in |X|$ of the formula

$$\prod_{y \in |f^{-1}(y)|} L(y, \underline{\mathcal{F}}_y, T) = L(x, (Rf_!\underline{\mathcal{F}})_x, T). \tag{3}$$

Suppose now that A is not necessarily reduced. Then (c) and (f) above will fail. The point is that in formula (3) we no longer have equality, but $=$ gets replaced by $\sim$, cf. Theorem 8.14. But if we take an infinite product over formulas of the type (3) with $\sim$ instead of $=$, we loose all control, since the infinite product over elements in $1 + T\mathfrak{n}_A[T]$ may well be a non-rational power series. The path taken in [8, Chap. 9] to overcome this difficulty is rather demanding. One can stratify X so that over the finitely many pieces of the stratification all sufficiently high twists of the initial τ-sheaf have a locally free representative. Using this representative, one shows that on each stratum formula (3) is in fact an equality for all but finitely many $x \in |X|$.

Chapter 10

Global L-functions for A-motives

So far we were mainly concerned with L-functions which when compared with the classical theory should be considered as local L-functions. In the following we present an approach, essentially due to D. Goss, to define global Carlitz type L-functions for arbitrary τ-sheaves over A or A-crystals with a characteristic function from their underlying scheme to the coefficient ring A (at least for certain A to be specified below). These global L-functions are continuous homomorphisms from $\mathbb{Z}_p$ to entire functions on $\mathbb{C}_\infty$. We state the main conjectures of Goss on these L-functions on meromorphy, entireness and algebraicity and indicate the proof from [3] of these conjectures which is based on the theory of [8] which was explained in the previous chapters. A different proof of Goss' conjectures was given in [43] at least for $A = \mathbb{F}_q[t]$.

Section 5, the last section of this chapter, consists of an extended example: We recall the explicit expression for the global Carlitz-Goss type ζ-function of the affine line. Then we derive formulas for the special values at negative integers $-n$ in terms of the cohomological formalism developed so far. These formulas can be evaluated in complexity $\mathcal{O}(\log|n|)$ by computing an explicit determinant. For $p = q$ this will also provide yet another approach on a conjecture of Goss on the distribution of zeros of the global L-functions evaluated at elements of $\mathbb{Z}_p$. The proof for arbitrary q was given by Sheats in [42] after previous special work for $p = q$ by Wan [49], Diaz-Vargaz [9] and Poonen. Yet another approach for $p = q$ is due to Thakur [48]. We end this section with some observations and questions based on computer experiments for the ζ-functions of some other affine curves.

Throughout this chapter, we fix the following notation:

- $\bar{C}/k$ will be a smooth projective geometrically irreducible curve over the finite field k of characteristic p.
- ∞ will be a marked closed point on $\bar{C}$ and $C := \bar{C} \smallsetminus \{\infty\}$.
- $K := k(\bar{C})$ is the function field of $\bar{C}$ and $A := \Gamma(C, \mathcal{O}_{\bar{C}})$ the coordinate ring of the affine curve C.
- K_∞ is the completion of K at ∞, $\mathcal{O}_\infty \subset K_\infty$ its ring of integers, $\pi_\infty \in \mathcal{O}_\infty$ a uniformizer, k_∞ the residue field at ∞ and $d_\infty := [k_\infty : k]$ the residue degree.

Suppose X is a scheme of finite type over k with a morphism $f\colon X \to \operatorname{Spec} A$ and that $\underline{\mathcal{F}}$ is a flat A-crystal over X. (Such an f would naturally arise in the case where $\underline{\mathcal{F}}$ comes from a Drinfeld A-module or an A-motive; there one could take f as the characteristic of this object.) For every closed point $x \in |X|$, its image $\mathfrak{p}_x := f(x) \in \operatorname{Spec}(A)$ is a maximal ideal and one clearly has the divisibility $d_{\mathfrak{p}_x} | d_x$ for the residue degrees. One would like to define

$$L^{\mathrm{glob}}(X, \underline{\mathcal{F}}, s) := \prod_{x \in |X|} L(x, \underline{\mathcal{F}}_x, T)_{|T^{d_{\mathfrak{p}_x}} = \mathfrak{p}^{-s}}.$$

What is needed at this point is a good definition of $\mathfrak{p}_x^s$ and a characteristic p domain in which the exponents s will lie. Below, following Goss, we shall give such definitions.

A main conjecture of Goss was that these L-functions should be entire or meromorphic in a sense to be defined below. For $A = k[t]$ this conjecture was proved by Taguchi and Wan in [43]. The general conjecture was proved in [3] using crucially the theory of the previous chapters. As we shall see, and this goes back to [43], the special values of Goss L-functions agree with the L-functions of the previous chapters if one replaces $\underline{\mathcal{F}}$ by a suitable twist. The cohomological theory will prove that these special values are polynomials whose degree grows logarithmically. It was known to Goss that this would suffice to deduce the entireness (or meromorphy) of the global L-functions defined by him.

In fact, we will not completely follow Goss' approach. He defines L^{glob} as a function

$$L^{\mathrm{glob}}(X, \underline{\mathcal{F}}, \_\_)\colon \mathbb{Z}_p \times \mathbb{C}_\infty^* \longrightarrow \mathbb{C}_\infty.$$

Moving one copy of $\mathbb{C}_\infty$ from left to right and working with $T = z^{-1}$, we shall defined a global L-function

$$L^{\mathrm{glob}}(X, \underline{\mathcal{F}}, \_\_)\colon \mathbb{Z}_p \longrightarrow \Big(\mathbb{C}_\infty[[T]]_{\le c}\Big)^*,$$

where $\mathbb{C}_\infty[[T]]_{\le c}$ denotes the ring of power Series on $\mathbb{C}_\infty$ which converge on the closed disc of radius c and the superscript $*$ denotes it units. This ring is in a natural way a Banach space, and thus in particular a topological space.

1 Exponentiation of ideals

The case $A = k[t]$: Here any maximal ideal $\mathfrak{p}$ is generated by a unique irreducible monic polynomial $f_\mathfrak{p} \in k[t]$, and thus a natural definition of $\mathfrak{p}^s$, at least for $n = s \in \mathbb{Z}$, is $\mathfrak{p}^n := f_\mathfrak{p}^n$. As we have seen in the lectures of D. Thakur, it is also natural to consider the expression $f_\mathfrak{p}/t^{\deg f_\mathfrak{p}}$ which is a 1-unit in $\mathcal{O}_\infty$. For it the expression $(f_\mathfrak{p}/t^{\deg f_\mathfrak{p}})^n$ is well-defined for any $n \in \mathbb{Z}_p$. Goss defines $\mathfrak{p}^s$ for any $s = (n, z) \in \mathbb{Z}_p \times \mathbb{C}_\infty^*$, cf. [23, §8.2], by

$$\Big(f_\mathfrak{p}/t^{\deg f_\mathfrak{p}}\Big)^n z^{\deg \mathfrak{p}}.$$

Let us now recall the necessary definitions for arbitrary A – note that k_∞ is canonically contained in K_∞:

Definition 10.1. A continuous homomorphism $\mathrm{sign}\colon K_\infty^* \to k_\infty^*$ is a

- *sign-function* if $\mathrm{sign} = \mathrm{id}$ when restricted to k_∞^*;
- *twisted sign-function* if there exists $\sigma \in \mathrm{Gal}(k_\infty/k)$, such that $\mathrm{sign} = \sigma$ when restricted to k_∞^*.

Obviously any twisted sign function is of the form $\sigma \circ \mathrm{sign}'$ for some $\sigma \in \mathrm{Gal}(k_v/k)$ and some sign-function sign'. Any sign-function is trivial on the 1-units in $\mathcal{O}_\infty^*$. If π is a uniformizer of K_∞, then any sign-function is uniquely determined by the image of π in k_∞^*. Using that $K_\infty^* \cong \pi^{\mathbb{Z}} \times \mathcal{O}_\infty^*$ one finds that the number of sign-functions is equal to $\#k_\infty^*$, and the number of twisted sign-functions to $\#k_\infty^* \cdot d_\infty$.

We fix a sign-function sign for K_∞.

Definition 10.2. An element $x \in K_\infty^*$ is *positive (for* sign*)* iff $\mathrm{sign}(x) = 1$. An element $x \in K^*$ is positive if its image in K_∞^* is positive. We write K_+ for the positive elements of K^* and A_+ for $A \cap K_+$.

From now on, we also fix a positive element $\pi \in K^*$ which is a uniformizer of K_∞.

The *class group* $\mathrm{Cl}(A)$ *of* A is the quotient of the group of fractional ideals $\mathcal{I}_A$ of A modulo the subgroup of principal fractional ideals $\mathcal{P}$ of A.

Definition 10.3. The *strict class group* $\mathrm{Cl}^+(A)$ *of* A *(w.r.t.* sign*)* is the quotient of the group of fractional ideals of A modulo the subgroup of principal positively generated fractional ideals $\mathcal{P}^+$ of A. (A fractional ideal is principal positively generated if it is of the form Ax for $x \in K_+$.)

In particular one has an obvious short exact sequence $0 \to k_\infty^*/k^* \to \mathrm{Cl}^+(A) \to \mathrm{Cl}(A) \to 0$. Correspondingly, one defines the *strict Hilbert class* $H^+ \supset H$ field of K as the abelian extension of K which under the Artin-homomorphism has Galois group isomorphic to $\mathrm{Cl}^+(A)$. More precisely: under the reciprocity map of class field theory $H^+ \supset H \supset K$ correspond to

$$\prod_{v \text{ finite}} \mathcal{O}_v^* \times (\mathrm{Ker}(\mathrm{sign})) \subset \prod_{v \text{ finite}} \mathcal{O}_v^* \times K_\infty^* \;\subset\; \mathbb{A}_K^*.$$

Due to our choice of π_∞, we have $\mathrm{Ker}(\mathrm{sign}) = \pi_\infty^{\mathbb{Z}} \times (1 + \mathfrak{m}_\infty)$ where $\mathfrak{m}_\infty \subset \mathcal{O}_\infty$ is the maximal ideal.

Proposition 10.4 (Goss)**.** *Let U_1^{perf} denote the group of 1-units in the perfect closure of K_∞. There exists a unique homomorphism*

$$\langle \_ \rangle \colon \mathcal{I} \to U_1^{\mathrm{perf}}$$

such that for all $a \in K_+$ one has

$$\langle a \rangle = a \cdot \pi_\infty^{-v_\infty(a)}.$$

Proof. The idea of the proof is as follows: Let $h^+ := \#\mathrm{Cl}^+(A)$ denote the strict class number. Then for any fractional ideal $\mathfrak{a}$, the power $\mathfrak{a}^{h^+}$ is positively generated, and thus $\langle \mathfrak{a}^{h^+} \rangle$ is defined. To define $\langle \mathfrak{a} \rangle$, we take the unique h^+-th root within U_1^{perf} - note that for $m \in \mathbb{N}$ prime to p the m-th root of a 1-unit in K_∞ can simply be defined by the binomial series for $1/m$, and thus lies itself again in K_∞. However h^+ may have p as a divisor. This necessitates the use of U_1^{perf}. $\square$

The above definition and the following one of exponentiation arose in correspondence between D. Goss and D. Thakur.

Fix a d_∞-th root $\pi_* \in K^{\mathrm{alg}}$ of π_∞.

Definition 10.5. Foir $n \in \mathbb{Z}$ and $\mathfrak{a}$ a fractional ideal of A, one defines

$$\mathfrak{a}^n := \pi_*^{-n \deg \mathfrak{a}} \cdot \langle \mathfrak{a} \rangle^n \in K_\infty^{\mathrm{alg}}.$$

More generally, for $s := (n, z) \in \mathbb{Z}_p \times \mathbb{C}_\infty^*$ one sets

$$\mathfrak{a}^s := z^{\deg \mathfrak{a}} \cdot \langle \mathfrak{a} \rangle^n \in K_\infty^{\mathrm{alg}}.$$

Note that the first exponentiation is a special case of the second if one takes for s the pair (n, π_*^{-n}). Also, going through the definitions and choosing $\pi_* = \frac{1}{t}$ for $A = k[t]$, one may easily verify that in this case Definition 10.5 agrees with the exponentiation described at the beginning of this subsection.

Proposition 10.6 (Goss)**.** *Let $\mathbb{V}$ be the subfield of K_∞ generated by K and all the $\mathfrak{a}^n$ for all fractional ideals $\mathfrak{a}$ of A and all $n \in \mathbb{Z}$. Then $\mathbb{V}$ is a finite extension of K. Let $\mathcal{O}_{\mathbb{V}}$ denote its ring of integers (over A). Then for all ideals $\mathfrak{a} \subset A$ and all $n \in \mathbb{N}$, one has $\mathfrak{a}^n \in \mathcal{O}_{\mathbb{V}}$.*

2 Definition and basic properties of the global L-function

Let $X, f, \underline{\mathcal{F}}$ etc. be as above. At $x \in |X|$ the local L-factor $L(x, \underline{\mathcal{F}}_x, w)$ lies in $1 + w^{d_x} A[[w^{d_x}]] \subset 1 + w^{d_{\mathfrak{p}_x}} A[[w^{d_{\mathfrak{p}_x}}]]$. This shows that the following definition makes sense:

Definition 10.7. The global L-function of $(X, f, \underline{\mathcal{F}})$ is defined by

$$L^{\mathrm{glob}}(X, \underline{\mathcal{F}}, \_) \colon \mathbb{Z}_p \longrightarrow 1 + T\mathbb{C}_\infty[[T]] : n \mapsto \prod_{x \in |X|} L(x, \underline{\mathcal{F}}_x, w)_{w^{d_{\mathfrak{p}_x}} = T^{d_{\mathfrak{p}_x}} \langle \mathfrak{p}_x \rangle^n}$$

Remarks 10.8. (a) It is elementary to see that there is a constant $c \in \mathbb{Q}_{>0}$, independent of $n \in \mathbb{Z}_p$, such that any of the values

$$L^{\text{glob}}(X, \underline{\mathcal{F}}, n) \in 1 + T\mathbb{C}_\infty[[T]]$$

lies in the ring $\mathbb{C}_\infty[[T]]_{\le c}$ of convergent power series around 0 on the (closed) disc $\{T \in \mathbb{C}_\infty \mid |T| \le c\}$. Since at $T = 0$, the power series take the value 1, one can choose c such that for all $n \in \mathbb{Z}_p$ the power series $L^{\text{glob}}(X, \underline{\mathcal{F}}, n)$ are bounded away from zero uniformly, i.e., for each $n \in \mathbb{Z}_p$ they are units in $\mathbb{C}_\infty[[T]]_{\le c}$.

(b) The function $n \mapsto \langle \mathfrak{p} \rangle^n$ satisfies a very strong interpolation properties: The limit $\lim_{m\to\infty} \langle \mathfrak{p} \rangle^{p^m} = 1$ is uniform in $\mathfrak{p} \in \mathbf{Max}(A)$. Moreover $\mathbb{C}_\infty[[T]]_{\le c}$ is a Banach space under the norm given by

$$|| \sum_{m\ge 0} a_m T^m || = \sup_{m\ge 0} |a_m||c|^n.$$

Using the interpolation property of $n \mapsto \langle \mathfrak{p} \rangle^n$ it is not difficult to show that $L^{\text{glob}}(X, \underline{\mathcal{F}}, \_\_)$ is a continuous function from $\mathbb{Z}_p$ into this Banach space.

(c) In Goss' formulation, the variable T is substituted by z^{-1}. Thus one has a continuous function from $\mathbb{Z}_p$ into a Banach space of convergent power series in $1/z$ around ∞ of radius $1/c$.

(d) The definition of $L^{\text{glob}}(X, \underline{\mathcal{F}}, \_\_)$ depends the choice of $\pi_* \in K^{\text{alg}}$. This element is used to define $\langle \_\_ \rangle$ and its d_∞-th power is a uniformizer π_∞ of K_∞ which determines our sign function.

Let us draw a first conclusion from the cohomological theory of crystals for Goss' global L-functions:

Proposition 10.9. *Suppose $f\colon X \to \operatorname{Spec} A$ is a morphism of finite type and $\underline{\mathcal{F}}$ is a flat A-crystal on X. Then*

(a) The complex $Rf_!$ can be represented by a bounded complex $\underline{\mathcal{G}}^\bullet$ of flat A-crystals on $\operatorname{Spec} A$.

(b) With $\underline{\mathcal{G}}^\bullet$ from (a) one has

$$L^{\text{glob}}(X, \underline{\mathcal{F}}, n) = \prod_{i\in\mathbb{Z}} L^{\text{glob}}(X, \underline{\mathcal{G}}^i, n)^{(-1)^i}$$

Part (a) follows from Theorem 7.10, part (b) is an application of the trace formula Theorem 8.14 to each fiber $X \times_{\operatorname{Spec} A} \mathfrak{p}$, together with the proper base change, Theorem 6.6.

Thus to study general properties of global L-functions, it suffices to consider the case $X =$ $\operatorname{Spec} A$! We will do so from now on unless stated otherwise.

3 Global L-functions at negative integers

We now describe the link between L^{glob} and the L-function from the previous chapters. We learned this, in the case $A = k[t]$, from [43].

For any ring A there is an analog of the Carlitz module, the so called Drinfeld Hayes module. Let $\mathcal{O}^+$ denote the ring of integers over A of $H^+ \supset K$. In [28] Hayes shows that there are $\#\mathrm{Cl}^+(A)$ sign-normalized rank 1-Drinfeld modules

$$\psi_{DH,A}\colon A \to \mathcal{O}^+[\tau].$$

(We provide more details on these in Appendix 3.) Denote the structure morphism $\operatorname{Spec} \mathcal{O}^+ \to \operatorname{Spec} A$ by s and the A-motive associated to $\psi_{DH,A}$ by $\mathcal{H}_A$. Thus $\mathcal{H}_A$ is a locally free τ-sheaf on $\operatorname{Spec} \mathcal{O}^+$ over A of rank 1.

It is now possible and not too hard to compute the local L-factors of the tensor powers $\mathcal{H}^{\otimes n}$, $n \in \mathbb{N}_0$, cf. [3, Lem. 3.2]. Define

$$L_X^{\text{glob}}(n, T) := L^{\text{glob}}(X, (\mathcal{O}_{X\times \operatorname{Spec} A}, \tau_{\text{can}} = (\sigma \times \text{id})), n)$$

as the global Carlitz-Goss L-function for any A-scheme X of finite type, i.e., scheme with a structure morphism to $\operatorname{Spec} A$. We explicitly write the variable T on the left to stress that for each $n \in \mathbb{Z}_p$ the right hand term is a function in T. Similarly, for any character χ of the abelian group $\mathrm{Gal}(H^+/K)$ define $L_X^{\mathrm{glob}}(n, \chi, T)$ as the ζ-function on $\operatorname{Spec} A$ for the character χ. Using this intuitive notation, the following formula was first observed by Goss, see also Lemma 10.22:

Theorem 10.10 (Goss)**.** *For $n \in \mathbb{N}_0$ one has*

$$L_{\operatorname{Spec} \mathcal{O}^+}(-n, T) = L(\operatorname{Spec} A, s_*\mathcal{H}_A^{\otimes n}, T) = \prod_{\chi \in \widehat{\mathrm{Gal}(H^+.K)}} L_X^{\mathrm{glob}}(-n, \chi, T)|_{T = T\pi_*^n}.$$

By the symbol $(\_\_)|_{T=T\pi_*^n}$ we mean that one substitutes the term $T\pi_*^n$ for T.

The same computations and defining L-functions with characters for crystals, cf. [3, Cor. 3.8], yield

Theorem 10.11. *Suppose $\underline{\mathcal{F}}$ is a flat A-crystal on $\operatorname{Spec} A$. Then for $n \in \mathbb{N}_0$ one has*

$$L(\operatorname{Spec} A, \underline{\mathcal{F}} \otimes s_*\mathcal{H}_A^{\otimes n}, T) = \prod_{\chi \in \widehat{\mathrm{Gal}(H^+.K)}} L^{\mathrm{glob}}(\operatorname{Spec} A, \underline{\mathcal{F}}, \chi^n, -n)|_{T = T\pi_*^n}.$$

Remark 10.12. The importance of the previous theorem lies in the fact that the left hand side is, via the trace formula, Corollary 8.15, equal to the L-function of $H^1(C_{H^+}, j_!\underline{\mathcal{F}} \otimes s_*\mathcal{H}_A^{\otimes n})$ where C_{H^+} is the smooth projective geometrically irreducible curve over k_∞ with function field H^+ and where $j\colon \operatorname{Spec} \mathcal{O}^+ \hookrightarrow C_{H^+}$ is the canonical open immersion. The cohomology can be represented by a τ-sheaf whose underlying module if free of finite rank over A. To compute this τ-sheaf one may compute the coherent cohomology of the underlying sheaf together with the endomorphism induced by τ. In doing so, one has the freedom to replace the τ-sheaf which appears as the argument of cohomology by a nil-isomorphic one.

4 Meromorphy and entireness

One can prove the following simple but important lemma. Its proof follows closely the method described in Subsection 5 for $\underline{\mathcal{C}}^{\otimes n}$:

Lemma 10.13. *For a locally free τ-sheaf $\underline{\mathcal{F}}$ on $\operatorname{Spec} A$ over A, the function*

$$n \mapsto \deg_T L(\operatorname{Spec} A, \underline{\mathcal{F}} \otimes s_*\mathcal{H}_A^{\otimes n}, T)$$

is of order of growth at most $O(\log n)$.

Now for any $n \geq 0$ which is a multiple of $h^+ := \#\mathrm{Cl}^+(A)$, the characters χ^n, $\chi \in \widehat{\mathrm{Gal}(H^+.K)}$, are all trivial. Thus one deduces from the lemma and Theorem 10.11 that $L^{\mathrm{glob}}(\operatorname{Spec} A, \underline{\mathcal{F}}, -n)$ for such n is a polynomial in T whose degree grows at most like $O(\log n)$. Let $h^{(p)}$ be the prime-to-p part of h^+. By replacing $\underline{\mathcal{F}}$ by some kind of Frobenius twist on the base – this is similar to the observation made in Remark 10.20 (a) – one can show logarithmic degree growth for all negative integers which are a multiple of $h^{(p)}$. These lie dense in $\mathbb{Z}_p$ and the strong uniform interpolation property of $n \mapsto \langle \mathfrak{p} \rangle^n$ yields:

Theorem 10.14. *Suppose $\underline{\mathcal{F}}$ is a locally τ-sheaf on $\operatorname{Spec} A$ over A. Then $L^{\mathrm{glob}}(\operatorname{Spec} A, \underline{\mathcal{F}}, \_\_)$ is a continuous function $\mathbb{Z}_p \longrightarrow \mathbb{C}_\infty[[T]]^{\mathrm{ent}}$, where $\mathbb{C}_\infty[[T]]^{\mathrm{ent}}$ is the ring of entire power series with coefficients in $\mathbb{C}_\infty$ – it is in a natural way a Fréchet space, and thus in particular a metrizable topological space.*

Once the function $L^{\mathrm{glob}}(\operatorname{Spec} A, \underline{\mathcal{F}}, \_\_)$ is known to be entire, one deduces from Theorem 10.11 and Lemma 10.13, invoking Proposition 10.6, the following:

Theorem 10.15. *For all $n \in \mathbb{N}_0$, the values $L^{\mathrm{glob}}(\operatorname{Spec} A, \underline{\mathcal{F}}, -n)|_{T=T\pi_*^n}$ are polynomials in T with coefficients in $\mathcal{O}_{\mathbb{V}}$ and their degrees in T are of growth at most $O(\log n)$.*

Definition 10.16 (Goss)**.** Let f be a function $f\colon \mathbb{Z}_p \to 1 + T\mathbb{C}_\infty[[T]]$.

The function f is called *entire* if the image of f lies in $\mathbb{C}_\infty[[T]]^{\mathrm{ent}}$ and if f is continuous with respect to the natural topologies on $\mathbb{Z}_p$ and $\mathbb{C}_\infty[[T]]^{\mathrm{ent}}$.

The function f is called *meromorphic* if it is the quotient of two entire functions.

An entire function f is called *essentially algebraic*, if for all $n \in \mathbb{N}_0$, the value $f(-n)|_{T=T\pi_*^n}$ is a polynomial whose coefficients are integral over A and lie in a fixed finite extension of K, independent of n.

A meromorphic function f is called *essentially algebraic*, if it is the quotient of two entire essentially algebraic functions.

From Theorems 10.14 and 10.15 and Theorem 7.14 one deduces readily:

Theorem 10.17. *Suppose $\underline{\mathcal{F}}$ is a flat A-crystal on* $\operatorname{Spec} A$ *(or a finite complex of such). Then $L^{\mathrm{glob}}(\operatorname{Spec} A, \underline{\mathcal{F}}, \_\_)$ is meromorphic and essentially algebraic. If in addition $\underline{\mathcal{F}}$ has a locally free representative, then $L^{\mathrm{glob}}(\operatorname{Spec} A, \underline{\mathcal{F}}, \_\_)$ is entire and essentially algebraic.*

5 The global Carlitz-Goss L-function of the affine line

Throughout this subsection, we fix $A = k[t]$ and we identify $\mathbb{A}^1 = \operatorname{Spec} A$. Recall that

$$L^{\mathrm{glob}}_{\mathbb{A}^1}(n, T) := L^{\mathrm{glob}}(\operatorname{Spec} A, (\mathcal{O}_{\operatorname{Spec} A \times \operatorname{Spec} A}, \sigma \times \mathrm{id}), n) = \prod_{\mathfrak{p} \in \mathbf{Max}(A)} (1 - T^{d_\mathfrak{p}} \langle \mathfrak{p} \rangle^{-n})^{-1}.$$

We also define for $n \in \mathbb{Z}$ the more intuitive ζ-function

$$\zeta_A(n, T) := \prod_{a \in A_+,\, \mathrm{irred.}} (1 - T^{\deg(a)} a^{-n})^{-1} = \prod_{\mathfrak{p} \in \mathbf{Max}(A)} (1 - T^{d_\mathfrak{p}} \mathfrak{p}^{-n})^{-1} = \sum_{\mathfrak{a} \leq A} T^{\deg \mathfrak{a}} \mathfrak{a}^{-n} = \sum_{d=0}^{\infty} T^d \Big(\sum_{a \in A_{d+}} a^{-n} \Big);$$

by A_{d+} we denote the set of monic elements of degree d in $A = k[t]$. The agreement of the second and fifth term is immediate by expanding the Euler product. The third and fourth terms are term by term the same as the second and fifth where however we have used the exponentiation notation from Subsection 1.

