# A local-to-global principle for deformations of Galois representations

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**Abstract**

Given an absolutely irreducible Galois representation $\bar{\rho} : G_E \to \mathrm{GL}_N(k)$, $E$ a number field, $k$ a finite field of characteristic $l > 2$, and a finite set of places $Q$ of $E$ containing all places above $l$ and $\infty$ and all where $\bar{\rho}$ ramifies, there have been defined many functors representing strict equivalence classes of deformations of such a representation, e.g. by Mazur or Wiles in [15] or [26], with various conditions on the behaviour of the deformations at the places in $Q$ and with the condition that the deformations are unramified outside $Q$. Those functors are known to be representable. For $\bar{\rho}$ as above, our goal is to present a rather general class of global deformation functors that satisfy local deformation conditions and to investigate for those, under what conditions the global deformation functor is determined by the local deformation functors.

We will give precise conditions under which the local functors for all places in $Q$ are sufficient to describe the global functor, first in a coarse form, then in a refined form using auxiliary primes as done by Taylor and Wiles in [24]. This has several consequences. The strongest is that one can derive ring theoretic results for the universal deformation space by Mazur if one uses results of Diamond and Wiles, c.f. [11] and [26], and if one has a good understanding of all local situations. Furthermore it is easier to understand what happens under increasing the ramification as done by Boston and Ramakrishna in [6] and [20, 21]. Finally we shall reinterpret the results in the case of a tame representation $\bar{\rho}$ by directly considering presentations of certain pro-$l$ Galois groups and revisiting the prime-to-adjoint principle of Boston, c.f. [5].

## 1 Introduction

That in some sense the local deformation problems should govern the global universal deformation was certainly an important idea in many developments involving deformations of Galois representations, most notably in the work of Wiles, [26], and its continuation through others, e.g. [11], [13]. Yet a precise principle seems not have been defined anywhere. Our goal will be to show that, under suitable conditions, the universal deformation ring for a global problem can be cut out from a power series ring $W(k)[[x_1, \ldots, x_n]]$ by equations that naturally arise from local deformation problems. This principle will then be applied in several situations to derive ring-theoretic properties of deformation rings.

In our analysis we were guided by the following two ideas. The first becomes most apparent if one considers deformations of a tame Galois representation. There the deformation functor is described by sets of $H$-equivariant homomorphisms between a given pro-$l$ Galois group and

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a group of $N$ by $N$ matrices over a complete Noetherian local ring, where $H$ is a certain finite group of order prime to $l$, naturally acting on the two groups in question. The local-to-global principle should then come from a local-to-global principle for the presentation of the pro-$l$ Galois group mentioned above. Such presentations are indeed known, see [14], §11, provided a certain obstruction group vanishes. We will explain all this in greater detail in the last section.

The problem in this approach is the possible non-vanishing of the obstruction group. To overcome this, one can use auxiliary primes. This is now the second main point, as introduced in [24], see for example [10], Theorem 2.49. By considering not just the primes that can ramify, but also, in a controlled way, a few primes that do not ramify, one obtains a complete list of relations in a presentation of the pro-$l$ Galois group under consideration. Then one can control the global deformation functor by controlling all the local functors. This implies that all equations describing the universal deformation ring are local. If the representation is not tame, it is no more possible to use this method, because there is no description of the deformation functor as sets of equivariant homomorphisms, and because there are purely group theoretic obstructions to the lifting of certain deformations. As we will show, one can overcome this problem by using purely cohomological methods.

The organization is as follows. We will first revisit Mazur's theory of versal deformation rings, and in particular we shall reprove Proposition 2 of [15] without reducing modulo $l$. In the next section, Section 3, we will introduce local deformation problems. In general, the deformation problems are no more representable, as the image of the residual representation can be quite small. Therefore we can only obtain versal deformation rings. In the case of two-dimensional representations, however, we shall define rigid deformation problems as sets of equivariant homomorphisms, possibly with extra structure, whose universal deformation space is naturally isomorphic to the versal one of the usual deformation problem. We shall also collect what is known about the explicit shape of local versal deformation rings.