Lemma 10.18. *For $n \in \mathbb{N}_0$ one has*

$$L^{\mathrm{glob}}_{\mathbb{A}^1}(-n, T)|_{T=T\pi_*^n} = \zeta_A(-n, T)$$

Proof.

$$\begin{aligned}
L^{\mathrm{glob}}_{\mathbb{A}^1}(-n, T)|_{T=T\pi_*^n} &= \prod_{\mathfrak{p} \in \mathbf{Max}(A)} (1 - (T\pi_*^n)^{d_\mathfrak{p}} \langle \mathfrak{p} \rangle^{-n})^{-1} \\
&= \prod_{\mathfrak{p} \in \mathbf{Max}(A)} (1 - T^{d_\mathfrak{p}} (1/t)^{n d_\mathfrak{p}} \langle \mathfrak{p} \rangle^{-n})^{-1} \\
&= \prod_{a \in A_+,\, \mathrm{irred.}} (1 - T^{\deg(a)} t^{-n \deg(a)} (a/t^{\deg(a)})^{-n})^{-1} \\
&= \prod_{a \in A_+,\, \mathrm{irred.}} (1 - T^{\deg(a)} a^{-n})^{-1} \quad = \quad \zeta_A(-n, T).
\end{aligned}$$

□

Definition 10.19. For $n \in \mathbb{N}_0$ with q-digit expansion $n = a_0 + a_1 q + \ldots a_r q^r$ we define $\ell(n) := a_0 + a_1 + \ldots + a_r$ to be the sum over the digits in the base q expansion of n.

Remark 10.20. (a) As can be seen directly from the definition of $\zeta_A(n,T)$, one has: $\zeta_A(n,T)^p = \zeta(pn, T^p)$.

(b) By a theorem of Lee (a student of Carlitz) one has $\sum_{a\in A_{d+}} a^n = 0$ for $d > \ell(n)/(q-1)$, so that

$$\deg_T \zeta_A(-n,T) \le \ell(n)/(q-1).$$

(c) Combining (a) and (b), for $q = p^m$ one finds

$$\deg_T \zeta_A(-n,T) \le \frac{1}{q-1}\min\{\ell(n), \ell(pn), \ell(p^2n), \ldots, \ell(p^{m-1}n)\}.$$

Although not stated explicitly in [42], one can deduce from it easily the following result:

Proposition 10.21. *For $n \in \mathbb{N}_0$ and q an arbitrary prime power one has*

$$\deg_T \zeta_A(-n,T) = \Big\lfloor \frac{\min\{\ell(n), \ell(pn), \ell(p^2n), \ldots, \ell(p^{m-1}n)\}}{q-1}\Big\rfloor.$$

Lemma 10.22. *For $n \in \mathbb{N}_0$ one has $\zeta_A(-n,T) = L(\mathbb{A}^1, \underline{\mathcal{C}}^{\otimes n}, T)$.*

The lemma is Theorem 10.10 in its simplest case!

Proof. It suffices to show that for $f \in A_+$ irreducible and $\mathfrak{p} = (f)$ one has

$$1 - f^n T^{\deg(f)} \stackrel{!}{=} \det(1 - T\tau_{\mathcal{C}}^{\otimes n} \mid \mathcal{C}_{\mathfrak{p}}^{\otimes n}) = \det(1 - T^{\deg(f)}(\tau_{\mathcal{C}}^{\otimes n})^{\deg(f)} \mid \mathcal{C}_{\mathfrak{p}}^{\otimes n}).$$

Starting from $\tau^{\otimes n} = (t-\theta)^n(\sigma \times \mathrm{id})$ we compute

$$(\tau^{\otimes n})^{\deg(f)} = (t-\theta)^n(t-\theta^q)^n \ldots (t-\theta^{q^{\deg f-1}})^n(\sigma^{\deg f} \times \mathrm{id}).$$

On $k_{\mathfrak{p}} = k[\theta]/(f(\theta))$ we have $\sigma^{\deg f} = \mathrm{id}$. Writing $\bar\theta$ for the image of θ in $k_{\mathfrak{p}}$, the roots of f are precisely the elements $\bar\theta^{q^i}$, $i = 0, \ldots, \deg f - 1$, and so $f(t) = (t-\bar\theta)(t-\bar\theta^q)^n \ldots (t-\bar\theta^{q^{\deg f-1}})$. We deduce

$$(\tau^{\otimes n})^{\deg(f)} = f(t)^n \in k_{\mathfrak{p}}[t]$$

which completes the proof. ☐

The next aim is to find for each $n \in \mathbb{N}_0$ an explicit square matrix whose dual characteristic polynomial is $\zeta_A(-n,T)$. Moreover we would like to find such a matrix of size $\deg_T(\zeta_A(-n,T))$!

The straightforward use of the representative $j_!\underline{\mathcal{C}}^{\otimes n}$ found in Example 5.12 is not suitable for this, since it yields a matrix of size $\dim H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = -1-m > \frac{n}{q-1} - 1$ which is far larger than $\frac{\ell(n)}{q-1}$ – or the possibly even smaller true degree given in Proposition 10.21.

Definition 10.23. For $i \in \mathbb{N}_0$ define the τ-sheaf $\underline{\mathcal{C}}^{(q^i)}$ on $\mathbb{A}^1$ over A by the pair

$$(k[\theta,t], (t^{q^i} - \theta)(\sigma \times \mathrm{id})).$$

Lemma 10.24. *The τ-sheaf $\underline{\mathcal{C}}^{(q^i)}$ is nil-isomorphic to $\underline{\mathcal{C}}^{\otimes q^i}$.*

Proof. The pullback $(\sigma^i)^*\underline{\mathcal{C}}^{(q^i)}$ is isomorphic to $\underline{\mathcal{C}}^{\otimes q^i}$ and via τ^i it is nil-isomorphic to $\underline{\mathcal{C}}^{(q^i)}$. ☐

Corollary 10.25. *Suppose $n = a_0 + a_1 q + \ldots + a_r q^r$ with $0 \le a_i \le q-1$ is the base q expansion of $n \in \mathbb{N}_0$. Let $g(\theta) := \prod_{i=0}^r (t^{q^i} - \theta)^{a_i}$. Then $\underline{\mathcal{C}}^{\otimes n}$ is nil-isomorphic to*

$$\underline{\mathcal{C}}^{(n)} := \bigotimes_{i=0}^r (\underline{\mathcal{C}}^{(q^i)})^{\otimes a_i} = (k[\theta,t], g(\theta)(\sigma \times \mathrm{id})).$$

Let us now determine a representative of $j_!\underline{\mathcal{C}}^{(n)}$ for the open immersion $j\colon \mathbb{A}^1 \hookrightarrow \mathbb{P}^1$. Proceeding exactly as in Example 5.12, we find the following: As the underlying sheaf we take $\mathcal{O}_{\mathbb{P}^1}(-m\infty)\otimes_k A$ with $m = \lceil \frac{-1-\ell(n)}{q-1} \rceil$, and as τ_m we take $g(\theta)(\sigma \times \mathrm{id})$. Note that now

$$\mathrm{rank}_{k[t]}\, H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-m\infty)) = \mathrm{rank}_{k[t]}\, H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m-2\infty)) = m-1 = \lfloor \frac{\ell(n)}{q-1} \rfloor =: D.$$

Suppressing $\mathrm{d}\theta$ in the notation, we take $e_1 := 1, e_2 := \theta, \ldots, e_D := \theta^{D-1}$ as a basis over $k[t]$ of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m-2\infty))$. Then by Lemma 9.14 this a a nucleus in the sense of Anderson for $D(\underline{\mathcal{C}}^{(n)}) = k[\theta, t]\mathrm{d}\theta$. Note that the image of e_i under $D(\tau^{(n)})$ is $\kappa(\theta^{i-1}) = C_q(g(\theta)\theta^{i-1})$. To be more explicit, we define $g_j(t) \in k[t]$ by

$$g(\theta) := \sum_{j\geq 0} g_j(t)t^j$$

and $g_j = 0$ for $j < 0$. Then

$$\kappa(e_i) = C_q(\sum_{j\geq 0} g_j(t)\theta^{j+i-1}) = \sum_{j\geq 0, j+i\equiv 0 \,(\mathrm{mod}\ q)} g_j(t)\theta^{(j+i)/q-1} = \sum_{\ell\geq 0, j=-i+q\ell} g_{q\ell-i}(t)\theta^{\ell-1} = \sum_{\ell\geq 0, j=-i+q\ell} g_{q\ell-i}(t)e_\ell.$$

Define now M as the matrix

$$M := (m_{i,j}) := (g_{jq-i})_{i,j=1,\ldots,D}.$$

This is the matrix representing the action of κ on the nucleus of $D(\underline{\mathcal{C}}^{(n)})$. Hence its dual characteristic polynomial is the L-function of $\underline{\mathcal{C}}^{(n)}$. Since A is reduced, we have shown:

Proposition 10.26.

$$\det(1-TM) = \zeta_A(-n, T) \in 1 + TA[T].$$

Example 10.27. Let $q = 4$, $n = 181 = 2\cdot 64 + 3\cdot 16 + 1\cdot 4 + 1\cdot 1$. Thus $\ell(n) = 7$ and $D = \lfloor \frac{\ell(n)}{q-1} \rfloor = \lfloor \frac{7}{3} \rfloor = 2$. We have

$$g(\theta) = (t^{64}-\theta)^2(t^{16}-\theta)^3(t^4-\theta)(t-\theta) = (t^{128}-\theta^2)(t^{32}-\theta^2)(t^{16}-\theta)(t^4-\theta)(t-\theta) = \sum_{j\geq 0} g_j(t)\theta^j.$$

One finds the following expressions for the coefficients g_j:

$$\begin{aligned} g_7 &= 1. \\ g_6 &= t^{16}+t^4+t. \\ g_5 &= t^{128}+t^{32}+t^{20}+t^{17}+t^5. \\ g_3 &= t^{160}+t^{148}+t^{145}+t^{133}+t^{52}+t^{49}+t^{37}. \\ g_2 &= t^{176}+t^{164}+t^{161}+t^{149}+t^{53}. \end{aligned}$$

The matrix M is given by $\left(\begin{smallmatrix} g_3 & g_7 \\ g_2 & g_6 \end{smallmatrix}\right)$ and one finds

$$\zeta_A(-181, T) = \det(M) = 1 + T(\ldots) + T^2(t^{164}+t^{161} + \text{lower order terms}).$$

Definition 10.28. For $n \in \mathbb{N}_0$ we set $S_d(n) := \sum_{a\in A_{d+}} a^{-n}$, so that $\zeta_A(-n,T) = \sum_{d\geq 0} T^d S_d(n)$, and we define $s_d(n) := \deg_t S_d(n)$.

Using M we shall give yet another proof of the following theorem due to Wan [49], which was later reproved by Diaz-Vargas [9] and by Thakur [48].

Theorem 10.29 (Riemann hypothesis)**.** *Let $q = p$. Then the following hold:*

(a) For any $n \in \mathbb{N}_0$ one has

$$\deg_T \zeta_A(-n,T) = \lfloor \frac{\ell(n)}{q-1} \rfloor \text{ and } s_d(n) = \sum_{j=1}^{s} \deg_t(g_{j(q-1)}).$$

(b) The sequence $(\deg_t g_{j(q-1)})_{j\geq 0}$ is strictly decreasing, and thus the Newton polygon of $\zeta_A(-n,T)$ has $\lfloor \frac{\ell(n)}{q-1} \rfloor$ distinct slopes all of width one.

(c) For any $m \in \mathbb{Z}_p$, the entire function $L_A(m,T)$ in T has a Newton polygon whose slopes are all of width one and thus the roots of $T \mapsto L_A(m,T)$ lie in K_∞ they are simple and of pairwise distinct valuation.

Part (c) for $q = 4$ was proved by Poonen and for arbitrary q by Sheats in [42].

Recall that the Newton polygon of the polynomial $\deg_T \zeta_A(-n,T) \in A[T] \subset K_\infty[T]$ is the lower convex hull of the points $(d, -\deg_t(S_d(n)))_{d\geq 0}$. By (a), the slope between d and $d+1$ is $-s_{d+1}(n) - (-s_d(n)) = -\deg_t g_{d(q-1)}$. By the first assertion of (b), the sequence $\deg_t g_{j(q-1)}$ is strictly decreasing and it follows that the points $(d, -\deg_t(S_d(n)))_{d\geq 0}$ lie all on the lower convex hull and are break points, i.e., points where the slope changes. Thus the second part of (b) follows, once (a) and the first is shown. Part (c) is a simple formal consequence of (b) as explained in [49]. To prove the theorem, our first aim will be to compute the degrees of the polynomials g_j.

Let us first recall a lemma of Lucas.

Lemma 10.30. *For integers $a_0, \ldots, a_r$ and $l_0, \ldots, l_r$ in the interval $[0, p-1]$ one has*

$$\prod_{i=0}^{r} \binom{a_i}{l_i} = \binom{\sum_{i=0}^r a_i p^i}{\sum_{i=0}^r l_i p^i}.$$

An analogous formula holds for multinomial coefficients $\binom{m}{m_1\, m_2 \ldots m_\ell}$.

Lucas' formula follows easily from expanding both sides of the following equality by the binomial theorem

$$(1+T)^{\sum_{i=0}^r a_i p^i} = \prod_{i=0}^{r} (1+T^{p^i})^{a_i}.$$

Lucas' lemma holds for arbitrary q; however the formulation for p implies the lemma also for all p-powers.

Note that by the Lemma of Lucas the coefficient $\binom{n}{l}$ modulo p is non-zero if and only if, considering the base p expansions of n and l, each digit of the expansion of l is at most as large as the corresponding digit of n.

Lemma 10.31.

$$g_j = (-1)^j \sum_{\substack{l_0,\ldots,l_r,\, 0\leq l_i \leq a_i \\ \sum_i l_i = d-j}} \left(\prod_{i=0}^{r} \binom{a_i}{l_i} \right) t^{l_0 + l_1 q + \ldots + l_r q^r} = (-1)^j \sum_{\substack{l=0 \\ \ell(l) = d-j}}^{n} \binom{n}{l} t^l.$$

For each $m \in \mathbb{N}_0$ the sum contains at most one summand of degree m in t.

Proof. The first equality in Lemma 10.31 is proved by a multiple application of the binomial theorem and collecting all the coefficients of θ^j in $\sum_{g\geq 0} g_j \theta^j = \prod_{i\geq 0} (T^{q^i} - \theta)^{a_i}$. The second equality follows from Lucas's lemma. $\square$

Set $\mu(D) = r$ and $m_{\mu(D)} = 0$ and for $j \in \{0, \ldots, D-1\}$ define $\mu(j) \in \{0, \ldots, r\}$ and $m_{\mu(j)} \in \{1, \ldots, a_{\mu(j)}\}$ by

$$D - j = a_r + \ldots + a_{\mu(j)+1} + m_{\mu(j)}.$$

Lemma 10.32. *Suppose $q = p$. Then for each $j \in \{0, \dots, D\}$ one has*

$$\deg g_j = a_r q^r + \ldots + a_{\mu(j)+1} q^{\mu(j)+1} + m_{\mu(j)} q^{\mu(j)}.$$

Proof. Note that in the case $p = q$ the coefficients $\left(\prod_{i=0}^r \binom{a_i}{l_i}\right)$ of g_j are all non-zero. The formulas are now immediate from the definitions of μ and m_μ. $\square$

Suppose $q = p$. Then if j increases, $D - j$ decreases and thus $\deg g_j$ is strictly decreasing in j. Thus we have proved Theorem 10.29(b) once we have proved part (a). Moreover if ℓ and ℓ' from $\{0, \dots, D\}$ satisfy $\ell' \geq \ell + (q-1)$, then $\mu(\ell') \geq \mu(\ell) + 1$. This yields a precise result on the rate of the decrease of the $\deg_t g_j$:

Lemma 10.33. *Suppose $p = q$. Then for $0 \leq \ell, \ell' \leq D - 1$ and $\ell' \geq \ell + (q-1)$ one has*

$$0 < \deg g_\ell - \deg g_{\ell+1} = q^{\mu(\ell)} \leq \frac{1}{q} \cdot q^{\mu(\ell')} = \frac{1}{q} \cdot (\deg g_{\ell'} - \deg g_{\ell'+1}).$$

Proof of Theorem 10.29(a). Our aim is to compute the degree in t of the coefficients of the T^i in the expression

$$\det(1 - TM) = \sum_{\pi \in \Sigma_d} \operatorname{sign} \pi (\delta_{1\pi(1)} - T m_{1\pi(1)}) \cdot \ldots \cdot (\delta_{d\pi(d)} - T m_{d\pi(d)}).$$

Let us expand the inner products by the distributive law. If a product contributes to T^i, then in the distributed term, we need $d - j$ occurrences of terms not involving T, i.e. of 1's. The latter can only come from the diagonal. We deduce:

Up to sign, the terms contributing to T^i are those $j \times j$-minors of M which are symmetric, i.e. in which the same rows and columns were deleted from M.

Let J be a subset of $\{1, \dots, d\}$ (which may be empty) and let $\pi \in \Sigma_d$ be a permutation of the set $\{1, \dots, d\}$ which is the identity on J. Then for the pair J, π we define

$$\deg_{J,\pi} := \sum_{j \in J} \deg(m_{\pi(j),j}).$$

For fixed J, we shall show that the identity permutation is the unique one for which $\deg_{J,\pi}$ is maximal. The following lemma is the key step.

Lemma 10.34. *Suppose $q = p$ and fix $J \subset \{1, \dots, d\}$. Then for all $\pi \in \Sigma_d \smallsetminus \{\mathrm{id}\}$ fixing J one has* $\deg_{\pi,J} < \deg_{\mathrm{id},J}$. *In particular* $\operatorname{ord}(\det(M)) = \operatorname{ord}_{\mathrm{id}}$ *and thus* $\det(M)$ *is non-zero.*

Proof. We only give the proof for $J = \varnothing$. The other cases are totally analogous. To simplify notation, we abbreviate $\deg_\pi := \deg_{\varnothing,\pi}$.

Assume contrary to the assertion of the lemma, that for some $\pi \in \Sigma_d \smallsetminus \{\mathrm{id}\}$ we have $\deg_\pi = \max\{\deg_\tau \mid \tau \in \Sigma_d\}$. Since π is not the identity, the permutation matrix representing π has some entry above the diagonal. Let $j_0 \in \{1, \dots, d\}$ be maximal such that $\pi(j_0) < j_0$. In row j_0 let j_1 be the column which contains the non-zero entry of the permutation matrix of π, i.e., $j_1 = \pi^{-1}(j_0)$. By the maximality of j_0 we have we have

$$j_1 < j_0. \tag{1}$$

Consider the matrix

$$\begin{pmatrix} \ddots & & & & \\ & 0 & & m_{\pi(j_0),j_0} & \\ & & \ddots & & \\ & m_{j_0,j_1} & & 0 & \\ & & & & \ddots \end{pmatrix}$$

with entries $m_{\pi(j),j}$ at $(\pi(j), j)$ for $j = 1, \dots, d$ and zero otherwise. Define the permutation π' by

$$\pi'(j) = \pi(j) \text{ for } j \neq j_0, j_1, \quad \pi'(j_0) = j_0, \quad \pi'(j_1) = \pi(j_0).$$

Then

$$\begin{aligned}
\deg_{\pi'} - \deg_{\pi} &= \deg(m_{\pi(j_0),j_1}) + \deg(m_{j_0,j_0}) - \deg(m_{j_0,j_1}) - \deg(m_{\pi(j_0),j_0}) \\
&= \deg(g_{j_1 q-\pi(j_0)}) - \deg(g_{j_1 q - j_0}) + \deg(g_{j_0 q - j_0}) - \deg(g_{j_0 q-\pi(j_0)}) \\
&= \sum_{i=\pi(j_0)+1}^{j_0} (\deg(g_{j_0 q-i}) - \deg(g_{j_0 q-i+1})) - (\deg(g_{j_1 q-i}) - \deg(g_{j_1 q-i+1})) \\
&\geq \sum_{i=j_1+1}^{\pi(j_1)} (q-1) \quad \geq \quad (q-1).
\end{aligned}$$

Where the last inequality follows from formula (1) and Lemma 10.33(a). We reach the contradiction $\deg_{\pi'} > \deg_{\pi}$. □

For fixed J, the lemma tells us that id is the unique permutation for which $\deg_{J,\mathrm{id}}$ is maximal. Moreover from the definition of $\deg_{J,\pi}$, we see that

$$\deg_{J,\mathrm{id}} = \sum_{j\in J} \deg_t m_{jj} = \sum_{j \in J} \deg_t g_{j(q-1)}.$$

Since the degrees of the g_i are strictly decreasing, we find that among those J for which $\#J$ is fixed there is also a unique one for which $\deg_{J,\mathrm{id}}$ is maximal, namely $J = \{1, 2, \dots, \#J\}$.

It follows that the degree of the coefficient of T^i is equal to $\deg_{\{1,2,,\dots,i\},\mathrm{id}} = \sum_{j=1,\dots,i} \deg_t g_{j(q-1)}$. This completes the proof of Theorem 10.29(a) and thus of the theorem itself. □

Let us add some further observations regarding the degrees of the coefficients $S_d(n)$. Say we write

$$(a_r\, a_{r-1}\, \dots a_1\, a_0)_p$$

for the base p digit expansion of n. By the definition of $\mu(j)$ and $m_{\mu(j)}$, we have that for

$$(0\,0\, \dots\, 0\, \underbrace{a_{\mu(j)} - m_{\mu(j)}}_{\mu(j)}\, a_{\mu(j)-1}\, \dots\, a_1\, a_0)$$

the sum over its digit in base p is exactly j. We define

$$n_1 := (0\,0\, \dots\, \underbrace{a_{\mu(p-1)} - m_{\mu(p-1)}}_{\mu(p-1)}\, a_{\mu(p-1)-1}\, \dots\, a_1\, a_0)$$

$$n_2 := (0\,0\, \dots\, \underbrace{a_{\mu(2p-2)} - m_{\mu(2p-2)}}_{\mu(2p-2)}\, a_{\mu(2p-2)-1}\, \dots\, \underbrace{m_{\mu(p-1)}}_{\mu(p-1)}\, 0\, \dots\, 0)$$

$$n_3 := (0\,0\, \dots\, \underbrace{a_{\mu(3(p-1)} - m_{\mu(3(p-1))}}_{\mu(3(p-1))}\, a_{\mu(3(p-1))-1}\, \dots\, \underbrace{m_{\mu(2(p-1))}}_{\mu(2(p-1))}\, 0\, \dots\, 0) etc.$$

and m_d by $n = m_d + n_d + n_{d-1} + \ldots + n_1$. Thus n_1 is formed from the $p-1$ lowest digits of n, next n_2 is formed from the next $p-1$ lowest digits of n that haven't been used in forming n_1 etc.

We can now rephrase Theorem 10.29(a) as follows:

$s_1(n) = n - n_1,\ s_2(n) = (n-n_1) + (n-n_1-n_2),\ \dots, s_d(n) = (n-n_1) + (n-n_1-n_2) + \ldots + (n-n_1-\ldots-n_d)$ etc.

and $s_\ell(n) = -\infty$ for all $\ell \geq \ell_0 + 1$ where ℓ_0 is smallest so that $n - n_1 - \ldots - n_{\ell_0}$ has digit sum less than $p-1$ in its p-digit, or if no such $\ell_0 > 0$ exists, we set $\ell_0 = 0$.

We obtain an alternative proof of the following recursion from [48]:

Corollary 10.35 (Thakur). $s_d(n) = s_1(n) + s_{d-1}(s_1(n))$.

Proof.

$$\begin{aligned} s_d(n) &= (n-n_1) + (n-n_1-n_2) + \ldots + (n-n_1-\ldots-n_d) \\ &= (n-n_1) + ((n-n_1)-n_2) + ((n-n_1)-n_2-n_3) + \ldots + ((n-n_1)-\ldots-n_d) \\ &= s_1(n) + (s_1(n)-n_2) + (s_1(n)-n_2-n_3) + \ldots + (s_1(n)-n_2-\ldots-n_d) \\ &= s_1(n) + s_{d-1}(s_1(n)). \end{aligned}$$

Note that to define $n_2, n_3, \ldots, n_d$ it is not necessary to know n. It suffices to know $n - n_1 = s_1(n)$ as defined above. ☐

Open questions regarding the zero distributions of $\zeta_A(-n,T)$ for A different from $\mathbb{F}_q[T]$:

Due to some simple but remarkable examples, Thakur [46] showed that Theorem 10.29(c) cannot hold for general A: It is known that $\zeta_A(-n,T)$ has a zero $T=1$ if $q-1$ divides n. Such zeros are called *trivial zeros*. For $\mathbb{F}_q[t]$ all trivial zeros are simple and if n is not divisible by $q-1$ then $T=1$ is not a root of $\zeta_A(-n,T)$. In [46] Thakur shows by explicit computation that for some even n, i.e., n divisible by $q-1$, the root $T=1$ is a double root! Let us say that $\zeta_A(-n,T)$ has an **extra zero at** $-n$ if either $(T-1)^2$ divides it, or if $(T-1)$ divides it, but n is not a multiple of $p-1$.

The following patterns for negative integers $-n$ were **observed when computing**, using a computer algebra package, the Newton polygons for several rings A for the function $\zeta_A(-n,T) := \sum_{d\geq 0} T^d \sum_{a\in A_{d+}} a^n$ under the hypotheses $p = q$ **and** $d_\infty = 1$. Note that the Newton polygons all lie under or on the x-axis and start at $(0,0)$. Moreover all slopes are less or equal to zero.

Define $B \subset \mathbb{N}_0$ as

$$B := \{0\} \cup \{d \in \mathbb{N}_0 \mid \dim_k A_{(d+1)} > \dim A_d\}.$$

In particular, B contains every integer $d \geq 2g$, where g is the genus of A. The missing integers (in $\{0,\ldots,2g\}$) are precisely the the Weierstrass gaps for ∞.

- The x-coordinates of all the break and end points of all Newton polygons are in the set B and at every x-coordinate in B (along the Newton polygon) there is a break point.
- In particular, all slopes beyond the g-th one have width 1.
- There are no extra zeros for n not divisible by $p-1$. Thus extra zeros can only occur at horizontal slopes of width larger than one, i.e., among the first g slopes.
- Even if a slope is horizontal and of length 2, there may not be an extra zero.
- The degree in T of $\zeta_A(-n,T)$ is determined by the following rule: The number of slopes of the Newton polygon is exactly equal to $\lfloor \frac{\ell(n)}{p-1} \rfloor$.

Chapter 11

Relation to étale sheaves

Throughout this chapter we assume that A *is a finite k-algebra, and so in particular A is finite.* For such A, we shall set up a correspondence between A-crystals and étale sheaves of A-modules. As we shall see, the correspondence is modeled at the Artin-Schreier sequence in étale cohomology and Deligne's [51, Fonctions L].

One reason why one might be interested in finite rings A is to study geometric questions in positive characteristic via étale mod p cohomology. Another reason is the following: Suppose φ is a Drinfeld A-module and $\underline{\mathcal{M}}(\varphi)$ its associated A-motive. Then for all finite primes $\mathfrak{p}$ of A, the $\mathfrak{p}^n$-torsion of φ provides us with a Galois representation, or a lisse étale sheaf of $A/\mathfrak{p}^n$-modules on the base. On the side of A-motives, $A/\mathfrak{p}^n$-torsion is described by $\underline{\mathcal{M}}(\varphi) \otimes_A A/\mathfrak{p}^n$. Using the equivalence of categories introduced in the following subsection and the relation between the torsion points of φ and the above quotient of $\underline{\mathcal{M}}(\varphi)$, the following result is straightforward:

$$\mathrm{Hom}_A(A/\mathfrak{p}^n, \varphi[\mathfrak{p}^n]) \cong \epsilon(\underline{\mathcal{M}}(\varphi) \otimes_A A/\mathfrak{p}^n).$$

I.e., the dual of the module of $\mathfrak{p}^n$-torsion points is naturally associated to the motive modulo $\mathfrak{p}^n$.

In Section 1 we define the functor from A-crystals to constructible étale sheaves of A-modules and discuss its basic properties. Some proofs are given. In the subsequent Section 2 we use these results to reprove (in many but not all cases) a result of Goss and Sinnott which links properties of class groups to special values of Goss' L-functions.

1 An equivalence of categories

Our first aim is to define a functor

$$\epsilon\colon \mathbf{QCoh}_\tau(X, A) \to \mathbf{\acute{E}t}(X, A).$$

For this, we consider a τ-sheaf $\underline{\mathcal{F}}$. Using adjunction, we assume that it is given by a pair $(\mathcal{F}, \tau_{\mathcal{F}}\colon \mathcal{F} \to (\sigma \times \mathrm{id})_*\mathcal{F})$. Let $u\colon U \to X$ be any étale morphism. Pullback of τ along $u \times \mathrm{id}$ induces a homomorphism

$$(u \times \mathrm{id})^*\tau_{\mathcal{F}}\colon (u \times \mathrm{id})^*\mathcal{F} \to (u \times \mathrm{id})^*(\sigma \times \mathrm{id})_*\mathcal{F} \cong (\sigma \times \mathrm{id})_*(u \times \mathrm{id})^*\mathcal{F}.$$

Taking global section on $U \times C$ and observing that $\sigma \times \mathrm{id}$ is a topological isomorphism, we obtain a homomorphism of A-modules:

$$(u \times \mathrm{id})^*\tau_{\mathcal{F}}\colon ((u \times \mathrm{id})^*\mathcal{F})(U \times C) \longrightarrow ((\sigma \times \mathrm{id})_*(u \times \mathrm{id})^*\mathcal{F})(U \times C) = ((u \times \mathrm{id})^*\mathcal{F})(U \times C).$$

By slight abuse of notation, let us denote this homomorphism by τ_{et}. Then one verifies that

$$(u\colon U \to V) \;\longmapsto\; \mathrm{Ker}\left(1 - \tau_{\mathrm{et}}\colon ((u \times \mathrm{id})^*\mathcal{F})(U \times C) \longrightarrow ((u \times \mathrm{id})^*\mathcal{F})(U \times C)\right)$$

for $(u: U \to X)$ varying over the étale morphisms to X defines a sheaf of A-modules on the small étale site of X denoted $\epsilon(\underline{\mathcal{F}})$. A more concise way of defining ϵ is as follows. Let $\mathcal{F}_{\mathrm{et}}$ denote the étale sheaf associated to $\mathrm{pr}_{1*}\mathcal{F}$ by change of sites – this is what is done above, of one forgets about τ. Then $\tau_{\mathcal{F}}$ induces a homomorphism $\tau_{\mathrm{et}}\colon \mathcal{F}_{\mathrm{et}} \to \mathcal{F}_{\mathrm{et}}$ and

$$\epsilon(\underline{\mathcal{F}}) := \mathrm{Ker}\big(\mathrm{id} - \tau_{\mathrm{et}}\colon \mathcal{F}_{\mathrm{et}} \to \mathcal{F}_{\mathrm{et}}\big). \tag{1}$$

Clearly this construction is functorial in $\underline{\mathcal{F}}$, that is, to any homomorphism $\varphi\colon \underline{\mathcal{F}} \to \underline{\mathcal{G}}$ it associates a homomorphism $\epsilon(\varphi)\colon \epsilon(\underline{\mathcal{F}}) \to \epsilon(\underline{\mathcal{G}})$. Thus it defines an A-linear functor

$$\epsilon\colon \mathbf{QCoh}_\tau(X, A) \to \mathbf{\acute{E}t}(X, A). \tag{2}$$

Following its construction one finds that ϵ is left exact.

Example 11.1. Let $\mathbb{1}_{X,A}$ denote the τ-sheaf consisting of the structure sheaf $\mathcal{O}_{X\times C}$ together with its obvious τ given by

$$\tau = \sigma \times \mathrm{id}\colon \mathcal{O}_{X\times C} \to (\sigma \times \mathrm{id})_*\mathcal{O}_{X\times C}.$$

The étale sheaf associated to $\mathcal{O}_{X\times C}$ is simply $\mathcal{O}_{X_{\mathrm{et}}} \otimes A$ with τ_{et} the morphism $(u \otimes a) \mapsto u^q \otimes a$. Therefore $\epsilon(\mathbb{1}_{X,A}) \cong \underline{A}_X$, the constant étale sheaf on X with stalk A. In the special case $A = k$ we recover the sequence

$$0 \longrightarrow \underline{k}_X \longrightarrow \mathcal{O}_{X_{\mathrm{et}}} \xrightarrow{1-\sigma} \mathcal{O}_{X_{\mathrm{et}}}$$

from Artin-Schreier theory.

Lemma 11.2. *Let $\varphi: \underline{\mathcal{F}} \to \underline{\mathcal{G}}$ be a nil-isomorphism in $\mathbf{Coh}_\tau(X, A)$. Then the induced $\epsilon(\varphi): \epsilon(\underline{\mathcal{F}}) \to \epsilon(\underline{\mathcal{G}})$ is an isomorphism.*

Proof. Observe first that regarding τ as a homomorphism of τ-sheaves, we have $\epsilon(\tau) = \mathrm{id}$: This is so, because ϵ is precisely the operation on the étale sheaf associated to $\underline{\mathcal{F}}$ of taking fixed points under τ. Clearly τ is the identity on the set of these fixed points. Having clarified this, the proof of the proposition follows immediately from applying τ to the diagram (2). $\square$

The assertion of the lemma also holds for τ-sheaves whose underlying sheaf is only quasi-coherent. The proof however is much more subtle. In [8] it is shown that ϵ factors via the category of ind-coherent τ-sheaves, i.e., τ-sheaves which can be written as inductive limits of coherent τ-sheaves. Then an argument involving direct limits reduces one to the already proved case of the lemma. In total one obtains:

Proposition 11.3. *The functor ϵ induces a unique left exact A-linear functor*

$$\overline{\epsilon}\colon \mathbf{QCrys}(X, A) \to \mathbf{\acute{E}t}(X, A).$$

It is shown in [8] that the isomorphisms ϵ defined for all pairs (X, A) (with A finite) is compatible with the formation of functors on crystals and on the étale site, respectively:

Proposition 11.4. *For $f: Y \to X$ a morphism, $j: U \hookrightarrow X$ an open immersion and $h: C \to C'$ a base change homomorphism one has the following compatibilities:*

$$\epsilon \circ f^* \cong f^* \circ \epsilon, \quad \epsilon \circ (\_\_ \otimes \_\_) \cong (\_\_ \otimes \_\_) \circ \epsilon, , \quad \epsilon \circ (\_\_ \otimes_A A') \cong (\_\_ \otimes_A A') \circ \epsilon \quad \epsilon \circ f_* \cong f_* \circ \epsilon, \quad \epsilon \circ j_! \cong j_! \circ \epsilon.$$

Except for the very first compatibility, i.e. that of inverse image, the proofs are rather straight forward. Note that to the left of $\circ\epsilon$ the functors are functors on étale sheaves and to the right of $\epsilon\circ$ they are functors on A-crystals.

Let $\mathbf{\acute{E}t}_c(X, A) \subset \mathbf{\acute{E}t}(X, A)$ denote the subcategory of constructible étale sheaves. Recall that an étale sheaf of A-modules is constructible, if X has a finite stratification by locally closed subsets U_i such that the restriction of the sheaf to each U_i is locally constant. This in turn means that there exists a *finite* étale morphism $V_i \to U_i$ such that the pullback to V_i is a constant sheaf on a finite A-module.

Proposition 11.5. *The image of* $\mathbf{Crys}(X,A)$ *under* ϵ *lies in* $\mathbf{\acute{E}t}_c(X,A)$.

Proof. (Sketch) Since being constructible is independent of the A-action, we may restrict the proof to $\mathbf{Crys}(X,k)$. We also may assume that X is reduced – cf. Theorem 5.9. Let $\underline{\mathcal{F}}$ be a coherent τ-sheaf on X over k. Since we have the functors $j_!$, f_* and f^* at our disposal we can apply noetherian induction on X in order to show that $\epsilon(\underline{\mathcal{F}})$ is constructible. Thus it suffices to fix a generic point η of X and to prove that there exists an open neighborhood U of η such that $\epsilon(\underline{\mathcal{F}}|U)$ is locally constant.

We first choose a neighborhood U of η which is regular as a scheme. By [29, Thm. 4.1.1], it suffices to show that after possibly further shrinking U one can find a τ-sheaf $\underline{\mathcal{G}}$ which is nil-isomorphic to $\underline{\mathcal{F}}$ and such that $\underline{\mathcal{G}}$ is locally free and τ is an isomorphism on it. At the generic point both properties can be achieved by replacing $\underline{\mathcal{F}}$ by $\mathrm{Im}(\tau_{\mathcal{F}}^m)$ for m sufficiently large. And then one shows, using $A=k$, that this extends to an open neighborhood of η. $\square$

A main theorem of [8, Ch. 10] is the following

Theorem 11.6. *For A a finite k-algebra, the functor* $\epsilon\colon \mathbf{Crys}(X,A)\to\mathbf{\acute{E}t}_c(X,A)$ *is an equivalence of categories.*

Since it is compatible with all functors, the definition of flatness for both categories implies that ϵ induces an equivalence between the full subcategory of flat A-crystals and the full subcategory of flat constructible étale sheaves of A-modules:

$$\epsilon\colon \mathbf{Crys}^{\mathrm{flat}}(X,A)\xrightarrow{\cong}\mathbf{\acute{E}t}_c^{\mathrm{flat}}(X,A).$$

For flat A-crystals we have a definition of L-functions of X is of finite type over k. Under the same hypothesis on X, for flat constructible étale sheaves of A-modules such a definition is given in [51, Fonctions L, 2.1]. At a closed point x, it is the following:

Definition 11.7. The *L-function* of $\mathsf{F}\in\mathbf{\acute{E}t}_c^{\mathrm{flat}}(x,A)$ is

$$L(x,\mathsf{F},t)\ :=\ \det_A\bigl(\mathrm{id}-t^{d_x}\cdot\mathrm{Frob}_x^{-1}\bigm|\mathsf{F}_{\bar{x}}\bigr)^{-1}\ \in\ 1+t^{d_x}A[[t^{d_x}]].$$

The obvious extensions to schemes X of finite type over k and to complexes representable by bounded complexes of objects in $\mathbf{\acute{E}t}_c^{\mathrm{flat}}(X,A)$ are left to the reader.

It is a basic result that ϵ is compatible with the formation of L-functions:

Proposition 11.8. *For any* $\underline{\mathcal{F}}\in\mathbf{Crys}^{\mathrm{flat}}(X,A))$ *we have*

$$L(X,\epsilon(\underline{\mathcal{F}}),t)=L^{\mathrm{crys}}(X,\underline{\mathcal{F}},t).$$

As a consequence of Theorem 8.14 we find:

Theorem 11.9. *Let $f:Y\to X$ be a morphism of schemes of finite type over k and* $\mathsf{F}^\bullet\in\mathbf{D}^b(\mathbf{\acute{E}t}_c(Y,A))_{\mathrm{ftd}}$ *a complex representable by a bounded complex with objects in* $\mathbf{\acute{E}t}_c^{\mathrm{flat}}(X,A)$. *Then one has*

$$L(Y,\mathsf{F}^\bullet,t)\sim L(X,Rf_!\mathsf{F}^\bullet,t),$$

i.e., their quotient is a unipotent polynomial.

For reduced coefficient rings A, the above result was first proved by Deligne in [51, Fonctions L, Th. 2.2]. In [13, Thm. 1.5], Emerton and Kisin give a proof for arbitrary finite A of some characteristic p^m. By an inverse limit procedure, in [13, Cor. 1.8], they give a suitable generalization to any coefficient ring A which is a complete noetherian local $\mathbb{Z}_p$-algebra with finite residue field.

Remark 11.10. The category $\mathbf{\acute{E}t}_c(X,A)$ has no duality and $f^!$, f_* and an internal Hom are either not all defined or not well-behaved. Thus for the theory of A-crystals, we cannot hope for this either.

2 A result of Goss and Sinnott

In the following we shall use the correspondence between étale sheaves and crystals of the previous subsection to reprove a result of Goss and Sinnott – in many, but so far not all cases. The original proof of the result of Goss and Sinnott is based on the comparison of classical L-functions for function fields and Goss-Carlitz type L-functions for function fields. Our proof avoids all usage of classical result but instead uses the results from the previous subsections.

Class groups of Drinfeld-Hayes cyclotomic fields

We consider the following situation: Let K be a function field with constant field k, let ∞ be a place of K and A the ring of regular functions outside ∞. Let $H \subset H+$ be the (strict) Hilbert class field with ring of integers $\mathcal{O} \subset \mathcal{O}^+$. By the theory of Drinfeld-Hayes modules, there exist $[H : K]$ many sign normalized rank 1 Drinfeld-Hayes modules

$$\varphi \colon A \to \mathcal{O}^+[\tau], a \mapsto \varphi_a.$$

Let $\mathfrak{p}$ be a maximal ideal of A. Then the $\mathfrak{p}$-torsion points $\varphi[\mathfrak{p}](\bar{K})$ of φ over $\bar{K}$ form a free $A/\mathfrak{p}$-module of rank 1 carrying an A-linear action of $\mathrm{Gal}(\bar{K}/H^+)$. If $H_{\mathfrak{p}}^+$ denotes the fixes field of the kernel of this representation, then $G := \mathrm{Gal}(H_{\mathfrak{p}}^+/H^+)$ is isomorphic to $(A/\mathfrak{p})^*$. The extension $H_{\mathfrak{p}}^+/H^+$ is totally ramified at the places of $\mathcal{O}^+$ above $\mathfrak{p}$ and unramified above all other finite places. For all places above ∞ the decomposition and inertia groups agree and are isomorphic to the subgroup $k^* \subset (A/\mathfrak{p})^*$.

$$\begin{array}{c} H_{\mathfrak{p}}^+ \\ \big|{\scriptstyle G} \\ H^+ \\ \big| \\ K \end{array}$$

For $K = k(t)$ and $A = k[t]$ one has $H = K$ and the Drinfeld module φ is simply the Carlitz module.

Let us denote by $\chi \colon G \to (A/\mathfrak{p})^*$ the character of G the arises from the action of $\varphi[\mathfrak{p}]$. This is the analog of the mod p cyclotomic character in classical number theory. We introduce the following notation: By $\mathrm{Jac}_{K,\mathfrak{p}}$ we denote the Jacobian variety of the smooth projective geometrically irreducible curve $C_{K,\mathfrak{p}}$ with constant field k_∞ and function field H^+ and we let $\mathrm{Cl}(H_{\mathfrak{p}}^+)$ denote the class group of the field $H_{\mathfrak{p}}^+$. Then the p-torsion subgroup of $\mathrm{Cl}(H_{\mathfrak{p}}^+)$ is isomorphic to the invariant of $\mathrm{Gal}(\bar{k}_\infty/k_\infty)$ of the p-torsion group $\mathrm{Jac}_{K,\mathfrak{p}}[p](\bar{k})$. For $w \in \mathbb{Z}$ (it suffices $w \in \{1, 2, \ldots, \#(A/\mathfrak{p})^*\}$) we define the χ^w components of the above groups as

$$C(w) := (\mathrm{Cl}(H_{\mathfrak{p}}^+) \otimes_{\mathbb{F}_p} A/\mathfrak{p})_{\chi^w} \quad \widetilde{C}(w) := (\mathrm{Jac}_{K,\mathfrak{p}}[p](\bar{k}) \otimes_{\mathbb{F}_p} A/\mathfrak{p})_{\chi^w}.$$

We remark the following: The extension $\bar{k}H^+/H^+$ is purely algebraic, the extension $H_{\mathfrak{p}}^+/H^+$ purely geometric. Hence they are linearly disjoint. Moreover the group G is of order prime to p and thus its action on the p-group $\mathrm{Jac}_{K,\mathfrak{p}}[p](\bar{k})$ is exact, so that

$$\widetilde{C}(w)^{\mathrm{Gal}(\bar{k}_\infty/k_\infty)} = C(w).$$

Let $\tilde{h}_A+$ denote the number of places of H^+ above ∞, so that $\tilde{h}_A^+ = h_A \frac{\#k_\infty^*}{\#k^*}$. The following result (even under more general hypotheses) is due to Goss and Sinnott:

Theorem 11.11 (Goss, Sinnott)**.** *Let $w \in \mathbb{N}$. For $a, b \in \mathbb{N}$ define $\delta_{a|b}$ to be 1 if a is a divisor of b and zero otherwise. Then the following hold:*

(a) $\dim_{A/\mathfrak{p}} \widetilde{C}(w) = \deg_T(L_{\mathrm{Spec}\,\mathcal{O}^+}(w, T) \mod \mathfrak{p}) - \tilde{h}_A^+ \delta_{(q-1)|w}$.

(b) $C(w) \neq 0$ *if and only if* $\mathrm{ord}_{T=1}(L_{\mathrm{Spec}\,\mathcal{O}^+}(w, T) \mod \mathfrak{p}) > \tilde{h}_A^+ \delta_{(q-1)|w}$.

(c) $\dim_{A/\mathfrak{p}} C(w)$ *is the multiplicity of the eigenvalue* 1 *of the action of* τ *on* $H^1(C_{K,\mathfrak{p}}, (\underline{\mathcal{M}}(\varphi)^{\otimes w})^{\max}) \otimes_A A/\mathfrak{p}$ *– for the superscript* max*, see Definition 11.12.*

The proof in [24] uses congruences between L-functions of τ-sheaves and classical Hasse-Weil L-functions. This comparison is replaced by comparing the cohomology of a τ-sheaf and that of the étale sheaf associated to its mod $\mathfrak{p}$-reduction.

Before we can give the proof of the theorem, we need to introduce the concept of maximal extension of a τ-sheaf. Once this is understood, we shall present a proof of the above theorem different from that in [24].

The maximal extension of Gardeyn

The material on maximal extensions is based on work and ideas of F. Gardeyn from [15, §2].. We follow the approach in [4]. By B we denote a k-algebra which is essentially of finite type. Typically it is equal to A or to $A/\mathfrak{n}$ for some ideal $\mathfrak{n} \subset A$. We omit almost all proofs. The can be found either in [15, §2] or in [4, Ch. 8].

Throughout the discussion of maximal extensions, we fix an open immersion $j : U \hookrightarrow X$ and a complement $Z \subset X$ of U.

Definition 11.12 (Gardeyn)**.** Suppose $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U, B)$.

(a) A coherent τ-subsheaf $\underline{\mathcal{G}}$ of $j_*\underline{\mathcal{F}}$ with $j^*\underline{\mathcal{G}} = \underline{\mathcal{F}}$ is called an *extension of* $\underline{\mathcal{F}}$.

(b) The union of all extensions of $\underline{\mathcal{F}}$ is denoted by $j_\#\underline{\mathcal{F}} \subset j_*\underline{\mathcal{F}}$..

(c) If $j_\#\underline{\mathcal{F}}$ is coherent, it is called the *maximal extension of* $\underline{\mathcal{F}}$. It is also denoted by $\underline{\mathcal{F}}^{\max}$.

The assignment $\underline{\mathcal{F}} \mapsto j_\#\underline{\mathcal{F}}$ defines a functor $\mathbf{Coh}_\tau(U, B) \longrightarrow \mathbf{QCoh}_\tau(X, B)$. Note that if $j_*\mathcal{F}$ is not coherent, the same holds for $j_\#(\mathcal{F}, 0) = (j_*\mathcal{F}, 0)$ – consider for instance the case $\operatorname{Spec} R \hookrightarrow \operatorname{Spec} K$ where R is a discrete valuation ring with fraction field K.

We state some basic properties:

Proposition 11.13. *Any τ-sheaf $\underline{\mathcal{F}}$ has an extension to X which represents the crystal. $j_!\underline{\mathcal{F}}$.*

Proof. This follows from the part of the proof of Theorem 5.10 giving the existence of the crystal $j_!\underline{\mathcal{F}}$. ☐

The τ-sheaf $j_\#\underline{\mathcal{F}}$ has the following intrinsic characterization modeled after the Néron mapping property:

Proposition 11.14. *Suppose $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U, B)$ and $\underline{\mathcal{G}} \in \mathbf{QCoh}_\tau(X, B)$ such that $j^*\underline{\mathcal{G}} \cong \underline{\mathcal{F}}$. Then $\underline{\mathcal{G}}$ is isomorphic to $j_\#\underline{\mathcal{F}}$ if and only if the following conditions hold:*

(a) $\underline{\mathcal{G}}$ is an inductive limit of coherent τ-sheaves, and

(b) for all $\underline{\mathcal{H}} \in \mathbf{Coh}_\tau(X, B)$, the following canonical map is an isomorphism:

$$\operatorname{Hom}_{\mathbf{QCoh}_\tau(X,B)}(\underline{\mathcal{H}}, \underline{\mathcal{G}}) \longrightarrow \operatorname{Hom}_{\mathbf{Coh}_\tau(U,B)}(j^*\underline{\mathcal{H}}, \underline{\mathcal{F}})$$

Proposition 11.14 motivates the following axiomatic definition of maximal extension for crystals:

Definition 11.15. A crystal $\underline{\mathcal{G}} \in \mathbf{Crys}(X, B)$ is called an extension of $\underline{\mathcal{F}}$ if $j^*\underline{\mathcal{G}} \cong \underline{\mathcal{F}}$. It is called a *maximal extension* if in addition for all $\underline{\mathcal{H}} \in \mathbf{Crys}(X, B)$, the canonical map

$$\operatorname{Hom}_{\mathbf{Crys}(X,B)}(\underline{\mathcal{H}}, \underline{\mathcal{G}}) \longrightarrow \operatorname{Hom}_{\mathbf{Crys}(U,B)}(j^*\underline{\mathcal{H}}, \underline{\mathcal{F}})$$

is an isomorphism.

Proposition 11.16. *Let $\underline{\mathcal{F}}$ be in $\mathbf{Coh}_\tau(X,B)$. If $j_\#\underline{\mathcal{F}}$ is coherent, then the crystal represented by $\underline{\mathcal{F}}$ possesses a maximal extension and the latter is represented by $j_\#\underline{\mathcal{F}}$.*

Proposition 11.17. *The functor $j_\#$ is left exact on τ-sheaves. Moreover if one has a left exact sequence of crystals, such that the outer terms have a maximal extension, then so does the central term and the induced sequence of the maximal extensions is left exact.*

We now impose the following conditions sufficient for our intended applications. Under these, the main result on the existence of a maximal extension, Theorem 11.22, is due to Gardeyn.

(a) the ring B is finite over k or over A.

(b) X is a smooth geometrically irreducible curve over k and $U \subset X$ is dense.

Proposition 11.18. *For $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U,B)$ and $\underline{\mathcal{G}}$ an extension of $\underline{\mathcal{F}}$, the following assertions are equivalent:*

(a) $\underline{\mathcal{G}}$ is the maximal extension of $\underline{\mathcal{F}}$

(b) For any $x \in Z$ and $j_x\colon \operatorname{Spec}\mathcal{O}_{X,x} \hookrightarrow X$ the canonical morphism, the τ-sheaf $j_x^*\underline{\mathcal{G}}$ is the maximal extension of $i_\eta^*\underline{\mathcal{F}}$; here η is the generic point of X

This proposition allows one to reduce the problem of the existence of a maximal extension to the situation where X is a discrete valuation ring. The proof is a simple patching argument.

Definition 11.19 (Gardeyn)**.** Let $\underline{\mathcal{G}}$ be a locally free τ-sheaf on X over B. Then $\underline{\mathcal{G}}$ is called *good* at $x \in X$ if τ is injective on $i_x^*\underline{\mathcal{G}}$. It is called *generically good* if it is good at the generic point of X.

Note that if $\underline{\mathcal{G}}$ is generically good, then it is good for all x in a dense open subset.

Proposition 11.20. *Suppose $\underline{\mathcal{G}} \in \mathbf{Coh}_\tau(X,B)$ is an extension of $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(U,B)$ such that $\underline{\mathcal{G}}$ is good at all $x \in Z$, then $\underline{\mathcal{G}} = j_\#\underline{\mathcal{F}}$.*

The point is that after pulling back the situation to any $\operatorname{Spec}\mathcal{O}_{X,x}$ for $x \in Z$, the fact that $\underline{\mathcal{G}}$ is good at x easily implies that it is a maximal extension. Now one can apply Proposition 11.18.