The following section presents a definition of the global deformation problem that we shall study. Our definition contains all the previously defined deformation problems with local restriction, e.g. those in [11, 15, 26], and also those in [25] if we chose to work with more general group schemes as $\mathrm{GL}_N(k)$ (which would pose no further problems). In section 5, we shall formulate and prove our local-to-global principle. The principle requires the existence of auxiliary primes which is discussed in section 6. For two-dimensional representations we shall follow the construction given in [10], Theorem 2.49, that was originally given in [24], which gives an optimal result in the sense that the number of auxiliary primes is as small as possible. In general we shall give criteria for the existence of such primes in terms of the image of the residual representation and its cohomology with coefficients in the adjoint representation, i.e. conditions of a purely group-theoretical nature.

We shall then give several applications. We generalize the raising of the level results of Boston, [6], make some comments on the transition from minimal deformation problems to slightly more ramified ones as done by Wiles in [26], and reinterpret and improve some recent results of Ramakrishna, c.f. [20, 21] on raising the level in even cases. In particular we shall show that all universal deformation rings that arise through his method are finite flat over $\mathbb{Z}_p$ in the even case.

The central idea in all our applications to the study of universal deformation rings, which was already present in [2, 11, 24, 26], is the following. Suppose we want to show that a certain universal deformation ring $R = \mathbb{Z}_p[[x_1, \ldots, x_k]]/(f_1, \ldots, f_n)$ is a complete intersection and flat over $\mathbb{Z}_p$ (i.e. $p$-torsion free). Then one needs to find local conditions such that the universal ring $R'$, which satisfies those and the conditions of $R$, is finite flat over $\mathbb{Z}_p$, or alternatively that $R'/(p)$ is finite. We suppose these local conditions give rise to equations $f_i$ such that $R' = R/(f_{n+1}, \ldots, f_{n+m})$. Obstruction theory, and this is our main tool, usually gives bounds on $n$. The local conditions give bounds on $m$. To obtain our desired conclusion we need $k = n+m$. This is the main obstacle with the approach, namely to choose appropriate local conditions and (obviously) to prove that $R'$ has the desired properties. For the results in [2], the basis are the results in [11, 24]. For our improvements of [20, 21], the basis are results in [20], namely the fact that $R'$ is isomorphic to $\mathbb{Z}_p$ (or to $\mathbb{Z}_p[[T_1, \ldots, T_r]]$).

In the last section, Section 8, we shall explain how to obtain the above principle in the tame case, as a nice application of the prime-to-adjoint principle in [5]. We shall also be able to give a direct meaning to certain cohomology classes that appear in sections 5 and 6.

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## 2 Mazur's versal deformation rings

Here we recall some results of Mazur from [15] on deformations of representations of profinite groups. We shall in the sequel always assume that $l > 2$. In this section we shall assume that $\Pi$ is a profinite group having the property that the pro-$l$ completion of every open subgroup of it is topologically finitely generated. We assume that we are given a continuous representation

$$\bar{\rho} : \Pi \to \mathrm{GL}_N(k)$$

where $k$ is a finite field of characteristic $l$.

Let $\mathcal{C}$ be the category of complete noetherian local rings $R$ with residue field $k$ and local ring homomorphisms which induce the identity on residue fields. $\mathfrak{m}_R$ shall denote the maximal ideal of $R$ and $t_R$ its mod $l$ tangent space, i.e. $t_R = \mathfrak{m}_R/(l, \mathfrak{m}_R^2)$. The objects of $\mathcal{C}$ are naturally $W(k)$-algebras, where $W(k)$ is the ring of Witt vectors of $k$. For $\mathcal{O} \in \mathcal{C}$ we define $\mathcal{C}_\mathcal{O}$ to be the category of $\mathcal{O}$-algebras inside $\mathcal{C}$, and so $\mathcal{C} = \mathcal{C}_{W(k)}$. For an $\mathcal{O}$-algebra $R$ we define $t_{R,\mathcal{O}} = \mathfrak{m}_R/(\mathfrak{m}_\mathcal{O}, \mathfrak{m}_R^2)$. If $\mathcal{O}$ is a discrete valuation ring, we shall use $\lambda$ for its uniformizing parameter. Furthermore by $k[\varepsilon]$ we denote the ring $k[\varepsilon]/(\varepsilon^2)$. If we consider $\mathcal{O}$-algebras, the $\mathcal{O}$-algebra structure on $k[\varepsilon]$ shall be the one coming from $\mathcal{O} \to k \to k[\varepsilon]$.