Corollary 11.21. *The unit τ-sheaf $\underline{\mathbb{1}}_{X,A}$ is good at all $x \in X$. Suppose now that $\underline{\mathcal{G}} \in \mathbf{Coh}_\tau(X,B)$ is an extension of $\underline{\mathcal{F}} \in \mathbf{Coh}_\tau(X,B)$ such that $i_x^*\underline{\mathcal{G}} \cong \underline{\mathbb{1}}_{x,A}$ for all $x \in Z$. Then $\underline{\mathcal{G}} = j_\#\underline{\mathcal{F}}$ and moreover in $\mathbf{Crys}(X,B)$ the following sequence is exact:*

$$0 \longrightarrow j_!\underline{\mathcal{F}} \longrightarrow j_\#\underline{\mathcal{F}} \longrightarrow \oplus_{x\in Z}\underline{\mathbb{1}}_{x,A} \longrightarrow 0.$$

The following are the central results on maximal extensions:

Theorem 11.22 (Gardeyn)**.** *If $\underline{\mathcal{F}}$ is a locally free, generically good τ-sheaf on U over B, then $j_\#\underline{\mathcal{F}}$ is locally free.*

Theorem 11.23. *Suppose B is finite. Then $j_\# : \mathbf{Crys}(U,B) \to \mathbf{Crys}(X,B)$ is a well-defined functor. Moreover one has $\epsilon \circ j_\# \cong j_* \circ \epsilon$ where $\epsilon\colon \mathbf{Crys}(\ldots) \to \mathbf{\acute{E}t}_c(\ldots)$ is the functor Theorem 11.6.*

Another simple assertion along the lines of Corollary 11.21 is the following:

Proposition 11.24. *Suppose $\underline{\mathcal{F}} \in \mathrm{Coh}(U,A)$ has a maximal extension to X. Then the canonical homomorphism of crystals*

$$\underline{\mathcal{F}}^{\max} \otimes_A A/\mathfrak{p} \hookrightarrow (\underline{\mathcal{F}} \otimes_A A/\mathfrak{p})^{\max}$$

is injective. If $\underline{\mathcal{F}}^{\max} \otimes_A A/\mathfrak{o}$ has good reduction at all $x \in Z$, it is an isomorphism.

Proof of Theorem 11.11

Proof. It is well-known that the first étale cohomology of a curve for the constant sheaf $\mathbb{F}_p$ can be expressed in terms of the p-torsion group of its Jacobian: Denoting by the superscript $^\vee$ the formation of the $\mathbb{F}_p$-dual, i.e, $\mathrm{Hom}_{\mathbb{F}_p}(\_\_,\mathbb{F}_p)$, one has

$$\mathrm{Jac}_{K,\mathfrak{p}}[p](\bar{k}) \cong H^1_{\mathrm{et}}(C_{K,\mathfrak{p}}/\bar{k},\mathbb{F}_p)^\vee.$$

Both sides carry Galois actions of $\mathrm{Gal}(\bar{k}/k_\infty)$ and of G. The extension $H^+_{\mathfrak{p}}/H^+$ is totally ramified at all places above $\mathfrak{p}$. Therefore it is linearly disjoint from the constant field extension $\bar{k}H^+/H^+$, and hence the two actions commute. We tensor both sides with $A/\mathfrak{p}$ over $\mathbb{F}_p$. This allows to decompose them into isotypic components for the semisimple action of G, whenever desired. Observe that the isotypic components on the left are the groups $\widetilde{C}(w)$.

To analyze the right hand term, we introduce some notation. Let $o_\mathfrak{p}$ denote the order of $(A/\mathfrak{p})^*$. Denote by $f_\mathfrak{p}: C_{K,\mathfrak{p}} \to C_{H^+}$ the G-cover of smooth projective geometrically irreducible curves over k_∞ corresponding to $H^+_\mathfrak{p}/H^+$. Define $U_\mathfrak{p}$ to be $\mathrm{Spec}\,\mathcal{O}^+$ minus the finitely many places above $\mathfrak{p}$ and let $j_\mathfrak{p}$ denote the open immersion of $U_\mathfrak{p} \hookrightarrow C_{H^+}$. Over $U_\mathfrak{p}$ the representation of G on $\varphi[\mathfrak{p}]$ is unramified, and thus it defines a lisse étale sheaf of rank one over $A/\mathfrak{p}$ which we denote by $\underline{\varphi[\mathfrak{p}]}$. This sheaf and all its tensor powers become, after pullback along the finite étale cover $f_\mathfrak{p}^{-1}(U_\mathfrak{p}) \to U_\mathfrak{p}$, isomorphic to the constant sheaf $\underline{A/\mathfrak{p}}$ on $C_{K,\mathfrak{p}}$ with generic fiber $A/\mathfrak{p}$. Using simple representation theory, one deduces that

$$\big(f_{\mathfrak{p}*}\underline{A/\mathfrak{p}}\big)\big|_{U_\mathfrak{p}} \cong \bigoplus_{w\in\mathbb{Z}/o_\mathfrak{p}} \underline{\varphi[\mathfrak{p}]}^{\otimes w}.$$

From adjunction of j^* and j_* we deduce a homomorphism $f_{\mathfrak{p}*}\underline{A/\mathfrak{p}} \longrightarrow \bigoplus_{w\in\mathbb{Z}/o_\mathfrak{p}} j_{\mathfrak{p}*}\underline{\varphi[\mathfrak{p}]}^{\otimes w}$. On stalks one can verify that the map is an isomorphism: At points where the representation $\varphi[\mathfrak{p}](\bar{K})^{\otimes w}$ is ramified, the sheaf $j_{\mathfrak{p}*}\underline{\varphi[\mathfrak{p}]}^{\otimes w}$ is zero and so is the corresponding summand on the left. At the other (unramified) points above $\mathfrak{p},\infty$, the sheaf $j_{\mathfrak{p}*}\underline{\varphi[\mathfrak{p}]}^{\otimes w}$ is lisse, as is the corresponding summand on the left. Using $H^1_{\mathrm{et}}(C_{K,\mathfrak{p}},\_\_) \cong H^1_{\mathrm{et}}(C_{H^+}, f^*_\mathfrak{p}\_\_)$, we deduce

$$\mathrm{Jac}_{K,\mathfrak{p}}[p](\bar{k}) \otimes_{\mathbb{F}_p} A.\mathfrak{p} \cong H^1_{\mathrm{et}}\Big(C_{H^+}/\bar{k}, \bigoplus_{w\in\mathbb{Z}/o_\mathfrak{p}} j_{\mathfrak{p}*}\underline{\varphi[\mathfrak{p}]}^{\otimes w}\Big)^\vee.$$

Now we decompose the isomorphism into isotypic components – note that $^\vee$ changes the sign of w. This yields

$$\widetilde{C}(w) \cong H^1_{\mathrm{et}}\big(C_{H^+}/\bar{k}, j_{\mathfrak{p}*}\underline{\varphi[\mathfrak{p}]}^{\otimes(-w)}\big)^\vee.$$

Our next aim is to relate the coefficient sheaf to a tensor power of the τ-sheaf $\underline{\mathcal{M}}(\varphi)$ attached to the Drinfeld module φ. We observed earlier that $\epsilon(\underline{\mathcal{M}}(\varphi)\otimes_A A/\mathfrak{p})$ on $\mathrm{Spec}\,\mathcal{O}^+$ is dual to $\underline{\varphi}[\mathfrak{p}]$. Since $\varphi[\mathfrak{p}](\bar{K})$ is totally ramified at $\mathfrak{p}$, the same holds for tensor powers w, except if w is a multiple of $o_\mathfrak{p}$ – here $\epsilon(\underline{\mathcal{M}}(\varphi)\otimes_A A/\mathfrak{p})^{\otimes w}$ may be zero above $\mathfrak{p}$, while the representation defined by $\varphi[\mathfrak{p}](\bar{K})^{\otimes w}$ is trivial and hence lisse. Let $j:\mathrm{Spec}\,\mathcal{O}^+ \hookrightarrow C_{H+}$ denote the canonical open immersion. Using Theorem 11.23, for w not a multiple of $o_\mathfrak{p}$ we find

$$j_{\mathfrak{p}*}\underline{\varphi[\mathfrak{p}]}^{\otimes(-w)} \cong \epsilon\big(j_\#(\underline{\mathcal{M}}(\varphi)\otimes_A A/\mathfrak{p})^{\otimes w}\big)$$

One can either use that the representation defined by $\varphi[\mathfrak{p}](\bar{K})^{\otimes w}$ is unramified at the places above ∞ if and only if $(q-1)$ divides w – the ramification group at those places is $k^* \subset A/\mathfrak{p}^* \cong G$ – or a direct computation on the side of τ-sheaves to deduce from Corollary 11.21 that

$$j_!(\underline{\mathcal{M}}(\varphi)\otimes_A A/\mathfrak{p})^{\otimes w} \hookrightarrow j_\#(\underline{\mathcal{M}}(\varphi)\otimes_A A/\mathfrak{p})^{\otimes w}$$

is an isomorphism whenever w is not a multiple of $(q-1)$ and has cokernel $\bigoplus_{\infty'|\infty}\underline{\mathbb{1}}_{\infty',A/\mathfrak{p}}$ otherwise – the sum is over all places of H^+ above ∞. One can in fact also prove that $j_!\underline{\mathcal{M}}(\varphi)^{\otimes w} \hookrightarrow j_\#(\underline{\mathcal{M}}(\varphi))^{\otimes w}$ is an isomorphism for $(q-1)\nmid w$ and has cokernel $\bigoplus_{\infty'|\infty}\underline{\mathbb{1}}_{\infty',A}$ otherwise. Finally we use that ϵ commutes with all functors defined for for crystals, so that to compute H^1_{et} we may first compute H^1 for crystals and then apply ϵ. This yields

$$\widetilde{C}(w) \cong \epsilon\Big(H^1(C_{H^+}/\bar{k},\, \underline{\mathbb{1}}_{C_{H^+},A}) \otimes A/\mathfrak{p}\Big)(\operatorname{Spec}\bar{k}) \quad \text{for } o_{\mathfrak{p}}|w \tag{3}$$

$$\widetilde{C}(w) \cong \epsilon\Big(\big(H^1(C_{H^+}/\bar{k},\, j_!\underline{\mathcal{M}}(\varphi)^{\otimes w})/\textstyle\bigoplus_{\infty'|\infty}\underline{\mathbb{1}}_{\operatorname{Spec}\bar{k},A}\big) \otimes A/\mathfrak{p}\Big)(\operatorname{Spec}\bar{k}) \quad \text{for } (q-1)|w, o_{\mathfrak{p}}\not| w \text{ or } w=0 \tag{4}$$

$$\widetilde{C}(w) \cong \epsilon\Big(H^1(C_{H^+}/\bar{k},\, j_!\underline{\mathcal{M}}(\varphi)^{\otimes w} \otimes A/\mathfrak{p}\Big)(\operatorname{Spec}\bar{k}) \quad \text{for } (q-1)\not| w \tag{5}$$

Note that without the evaluation $(\operatorname{Spec}\bar{k})$ outside ϵ we would have a sheaf on the right hand side.

In either case, the expression inside $\epsilon(\ldots)$ is a τ-sheaf $\underline{\mathcal{G}}$ on $\operatorname{Spec}\bar{k}$ over $A/\mathfrak{p}$. By Proposition 7.16 it can be written as $\underline{\mathcal{G}} \cong \underline{\mathcal{G}}_{\mathrm{ss}} \oplus \underline{\mathcal{G}}_{\mathrm{nil}}$ where on the first summand τ is bijective and on the second nilpotent. The underlying modules in both cases are finitely generated projective over $k \otimes_k A/\mathfrak{p}$. By the theory of the Lang torsor the τ-fixed points of the first summand form free $A/\mathfrak{p}$ vector space of dimension equal to $\operatorname{rank}_{\bar{k}\otimes A/\mathfrak{p}} \mathcal{G}_{\mathrm{ss}}$; those of the second summand are clearly zero. Moreover computing ϵ in the case at hand, cf. (1), is precisely the operation of taking τ-fixed points – the only relevant étale morphism that there is to $\operatorname{Spec}\bar{k}$ is the identity.

At the same time, the dual characteristic polynomial of $\big(H^1(C_{H^+},\, j_!\underline{\mathcal{M}}(\varphi)^{\otimes w} \otimes A/\mathfrak{p})$ has degree precisely equal to $\operatorname{rank}_{\bar{k}\otimes A/\mathfrak{p}} \mathcal{G}_{\mathrm{ss}}$. By Theorem 10.10 and Remark 10.12 this rank is the degree of $L_{\mathcal{O}^+}(w,T) \bmod \mathfrak{p}$. Thus we have now proved Theorem 11.11 (a). One may wonder about the case $o_{\mathfrak{p}}|w$ and $w \neq 0$. There are two answers why this case is covered as well: The formal answer is that the L-functions mod $\mathfrak{p}$ for w and w' in $-\mathbb{N}_0$ are equal whenever $w \equiv w' \pmod{o_{\mathfrak{p}}}$, and so it suffices to understand the case $w = 0$. An answer obtained by looking closer at what is happening goes as follows: The places above $\mathfrak{p}$ have L-factors congruent module 1 module $\mathfrak{p}$. So it doesn't matter whether we leave them in or not, i.e. whether we compute via the trace formula with $H^1(C_{H^+}/\bar{k},\, j_*\underline{\mathcal{M}}(\varphi)^{\otimes w} \otimes A/\mathfrak{p})$ or with $H^1(C_{H^+}/\bar{k},\, j_{\mathfrak{p}*}\underline{\mathcal{M}}(\varphi)^{\otimes w}|_{U_{\mathfrak{p}}} \otimes A/\mathfrak{p})$.

To prove (b) and (c) observe that we obtain formulas for $C(w)$ by taking invariants under $\operatorname{Gal}(\bar{k}/k_\infty)$ in the isomorphisms (3) to (5). The effect on the right hand sides is that we replace the curve $C_{H^+}/\bar{k}$ by $C_{H^+}/k_\infty = C_{H^+}$ and that for the resulting sheaf we compute global sections over $\operatorname{Spec} k_\infty$. This amounts to the same as computing the fixed points under τ of the expressions inside the brackets $\epsilon(\ldots)$. Since the τ-fixed points being non-zero is the same as the assertion that 1 is an eigenvalue of the τ-action, part (b) is now clear. – Note that $\bigoplus_{\infty'|\infty}\underline{\mathbb{1}}_{\operatorname{Spec}\bar{k},A}$ being a subcrystal of $M := H^1(C_{H^+}/\bar{k},\, j_!\underline{\mathcal{M}}(\varphi)^{\otimes w})$ in (4) means that $(T-1)^{\tilde{h}^+_A}$ is a factor of the L-function of M. Part (c) simply says that the dimension of the eigenspace of the eigenvalue 1 for the τ action is precisely the dimension of the space of τ-invariants, and the latter is $C(w)$. $\square$

Chapter 12

Drinfeld modular forms

The aim of this chapter is to give a description of Drinfeld modular forms via the cohomology of certain universal crystals on moduli spaces of rank 2 Drinfeld modules.

The basic definition of Drinfeld modular forms goes back to Goss, [19, 20]. Many important contributions are due to Gekeler, e.g. [17]. Moreover in [16] Gekeler obtains foundational results on Drinfeld modular curves. The work of Gekeler and Goss gives a satisfactory description of Drinfeld modular forms as rigid analytic functions on the Drinfeld analog of the upper half plane. The important work [45] of Teitelbaum. links this to harmonic cochains on the Bruhat Tits tree underlying the Drinfeld symmetric space. As shown in [4] the latter provides the link to a description of modular forms via crystals.

After introducing a moduli problem for Drinfeld modules of arbitrary rank (with a full level structure) in Section 1, in Section 2 we give equations for a particular example of such a moduli space. The universal Drinfeld module on it will give rise to a crystal via Anderson's correspondence between A-modules and A-motives. Following the classical case, this crystal yields a natural candidate for a cohomological description of Drinfeld modular forms, cf. Section 3. The cohomological object so obtained plays the role of a motive for the space of forms of fixed weight and level. It has various realizations: Its analytic realization is directly linked to Teitelbaum's description of Drinfeld modular forms via harmonic cochains. In Section 4 we consider its étale realizations. They allow one to attach Galois representations to Drinfeld Hecke eigenforms as in the classical case. Unlike in the classical case, the representations are 1-dimensional! The following Section 5 gives some discussion of ramification properties of the Galois representations so obtained. In Section 6 we indicate in what sense these compatible systems of 1-dimensional Galois representations associated to a cuspidal Drinfeld Hecke eigenform arise from a (suitably defined) Hecke character. So far the nature of these characters is still rather mysterious. We conclude with Section 7 which contains an extended example of the computation of the crystals associated to some low weight modular forms for $\mathbb{F}_q[t]$ and a particular level. It provides exemplary answers to many natural questions and points to open problems. Due to lack of time and space, we omit many details. We refer the reader to [4].

We recall the following notation: By $C = \operatorname{Spec} A$ we denote an irreducible smooth affine curve over k whose smooth compactification is obtained by adjoining precisely one closed point ∞. We define K as the fraction field of A, K_∞ as the completion of K at ∞ and $\mathbb{C}_\infty$ as the completion of the algebraic closure of K_∞. Similarly, for any place v of K we denote by K_v the completion of K at v and by $\mathcal{O}_v$ the ring of integers of K_v and by k_v the residue field of K_v. Often A will simply be $k[t]$. We fix a non-zero ideal $\mathfrak{n}$ of A. By $A[1/\mathfrak{n}]$ we denote the localization of A at all elements which have poles at most at $\mathfrak{n}$. The weight of a form will usually be denoted by n (or $n+2$), the letter k being taken as the name of the finite base field.

1 A moduli space for Drinfeld modules

Let S be a scheme over $\operatorname{Spec} A[1/\mathfrak{n}]$. Let $\underline{\varphi} := (L, \varphi)$ be a Drinfeld A-module on S of rank r, i.e., the line bundle L considered as a scheme of k-vector spaces over S is equipped with an endomorphism $\varphi \colon A \to \operatorname{End}(L), a \mapsto \varphi_a$. For any $a \in A$, the morphism $\varphi_a\, L \to L$ is finite flat of degree $\#(A/a)^r$ and hence its kernel

$$\varphi[(a)] := \operatorname{Ker}(\varphi_a \colon L \to L)$$

is a finite flat A-module scheme over S. Suppose that all prime factors of the ideal aA are the contained in the prime factors of $\mathfrak{n}$. Working locally on affine charts, it follows that the derivative of φ_a is a unit and thus $\varphi_a(z)$ is a separable polynomial. It follows that $\varphi[(a)]$ is étale over S. As a consequence the subscheme

$$\varphi[\mathfrak{n}] := \bigcap_{a \in \mathfrak{n} \smallsetminus \{0\}} \varphi[(a)]$$

is finite étale over S and of degree equal to $\#(A/\mathfrak{n})^r$. A level $\mathfrak{n}$-structure on $\underline{\varphi}$ is an isomorphism

$$\psi \colon \underline{(A/\mathfrak{n})^r}_S \xrightarrow{\cong} \varphi[\mathfrak{n}]$$

of finite étale group schemes over S, where $\underline{(A/\mathfrak{n})^r}_S$ denotes the constant group scheme on S with fiber $(A/\mathfrak{n})^r$.

Definition 12.1. Let $\mathcal{M}_r(\mathfrak{n})$ denote the functor on $A[1/\mathfrak{n}]$-schemes X given by

$$S \mapsto \{(\underline{\varphi}, \psi) \mid \underline{\varphi} = (L, \varphi) \text{ is a rank } r \text{ Drinfeld } A\text{-module on } X,\ \psi \text{ is a level } \mathfrak{n}\text{-structure on } \underline{\varphi}\}/\cong,$$

i.e., we consider such triples up to isomorphisms.

One has the following important theorem from [12]:

Theorem 12.2 (Drinfeld)**.** *Suppose $0 \neq \mathfrak{n} \subsetneq A$. Then the functor $\mathcal{M}_r(\mathfrak{n})$ is represented by an affine scheme $\mathfrak{M}_r(\mathfrak{n})$ which is smooth of finite type and relative dimension $r-1$ over* $\operatorname{Spec} A[1/\mathfrak{n}]$

Remark 12.3. In [12], Drinfeld also defines levels structures for levels dividing the characteristic of the Drinfeld module. Using these, he obtains a more general theorem as above: A universal Drinfeld module with level $\mathfrak{n}$-structure exists for A-schemes provided that $\mathfrak{n}$ has at least two distinct prime divisors. The universal space is regular of absolute dimension r. Its pullback to $\operatorname{Spec} A[1/\mathfrak{n}]$ is the space $\mathfrak{M}_r(\mathfrak{n})$.

Remark 12.4. Let $(L^{\mathrm{univ}}, \varphi^{\mathrm{univ}}, \psi^{\mathrm{univ}}))$ denote the universal object on $\mathfrak{M}_r(\mathfrak{n}) = \operatorname{Spec} R^{\mathrm{univ}}$. Then in fact L^{univ} is the trivial bundle on $\operatorname{Spec} R^{\mathrm{univ}}$. The reason is that the image under ψ^{univ} of any non-zero element in $(A/\mathfrak{n})^r$ is a section of L which is everywhere different from the zero section, i.e., it is a nowhere vanishing global section.

For this reason, we shall in the universal situation always assume that $L^{\mathrm{univ}} = \mathcal{O}_{\mathfrak{M}_r(\mathfrak{n})}$. Moreover in this affine situation we shall assume that the universal Drinfeld module is given in standard form, i.e., such that

$$A \to R^{\mathrm{univ}}[\tau] : a \mapsto \varphi_a = \alpha_0(a) + \alpha_1(a)\tau + \ldots + \alpha_{r \deg(a)}\tau^{r \deg(a)}$$

with $\alpha_{r \deg a}(a) \in (R^{\mathrm{univ}})^*$.

Exercise 12.5. Let φ be a Drinfeld A-module of rank r in standard form with $L = \mathbb{G}_a$, i.e., a ring homomorphism

$$A \to R[\tau], a \mapsto \varphi_a = \alpha_0(a) + \alpha_1(a)\tau + \ldots + \alpha_{r \deg(a)}\tau^{r \deg(a)}$$

for some A-algebra R. Suppose $s \in R$ is an a-torsion point which is non-zero on any component of R, i.e., $\alpha_0(a)s + \alpha_1(a)s^q + \ldots + \alpha_{r \deg(a)}s^{q^{r \deg(a)}} = 0$. Show that $a \mapsto s\varphi_a s^{-1}$ defines an isomorphic Drinfeld A-module such that 1 is an a-torsion point.

2 An explicit example

To make the above example more explicit, we consider the following special case: Let $A = k[t]$ and $\mathfrak{n} = (t)$. Then any Drinfeld A-module of rank r over an affine basis $\operatorname{Spec} R$ in standard form is described by the image of $t \in A$ in $R[\tau]$. This image is a polynomial of degree r which we denote by

$$\alpha_0 + \alpha_a \tau + \ldots + \alpha_r \tau^r$$

with $\alpha_r, \alpha_0 \in R^*$. The t-torsion points of $\underline{\varphi}$ are the solutions of $\varphi_t = 0$. Suppose we have a basis of t-torsion points $s_1, \ldots, s_r$ defined over $\operatorname{Spec} R$. We trivialize the bundle L via the section s_1. This means that we have $s_1 = 1$ on $L(\operatorname{Spec} R) \overset{\text{via } s_1}{\cong} \mathbb{G}_a(\operatorname{Spec} R) = (R, +)$.

The set of all t-torsion points is thus the set $\sum_{i=1}^r s_i \alpha_i$ where the α_i range over all elements of k. Since these points are precisely the roots of φ_t, we find

$$\varphi_t(z) = c \cdot \prod_{\underline{\alpha} \in k^r} \Big(z - \sum_{i=1}^r s_i \alpha_i \Big). \tag{1}$$

Recall the following result from [23, 1.3.7]

Proposition 12.6 (Moore determinant)**.** *Suppose $w_1, \ldots, w_r$ lie in an $\mathbb{F}_q$-algebra. Then*

$$\begin{vmatrix} w_1 & w_1^q & w_1^{q^2} & \ldots & w_1^{q^{r-1}} \\ w_2 & {w_2}^q & {w_2}^{q^2} & \ldots & {w_2}^{q^{r-1}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_r & {w_r}^q & {w_r}^{q^2} & \ldots & {w_r}^{q^{r-1}} \end{vmatrix} = \prod_{i=1}^r \prod_{(\ell_{i-1}, \ldots, k\ell_1) \in k^{i-1}} \Big(w_i + \ell_{i-1} w_{i-1} + \ldots + \ell_1 w_1 \Big).$$

By the theory of the Moore determinant, we obtain

$$\varphi_t(z) = c \cdot \begin{vmatrix} z & z^q & z^{q^2} & \ldots & z^{q^r} \\ s_r & {s_r}^q & {s_r}^{q^2} & \ldots & {s_r}^{q^r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_2 & {s_2}^q & {s_2}^{q^2} & \ldots & {s_2}^{q^r} \\ 1 & 1 & 1 & \ldots & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} s_r & {s_r}^q & \ldots & {s_r}^{q^{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ s_2 & {s_2}^q & \ldots & {s_2}^{q^{r-1}} \\ 1 & 1 & \ldots & 1 \end{vmatrix}. \tag{2}$$

Since the constant term of φ_t, i.e., the coefficient of z, is θ, the image of t under $A[1/t] \to R$, we can solve for c by computing the coefficient of z on the right hand side. It yields

$$\theta = c \cdot \begin{vmatrix} {s_r}^q & {s_r}^{q^2} & \ldots & {s_r}^{q^r} \\ \vdots & \vdots & \ddots & \vdots \\ {s_2}^q & {s_2}^{q^2} & \ldots & {s_2}^{q^r} \\ 1 & 1 & \ldots & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} s_r & {s_r}^q & \ldots & {s_r}^{q^{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ s_2 & {s_2}^q & \ldots & {s_2}^{q^{r-1}} \\ 1 & 1 & \ldots & 1 \end{vmatrix} = c \cdot \begin{vmatrix} s_r & {s_r}^q & \ldots & {s_r}^{q^{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ s_2 & {s_2}^q & \ldots & {s_2}^{q^{r-1}} \\ 1 & 1 & \ldots & 1 \end{vmatrix}^{q-1}$$

Proposition 12.7. *Let* $R = k\left[\theta^{\pm 1}, s_2, \ldots, s_r, \begin{vmatrix} s_r & {s_r}^q & \ldots & {s_r}^{q^{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ s_2 & {s_2}^q & \ldots & {s_2}^{q^{r-1}} \\ 1 & 1 & \ldots & 1 \end{vmatrix}^{-1}\right]$ *for indeterminates $\theta, s_2, \ldots, s_r$ and let $\varphi\colon k[t] \to R$ be the rank r Drinfeld module where φ_t is defined by (2). Then $\mathfrak{M}_r(t) \cong \operatorname{Spec} R$ and the universal triple is $(\varphi, \mathbb{G}_{a,R}, \psi)$ where $\psi\colon (k[t]/(t))^r \to \varphi[t]$ is defined by mapping the i-th basis vector on the left to s_i with the convention that $s_1 = 1$.*

Proof. Let (L',φ',ψ') be a Drinfeld module with a full level t-structure on a scheme $S=\operatorname{Spec} R'$. Assume first that S is affine. As in the above case we may take the section $\psi'(1,0,\ldots,0)$ of L to trivialize it. By the construction of R, there is a homomorphism from $R\to R'$ over $\mathbb{F}_q[\theta^{\pm1}]$-algebras sending s_i to the torsion point $s_i' := \psi'(0,\ldots,\underbrace{1}_{i},0,\ldots)$. The s_i' determine, in the same way as the s_i the function φ_t'. Hence (L',φ',ψ') is the pullback of (L,φ,ψ) under the morphism $\operatorname{Spec} R'\to\operatorname{Spec} R$. Moreover the morphism $R\to R'$ with this property is unique: The element s_1' determines the a unique isomorphism $L\to\mathbb{G}_a$. With respect to the coordinates of $\mathbb{G}_a$ given by $s_1'=1$, the sections $s_2,\ldots,s_r$ are uniquely determined from (L',φ',ψ') and hence $R\to R'$ is unique.

Now, let S be arbitrary. Fix an affine cover $\{\operatorname{Spec} R_i\}_i$. By the preceding paragraph we have unique morphisms $\operatorname{Spec} R_i\to\operatorname{Spec} R$. However by the uniqueness it also follows that on any affine subscheme of $\operatorname{Spec} R_i\cap\operatorname{Spec} R_j$, the two restriction to this subscheme agree. This in turn means the the local morphisms patch to a morphism $S\to\operatorname{Spec} R$ under which (L',φ',ψ') is the pullback of (L,φ,ψ). The uniqueness is true locally, hence also globally. This completes the proof of the representability of the functor $\mathcal{M}_r(\mathfrak{n})$. ☐

Remark 12.8. Geometrically $\operatorname{Spec} R$ can be described as follows: It is the affine space $\mathbb{A}^{r-1}_{\operatorname{Spec} k[\theta^{\pm1}]}$ with all the $\#k^{r-1}$ hyperplanes with coordinates in k removed. To see this, observe that $s_1=1$ and that by applying the Moore determinant, one has

$$\begin{vmatrix} s_r & s_r{}^q & \ldots & s_r{}^{q^{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ s_2 & s_2{}^q & \ldots & s_2{}^{q^{r-1}} \\ 1 & 1 & \ldots & 1 \end{vmatrix} = \prod_{i=1}^{r}\prod_{(\ell_{i-1},\ldots,\ell_1)\in k^{i-1}}\Big(s_i+\ell_{i-1}s_{i-1}+\ldots+\ell_1 s_1\Big).$$

Proposition 12.9. *We keep the notation of Proposition 12.7. The t-motive on* $\operatorname{Spec} R$ *corresponding to the universal Drinfeld module is isomorphic to the pair*

$$\underline{\mathcal{F}} := \left(R[t]^r, \tau = \begin{pmatrix} 0 & 0 & \ldots & 0 & \frac{t-\theta}{\alpha_r} \\ 1 & 0 & \ldots & 0 & \frac{-\alpha_1}{\alpha_r} \\ 0 & 1 & \ldots & 0 & \frac{-\alpha_2}{\alpha_r} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \ldots & 1 & \frac{-\alpha_{r-1}}{\alpha_r}\end{pmatrix}(\sigma_R\times \mathrm{id}_t)\right)$$

where $\alpha_i=\theta\cdot\begin{vmatrix} s_r & \ldots & s_r{}^{q^{i-1}} & s_r{}^{q^{i+1}} & \ldots & s_r{}^{q^r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ s_2 & \ldots & s_2{}^{q^{i-1}} & s_2{}^{q^{i+1}} & \ldots & s_2{}^{q^r} \\ 1 & \ldots & 1 & 1 & \ldots & 1\end{vmatrix} \Bigg/ \begin{vmatrix} s_r & s_r{}^q & \ldots & s_r{}^{q^{r-1}} \\ \vdots & \vdots & \ddots & \vdots \\ s_2 & s_2{}^q & \ldots & s_2{}^{q^{r-1}} \\ 1 & 1 & \ldots & 1\end{vmatrix}^q$ *for* $i=0,\ldots,r$.

Proof. The shape of τ is determined as in Exercise 3.10. The formulas for the coefficients result easily from (2) by first eliminating c and then expanding the determinant. in the numerator of (2) according to the first row. ☐

Let us, for some computations below, describe the case $n=2$ in greater detail. For simplicity, we write $s:=s_1$. In this case

$$R=k[\theta^{\pm1},s,(s^q-s)^{-1}],\quad \underline{\mathcal{F}}=\left(R[t]^2,\tau=\begin{pmatrix} 0 & (t/\theta-1)(s-s^q)^{q-1} \\ 1 & (s-s^{q^2})(s-s^q)^{-1}\end{pmatrix}(\sigma_R\times\mathrm{id}_t)\right)$$

Substituting $u:=s^q-s$ and observing that $s-s^{q_2}=u+u^q$, we obtain

$$R_u := k[\theta^{\pm1},u^{\pm1}],\quad \underline{\mathcal{F}}=\left(R_u[t]^2,\tau=\begin{pmatrix} 0 & (t/\theta-1)u^{q-1} \\ 1 & 1+u^{q-1}\end{pmatrix}(\sigma_{R_u}\times\mathrm{id}_t)\right) \tag{3}$$

The introduction of u corresponds to a cover $\operatorname{Spec} R_s\to\operatorname{Spec} R_u$ of degree q. The space $\operatorname{Spec} R_u$ is a moduli space for Drinfeld modules with a level $\Gamma_1(t)$-structure, where $\Gamma_1(t)$ is the set of matrices $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{GL}_2(k[t])$

such that $a, d \equiv 1 \pmod t$ and $c \equiv 0 \pmod t$. Note that the moduli correspond to a choice of two t-torsion points $1, u$ where u is only determined up to adding a multiple of 1. (Due to our choice of coordinates for the line bundle underlying the rank 2 Drinfeld module, the first torsion point is 1).

In the sequel, the symmetric powers $\mathrm{Sym}^n \underline{\mathcal{F}}$ and their extension by zero to a compactification of $\mathfrak{M}_r(\mathfrak{n})$ will play an important role. We make this explicit in the setting of (3): Here a smooth compactification of $\operatorname{Spec} R_u = \mathbb{A}^1_{k[\theta^{\pm 1}]}$ is $\mathbb{P}^1_{k[\theta^{\pm 1}]}$. One simply has to extend $\underline{\mathcal{F}}$ to 0 and ∞. (Taking symmetric powers is compatible with this extension process.)

At $u = 0$, the matrix $\left(\begin{smallmatrix} 0 & (t/\theta-1)u^{q-1} \\ 1 & 1+u^{q-1} \end{smallmatrix}\right)$ specializes to $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, i.e., the extension is defined but not zero. At $u = \infty$ specializing the matrix leads to poles. To analyze the situation, we introduce $v = 1/u$, so that the matrix describing τ becomes $\left(\begin{smallmatrix} 0 & (t/\theta-1)v^{1-q} \\ 1 & 1+v^{1-q} \end{smallmatrix}\right)$. Next we multiply the standard basis e_1, e_2 of $R_u[t]^2$ by v. Then the action of τ for this new basis is given by

$$v^{-1} \begin{pmatrix} 0 & (t/\theta - 1)v^{1-q} \\ 1 & 1 + v^{1-q} \end{pmatrix} (\sigma_{R_u} \times \mathrm{id}_t) v = v^{-1} \begin{pmatrix} 0 & (t/\theta - 1)v^{q-1} \\ 1 & 1 + v^{q-1} \end{pmatrix} v^q = \begin{pmatrix} 0 & (t/\theta - 1) \\ v^{1-q} & v^{1-q} + 1 \end{pmatrix}.$$

The following result summarizes the above discussion

Proposition 12.10. *Consider $R_u = k[\theta^{\pm 1}, u^{\pm 1}]$ as an algebra over $A[1/\theta] = k[\theta^{\pm 1}]$.*

(a) The moduli space of rank 2 Drinfeld modules with level $\Gamma_1(t)$-structure is isomorphic to $\operatorname{Spec} R$ as a scheme over $\operatorname{Spec} A[1/\theta]$.

(b) A relative smooth compactification of $\operatorname{Spec} R$ is the projective line $\mathbb{P}^1_{A[1/\theta]}$.

(c) The A-motive of attached to the universal Drinfeld A-module over $\operatorname{Spec} R$ is given by

$$\underline{\mathcal{F}} = \left(R_u[t]^2, \tau = \begin{pmatrix} 0 & (t/\theta - 1)u^{q-1} \\ 1 & 1 + u^{q-1} \end{pmatrix} (\sigma_{R_u} \times \mathrm{id}_t) \right)$$

(d)

$$j_{\#}\underline{\mathcal{F}} := \left(\mathcal{O}^{\oplus 2}_{\mathbb{P}^1_{k[\theta^{\pm 1}]}}(-1 \cdot [\infty]), \tau\right) \quad \text{and} \quad j_!\underline{\mathcal{F}} := \left(\mathcal{O}^{\oplus 2}_{\mathbb{P}^1_{k[\theta^{\pm 1}]}}(-2 \cdot [\infty] - 1 \cdot [0]), \tau\right)$$

are a coherent extension of $\underline{\mathcal{F}}$ to $\mathbb{P}^1_{A[1/\theta]}$ and an extension by zero, respectively. Moreover one has a canonical monomorphism $j_!\underline{\mathcal{F}} \hookrightarrow j_{\#}\underline{\mathcal{F}}$ whose cokernel is a skyscraper sheaf supported on $\{0, \infty\}$.

To compute the cohomology of $j_!\underline{\mathcal{F}}$ it suffices to compute that of $j_{\#}\underline{\mathcal{F}}$, since the discrepancy is easy to describe by the cokernel of $j_!\underline{\mathcal{F}} \hookrightarrow j_{\#}\underline{\mathcal{F}}$. This simplifies the computation of the cohomology of the crystal $j_! \mathrm{Sym}^n \underline{\mathcal{F}}$ significantly, since the coherent cohomology under $\mathbb{P}^1_{k[\theta^{\pm 1}]} \to \operatorname{Spec} k[\theta^{\pm 1}]$ (with the induced τ) is much easier to carry out for $\mathrm{Sym}^n j_{\#}\underline{\mathcal{F}}$ than for $\mathrm{Sym}^n j_!\underline{\mathcal{F}} = j_! \mathrm{Sym}^n \underline{\mathcal{F}}$.

3 Drinfeld modular forms via cohomology

Let us now return to the general situation over $\mathfrak{M}_r(\mathfrak{n})$. **Assume that** $r = 2$. For $r > 2$, the material below has not been carried out. Only recently Pink has constructed compactifications of the moduli spaces $\mathfrak{M}_r(\mathfrak{n})$ with good properties, cf. [40] (and also [41]).

Let

$$f \colon \mathfrak{M}_r(\mathfrak{n}) \to \operatorname{Spec} A[1/n] \tag{4}$$

denote the structure morphism and define

$$\underline{\mathcal{S}}_n(\mathfrak{n}) := R^1 f_! \,\mathrm{Sym}^n(j_!\underline{\mathcal{F}}).$$

(For $i \neq 1$ one has $R^i f_! \,\mathrm{Sym}^n(j_!\underline{\mathcal{F}}) = 0$.) Here are some basic facts on $\underline{\mathcal{S}}_n(\mathfrak{n})$:

(a) By general theory, since $\underline{\mathcal{F}}$ is flat as a crystal, we deduce that $\underline{\mathcal{S}}_n(\mathfrak{n})$ is flat as an A-crystal. This implies that on an open subscheme of $A[1/n]$ it has a free representative. However one has better representability results:

(b) Since $\underline{\mathcal{F}}$ is of pullback type (it is a τ-sheaf on an affine scheme), $R^i f_! \operatorname{Sym}^n(j_!\underline{\mathcal{F}}) = 0$ is of pullback type. From this one can deduce that it has a free representative as a τ-sheaf. This representative does have, however, the disadvantage that the action of τ may be highly nilpotent.

(c) Using Gardeyn's theory of maximal extensions, one can also construct a representing τ-sheaf whose underlying sheaf is locally free *and* on which τ^{lin} is injective. We denote it by $\underline{\mathcal{S}}_n(\mathfrak{n})^{\max}$.

Our first aim is to give an interpretation for the *analytic realization* of $\underline{\mathcal{S}}_n(\mathfrak{n})$. For this fix a homomorphism $k[t] \to A$ such that $\mathrm{Im}(t) \in A \smallsetminus k$. Denote by $K_\infty\{t\}$ the entire power series over K_∞. Define

$$\underline{\mathcal{S}}_n^{\mathrm{an}}(\mathfrak{n}) := (\underline{\mathcal{S}}_n(\mathfrak{n})/K_\infty) \otimes_{K_\infty[t]} K_\infty\{t\} \quad \text{and} \quad M_n^B(\mathfrak{n}) := (\underline{\mathcal{S}}_n^{\mathrm{an}}(\mathfrak{n}))^\tau$$

where by $(\underline{\mathcal{S}}_n(\mathfrak{n})/K_\infty)$ we mean the base change of $\underline{\mathcal{S}}_n(\mathfrak{n})$ under $\operatorname{Spec} K_\infty \to \operatorname{Spec} A(1/n)$. In the simplest case $A = k[t]$, the pair defining $\underline{\mathcal{S}}_n^{\mathrm{an}}(\mathfrak{n})$ is a free sheaf on the rigid analytic $\mathbb{A}^1$ and τ defines a semilinear endomorphism on it of which one could think of as a system of differential equations. Then $M_n^B(\mathfrak{n})$ is the solution set of this system. It is not hard to see that $M_n^B(\mathfrak{n})$ is independent of the chosen representative of the crystal $\underline{\mathcal{S}}_n(\mathfrak{n})$. From this it is not hard to see that it is free over A of rank at most the rank of $\underline{\mathcal{S}}_n(\mathfrak{n})^{\max}$.

Denote by $S_n(\Gamma(\mathfrak{n}))$ the space of Drinfeld modular forms of full level $\mathfrak{n}$ and by $F_\mathfrak{n}$ the ray class group of F with conductor $\mathfrak{n}$. One of the central results of [4] is the following:

Theorem 12.11. *There is a Hecke-equivariant isomorphism*

$$\left(M_n^B(\mathfrak{n})\right)^\vee \otimes_A \mathbb{C}_\infty \xrightarrow{\cong} S_n(\Gamma(\mathfrak{n}))^{\mathrm{Gal}(F_\mathfrak{n}/F)}.$$

In the above theorem the cardinality of $\mathrm{Gal}(F_\mathfrak{n}/F)$ describes the number of connected components of $\mathcal{M}_2(\mathfrak{n})$ over the algebraically closure of K. In an adelic description of Drinfeld modular forms no such exponent is necessary.

We recall for the convenience of the reader the definition of the Hecke action on the crystal $\underline{\mathcal{S}}_n(\mathfrak{n})$. (This induces the action on $M_n^B(\mathfrak{n})$.) The action on $S_n(\Gamma(\mathfrak{n}))^{\mathrm{Gal}(F_\mathfrak{n}/F)}$ can be similarly defined. In the case where A does not have class number one, this definition should only be given adelically. Since we have not developed the corresponding language, we will not give the definition and simply refer to [4]. To define the Hecke action, let for any prime $\mathfrak{p}$ not dividing $\mathfrak{n}$ denote by $\mathfrak{M}_2(\mathfrak{n},\mathfrak{p})$ the moduli scheme for quadruples (L,φ,ψ,C) where (L,φ,ψ) is a Drinfeld A-module of rank r with full level $\mathfrak{n}$-structure and C is a cyclic $\mathfrak{p}$-torsion subscheme of L. (in the sense of Drinfeld if $\mathfrak{p}$ is the characteristic of the base scheme). Consider

$$\mathfrak{M}_2(\mathfrak{n}) \xleftarrow[\pi_1]{} \mathfrak{M}_2(\mathfrak{n},\mathfrak{p}) \xrightarrow{\pi_2} \mathfrak{M}_2(\mathfrak{n})$$

with $\pi_1((L,\varphi,\psi,C)) = (L,\varphi,\psi)$ and $\pi_2((L,\varphi,\psi,C)) = (L/C,\varphi/C,\psi/C)$. Denote by $\underline{\mathcal{G}}$ the τ-sheaf $\operatorname{Sym}^n \underline{\mathcal{F}}$ and by $\underline{\mathcal{G}}_\mathfrak{p}$ the n-th symmetric power of the A-motive on $\mathfrak{M}_2(\mathfrak{n},\mathfrak{p})$ associated to its tautological Drinfeld module. By the universal property of $\mathfrak{M}_2(\mathfrak{n})$ it follows that there are canonical isomorphisms

$$\pi_!^*\underline{\mathcal{G}} \cong \underline{\mathcal{G}}_\mathfrak{o} \cong \pi_1^*\underline{\mathcal{G}}. \tag{5}$$

Adjunction yields a natural homomorphism

$$\underline{\mathcal{G}} \to \pi_{1*}\pi_1^*\underline{\mathcal{G}}. \tag{6}$$

Since π_2 is finite flat of degree $\deg\mathfrak{p}+1$ there also is a trace homomorphism

$$\mathrm{Tr}\colon \pi_{2*}\pi_2^*\underline{\mathcal{G}} \to \underline{\mathcal{G}}. \tag{7}$$

The above isomorphisms and homomorphisms extend to $\overline{\mathfrak{M}_2(\mathfrak{n})}$ for $j_!\underline{\mathcal{G}}$ and $j\colon \mathfrak{M}_2(\mathfrak{n}) \hookrightarrow \overline{\mathfrak{M}_2(\mathfrak{n})}$ a compactification over $\operatorname{Spec} A(1/n)$. In analogy to (4), we denote the structure homomorphism $\mathfrak{M}_2(\mathfrak{n},\mathfrak{p})$ by $f_\mathfrak{p}$. Then the following chain of homomorphisms then defines the Hecke-operator $T_\mathfrak{p}$:

$$R^1 f_!\underline{\mathcal{G}} \xrightarrow{(6)} R^1 f_!\pi_{1*}\pi_1^*\underline{\mathcal{G}} \xrightarrow{\text{can. isom}} R^1 f_{\mathfrak{p}!}\pi_1^*\underline{\mathcal{G}} \xrightarrow{(5)} R^1 f_{\mathfrak{p}!}\pi_2^*\underline{\mathcal{G}} \xrightarrow{\text{can. isom}} R^1 f_!\pi_{2*}\pi_2^*\underline{\mathcal{G}} \xrightarrow{(7)} R^1 f_!\underline{\mathcal{G}}$$

This is the standard way to make a correspondence act on a cohomology. It is also applies to the definition of Drinfeld modular forms as global sections of a suitable line bundle (which depends on the weight) and agrees there with other common definitions of the Hecke operator $T_{\mathfrak{p}}$.

Sketch of proof of Theorem 12.11. The proof given in [4] has its basic structure modeled at the classical proof by Shimura. Some details seem to be quite different however. Here we shall only give a rough sketch of the individual steps of the proof:

(a) By a theorem of Teitelbaum, the right hand side of the isomorphism in Theorem 12.11 is isomorphic to the space of harmonic cochains on the Bruhat-Tits tree for PGL_2 $C^{\mathrm{har}}(\Gamma(\mathfrak{n}), M_n(\mathbb{C}_\infty))$. These are $\Gamma(\mathfrak{n})$-equivariant function on the edges of this tree into a certain $\mathbb{C}_\infty[\Gamma(\mathfrak{n})]$-module $M_n(\mathbb{F}_\infty)$. In fact, the $\Gamma(\mathfrak{n})$-module can be naturally obtained by coefficient change from a $F(\Gamma(\mathfrak{n}))$-module $M_n(F)$ defined already over F. Bet even more is true: One can define a local system $M_n(A)$ of free A-modules on the edges of the tree that naturally carries a $\Gamma(\mathfrak{n})$-action, such that this local system is an A-structure from the local system given by $M_n(F)$. Concretely, for every edge of the tree, the local system $M_n(A)$ is given by a projective A-submodule of $M_n(F)$ of rank equal to $\dim_F M_n(F)$ and such that under the action of $\Gamma(\mathfrak{n})$ on the tree there is a corresponding compatible action on these A-submodules of $M_n(F)$. An $M_n(A)$-valued $\Gamma(\mathfrak{n})$-invariant harmonic cochain is now a map which to any edge of the tree assigns a value in the A-module defined for this edge and such that the values are $\Gamma(\mathfrak{n})$-equivariant. Thus we have $S_n(\Gamma(\mathfrak{n})) \cong C_{\mathrm{har}}(\Gamma(\mathfrak{n}), M_n(A)) \otimes_A \mathbb{C}_\infty$. It this suffices to prove that there is a natural Hecke-equivariant isomorphism

$$\left(M_n^B(\mathfrak{n})\right)^\vee \xrightarrow{\cong} C_{\mathrm{har}}(\Gamma(\mathfrak{n}), M_n(A))^{\mathrm{Gal}(F_{\mathfrak{n}}/F)}.$$

(b) Let us now consider the right hand side $M_n^B(\mathfrak{n})$. It was obtained by base change of the crystal $\underline{\mathcal{S}}_n(\mathfrak{n})$ to K_∞, passing to analytic coefficients and then taking τ-invariants. Now parallel to the algebraic theory of A-crystals over an algebraic base including the functors defined there, one can develop a theory of crystals over rigid analytic spaces and with analytic coefficients. Moreover one can define a natural rigidification functor from the algebraic to the rigid analytic setting which is compatible with all functors. This allows one to recover $\underline{\mathcal{S}}_n(\mathfrak{n})/F_\infty \otimes_{F_\infty[t]} F_\infty\{t\}$ as follows: Denote by $\mathfrak{M}_2(\mathfrak{n})^{\mathrm{an}}_{/F_\infty}$ the rigidification of the scheme $\mathfrak{M}_2(\mathfrak{n})$ after base change from $A[1/n]$ to F_∞. This rigid analytic space is (after finite extensions of the base, e.g. from F_∞ to $F_{\mathfrak{n},\infty}$, isomorphic to $\Gamma(\mathfrak{n})\backslash\Omega^{\mathrm{Gal}(F_{\mathfrak{n}}/F)}$ where Ω is the Drinfeld symmetric space of dimension one over F_∞. In particular this rigid analytic curve has a module interpretation. Let $\mathrm{Sym}^n \underline{\mathcal{F}}^{\mathrm{an}}$ denote the rigid τ-sheaf on $\Gamma(\mathfrak{n})\backslash\Omega$ with $F_\infty\{t\}$-coefficients associated to $\mathrm{Sym}^n \underline{\mathcal{F}}$. The sheaf $\underline{\mathcal{F}}^{\mathrm{an}}$ can also be obtained purely from the universal analytic Drinfeld-module over $\Gamma(\mathfrak{n})\backslash\Omega$. Extension by zero leads to an extension by zero in the rigid setting where however it is important that one rigidified an algebraic compactification. Let us denote by $H^1_{\mathrm{an},c}$ the cohomology with compact support on this rigid analytic site of τ-sheaves (or crystals). Then there is a natural isomorphism

$$M^B(\mathfrak{n}) \cong H^1_{\mathrm{an},c}(\overline{\mathfrak{M}_2(\mathfrak{n})}^{\mathrm{an}}_{/F_\infty}, j_! \,\mathrm{Sym}^n \underline{\mathcal{F}}^{\mathrm{an}})^\tau.$$

Thus it now suffices to construct a natural isomorphism

$$\left(H^1_{\mathrm{an},c}(\Gamma(\mathfrak{n})\backslash\ \Omega^*, j_! \,\mathrm{Sym}^n \underline{\mathcal{F}}^{\mathrm{an}})^\tau\right)^\vee \xrightarrow{\cong} C_{\mathrm{har}}(\Gamma(\mathfrak{n}), M_n(A)) \tag{8}$$

where $\Omega^* = \Omega \cup \mathbb{P}^1(F)$ ("suitably topologized") and where the base of Ω on the left is sufficiently large and lies between F and $\mathbb{C}_\infty$. Moreover one needs to prove the Hecke-compatibility of this isomorphism if extended in a natural way to its $\#\,\mathrm{Gal}(F_{\mathfrak{n}}/F)$-fold sum.

(c) To prove the isomorphism (8), one compute the left hand side using an explicit Čech cover of $\Gamma(\mathfrak{n})\backslash\Omega^*$: The cover is obtained as follows: There is a well-known reduction map $\Omega \to \mathcal{T}$ where $\mathcal{T}$ is the Bruhat-Tits tree for $\mathrm{PGL}_2(F_\infty)$. The cover is equivariant with respect to an action of $\Gamma(\mathfrak{n})$, and there is an induced reduction map

$$\Gamma(\mathfrak{n})\backslash\Omega \to \Gamma(\mathfrak{n})\backslash\mathcal{T}.$$

There is a finite number of orbits of vertices $\Gamma(\mathfrak{n})v_i$, $i = 1, \ldots, n$, as well as of edges $\Gamma(\mathfrak{n})e_j$, $j \in \{1, \ldots, m\}$, in $\Gamma\backslash\mathcal{T}$ on which the action of $\Gamma(\mathfrak{n})$ is free. If these orbits are removed the remaining graph becomes a disjoint union of subgraphs c_ℓ which contain no loops and are in a 1-1 correspondence with the cusps of $(\Gamma(\mathfrak{n})\backslash\ \Omega$. The preimage of a closed ε-neighborhood of any $\Gamma(\mathfrak{n})v_i$ is an affinoid subset $\mathfrak{U}_i \subset \Gamma(\mathfrak{n})\backslash\Omega$, a disc minus q open subdiscs. The preimage of a closed ε-neighborhood of any c_ℓ after adding one puncture is an affinoid subset $\mathfrak{W}_\ell \subset \Gamma(\mathfrak{n})\backslash\Omega^*$ of the cusp. The preimage of any edge orbit $\Gamma(\mathfrak{n})e_j$ minus closed $\varepsilon/2$-neighborhoods at each end is an annulus $\mathfrak{V}_j \subset \Gamma(\mathfrak{n})\backslash\Omega$. Using the **uniformizability** of the universal Drinfeld module over the $\mathfrak{U}_i$ and the $\mathfrak{V}_j$ and the fact that $\mathrm{Sym}^n \underline{\mathcal{F}}^{\mathrm{an}}$ is extended by zero to the cusps one finds

$$H^0_{\mathrm{an},c}(\mathfrak{V}_j, j_! \,\mathrm{Sym}^n \underline{\mathcal{F}}^{\mathrm{an}})^\tau \cong M_n(A)^\vee|_{e_j}, \quad H^0_{\mathrm{an},c}(\mathfrak{W}_\ell, j_! \,\mathrm{Sym}^n \underline{\mathcal{F}}^{\mathrm{an}})^\tau = 0$$

and if $e_{j'}$ is any edge neighboring v_i, then $H^0_{\mathrm{an},c}(\mathfrak{U}_j, j_! \,\mathrm{Sym}^n \underline{\mathcal{F}}^{\mathrm{an}})^\tau \cong M_n(A)^\vee|_{e_{j'}}$. The Čech complex is particularly simple, since any triple intersections of distinct sets of the covering are empty and any non-empty double intersections are given by a small annulus on some $\mathfrak{V}_j$ where the double intersection is by intersecting $\mathfrak{V}_j$ and some adjacent $\mathfrak{U}_i$. By explicit inspection, one can show that the Čech complex is dual to the stable complex given by Teitelbaum to compute harmonic cochains. This yields the asserted isomorphism (8).