For $R$ in $\mathcal{C}$ we define $\Gamma_N(R) := \ker(\mathrm{GL}_N(R) \to \mathrm{GL}_N(k))$. Two liftings $\rho, \rho' : \Pi \to \mathrm{GL}_N(R)$ of $\bar{\rho}$ are called *strictly equivalent* if there is an $M \in \Gamma_N(R)$ such that $\rho = M\rho' M^{-1}$. A strict equivalence class $[\rho]$ of lifts of $\bar{\rho}$ to $R$ is called a *deformation*.

We consider the functor

$$\mathrm{Def}_{\mathcal{O},\Pi} : \mathcal{C}_\mathcal{O} \to \mathit{Sets} : R \mapsto \{\text{deformations } [\rho] \text{ of } \bar{\rho} \text{ to } R\}$$

For $\mathcal{O} = W(k)$, we shall drop the index $\mathcal{O}$. In [15] and [19] the following is shown.

**Theorem 2.1** *Given $\bar{\rho}$, a versal deformation of $\mathrm{Def}_{\mathcal{O},\Pi}$ exists. This means that there exists $R^v_\mathcal{O} \in \mathcal{C}_\mathcal{O}$ and a lift $\rho^v_\mathcal{O}$ of $\bar{\rho}$ to $R$ such that the following holds. Given any deformation $\rho$ of $\bar{\rho}$ to $A \in \mathcal{C}_\mathcal{O}$, there is a morphism $u : R^v_\mathcal{O} \to A$ such that $[\rho] = [u \circ \rho^v_\mathcal{O}]$. $u$ is not necessarily unique and neither is $[\rho^v_\mathcal{O}]$. However $u$ is unique for $A = k[\varepsilon]$. The latter is a minimality condition on $R^v_\mathcal{O}$ and implies in particular that one can identify the tangent space*

$t_{\mathrm{Def}_{\mathcal{O},\Pi}} := \mathrm{Def}_{\mathcal{O},\Pi}(k[\varepsilon])$ *with the mod* $\mathfrak{m}_{\mathcal{O}}$ *tangent space* $t_{R^v_{\mathcal{O}},\mathcal{O}}$ *of* $R^v_{\mathcal{O}}$*. The ring* $R^v_{\mathcal{O}}$ *is unique up to non-canonical isomorphism.* $(R^v_{\mathcal{O}}, \rho^v_{\mathcal{O}})$ *is called a hull.*

*If the centralizer of* $Im(\bar{\rho})$ *is the set of homotheties, then the above pair* $(R^v_{\mathcal{O}}, \rho^v_{\mathcal{O}})$ *is universal, i.e. the morphism* $u$ *above is always unique,* $R^v$ *is unique up to canonical isomorphism and* $[\rho^v_{\mathcal{O}}]$ *is unique.*

The proof in [15] procedes by verifying the axioms $(H_1), (H_2), (H_3)$ and for the second half also $(H_4)$ of the criterion of Schlessinger in Theorem 2.11 of [23].

**Remark 2.2** We note that [23] is generally concerned with the representability of functors $F : \mathcal{C}_{\mathcal{O}} \to$ *Sets*. As shown in [19], any subfunctor $G$ of a functor $F$ has a hull if it satisfies property $(H_1)$ of [23], if $G(k)$ is a one-point set, and if $F$ has a hull. If moreover $F$ is representable, then so is $G$.

Condition $(H_1)$ for $F$ says that whenever one has artinian rings $A_i, i = 0, 1, 2$ in $\mathcal{C}$, and maps $f_i : A_i \to A_0, i = 1, 2$ in $\mathcal{C}$ such that $\ker(f_1)$ is generated by one element $t$ such that $\mathfrak{m}_{A_1} t = 0$, then

$$F(A_1 \times_{A_0} A_2) \to F(A_1) \times_{F(A_0)} F(A_2)$$

is surjective. A map like $f_1$ is called a small surjection.

For a pair consisting of a functor $F$ and a subfunctor $G$, both from $\mathcal{C}_{\mathcal{O}}$ to (Sets), the following condition, we call it $(*)$, will be important in the sequel:
$G(k) \neq \emptyset$ and for all small surjections $f_1 : A_1 \to A_0$ and maps $f_2 : A_2 \to A_0$ of artinian rings $A_i$, $i = 0, 1, 2$ in $\mathcal{C}_{\mathcal{O}}$, the following diagram is a pullback diagram:

$$\begin{array}{ccc} G(A_1 \times_{A_0} A_2) & \longrightarrow & G(A_1) \times_{G(A_0)} G(A_2) \\ \downarrow & & \downarrow \\ F(A_1 \times_{A_0} A_2) & \longrightarrow & F(A_1) \times_{F(A_0)} F(A_2) \end{array} \tag{1}$$

The importance of property $(*)$ is explained by the following lemma, whose proof is a simple diagram chase. This property will be crucial for constructing well-behaved global deformation functors satisfying certain local properties.