(d) Finally one verifies the Hecke-equivariance by comparing explicit formulas for Hecke operators on the Čech cover and on harmonic cochains.

□

4 Galois representations associated to Drinfeld modular forms

One can also study the étale realizations of the crystal $\underline{\mathcal{S}}_n(\mathfrak{n})$. For this we fix a maximal ideal $\mathfrak{p}$ of the coefficient ring A. The functor ϵ yields the following inverse system of étale sheaves

$$\Big\{\epsilon\big(\underline{\mathcal{S}}_n(\mathfrak{n}) \otimes_A A/\mathfrak{p}^m\big)\Big\}_{m\in\mathbb{N}}$$

on $\mathrm{Spec}\, A(\mathfrak{n})$. The resulting inverse limit is an étale $A_\mathfrak{p}$-sheaf $\underline{\mathcal{S}}_n(\mathfrak{n})_{\mathrm{et},\mathfrak{p}}$ over $\mathrm{Spec}\, A[1/\mathfrak{n}]$. We shall discuss later that it can be ramified at a finite number of places of $\mathrm{Spec}\, A[1/\mathfrak{n}]$. However we have the following result:

Theorem 12.12. *The étale sheaf* $\epsilon\big(\underline{\mathcal{S}}_n(\mathfrak{n}) \otimes_{A[1/\mathfrak{n}]\otimes A} (F \otimes A/\mathfrak{p}^m)\big)$ *has rank* $s_n(\mathfrak{n}) := \dim S_n(\mathfrak{n}) \cdot \#\,\mathrm{Gal}(F_\mathfrak{n}/F)$.

Sketch of proof: Recall that the functor from crystals to étale sheaves is compatible with all functors. Thus the sheaf given in the theorem is isomorphic to

$$H^1_{\mathrm{et},c}(\mathfrak{M}_2(\mathfrak{n})/F, \mathrm{Sym}^n \underline{\mathcal{F}} \otimes_A A/\mathfrak{p}^m) \cong H^1_{\mathrm{et},c}(\mathfrak{M}_2(\mathfrak{n})/F, \mathrm{Sym}^n \varphi[\mathfrak{p}^m]^\vee)$$

where φ is the universal Drinfeld module on $\mathfrak{M}_2(\mathfrak{n})/F$. By a result of Gekeler, [16], the modular curve $\overline{\mathfrak{M}_2(\mathfrak{n})}/F$ is ordinary for any $\mathfrak{n}$. By a result of Pink, [39], there is Grothendieck-Ogg-Shafarevich type formula for the $\mathbb{F}_p$-dimension of the cohomology of étale $\mathbb{F}_p$-sheaves on curves, provided the curve has an ordinary cover over which the monodromy of the étale sheaf is unipotent. In the case at hand, by Gekeler's result we can take the cover $\overline{\mathfrak{M}_2(\mathfrak{n}\mathfrak{p}^m)}$. The formula of Pink shows

$$\dim_{\mathbb{F}_p} \epsilon\big(\underline{\mathcal{S}}_n(\mathfrak{n}) \otimes_{A[1/\mathfrak{n}]\otimes A} (F \otimes A/\mathfrak{p}^m)\big) = \dim_{\mathbb{F}_p} A/\mathbb{F}_p^m s_n(\mathfrak{n})$$

for any m. From this the theorem easily follows – the proof is essentially the same as the proof that the $\mathfrak{n}$-torsion of a Drinfeld module away from the characteristic is equal to $A/\mathfrak{n}^r$. □

It follows that $\underline{\mathcal{S}}_n(\mathfrak{n})_{\mathrm{et},\mathfrak{p}}/F$ defines a continuous homomorphism

$$\rho_{A,\mathfrak{n}} \colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathrm{GL}_{s_n(\mathfrak{n})}(A_\mathfrak{p}).$$

Moreover $\underline{\mathcal{S}}_n(\mathfrak{n})_{\mathrm{et},\mathfrak{p}}$ carries the Hecke action induced from $\underline{\mathcal{S}}_n(\mathfrak{n})$.

In [4] the following result is proved:

Theorem 12.13. *For any prime $\mathfrak{q}$ different from $\mathfrak{n}$ and $\mathfrak{p}$, the actions of $T_{\mathfrak{q}}$ and of $\mathrm{Frob}_{\mathfrak{q}}$ are the same on the reduction of $\underline{\mathcal{S}}_n(\mathfrak{n})_{\mathrm{et},\mathfrak{p}}$ from $\operatorname{Spec} A[1/\mathfrak{p}]$ to $\operatorname{Spec} A/\mathfrak{q}$ agree.*

Since the Hecke-operators commute among each other and since the $\mathrm{Frob}_{\mathfrak{q}}$ where $\mathfrak{q}$ runs through all maximal ideal prime ideals of $\operatorname{Spec} A[1/\mathfrak{n}\mathfrak{p}]$ are dense in $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ we deduce:

Corollary 12.14. *The image of $\rho_{A,\mathfrak{n}}$ is abelian.*

The crystal $\underline{\mathcal{S}}_n(\mathfrak{n})$ has given rise to two realizations, an analytic one and for each maximal ideal of A an étale one:

$$(\underline{\mathcal{S}}_n(\mathfrak{n})^{\mathrm{an}}_{/K_\infty})^\tau \dashleftarrow \underline{\mathcal{S}}_n(\mathfrak{n}) \dashrightarrow \underline{\mathcal{S}}_n(\mathfrak{n})_{\mathrm{et},\mathfrak{p}}$$

In each case, the realizations inherited a Hecke action. As examples show, cf. [33], the Hecke action may not be semisimple. So we pass in both cases to the semisimplification and decompose $\underline{\mathcal{S}}_n(\mathfrak{n})$ into Hecke eigenspaces (if necessary after inverting some elements in the coefficient ring A.). This yields a correspondence between Hecke-eigensystems of Drinfeld modular forms and simple abelian Galois representations. Before we give the precise statement from [4], we recall the following classical theorem due to Goss:

Theorem 12.15 (Goss)**.** *Let f be a Hecke eigenform of weight n and level $\mathfrak{n}$ with Hecke eigenvalues $a_{\mathfrak{p}}(f)$ for all $\mathfrak{p}$ not dividing $\mathfrak{n}$. Then all $a_{\mathfrak{p}}(f)$ are integral over A and the field $F_f := F(a_{\mathfrak{p}}(f) \mid \mathfrak{p} \in \operatorname{Spec} A[1/\mathfrak{n}])$ is a finite extension of F. Denote by $\mathcal{O}_f$ the ring of integers of F_f.*

Theorem 12.16 (B.)**.** *Let f be as above and suppose f is cuspidal. Then there exists a system of Galois representations*

$$\rho_{f,\mathfrak{P}} \colon \mathrm{Gal}(F^{\mathrm{sep}}/F) \to \mathrm{GL1}(\widehat{\mathcal{O}}^{\mathfrak{P}})_{\mathfrak{P}\in\mathbf{Max}(\mathcal{O}_f)}$$

uniquely characterized by the condition that for each fixed $\mathfrak{P}$, one has for almost all $\mathfrak{q}$ prime to $\mathfrak{P}\mathfrak{n}$ the equation

$$\rho_{f,\mathfrak{P}}(\mathrm{Frob}_{\mathfrak{q}}) = a_{\mathfrak{q}}(f),$$

so that the right hand side is independent of the prime $\mathfrak{P}$.

Remark 12.17. (a) It is not clear whether there is a theory of old new forms for Drinfeld modular forms. So one cannot proceed as in the classical case.

(b) There are various counterexamples to a strong multiplicity one theorem, by Gekeler and Gekeler-Reversat, e.g. [18, Ex. 9.7.4] for an example in weight 2.

(c) As far as we know there are no counterexamples to multiplicity one for $S_n(\Gamma_0(\mathfrak{p}))$ for fixed n and a prime $\mathfrak{p}$ of $\operatorname{Spec} A$.

(d) Despite the results of the following section, the ramification locus of the system $(\rho_{f,\mathfrak{P}})_{\mathfrak{P}\in\mathbf{Max}(\mathcal{O}_f)}$ is rather mysterious.

5 Ramification or Galois representations associated to Drinfeld modular forms

Theorem 12.18. *Let $\underline{\mathcal{S}}_n(\mathfrak{n})^{\max}$ be the maximal extension in the sense of Gardeyn representing the same-named crystal. Define D as the support of the cokernel of the injective homomorphism $\tau^{\mathrm{lin}} : (\sigma \times \mathrm{id})^*\underline{\mathcal{S}}_n(\mathfrak{n})^{\max} \to \underline{\mathcal{S}}_n(\mathfrak{n})^{\max}$. Then from a result of Katz, [30] one easily deduces the following.*

Let f be a Drinfeld modular form of level $\mathfrak{n}$ and weight n. Let $\mathfrak{P}$ be a maximal ideal of $\mathcal{O}_f$ and $\mathfrak{p}$ its contraction to A. Then $\rho_{f,\mathfrak{P}}$ is unramified at all primes $\mathfrak{q}$ of A such that $(\mathfrak{q},\mathfrak{p})$ is not in D. Moreover for such $\mathfrak{q}$ one has

$$\det(1 - T\rho_{f,\mathfrak{P}}(\mathrm{Frob}_{\mathfrak{q}})) = 1 - Ta_{\mathfrak{q}}(f),$$

$$\det(1 - T\rho_{A,\mathfrak{n}}(\mathrm{Frob}_{\mathfrak{q}})) = \det(1 - TT_{\mathfrak{q}}^{\mathrm{ss}}|S_n(\mathfrak{n})),$$

where $T_{\mathfrak{q}}^{\mathrm{ss}}$ is the semisimplification of $T_{\mathfrak{q}}$ acting on the analytic space of modular forms $S_n(\mathfrak{n})$ of weight n and level $\mathfrak{n}$.

Remark 12.19. After replacing the coefficient ring A by a larger ring A' which is a localization of A at a suitable element, one can in fact decompose $\underline{\mathcal{S}}_n(\mathfrak{n})$ into components corresponding to generalized eigenforms under the Hecke action. By enlarging A' to a ring A'', one may furthermore assume that A'' contains all the Hecke eigenvalues of all eigenforms. Then over A'', to any eigenform f one has a corresponding subcrystal of $\underline{\mathcal{S}}_f$ of $\underline{\mathcal{S}}_n(\mathfrak{n}) \otimes_A A''$. Katz' criterion then applies to the Gardeyn maximal model of $\underline{\mathcal{S}}_f$. This gives, in theory, a precise description of the ramification locus of $\rho_{f,\mathfrak{P}}$ – provided that $\mathfrak{P}$ is in $\mathbf{Max}(A'')$ –, given by a divisor D_f on $\operatorname{Spec} A \times \operatorname{Spec} A''$. The definition of D in the previous theorem is coarser. It gives a bound on ramification for all eigenforms f simultaneously.

While Galois representations $\rho_{f,\mathfrak{P}}$ for eigenforms f of level $\mathfrak{n}$ tend to be ramified at the places above $\mathfrak{n}$, it is not clear to me how the additional places of ramification are linked to $\mathfrak{P}$. In an abstract sense, the answer is that this link is given by D – or, more precisely, by D_f. Concretely, we do not know how to determine D_f from f or D from n and $\mathfrak{n}$. Below some explicit examples are given. One clue to the ramification locus is given by Theorems 12.21 and 12.22 given below. They describe a link between places of bad reduction of modular curves and the ramification of modular forms. The following result might serve as a motivation:

Let K be a local field of characteristic p with ring of integers $\mathcal{O}$ and residue field k. Let $\mathcal{A}/\mathcal{O}$ be an abelian scheme with generic fiber A/K of dimension g and special fiber $\mathcal{A}/k$. The p^n-torsion subscheme of $\mathcal{A}$ (or A) is denoted by $\mathcal{A}[p^n]$ (or $A[p^n]$, respectively,) and for any field $L \supset K$, we write $A[p^n](L)$ for the group of L-valued points of $A[p^n]$. Consider the p-adic Tate module

$$\mathrm{Tate}_p A := \varprojlim_n A[p^n](K^{\mathrm{sep}})$$

of A. The module underlying $\mathrm{Tate}_p A$ is free over $\mathbb{Z}_p$. One defines the *p-rank of A* as $\mathrm{rank}_p A := \mathrm{rank}_{\mathbb{Z}_p} \mathrm{Tate}_p A$. It satisfies $0 \leq \mathrm{rank}_p A \leq g$. The action of $G_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$ on $\mathrm{Tate}_p A$ is $\mathbb{Z}_p$-linear and thus with respect to some $\mathbb{Z}_p$-basis yields a Galois representation

$$\rho_{A,p} \colon G_K \longrightarrow \mathrm{Aut}_{\mathbb{Z}_p}(\mathrm{Tate}_p A) \cong \mathrm{GL}_{\mathrm{rank}_p A}(\mathbb{Z}_p).$$

By Hensel's Lemma any p^n-torsion point of $\mathcal{A}/k$ will lift to a unique p^n-torsion point of A/K. Thus

$$\mathrm{rank}_p \mathcal{A}/k \leq \mathrm{rank}_p A,$$

i.e., the p-rank can only decrease under reduction. The following result from [6] links the ramification of $\rho_{\mathcal{A}.p}$ to the p-rank:

Theorem 12.20. *The p-rank is invariant under reduction if and only if the action of G_K on $\mathrm{Tate}_p A$ is unramified.*

Let us, after this short interlude come back to the ramification of Drinfeld modular forms: Let f is a doubly cuspidal Drinfeld Hecke eigenform of weight 2 and level $\mathfrak{n}$. Let $J_{\mathfrak{n},f}$ denote the maximal abelian quotient of the Jacobian of the Drinfeld modular curve for level $\mathfrak{n}$ such that the semisimplification of the Hecke action on the p-torsion$J_{\mathfrak{n},f}[p]$ has the same Hecke eigenvalues as f. Then the following theorem is shown in [6]:

Theorem 12.21. *Suppose f is a doubly cuspidal Drinfeld Hecke eigenform of weight 2 and level $\mathfrak{n}$. Then for a prime $\mathfrak{q}$ not dividing $\mathfrak{n}$, the following are equivalent:*

(a) *For any (or for all)* $\mathfrak{P} \in \mathbf{Max}(\mathcal{O}_f)$ *the representation* $\rho_{f,\mathfrak{P}}$ *is ramified at* $\mathfrak{q}$.

(b) *The Hecke-eigenvalue* $a_{\mathfrak{q}}(f)$ *of* f *at* $\mathfrak{q}$ *is zero.*

(c) *The abelian variety* $J_{\mathfrak{n},f}$ *has supersingular reduction modulo* $\mathfrak{q}$.

Note that in the case of weight 2, there is a representation ρ_f from G_K into GL_1 over a finite field $\mathbb{F}$ contained in $\mathcal{O}$ such that $\rho_{f,\mathfrak{P}} = \rho_f \otimes_{\mathbb{F}} \mathcal{O}_{\mathfrak{P}}$ for all $\mathfrak{P} \in \mathbf{Max}(\mathcal{O}_f)$. Hence the set of primes ramified outside $\mathfrak{n}$ is independent of the choice of $\mathfrak{P}$.

Suppose now that f has weight larger than two. and consider a representation $\rho_{f,\mathfrak{P}}$ for $\mathfrak{P} \in \mathbf{Max}(\mathcal{O}_f)$ over $\mathfrak{p} \in \mathbf{Max}(A)$. As $\rho_{f,\mathfrak{P}}$ is associated to a Hecke character, see 12.25, and because it is known that such have square free levels, it follows that $\rho_{f,\mathfrak{P}}$ is ramified at $\mathfrak{q}$ if and only if its reduction mod $\mathfrak{P}$ is so. As in the case of classical modular forms the reduction mod $\mathfrak{P}$ is congruent to the representation of a form of weight 2 and level $\mathfrak{n}\mathfrak{p}$. To the latter one can apply the previous result. This yields:

Theorem 12.22. *Suppose* f *is a doubly cuspidal Drinfeld Hecke eigenform of weight* $n \geq 3$ *and level* $\mathfrak{n}$. *Let* $\mathfrak{P}$ *be in* $\mathbf{Max}(\mathcal{O}_f)$ *with contraction* $\mathfrak{p} \in \mathbf{Max}(A)$. *Then for a prime* $\mathfrak{q}$ *not dividing* $\mathfrak{n}\mathfrak{q}$, *the following are equivalent:*

(a) *The representation* $\rho_{f,\mathfrak{P}}$ *is ramified at* $\mathfrak{q}$.

(b) *The representation* $\rho_{f,\mathfrak{P}}$ $\pmod{\mathfrak{P}}$ *is ramified at* $\mathfrak{q}$.

(c) *The Hecke-eigenvalue* $a_{\mathfrak{q}}(f)$ *of* f *at* $\mathfrak{q}$ *is zero modulo* $\mathfrak{P}$.

(d) *The abelian variety* $J_{\mathfrak{n}\mathfrak{p},f}$ *has supersingular reduction modulo* $\mathfrak{q}$.

Note that in known examples, e.g. [6], the places of ramification of $\rho_{f,\mathfrak{P}}$ which are prime to the level $\mathfrak{n}$ do typically depend on $\mathfrak{P}$ unlike in the case of weight 2.

Question 12.23. For classical modular forms it is simple to list all the primes which are ramified for the associated Galois representations. By the theory of new forms these primes are those dividing the minimal level associated to the modular form together with the place p (or the places above p) if one considers p-adic Galois representations.

Because of this simplicity one wonders if there is also a simple recipe in the case of Drinfeld modular forms. The numerical data seems too little to make any predictions. This deserves to be studied more systematically. Because of Theorem 12.21 this question is directly linked to the reduction behavior of Drinfeld modular curves (at primes of good reduction!) and their associated Jacobians.

6 Drinfeld modular forms and Hecke characters

In [7] we introduce a notion of Hecke character that was more general than previous definitions due to Gross [25] and others. Our motivation was a question of Serre and independently Goss which asked whether Drinfeld modular forms are linked to Hecke characters. In [6] this question was answered in the affirmative. We will briefly indicate this result.

Definition 12.24. Let F be a global function field over $\mathbb{F}_p$. A homomorphism

$$\chi \colon \mathbb{A}_F^* \longrightarrow \overline{\mathbb{F}_p(t)}^*,$$

where $\mathbb{A}_F^*$ denotes the ideles of F and $\overline{\mathbb{F}_p(t)}$ is discretely topologized, is a **Hecke character (of type** Σ**)** if

(a) χ is continuous (i.e., trivial on a compact open subgroup of $\mathbb{A}_F^*$) and

(b) there exists a finite subset $\Sigma = \{\sigma_1, \dots, \sigma_r\}$ of field homomorphisms $\sigma_i : F \hookrightarrow \overline{\mathbb{F}_p(t)}$ and $n_i \in \mathbb{Z}$ for $i = 1, \dots, r$, such that

$$\chi(\alpha, \alpha, \dots, \alpha) = \sigma_1(\alpha)^{n_1} \cdot \ldots \cdot \sigma_r(\alpha)^{n_r}.$$

Note that for any compact open subgroup $U \subset \mathbb{A}_F^*$ the coset space $F^*\backslash\mathbb{A}_F^*/U$ admits a surjective degree map to $\mathbb{Z}$ whose kernel is finite and may be interpreted as a class group. It is an easy consequence of the above definition, noted first by Goss [21], that Hecke characters as above have square free conductors.

The main result on Hecke characters and modular forms is the following.

Theorem 12.25. *For any cuspidal Drinfeld Hecke eigenform f with eigenvalues $(a_{\mathfrak{p}}(f))_{\mathfrak{p}\in\mathbf{Max}(A)}$ there exists a unique Hecke character*

$$\chi_f \colon \mathbb{A}_F^* \longrightarrow K_f^*$$

such that

$$a_{\mathfrak{p}}(f) = \chi_f(1,\ldots,1,\underbrace{\varpi_{\mathfrak{p}}}_{\text{at }\mathfrak{p}},1,\ldots,1) \quad \textit{for almost all } \mathfrak{p}\in\mathbf{Max}(\mathcal{O}_f).$$

Unfortunately, the set Σ_f in Definition 12.24 for the Hecke character χ_f remains completely mysterious. The proof of the theorem sheds no light on it. What is however not so hard to see is that the ramification divisor D_f introduced in 12.19 is equal to $\bigcup_{\sigma\in\Sigma_f}\mathrm{Graph}(\sigma)$ where the $\sigma\in\Sigma_f$ are viewed as morphisms of algebraic curves.

Example 12.26. The following Hecke characters are taken from [6]. The are associated to Drinfeld modular forms. Let $F = \mathbb{F}_q(\theta)$, let n be in $\{2,\ldots,p\}$ and consider $\sigma\colon \mathbb{F}_q(\theta)\longrightarrow\mathbb{F}_q(t) : \theta\mapsto(1-k)t$. . Define

$$U := \big(1+\theta\mathbb{F}_q[\theta]\big)\times\prod_{\nu\nmid 0,\infty}\mathcal{O}_v^*\times\mathbb{F}_q\big(\big(\frac{1}{\theta}\big)\big)^*.$$

Then the natural homomorphism

$$\mathbb{F}_q(\theta)^* \xrightarrow{\simeq} \mathbb{A}_F^*/U$$

is an isomorphism. Hence there exists a unique Hecke character

$$\chi_n\colon \mathbb{A}_F^*\to\overline{\mathbb{F}_p(t)}^*$$

such that χ_n is trivial on U and such that it agrees with σ on $\mathbb{F}_q(\theta)$.

Remark 12.27. The Hecke character χ_f provides a compact way of storing essential information about the cuspidal Drinfeld Hecke eigenform f. To explain this, suppose that $A=\mathbb{F}_q[t]$, that the weight of f is n and that we have computed its Hecke eigenvalues $a_{\mathfrak{p}}(t)$ for many primes $\mathfrak{p}$ not dividing $\mathfrak{n}$. The conductor of χ_f consists of those primes $\mathfrak{p}\nmid\mathfrak{n}$ for which $T_{\mathfrak{p}}$ acts as zero and some primes dividing $\mathfrak{n}$. Having computed many eigenvalues, we may thus hope to know the prime to $\mathfrak{n}$-part of the conductor of χ_f and thus a lower bound for the conductor $\mathfrak{m}_f\subset A$ of χ_f by which we mean the largest square-free ideal such that χ_f is trivial on the group U_f of all ideles congruent to 1 modulo $\mathfrak{m}_f$. Suppose furthermore that we know the coefficient field K_f of f. For theoretical reasons the character χ_f is trivial on F_∞^* (the image of the decomposition group at ∞ of any $\rho_{f,\mathfrak{P}}$ is trivial).

If the weight n is 2, then χ_f is of finite order and in particular Σ is empty. Knowing a bound on $\mathfrak{m}_f$ and that χ_f is trivial on F_∞^*, by computing sufficiently many Hecke eigenvalues, we can completely determine χ_f as a function on $F^*\backslash\mathbb{A}_F^*/U_fF_\infty^*$.

If the weight n is larger than 2, it is necessary to find the embeddings $\sigma_i\colon\mathbb{F}_q(\theta)\to K_f$, $i=1,\ldots,r$ and their exponents n_i. Denote by $b_i\in K_f$ the image of θ under σ_i. If g is any element of $A=\mathbb{F}_q[\theta]$ which is congruent to 1 modulo $\mathfrak{m}_f$, then T_g acts on f as $\prod_i g(b_i)^{n_i}$. At the same time, if $(f)=\prod_{\mathfrak{p}}\mathfrak{p}^{m_{\mathfrak{p}}}$ for exponents $m_{\mathfrak{p}}=\mathrm{ord}_{\mathfrak{p}}(g)\in\mathbb{N}_0$, then we have the equation

$$\prod_{\mathfrak{p}}(a_{\mathfrak{p}}(f))^{m_{\mathfrak{p}}}=\prod_i g(b_i)^{n_i}.$$

The number r, the exponents n_i and the b_i can be determined by the following algorithm: Let n run through the positive integers, let (n_i) run through all (unordered) partitions of n. For each partition determine the solution set of

$$\prod_{\mathfrak{p}}(a_{\mathfrak{p}}(f))^{m_{\mathfrak{p}}}=\prod_i g(x_i)^{n_i}.$$

while g runs through many polynomials congruent to 1 modulo $\mathfrak{m}_f$. The algorithm terminates if a solution is found. The algorithm will terminate because of the above theorem. If it terminates and if sufficiently many g have been tested, the solution can assumed to be correct. In all explicitly known case one has $r = 1$ and $n_1 = 1$ – but this may be due to the fact that not so many examples are known.

7 An extended example

In this section we will carry out the explicit computation of the cohomology of certain crystals associated to Drinfeld cusp forms of low weight and level $\Gamma_1(t)$. We consider $R_u = k[\theta^{\pm 1}, u^{\pm 1}]$ as an algebra over $A[1/\theta] = k[\theta^{\pm 1}]$ as in Proposition 12.10 and let f be the morphism of the corresponding schemes. Let furthermore $\bar{f}: \mathbb{P}^1_{A[1/\theta]} \to \operatorname{Spec} A[1/\theta]$ be its relative compactification. We consider the τ-sheaf

$$\underline{\mathcal{F}} = \left(R_u[t]^2, \tau = \begin{pmatrix} 0 & (t/\theta - 1)u^{q-1} \\ 1 & 1 + u^{q-1} \end{pmatrix} (\sigma_{R_u} \times \mathrm{id}_t) \right)$$

and wish to compute $R^1 \bar{f}_*$ of

$$\operatorname{Sym}^n j_\# \underline{\mathcal{F}} := \operatorname{Sym}^n \left(\mathcal{O}^{\oplus 2}_{\mathbb{P}^1_{A[1/\theta]}}(-1 \cdot [\infty]), \tau \right)$$

as a crystal. Abbreviating $b := (t/\theta - 1)u^{q-1}$ and $c := 1 + u^{q-1}$, the endomorphism $\operatorname{Sym}^n \tau$ on $\operatorname{Sym}^n R_u[t]^2 = R_u[t]^{n+1}$ is given by

$$\alpha_n(\sigma \times \mathrm{id}) \quad \text{where} \quad \alpha_n := \begin{pmatrix} &&&&& b^n \\ &&&& \iddots & \vdots \\ &&& b^3 & \cdots & b^3 c^{n-3}\binom{n}{3} \\ && b^2 & b^2 c\binom{3}{2} & \cdots & b^2 c^{n-2}\binom{n}{2} \\ & b & bc\binom{2}{1} & bc^2\binom{3}{1} & \cdots & bc^{n-1}\binom{n}{1} \\ 1 & c & c^2 & c^3 & \cdots & c^n \end{pmatrix}$$

The corresponding basis of $R[u]^{n+1}$ we denote by $e_j, j = 0, \ldots, n$.

Next recall that we can compute the cohomology of a coherent sheaf $\mathcal{G}$ on $\mathbb{P}^1_S$ (over any affine base S) as follows: Let $\mathbb{A}_S \subset \mathbb{P}^1_S$ be the standard affine line contained in $\mathbb{P}^1$. Let $\mathcal{O}_{\infty,S}$ be the affine coordinate ring of the completion of $\mathbb{P}^!_S$ along the section $\infty \times S$ at ∞ and let $K_{\infty,S}$ be the ring obtained from $\mathcal{O}_{\infty,S}$ by inverting the section at ∞ . Then one has the short exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^1_S, \mathcal{G}) \longrightarrow H^0(\mathbb{A}^1_S, \mathcal{G}) \oplus \mathcal{G}_{|\mathcal{O}_{\infty,S}} \longrightarrow \mathcal{G}_{|K_{\infty,S}} \longrightarrow H^0(\mathbb{P}^1_S, \mathcal{G}) \longrightarrow 0.$$

The sequence is obtained as the direct limit over U over the sequences for the computation of Čech cohomology where $\mathbb{P}^1$ is covered $\mathbb{A}^1$ and a second affine set U containing $\infty \times S$. We apply this to the base $\operatorname{Spec} \mathcal{A}$ with $\mathcal{A} = k[\theta^{\pm 1}, t]$ and the sheaf $\operatorname{Sym}^n \mathcal{F}$. Disregarding τ we obtain the short exact sequence

$$0 \longrightarrow \mathcal{A}[u]^{n+1} \oplus \left(\frac{1}{u}\right)^n n\Big[\Big[\frac{1}{u}\Big]\Big] \otimes_k \mathcal{A}^{n+1} \longrightarrow k\Big(\Big(\frac{1}{u}\Big)\Big) \otimes_k \mathcal{A}^{n+1} \longrightarrow \mathrm{Coker}_n \longrightarrow 0.$$

This show that $\mathrm{Coker}_n = H^1(\mathbb{P}^1, \operatorname{Sym}^n j_\# \mathcal{F})$ is a free $\mathcal{A}$-module with basis $\{u^{-i} e_j \mid i = 1, \ldots, n-1; j = 0, \ldots, n\}$. Let us write the elements of the cokernel as

$$u^{-1} v_1 + \ldots + u^{1-n} v_{n-1}$$

where the v_j are column vectors over $\mathcal{A}$ of length $n+1$. (We can write them in the basis $e_0, \ldots, e_n$.) Applying τ to the summands $u^{-i} v_i$ yields

$$\tau(u^{-i} v_i) = u^{-iq} \alpha_n v_i = \left(u^{-i(q-1)} \alpha_n\right) u^{-i} v_i.$$

Now observe that α_n lies in $\mathcal{A}[u^{q-1}]$. Thus $u^{-i(q-1)}$ shifts the pole order at $u=0$ (and $u=\infty$) by multiples of $(q-1)$. We define matrices $\alpha_{n,i} \in M_{(n+1)\times(n+1)}(\mathcal{A})$ so that

$$\alpha_n = \sum_{i\geq 0} \alpha_{n,i} u^{i(q-1)}.$$

Assumption 12.28. We now assume that the weight n lies in the interval $\{0,\dots,q\}$.

Because i lies in $\{-1,\dots,1-n\}$, the absolute value of the difference of two such i is at most $q-2$. Therefore all but at most one summand in

$$\tau(u^{-i}v_i) = \Big(\sum_{i'\geq 0} u^{(i'-i)(q-1)}\alpha_{n,i'}\Big)u^{-i}v_i$$

is non-zero in Coker_n, namely that for $i'=i$. Let $\beta = \left(\frac{t}{\theta}-1\right)$ and abbreviate $x = u^{q-1}$. We enumerate the rows and columns by $r,s \in \{0,\dots,n\}$. We let $\widetilde{r} := n-r$, so that this variable counts rows from the bottom starting at zero.Then the (r,s)-coefficient of α_n is

$$c^{r+s-n}b^{n-r}\binom{s}{n-r} = c^{s-\widetilde{r}}b^{\widetilde{r}}\binom{s}{\widetilde{r}} = (1+x)^{s-\widetilde{r}}x^{\widetilde{r}}\beta^{\widetilde{r}}\binom{s}{\widetilde{r}} = \sum_{\ell=0}^{s-\widetilde{r}} x^{\ell+\widetilde{r}}\beta^{\widetilde{r}}\binom{s-\widetilde{r}}{\ell}\binom{s}{\widetilde{r}}.$$

The (s,r)-coefficient of $\alpha_{n,i}$ is the coefficient of x^i in the previous line. Thus it is the summand for $\ell = i-\widetilde{r}$, i.e.,

$$\beta^{\widetilde{r}}\binom{s-\widetilde{r}}{i-\widetilde{r}}\binom{s}{\widetilde{r}} = \beta^{\widetilde{r}}\frac{(s-\widetilde{r})!}{(i-\widetilde{r})!(s-i)!}\frac{s!}{(s-\widetilde{r})!\widetilde{r}!} = \beta^{\widetilde{r}}\frac{i!}{(i-\widetilde{r})!\widetilde{r}!}\frac{s!}{(s-i)!i!} = \beta^{\widetilde{r}}\binom{i}{i-\widetilde{r}}\binom{s}{i}.$$

Let w_i be the transpose of the row vector $\Big(0,\dots,0,\binom{i}{0}\beta^i,,\binom{i}{1}\beta^{i-1},\dots,,\binom{i}{i}\beta^0\Big)$ and let x_i be the row vector $\Big(0,\dots,0,\binom{i}{i},,\binom{i+1}{i},\dots,,\binom{n}{i}\Big)$. Then $\alpha_{n,i} = w_i \otimes x_i$ and so

$$\tau(u^{-i}v_i) = u^{-i}w_i\cdot(x_iv_i).$$

We deduce the following: As a τ-sheaf Coker_n is the direct sum of the sub-$\mathcal{A}$-modules W_i spanned by $u^{-i}e_j$, $j=0,\dots,n$. The τ-submodule W_i contains itself the τ submodule $\mathcal{A}u^{-i}w_i$, and because the image of W_i under τ is contained in $\mathcal{A}u^{-i}w_i$, it is nil-isomorphic to W_i. Thus we find that

$$\oplus_{j=1}^{n-1}(\mathcal{A}u^{-i}w_i, \tau_{|\mathcal{A}u^{-i}w_i}) \longrightarrow (\mathrm{Coker}_n, \tau)$$

is a nil-isomorphism. We compute the τ-action on $u^{-i}w_i$:

$$\tau(u^{-i}w_i) = u^{-i}w_i\Big(x_i\cdot(\sigma\times\mathrm{id})w_i\Big) = (u^{-i}w_i)\sum_{\ell=0}^{\min\{i,n-i\}}\binom{i}{\ell}\binom{n-\ell}{i}\widetilde{\beta}^{\ell}$$

with $\widetilde{\beta} = \left(\frac{t}{\theta^q}-1\right)$. Set $\gamma_{n,i} := \sum_{\ell=0}^{\min\{i,n-i\}}\binom{i}{\ell}\binom{n-\ell}{i}\beta^\ell$ and

$$\underline{\mathcal{L}}_{n,i} := (k[\theta^{\pm 1},t], \gamma_{n,i}(\sigma\times\mathrm{id})).$$

Then $(\mathcal{A}u^{-i}w_i, \tau_{|\mathcal{A}w_i}) \cong \sigma^*\underline{\mathcal{L}}_{n.i}$ and we have thus shown that as A-crystals we have

Proposition 12.29. *Suppose* $0\leq n\leq q$. *Then*

$$\underline{\mathcal{S}}_{n+2}(\Gamma_1(t)) = R^1\bar{f}_*\,\mathrm{Sym}^n\, j_\#\underline{\mathcal{F}} \cong \bigoplus_{i=1}^{n-1}\underline{\mathcal{L}}_{n,i}.$$

Moreover $\underline{\mathcal{L}}_{n,i} = \underline{\mathcal{L}}_{n,n-i}$ *and* $\underline{\mathcal{S}}_{n+2}(\Gamma_1(t)) = 0$ *for* $n=0,1$.

In Remark 12.33 we shall compare the above formula for $\gamma_{n,i}$ to a similar expression in [33, Formula (7.3)].

Example 12.30. For $i=1$ and $2 \le n \le q$ one has $\gamma_{n,1} = \binom{1}{0}\binom{n}{1}\beta^0 + \binom{1}{1}\binom{n-1}{1}\beta = 1 + (n-1)\frac{t}{\theta}$. The corresponding ramification divisor in the sense of Remark 12.19 is defined by $\theta = (1-n)t$ on $\operatorname{Spec} k[\theta^{\pm 1}, t]$. This leads to the Hecke character described in Example 12.26.

Example 12.31. Next we compute the local L-factors of $\underline{\mathcal{L}}_{n,1}$. The base scheme is $X := \mathbb{A}_k \smallsetminus \{0\}$ in the coordinate θ. Let $\mathfrak{p}$ be a place of X defined by the irreducible polynomial $h(\theta) \in k[\theta]$, We normalize it so that $h(0) = 1$. This is possible because $\mathfrak{p} \neq 0$. The residue field at $\mathfrak{p}$ is $k_\theta = k[\theta]/(h(\theta)) = k[\bar\theta]$ with $\bar\theta$ a root of h over $\bar k$. In $k_{\mathfrak{p}}[t]$ we have

$$h(t) = \Big(1 - \frac{t}{\bar\theta}\Big) \cdot \Big(1 - \frac{t}{\bar\theta^q}\Big) \cdot \ldots \cdot \Big(1 - \frac{t}{\bar\theta^{q^{\deg h - 1}}}\Big).$$

Thus

$$\begin{aligned} \det(1 - \tau_{\mathcal{L}_{n,1}} T^{\deg \mathfrak{p}}) &= \det\Big(1 - \Big(1 + (n-1)\frac{t}{\bar\theta}\Big) \cdot \Big(1 + (n-1)\frac{t}{\bar\theta^q}\Big) \cdot \ldots \cdot \Big(1 + (n-1)\frac{t}{\bar\theta^{q^{\deg h - 1}}}\Big) T^{\deg h}\Big) \\ &= 1 - h\big((1-n)t\big) T^{\deg h} \end{aligned}$$

Thus, if we denote by $g_{n,1}$ the cuspidal Drinfeld Hecke eigenform corresponding to $\underline{\mathcal{L}}_{n,1}$, then it eigenvalue at $\mathfrak{p} = (h)$ is $h\big((1-n)t\big)$. A similar but more involved computation yields the eigenvalue system for the form $g_{n,i}$ corresponding to $\underline{\mathcal{L}}_{n,i}$ and any $1 \le i \le n-1$.

Example 12.32. For classical as well as Drinfeld modular forms their automorphic weight, say n, is the exponent of the automorphy factor in the transformation formulas for the action of the congruence subgroup defined by the level of the form. In the classical case there is naturally a weight attached to the motive associated with a cuspidal Hecke eigenform. This *motivic weight* is the exponent of the complex absolute values of the roots of the characteristic polynomials defined by Hecke action. By the proof of the Ramanujan Peterson conjecture due to Deligne this weight is $(n-1)/2$. This weight occurs in the formula for the absolute values of the p-the Hecke eigenvalue of a classical modular form f:

$$|a_p(f)|_{\mathbb{C}} \le 2p^{(n-1)/2}.$$

It is therefore natural to also ask for a motivic weight of a cuspidal Drinfeld Hecke eigenform. Does it exist and how is it related to the weight that occurs in the exponent of the automorphy factor in the transformation law of the form? In the Drinfeld modular case, the characteristic polynomial arising from Hecke operators at $\mathfrak{p}$ is 1-dimensional. Therefore the motivic weight of a Drinfeld Hecke eigenform f is (if it exists) the exponent $q \in \mathbb{Q}$ such that

$$v_\infty(a_{\mathfrak{p}}(f)|) = -q \deg \mathfrak{p} \text{ for almost all } \mathfrak{p} \in \mathbf{Max}(A).$$

This weight is modeled after Anderson's definitions of purity and weights for t-motives, [1, 1.9 and 1.10].

The τ-sheaves $\underline{\mathcal{L}}_{n,i}$ defined in Proposition 12.29 posses a motivic weight. It is equal to $\deg_t \gamma_{n,i}$ since by computations as in the previous example one shows that $v_\infty(a_{\mathfrak{p}}(g_{n,i})|) = -\deg \gamma_{n,i} \cdot \deg \mathfrak{p}$. For $q = p$ the formulas in Remark 12.33 yield $\deg_{n,i} = \min\{n-i, i\}$. Thus, for a given n, any weight in $\{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ occurs. For $q \neq p$, the possible weights are more difficult to analyze because the expression $\binom{i}{m}\binom{n-i}{m}$ can vanish $m = \min\{i, n-i\}$ (and also for many values less than this minimum – see Lemma 10.30.

We expect but have no proof that the motives for all cuspidal Drinfeld Hecke eigenforms are pure, i.e., that they have a motivic weight. If this is true, it can be shown from the cohomological formalism in [8] that this weight lies, for given n, in the interval $\{0, \ldots, [\frac{n}{2}]\}$. That the range is optimal is shown by the above examples. Moreover the example shows that it is not possible to compute the motivic from the automorphic weight.

Remark 12.33. In [33, Formula (7.3)] a differently looking formula is given from which their the Hecke eigenvalue systems in for the forms $g_{n,i}$ are computed (for primes of degree one). From the following claim it follows that the formulas given there agree with those here (and thus with those given in [4]). **Claim:**

$$\gamma_{n,i} = \sum_{m \ge 0} \binom{i}{m}\binom{n-i}{m}\Big(\frac{t}{\theta}\Big)^m$$

where we recall that $\gamma_{n,i} := \sum_{\ell=0}^{\min\{i,n-i\}} \binom{i}{\ell}\binom{n-\ell}{i}\beta^{\ell}$ with $\beta = \frac{t}{\theta} - 1$. The proof follows by the use of generating series in a standard fashion. The key steps are

$$\begin{aligned}
\sum_{n\geq 0} \gamma_{n,i} X^n &= \sum_{\ell\geq 0} \binom{i}{\ell} (\beta X)^{\ell} \sum_{n\geq i+\ell} \binom{n-\ell}{i} X^{n-\ell} \\
&= \sum_{\ell\geq 0} \binom{i}{\ell} (\beta X)^{\ell} (1-X)^{-i} X^i \;=\; \left(\frac{1+\beta X}{1-X}\right)^i X^i \\
&= \left(1 + \frac{\frac{t}{\theta} X}{1-X}\right)^i X^i \;=\; \sum_{m\geq 0} \binom{i}{m} \left(\frac{t}{\theta} X\right)^m (1-X)^{-m} X^i \\
&= \sum_{m\geq 0} \binom{i}{m} \left(\frac{t}{\theta}\right)^m \sum_{n\geq i+m} \binom{n-i}{m} X^n \;=\; \sum_{n\geq 0} X^n \sum_{m\geq 0} \binom{i}{m} \binom{n-i}{m} \left(\frac{t}{\theta}\right)^m
\end{aligned}$$

Appendix A

Further results on Drinfeld modules

In this appendix we collect as a reference some further results on Drinfeld modules which were used in parts of the lecture notes. Throughout the appendix we denote by ι the canonical embedding $A \hookrightarrow \mathbb{C}_\infty$.

1 Drinfeld A-modules over $\mathbb{C}_\infty$

Important examples of Drinfeld A-modules are obtainable over $\mathbb{C}_\infty$ via a uniformization theory modeled after that of elliptic curves.

For Drinfeld modules over $X = \operatorname{Spec} \mathbb{C}_\infty$ one has the following result of Drinfeld [12]

Theorem A.1 (Drinfeld)**.** *Let φ be a Drinfeld module over $\mathbb{C}_\infty$ with $d\varphi\colon A \hookrightarrow \mathbb{C}_\infty$ equal to ι. Then there exists a unique entire function*

$$e_\varphi\colon \mathbb{C}_\infty \to \mathbb{C}_\infty : x \mapsto x + \sum_{i\geq 1} a_i x^{q^i} \quad (a_i \in \mathbb{C}_\infty)$$

such that for all $a \in A$ the following diagram is commutative:

$$\begin{tikzcd} \mathbb{C}_\infty \arrow[r, "e_\varphi"] \arrow[d, "x\mapsto ax"'] & \mathbb{C}_\infty \arrow[d, "x\mapsto\varphi_a(x)"] \\ \mathbb{C}_\infty \arrow[r, "e_\varphi"] & \mathbb{C}_\infty \end{tikzcd} \tag{1}$$

Moreover e_φ is a k-linear epimorphism and its kernel is a projective A-module of rank equal to the rank of the Drinfeld A-module which is discrete in $\mathbb{C}_\infty$.

Conversely to any discrete projective A-submodule $\Lambda \subset \mathbb{C}_\infty$ of rank r one can associate a unique exponential function

$$e_\Lambda(x) = x + \sum_{i\geq 1} a_i x^{q^i} \ \Big(= x \prod_{\lambda\in\Lambda\smallsetminus\{0\}} (1 - \frac{x}{\lambda})\Big),$$

whose set of roots is the divisor Λ, and a unique Drinfeld A-module φ_Λ of rank r over $\mathbb{C}_\infty$ such that (1) commutes for $\varphi = \varphi_\Lambda$ and $e_\varphi = e_\Lambda$. The characteristic of φ_Λ is the canonical inclusion $A \hookrightarrow \mathbb{C}_\infty$.

Moreover Drinfeld A-modules φ_Λ and $\varphi_{\Lambda'}$ are isomorphic (over $\mathbb{C}_\infty$) if and only if there is a scalar $\lambda \in \mathbb{C}_\infty^*$ such that $\lambda\Lambda = \Lambda'$. They are isogenous if there exists $\lambda \in \mathbb{C}_\infty^*$ and $a \in A \smallsetminus \{0\}$ such that $a\Lambda' \subset \lambda\Lambda \subset \Lambda'$.

In particular there exist Drinfeld A-modules of all ranks $r \in \mathbb{N} = \{1, 2, 3, \ldots\}$. For any rank $r \geq 2$ there exist infinitely many non-isomorphic Drinfeld A-modules of rank r over $\mathbb{C}_\infty$. In the rank 1 case, the number of isomorphism classes of Drinfeld A-modules (over $\mathbb{C}_\infty$) is equal to the number of projective A-modules of rank 1, i.e., to the cardinality of the class group $\mathrm{Cl}(A) = \mathrm{Pic}(A)$ of A. In fact within each isomorphism class there is one representative which is defined over the class field H of K with respect to ∞. There will be more on this in Appendix 3.

Indication of proof of Theorem A.1. Let φ be given and fix $a \in A \smallsetminus k$ and write $\varphi_a = a + \sum_{j=1}^t u_j \tau^j$ ($t = r \deg a$). Then (1) yields the recursion

$$a_i(a^{q^i} - a) = \sum_{j=1}^{t} u_j a_{i-j}^{q^j}$$

for the coefficients of e_φ where we set $a_i = 0$ for $i < 0$. Let v denote the valuation on $\mathbb{C}_\infty$ such that $v(\pi) = 1$ for π a uniformizer π of F at ∞. Let $C := \min_{j=1,\ldots,t} v(u_j)$. Then from $v(a) < 0$ one deduces

$$\frac{v(a_i)}{q^i} \geq \Big(\frac{C}{q^i} - v(a)\Big) + \min_{j=1,\ldots,t} \frac{v(a_{i-j})}{q^{i-j}}.$$

Choose $0 < \theta < v(a)$ so that for $i \gg 0$ one has $v(a) - \frac{C}{q^i} \geq \theta$. Setting $A_i := \min_{j=1,\ldots,t} \frac{v(a_{i-j})}{q^{i-j}}$ it follows that there is some $i_0 > 0$ such that for all $i \geq i_0$ one has

$$B_{i+1} \geq B_i, \quad \text{and} \quad B_{i+t} \geq B_i + \theta.$$

Thus (B_i) converges to ∞ and hence $\lim_{i\to\infty} \frac{v(a_i)}{q^i} = \infty$. This shows that e_φ has infinite radius of convergence, i.e., that it is entire. (The uniqueness of e_φ for a follows from the initial condition $de_\varphi = 1$.)

To show that e_φ is independent of a write $e_\varphi = \sum_{i\geq 0} a_i \tau^i$ as a formal power series in τ over $\mathbb{C}_\infty$. Then for any $b \in A$ the expression $e_\varphi^{-1} \varphi_b e_\varphi$ is again such a power series. Since the image of A under φ is commutative, it follows that $e_\varphi^{-1} \varphi_b e_\varphi$ commutes with $a = e_\varphi^{-1} \varphi_a e_\varphi$ as an element in the formal non-commutative power series ring $\mathbb{C}_\infty[[\tau]]$ over τ. One deduces that $e_\varphi^{-1} \varphi_b e_\varphi$ is a constant and by taking derivatives that $b = e_\varphi^{-1} \varphi_b e_\varphi$. Hence (1) commutes for any $b \in A$.

That $\mathrm{Ker}(e_\varphi)$ is an A-module is immediate from the commutativity of (1). The discreteness follows as in complex analysis from the entireness and non-constancy. That the roots have multiplicity one is deduced from the root at 0 having multiplicity one. The rank of $\mathrm{Ker}(e_\varphi)$ as an A-module is obtained by considering torsion points (introduced in the following section) and exploiting their relation to the rank. All further assertions are rather straight forward. □

2 Torsion points and isogenies of Drinfeld modules

Theorem A.1 indicates that Drinfeld modules are characteristic p analogs of elliptic curves. This suggests that torsion points of Drinfeld modules carry an interesting Galois action. Formally one defines modules of torsion points (or torsion schemes) as follows:

Fix a non-zero ideal $\mathfrak{a} \subset A$ and a Drinfeld A-module φ on X. For any $a \in A \smallsetminus \{0\}$ the kernel of $\varphi_a \colon L \to L$ is a finite flat group scheme over X of rank equal to $q^{r \deg a}$. (Passing to local coordinates it suffices to verify this for Drinfeld modules of standard type – where it is rather easy.) From this one deduces that the $\mathfrak{a}$-torsion scheme of φ, defined as

$$\varphi[\mathfrak{a}] := \cap_{a\in\mathfrak{a}} \mathrm{Ker}(\varphi_a)$$

is a finite flat A-module scheme over X of rank $q^{r \deg(\mathfrak{a})}$. If $\mathfrak{a}$ is prime to the characteristic of φ, then $\varphi[\mathfrak{a}]$ is finite étale over X. In the case $X = \mathrm{Spec}\, F$ for a field F, the finite group scheme $\varphi[\mathfrak{a}]$ becomes trivial over a finite Galois extension L of F. As an A-module, the group $\varphi[\mathfrak{a}](L)$ is isomorphic to $(A/\mathfrak{a})^r$ provided that $\mathfrak{a}$ is prime to the characteristic of φ. The Galois and the A-action commute on $\varphi[\mathfrak{a}]$. This yields the following first result:

Theorem A.2. *Let φ be a Drinfeld A-module of rank r on $\operatorname{Spec} F$ for a field F with algebraic closure $\overline{F}$. Suppose $\mathfrak{a}$ is prime to the characteristic of φ. Then the action of $\operatorname{Gal}(\overline{F}/F)$ on $\varphi[\mathfrak{a}](\overline{F})$ induces a representation*

$$\operatorname{Gal}(\overline{F}/F) \longrightarrow \operatorname{GL}_r(A/\mathfrak{a}).$$

We now turn to the relation between isogenies from a Drinfeld A-module (L,φ) and subschemes of $\varphi[\mathfrak{a}]$: If $\psi\colon (L,\varphi) \to (L',\varphi')$ is an isogeny then its kernel is finite. Hence there exists $a \in A \smallsetminus k$ which annihilates this kernel. One deduces that $\operatorname{Ker}(\psi) \subset \varphi[(a)]$ is a finite flat subscheme whose connected component agrees with that of $\varphi[\mathfrak{q}^n]$ for some $n \in \mathbb{N}_0$ and $\mathfrak{q}$ the characteristic of φ. Conversely given any flat A-module subscheme of $\varphi[\mathfrak{a}]$ satisfying the latter condition on its connected component, Drinfeld shows that there is up to isomorphism a unique Drinfeld A-module (L',φ') and an isogeny $(L,\varphi) \to (L',\varphi')$ whose kernel is that subscheme, cf. [12, §5 D, p. 577]. We indicate a proof for the finite flat subgroup scheme $\varphi[\mathfrak{a}] \subset L$ under the simplifying hypothesis that $X = \operatorname{Spec} R$ for an integrally closed domain R and that L is isomorphic to $\operatorname{Spec} R[x]$. (In particular this proves the assertion for all integral normal schemes X over A.)