**Lemma 2.3** *Suppose we have functors* $F_i, G_i, F : \mathcal{C}_{\mathcal{O}} \to$ *Sets fitting in a diagram*

$$\begin{array}{ccc} G & \dashrightarrow & \prod G_i \\ \downarrow & & \downarrow \\ F & \longrightarrow & \prod F_i \end{array}$$

*such that the* $G_i$ *are subfunctors of the* $F_i$*,* $F$ *and the* $F_i$ *have a hull, and* $G$ *is the pullback of the diagram. Suppose that all pairs* $G_i \subset F_i$ *satisfy the property* $(*)$ *above – in particular this condition implies that the functors* $G_i$ *satisfy* $(H_1)$ *and thus have a hull. Then* $G$ *has a hull. If furthermore* $F$ *is representable, then so is* $G$*.*

In Lemma 3.3, we shall give a list of pairs of functors and subfunctors satisfying property $(*)$ in situations relevant to us.

We now present a slight generalization of Proposition 2 in [15] that is shown by exactly the same deformation-theoretic methods as the original proof. By $\mathrm{ad}_{\bar{\rho}}$ we shall denote the representation of $\Pi$ on $M_N(k)$ obtained by composing $\bar{\rho}$ with the conjugation action of $\mathrm{GL}_N(k)$ on $M_N(k)$. By $\mathrm{ad}^0_{\bar{\rho}}$ we denote the subrepresentation on trace zero matrices inside $M_N(k)$. We note that for a versal deformation $R^v$ as above, one has an isomorphism as in [15], $H^1(\Pi, \mathrm{ad}_{\bar{\rho}})^* \cong t_{R^v}$.

**Theorem 2.4** *Given $\bar{\rho} : \Pi \to \mathrm{GL}_N(k)$ as above and $\mathcal{O} \in \mathcal{C}$. Then there is a presentation*

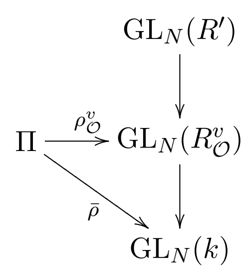
$$0 \to J \to \hat{R} = \mathcal{O}[[x_1, \ldots, x_n]] \to R^v_{\mathcal{O}} \to 0$$

*where $n = \dim_k H^1(\Pi, \mathrm{ad}_{\bar{\rho}})$ and $J$ is minimally generated by at most $m = \dim_k H^2(\Pi, \mathrm{ad}_{\bar{\rho}})$ elements.*

Proof: As we know that $t_{R^v}$ is isomorphic to the dual of $H^1(\Pi, \mathrm{ad}_{\bar{\rho}})$, we certainly can construct a presentation as above, where however a priori we have no bound on the number of generators of $I$. We will now derive this bound following the proof in [15]. Consider

$$0 \to J' = J/(J\mathfrak{m}_{\hat{R}}) \to R' = \hat{R}/(J\mathfrak{m}_{\hat{R}}) \to R^v_{\mathcal{O}} \to 0$$

As $\hat{R}$ is Noetherian, $J$ is finitely generated, and hence $J'$ is a finite dimensional vector space over $k \cong \hat{R}/\mathfrak{m}_{\hat{R}}$ whose dimension is the number of generators of $J$. We consider the following diagram, where $\rho^v_{\mathcal{O}}$ is a versal representation $\Pi \to \mathrm{GL}_N(R^v_{\mathcal{O}})$ as given in Theorem 2.1.




The obstruction for $\rho^v_{\mathcal{O}}$ to have a lift to $\mathrm{GL}_N(R')$ is given by a class $\theta$ in $H^2(\Pi, \mathrm{ad}_{\rho^v_{\mathcal{O}}} \otimes J')$. The class $\theta$ depends only on the deformation class of $\rho^v_{\mathcal{O}}$ and not on the chosen representation $\rho^v_{\mathcal{O}}$, even if $R^v_{\mathcal{O}}$ is not universal.