Since R is a domain, φ is automatically in standard form. Let E be the fraction field of R and denote by $\overline{E}$ a fixed algebraic closure. The ring $E[\tau]$ is left Euclidian and thus there exists a monic generator $\varphi_{\mathfrak{a}}$ of the left $E[\tau]$ ideal $\mathfrak{I}$ generated by $\{\varphi_a \mid a \in \mathfrak{a}\}$. The roots of $\varphi_{\mathfrak{a}}(z)$ in $\overline{E}$ are precisely the common roots of all the φ_a, $a \in \mathfrak{a}$. If h is the height of φ (over E), then $\varphi_{\mathfrak{a}}$ is a multiple of a power of τ^h times an element in $R[\tau]$ with non-vanishing constant term; i.e., the multiplicity of the roots is a power of q^h. By the definition of $\varphi_{\mathfrak{a}}$, its coefficients lie in E. Let R' be the ring obtained from R by adjoining all these roots. As these are roots of all the polynomials φ_a, $a \in \mathfrak{a}$, the ring R' is finite over R. Clearly the polynomial $\varphi_{\mathfrak{a}}$ has its coefficients in R'. Using that R is integrally closed, it follows that $\varphi_{\mathfrak{a}}$ lies in fact in $R[\tau]$.

Note that because $\varphi_{\mathfrak{a}}(x) \in R[x]$ is monic (of degree $q^{\deg_\tau \varphi_{\mathfrak{a}}}$) it defines a finite flat subscheme $\operatorname{Spec} R[x]/(\varphi_{\mathfrak{a}}(x)) = \varphi[\mathfrak{a}]$ of $L = \operatorname{Spec} R[x]$. The A-module structure on L induces an A-module structure on $\varphi[\mathfrak{a}]$.

Next, let $b \in A$ be arbitrary. Then $\varphi_{\mathfrak{a}} \circ \varphi_b$ lies in $\mathfrak{I}$ because it annihilates all elements of $\varphi[\mathfrak{a}]$. It can thus be written in the form $\varphi'_b\varphi_{\mathfrak{a}}$ for some $\varphi'_b \in E[\tau]$. But again, the roots of $\varphi_{\mathfrak{a}}\varphi_b$ are integral over R and thus so are the coefficients of φ'_b. By normality of R it follows that $\varphi'_b \in R[\tau]$. Using the equation $\varphi_{\mathfrak{a}} \circ \varphi_b = \varphi'_b \circ \varphi_{\mathfrak{a}}$ it is easy to extract the constant coefficient of φ'_b. It is the constant coefficient of φ_b raised to a certain power. It is here where the size of the connected component of the scheme $\varphi[\mathfrak{a}]$ is important, as it determines the height of $\varphi_{\mathfrak{a}}$. In the case at hand this implies that the constant coefficients of φ'_b and φ_b agree, so that φ' defines a Drinfeld A-module (in standard form) over $\operatorname{Spec} R$ of the same characteristic as φ. Therefore $\varphi_{\mathfrak{a}}$ defines an isogeny $\varphi \to \varphi'$. We depict this in the following diagrams on the ring and on scheme levels, writing $\mathbb{G}_{a,R}$ for $\operatorname{Spec} R[x]$:

$$\begin{tikzcd}
R[x] \arrow[r, "\varphi_{\mathfrak{a}}"'] \arrow[d, "\varphi'_b"'] & R[x] \arrow[d, "\varphi_b"] \\
R[x] \arrow[r, "\varphi_{\mathfrak{a}}"'] & R[x]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{G}_{a,R} & \mathbb{G}_{a,R} \arrow[l, "\varphi_{\mathfrak{a}}"'] \\
\mathbb{G}_{a,R} \arrow[u, "\varphi'_b"'] & \mathbb{G}_{a,R}. \arrow[l, "\varphi_{\mathfrak{a}}"'] \arrow[u, "\varphi_b"']
\end{tikzcd}$$

We denote the Drinfeld A-module $(\mathbb{G}_{a,R},\varphi')$ also by $(\mathfrak{a} * L, \mathfrak{a} * \varphi)$ (or simply $\mathfrak{a} * \varphi$). If $\mathfrak{a}$ is principal and generated by $a \in A \smallsetminus \{0\}$, then

$$\varphi_{(a)} = \mu_\varphi(a)^{-1}\varphi_a \quad \text{for } \mu_\varphi(a) \text{ the leading coefficient of } \varphi_a. \tag{2}$$

A short computation yields the following formula for the leading term of $(a) * \varphi$:

$$\mu_{(a)*\varphi}(b) = \mu_\varphi(b) \cdot \mu_\varphi(a)^{1-q^{\deg b}} \quad \forall b \in A. \tag{3}$$

Using the explicit form of the action for principal ideals, it is easy to extend the $*$-operation to all fractional ideal $\mathfrak{a}$ of A. It also shows that $(a) * \varphi$ is isomorphic to φ – the isomorphism is given by $\mu_\varphi(a) \in R^* \subset R[\tau]$. It follows that $\operatorname{Cl}(A)$ acts on the isomorphism classes of Drinfeld A-modules of fixed rank r (over any fixed base).

In the particular case where the base is $\operatorname{Spec} \mathbb{C}_\infty$, the characteristic is the canonical embedding $\iota\colon A \hookrightarrow \mathbb{C}_\infty$ and $r = 1$, one can say more. By Theorem A.1 any such module is given by a rank 1 A-lattice in $\mathbb{C}_\infty$. Homotheties induce isomorphisms of Drinfeld modules and any two lattices are isogenous up to rescaling. Making all identifications explicit, one obtains the following result

Proposition A.3 (Drinfeld, Hayes). *The set of isomorphism classes of Drinfeld A-modules of rank 1 over $\mathbb{C}_\infty$ with $d\varphi = \iota$ is a principal homogeneous space under the $*$-operation of $\mathrm{Cl}(A)$.*

3 Drinfeld Hayes modules

In this section we collect some results on Drinfeld-Hayes modules. The reader is advised to recall the definitions of a sign-function and the corresponding strict class group from Definitions 10.1 and 10.3. As in (2), the leading term of a Drinfeld module φ is denoted μ_φ. We fix a sign-function sign throughout this section.

Definition A.4. A rank 1 sign-normalized Drinfeld module or simply a *Drinfeld-Hayes module (for* sign*)* is a rank 1 Drinfeld-module φ over $\mathbb{C}_\infty$ with $d\varphi = \iota$ whose leading term μ_φ agree with the restriction of a twisted sign function (sign) to $A \subset K_\infty$.

A good reference of the following result is [23, Thm. 7.2.15].

Theorem A.5 (Hayes). *Every rank 1 Drinfeld-module over $\mathbb{C}_\infty$ with characteristic ι is isomorphic to a Drinfeld-Hayes module.*

Indication of proof. Define the $\mathbb{Z}$-graded ring $\mathrm{gr}(K_\infty) = \oplus_{i\in\mathbb{Z}} M^i/M^{i-1}$ using the filtration $M^i := \{x \in K_\infty : v_\infty(x) \geq -i\}$. Let L be a subfield of $\mathbb{C}_\infty$ and let $\varphi\colon A \to L\{\tau\}$ be a rank 1-Drinfeld module with $d\varphi = \iota$.

Sublemma A.6. *There exists a unique map $\lambda_\varphi\colon \mathrm{gr}(K_\infty) \to L^*$ with the following properties*

(a) For all $a \in A$ with image $\bar{a}$ in $M^{v_\infty(a)}/M^{v_\infty(a)-1} \subset \mathrm{gr}(K_\infty)$ one has $\lambda_\varphi(\bar{a}) = \mu_\varphi(a)$.

(b) For all $\alpha, \beta \in \mathrm{gr}(K_\infty)$ one has $\lambda_\varphi(\alpha\beta) = \lambda_\varphi(\alpha)^{q^{\deg\beta}}\lambda_\varphi(\beta)$.

One first observes that $\mu_\varphi(ab) = \mu_\varphi(a)^{q^{\deg b}}\mu_\varphi(b)$ for any $a, b \in A$. Then one uses property (b) to extend the definition of λ_φ on the images $\bar{a}$ for $a \in A$ to all of $\mathrm{gr}(K_\infty)$. One also observes that λ_φ is the identity on k and a Galois automorphism when restricted to k_∞. The details are left to the reader.

The next result, whose proof we leave again to the reader, describes the change of λ_φ under isomorphism:

Sublemma A.7. *Suppose $\varphi' = \alpha\varphi\alpha^{-1}$ for some $\alpha \in L^*$. Then $\lambda_{\varphi'}(x) = \lambda_\varphi(x)\alpha^{(1-q^{-\deg(x)})}$.*

Now given φ, choose $\alpha \in \mathbb{C}^*$ such that $\alpha^{q-1} = \lambda_\varphi(\pi)$, so that $\varphi' = \alpha\varphi\alpha^{-1}$ satisfies $\lambda_{\varphi'}(\pi) = 1$. Because $\lambda_{\varphi'}$ is given by some $\sigma \in \mathrm{Gal}(k_\infty/k)$ when restricted to k_∞, one deduces that φ' is sign-normalized. □

Denote by $\mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$ the set of sign-normalized rank 1-modules over $\mathbb{C}_\infty$ Since the number of isomorphism classes of Drinfeld A-modules of rank 1 over $\mathbb{C}_\infty$ with characteristic ι is finite and equal to the class number of A and since the number of choices for α in the previous proof is finite, the set $\mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$ is finite. Recall the action of fractional ideals $\mathfrak{a}$ on Drinfeld modules from the paragraphs above Proposition A.3. The following is from [23, §7.2]:

Theorem A.8. *The action of $\mathcal{I}_A$ on rank 1-Drinfeld A-modules preserves the sign-normalization and thus defines a well-defined action*

$$\mathcal{I}_A \times \mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty) \to \mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty) : (\mathfrak{a}, \varphi) \mapsto \mathfrak{a} * \varphi.$$

The action of $\mathcal{P}^+$ is trivial. Under the induced action

$$\mathrm{Cl}^+(A) \times \mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty) \to \mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty) : ([\mathfrak{a}], \varphi) \mapsto \mathfrak{a} * \varphi$$

the set $\mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$ becomes a principal homogeneous space under $\mathrm{Cl}^+(A)$, i.e., the action is simply transitive and the stabilizer of any element is trivial.

Indication of proof. It is easy to see that the $*$-action does not affect sign-normalization: The action is of the form $y \mapsto y^{q^i}$ for some $i \in \mathbb{N}_0$ on the leading coefficient.

For a principal ideal (a) the $*$-action on the leading term was determined in (3). It is

$$\mu_{(a)*\varphi}(b) = \mu_\varphi(b) \cdot \mu_\varphi(a)^{1-q^{\deg b}} \quad \forall b \in A.$$

Thus if φ is sign-normalized and if a is positive the effect on leading terms is trivial because of $\mu_\varphi(a) = \mathrm{sign}(a) = 1$. In particular $\mathcal{P}^+$ acts trivially.

Next we show that $\mathrm{Cl}(A)^+$ acts faithfully: Suppose $\mathfrak{a} * \varphi = \varphi$. Since this implies in particular that $\mathfrak{a} * \varphi$ is isomorphic to φ we deduce that $\mathfrak{a}$ is principal, say equal to (a). In this case we can use the formula displayed above. It implies that for all $b \in A$ we must have

$$\mu_\varphi(a)^{1-q^{\deg b}} = 1.$$

Since the gcd of the $\deg b$ is 1, it follows that $\mu_\varphi(a)^{q-1} = 1$, i.e., that $\alpha := \mu_\varphi(a) \in k^*$. But then $a\alpha^{-1}$ is a positive generator of $\mathfrak{a}$ and the faithfulness of the action is shown.

Finally by determining the cardinalities of $\mathrm{Cl}(A)^+$ and of $\mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$ the proof is complete: $\#\mathrm{Cl}(A)^+ = \#\mathrm{Cl}(A) \cdot \#(\mathcal{P}/\mathcal{P}^+)$. But all elements of k^*_∞ occur as signs of some $\alpha \in K$ and principal ideals αA are positively generated if and only if $\alpha \in k^*$. Hence $\#\mathrm{Cl}(A)^+ = \#\mathrm{Cl}(A) \cdot \#k^*_\infty/\#k^*$. Next we observe that all isomorphism classes of rank 1 Drinfeld A-modules over $\mathbb{C}_\infty$ of characteristic ι are represented in $\mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$. We count the number of times a class occurs in $\mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$. If φ is a Drinfeld-Hayes module for sign, and if the same holds for $\alpha\varphi\alpha^{-1}$, then one shows $\alpha \in k^*_\infty$. Moreover the two are equal if and only if $\alpha \in k^*$. Hence the cardinality of $\mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$ is equal to the number of isomorphism classes of rank 1 Drinfeld A-modules over $\mathbb{C}_\infty$ of characteristic ι times $\#k^*_\infty/\#k^*$, and thus equal to $\#\mathrm{Cl}(A)^+ = \#\mathrm{Cl}(A) \cdot \#k^*_\infty/\#k^*$ by Theorem A.3. ☐

One now argues as in the case of CM elliptic curves to deduce that every $\varphi \in \mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$ is defined over H^+: Let $\tilde{H} \subset \mathbb{C}_\infty$ be the field of definition of ψ. Since $\mathrm{Aut}(\mathbb{C}_\infty/K)$ preserves $\mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$, the extension $\tilde{H}/K$ is finite. Considering the infinite place it follows that $\tilde{H}/K$ is separable. Using that the automorphisms commute with the $*$-operation, one shows that the extension is abelian. The $*$-action also shows that $\tilde{H}$ is independent of φ.

Next one shows that φ has its coefficients in $\tilde{\mathcal{O}}$, the normal closure of A in $\tilde{H}$: The Drinfeld-module has potentially good reduction everywhere. But the leading coefficient is a unit, and thus the Drinfeld-module can be reduced without twist. This allows one to use reduction modulo any prime of $\tilde{\mathcal{O}}$ as a tool. It is not hard to see that to test equality of sign-normalized rank 1 Drinfeld A-modules it suffices to test it modulo any prime of $\tilde{\mathcal{O}}$. This implies that the inertia group at any finite places is trivial. In particular the Artin-symbol $\sigma_{\mathfrak{p}}$ is defined at any prime of $\tilde{\mathcal{O}}$. One deduces the following Shimura type reciprocity law:

Theorem A.9. *If σ_I denotes the Artin-symbol of a fractional ideal I of A, then $\sigma_I\varphi = I * \varphi$. Thus $\tilde{H} = H^+$. Moreover every Drinfeld-Hayes module is defined over the ring of integers $\mathcal{O}^+$ of H relative to $A \subset K$.*

The reciprocity identifies the Galois action with the $*$-action on $\mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$ (it is rather trivial to see that any type Galois action preserves sign normalization.) One easily deduces:

Corollary A.10. $\mathrm{Gal}(H^+/H) \cong \mathbb{F}^*_\infty/\mathbb{F}^*$ *is totally and tamely ramified at ∞. It is unramified outside ∞.*

Remark A.11. One can show that any rank 1 Drinfeld module with characteristic ι can be defined over the ring of integers of the Hilbert class field H. However this representative within the isomorphism class is less canonical and its leading coefficient has no simple description.

Torsion points of Drinfeld-Hayes modules

Fix φ a sign-normalized rank 1 Drinfeld module over $\mathbb{C}_\infty$. For $I \in \mathcal{I}_A$, let φ_I denote the isogeny $\varphi \to I * \varphi$. Recall that $\varphi[I]$ denotes the A-module of I-torsion points of φ. Denote by $\mathcal{M}^1_{I,\mathrm{sign}}(\mathbb{C}_\infty)$ the set of isomorphism classes of pairs (φ, λ) where $\varphi \in \mathcal{M}^1_{\mathrm{sign}}(\mathbb{C}_\infty)$ and λ is a primitive I-torsion point of φ. Let $\mathcal{I}_A(I)$ denote the set of fractional ideals prime to I.

Theorem A.12. *Let $I \subset A$ be an ideal. The action of $\mathcal{I}_A$ on $\mathcal{M}^1_{I,\mathrm{sign}}(\mathbb{C}_\infty)$ given by*

$$J * (\varphi, \lambda) = (J * \varphi, \varphi_J(\lambda))$$

is well defined and transitive. The stabilizer of any pair (φ, λ) is the subgroup $\mathcal{P}^+_I \subset \mathcal{P}^+$ of positively generated fractional ideals prime to I. The set $\mathcal{M}^1_{I,\mathrm{sign}}(\mathbb{C}_\infty)$ is a principal homogeneous space under the induced action of $\mathrm{Cl}(A, I) := \mathcal{I}_A(I)/\mathcal{P}^+_I$. *One has an exact sequence*

$$0 \to (A/I)^* \to \mathrm{Cl}(A, I) \to \mathrm{Cl}^+(A) \to 0.$$

The field $H^+(\varphi[I])$ is the ray class field of K of conductor I at the finite places and which at ∞ is totally split above H^+. One has $\mathrm{Gal}(H^+(\varphi[I])/K) \cong \mathrm{Cl}(A, I)$. *Let σ_J denote the Galois automorphism which under the Artin reciprocity map corresponds to $J \in \mathcal{I}_A(I)$. The Shimura type reciprocity law reads: For any $J \in \mathcal{I}_A(I)$:*

$$\sigma_J(\varphi, \lambda) = J * (\varphi, \lambda)$$

One has the following ramification properties:

Theorem A.13. *Let $P \subset A$ be a prime ideal and $\lambda \in \varphi[P]$ be a primitive element.*

The extension $H^+(\varphi[P^i])/H^+$ is totally ramified at P unramified at all other finite places. It is Galois with Galois group $\mathrm{Gal}(H^+(\varphi[P^i])/H^+) \cong (A/P^i)^*$. *The action of this group on $\psi[P]$ is given by the character*

$$\sigma \mapsto \left(\frac{\sigma(\lambda)}{\lambda}\right).$$

The extension H^+/H is totally ramified at ∞ and unramified at all other places. It is Galois with $\mathrm{Gal}(H^+/H) \cong \mathbb{F}^*_\infty/\mathbb{F}^*_q$. *The decomposition D_∞ group at ∞ in* $\mathrm{Gal}(H^+(\varphi[P^i])/H)$ *is isomorphic to $\mathbb{F}^*_\infty$. The action of $\alpha \in \mathbb{F}^*_q \cong D_\infty$ on $\varphi[P]$ is given by*

$$\sigma_\alpha(\lambda) = \mu_\psi(\alpha)^{-1}(\lambda) = \alpha^{-1}(\lambda).$$

Bibliography

[1] Anderson, G., t-motives, *Duke Math. J.* **53** (1986), 457–502.

[2] Anderson, G., An elementary approach to L-functions mod p, *J. Number Th.* **80** no. 2 (2000), 291–303.

[3] Böckle, G., *Global L-functions over function fields*, Math. Ann. **323** (2002), no. 4, 737–795.

[4] Böckle, G., *An Eichler-Shimura isomorphism over function fields between Drinfeld modular forms and cohomology classes of crystals*, preprint 2004, http://www.uni-due.de/arith-geom/boeckle/preprints.html

[5] Böckle, G., Arithmetic over function fields: a cohomological approach, in: *Number fields and function fields—two parallel worlds*, 1–38, Progr. Math., 239, Birkhäuser Boston, Boston, MA, 2005.

[6] Böckle, G., *Hecke characters associated to Drinfeld modular forms*, preprint.

[7] Böckle, G., *Algebraic Hecke characters and compatible systems of abelian mod p Galois representations over global fields*, preprint.

[8] Böckle, G., Pink, R., *Cohomological theory of crystals over function fields*, Tracts in Mathematics **5**, European Mathematical Society, 2009.

[9] Diaz-Vargas, J., *Riemann hypothesis for* $\mathbb{F}_q[t]$, J. Number Th. **59** (2) (1996), 313–318.

[10] Deligne, P., *Cohomologie a support propre et construction du foncteur* $f^!$, Appendix of: Hartshorne, R., Residues and Duality, Lect. Notes Math. **20**, Berlin etc.: Springer (1966), 404–421.

[11] Deligne, P., *La conjecture de Weil. I*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307.

[12] Drinfeld, V. G., *Elliptic modules*, Math. USSR-Sb. **23** (1974), no. 4, 561–592.

[13] Emerton, M., Kisin, M., *A Riemann-Hilbert correspondence for unit F-crystals*, Astérisque **293** (2004).

[14] Freitag, E., Kiehl, R. Étale cohomology and the Weil conjecture, Ergebnisse der Mathematik und ihrer Grenzgebiete **13**, Springer, Berlin, 1988.

[15] Gardeyn, F., *The structure of analytic* τ*-sheaves*, J. Number Th. **100** (2003), 332-362.

[16] Gekeler, E.-U., *Drinfeld modular curves*, Lect. Notes Math. **1231**, Springer, Berlin-New York, 1986.

[17] Gekeler, E.-U., *On the coefficients of Drinfeld modular formes*, Invent. Math. **93** (1988), 667–700.

[18] Gekeler, E.-U., Reversat, M., *Jacobians of Drinfeld modular forms*, J. reine angew. Math. **476** (1996), 27–93.

[19] Goss, D., $\bar{\pi}$*-adic Eisenstein series for function fields*, Compositio Math. **41** (1980), 3–38.

[20] Goss, D., *Modular forms for* $\mathbb{F}_r$, J. Reine Angew. Math. **317** (1980), 16–39.

[21] Goss, D., *L-series of Grössencharakters of type A_0 for function fields*. In Contemp. Math. **133**, "p-adic methods in number theory and algebraic geometry", AMS, 1992, 119–139.

[22] Goss, D., L-series of t-motives and Drinfeld Modules, in: *The Arithmetic of Function Fields*, Goss, D., et al. (Eds.) Proceedings of the workshop at Ohio State University 1991. Berlin: Walter de Gruyter (1992), 313–402.

[23] Goss, D., *Basic Structures of Function Field Arithmetic*, Ergebnisse **35**, Berlin etc.: Springer (1996).

[24] Goss, D., Sinnott, W., *Class-groups of function fields*. Duke Math. J. **528** (1985), :507–516.

[25] B. Gross. *Algebraic Hecke characters for function fields*. Seminar on Number Theory, Paris 1980-81, pp. 87–90, Progr. Math. **22**, Birkhäuser.

[26] Hartshorne, R., *Residues and Duality*, Lect. Notes Math. **20**, Berlin etc.: Springer (1966).

[27] Hartshorne, R., *Algebraic Geometry*, GTM **52**, New York etc.: Springer (1977).

[28] Hayes, D., *A brief introduction to Drinfel'd modules*, in "The arithmetic of function fields", eds. Goss, D. et al., Ohio State University Math. Res. Inst. Publications, de Gruyter, Berlin, 1992.

[29] Katz, N., *Nilpotent connections and the monodromy theorem: applications of a theorem of Turittin*, Publ. Math. IHES **39** (1970 175–232.

[30] Katz, N., *p-adic properties of modular schemes and modular varieties*, in: Modular Functions of One Variable III, NM **350**, Springer, Berlin-New York, 1973, 69–191.

[31] Kiehl, R., Weissauer, R. *Weil conjectures, perverse sheaves and ℓ-adic Fourier transform*, Ergebnisse der Mathematik und ihrer Grenzgebiete. **42**, Springer, Berlin, 2001.

[32] Lafforgue, V. *Valeurs spéciales des fonctions L en caractéristique p*, J. Number Th. **129** (2009), no. 10, 2600–2634.

[33] Li, W., Meemark, Y., Hecke operators on Drinfeld cusp forms, J. Number Th. **128** (2008), 1941–1965.

[34] Lütkebohmert, W., On compactification of schemes. *Manuscripta Math.* **80** (1993), no. 1, 95–111.

[35] Manin, Y. I., Panchishkin A. A., *Introduction to Modern Number Theory*, Springer-Verlag 2005.

[36] Milne, J. Étale cohomology, Princeton Math. Series **33**, Princeton University Press, Princeton, 1980.

[37] Nagata, M., Imbedding of an abstract variety in a complete variety. *J. Math. Kyoto Univ.* **2** (1962), 1–10.

[38] Nagata, M., A generalization of the imbedding problem of an abstract variety in a complete variety. *J. Math. Kyoto Univ.* **3** (1963), 89–102.

[39] Pink, R., *Euler-Poincaré Formula in equal characteristic under ordinariness assumptions*, Manuscripta Math. **102** (2000), no. 1, 1–24.

[40] Pink, R., *Compactification of Drinfeld modular varieties and Drinfeld Modular Forms of Arbitrary Rank.* http://uk.arxiv.org/pdf/1008.0013.pdf

[41] Pink, R., Schieder S., *Compactification of a Drinfeld Period Domain over a Finite Field.* http://uk.arxiv.org/pdf/1007.4796.pdf

[42] Sheats, J., *The Riemann hypothesis for the Goss zeta function for* $\mathbb{F}_q[T]$, J. Number Th. **71** (1998), no. 1, 121–157.

[43] Taguchi, Y., Wan, D., *L-functions of φ-sheaves and Drinfeld modules*, J. Amer. Math. Soc. **9** (1996), no. 3, 755–781.

[44] Taguchi, Y., Wan, D., *Entireness of L-functions of φ-sheaves on affine complete intersections*, J. Number Th. **63** (1997), no. 1, 170–179.

[45] Teitelbaum, J. *The Poisson kernel for Drinfeld modular curves*, J. Amer. Math. Soc. **4** (1991), no.3, 491–511.

[46] Thakur, D., *On characteristic p zeta functions*, Compositio Math. **99** (1995), no. 3, 231–247.

[47] Thakur, D., *Function Field Arithmetic*, World Scientific (2004).

[48] Thakur, D., *Power sums with applications to multizeta and zeta zero distribution for* $\mathbb{F}_q[t]$, Finite Fields and Their Applications
15 (2009) 534–552.

[49] Wan, D., *On the Riemann hypothesis for the characteristic p zeta function*, J. Number Th. **58** (1) (1996), 196–212.

[50] Weibel, C., *An introduction to homological algebra*, Cambridge: Cambridge Univ. Press (1994).

[51] Deligne, P., et al., *Cohomologie Étale*, Séminaire de Géométrie Algébrique du Bois Marie, SGA4$\frac{1}{2}$, Lect. Notes Math. **569**, Berlin etc.: Springer (1977).