In the above situation $\mathrm{ad}_{\rho^v_{\mathcal{O}}} \otimes J' \cong \mathrm{ad}_{\bar{\rho}} \otimes J'$, as $\mathfrak{m}_{R'} J' = 0$, and hence the obstruction class is in $H^2(\Pi, \mathrm{ad}_{\bar{\rho}}) \otimes J'$. For every one-dimensional quotient $J''$ of $J'$, one has the corresponding obstruction class in $H^2(\Pi, \mathrm{ad}_{\bar{\rho}}) \otimes J'' \cong H^2(\Pi, \mathrm{ad}_{\bar{\rho}})$. Such a quotient corresponds to an element in $(Hom_k(J', k) - \{0\})/k^*$. We obtain a map

$$Hom_k(J', k) \to H^2(\Pi, \mathrm{ad}_{\bar{\rho}}) : f \to (1 \otimes f)\theta.$$

We claim that this map is injective. From this the theorem follows readily.

We assume otherwise and let $f \in Hom_k(J', k)$ be an element that maps to zero. Taking $J''$ to be $J$ modulo the kernel of $f$, this implies that the obstruction class in $H^2(\Pi, \mathrm{ad}_{\bar{\rho}}) \otimes J'' \cong H^2(\Pi, \mathrm{ad}_{\bar{\rho}})$ is zero. Hence there is a lift $\rho''$ of $\rho^v_{\mathcal{O}}$ to $R'' = R'/\ker(f)$. This is a lift of $\bar{\rho}$, and so by the versality of $R^v_{\mathcal{O}}$, there is a map $s : R^v_{\mathcal{O}} \to R''$.

By construction the map $\pi : R'' \to R^v_{\mathcal{O}}$ is surjective and an isomorphism on mod $\mathfrak{m}_{\mathcal{O}}$ tangent spaces. By the property that $R^v_{\mathcal{O}}$ is universal for deformations to $k[\varepsilon]/(\varepsilon^2)$ and the construction of $s$, it follows that $s : R^v_{\mathcal{O}} \to R''$ induces an isomorphism on mod $\mathfrak{m}_{\mathcal{O}}$ tangent spaces. By Lemma 1.1 in [23] this implies that $s$ is surjective. Thus $s\pi$ and $\pi s$ are surjective ring endomorphisms of Noetherian local rings, and hence isomorphisms[1]. But this implies that $\pi$ is an isomorphism, clearly contradicting $J'' = \ker(\pi) \neq 0$. ∎

# 3 Local deformation problems

We now specialize the above to the situation where $\Pi$ is the absolute Galois group $G_K$ of a local field $K$ of characteristic zero and residue characteristic $p$. $p$ can be equal or different from $l$. Let $\bar{\tau} : G_K \to \mathrm{GL}_N(k)$ be a given Galois representation. We fix $\mathcal{O}$ in $\mathcal{C}$. Let $X_K$ be a condition on Galois representations $G_K \to \mathrm{GL}_N(R)$ for $R \in \mathcal{C}_{\mathcal{O}}$ that holds for $\bar{\tau}$ and that satisfies the following properties:

(i) $X_K$ is invariant under strict equivalence.

(ii) $X_K$ is preserved under composition with morphisms $f : R \to S$ in $\mathcal{C}_{\mathcal{O}}$.

(iii) The condition $(H_1)$ of Schlessinger, see Remark 2.2, is satisfied, i.e. given three Artinian rings $A_i, f_i$ as in 2.2 and a pair of representations $\tau_i : G_K \to \mathrm{GL}_N(A_i)$ $(i = 1, 2)$ that map to the same representation $\tau_0$ under the respective $f_i$, such that $\tau_0, \tau_1, \tau_2$ satisfy $X_K$, then there is a representation $\tau_3 : G_K \to \mathrm{GL}_N(A_1 \times_{A_0} A_2)$ which satisfies $X_K$ and specializes, up to strict equivalence, under the canonical maps to the other two.

We define $\mathrm{Def}_{\mathcal{O},X_K}$ to be the subfunctor of $\mathrm{Def}_{\mathcal{O},G_K}$ of those deformations that satisfy the property $X_K$. We shall omit $\mathcal{O}$ in the notation if $\mathcal{O} = W(k)$. In the special case that $X_K$ is vacuous, we shall write $X_K^o$. We define $\mathcal{L}_{X_K} \subset H^1(G_K, \mathrm{ad}_{\bar{\tau}})$ to be the $k$ sub vector space that corresponds to $\mathrm{Def}_{\mathcal{O},X_K}(k[\varepsilon])$. Its dimension is denoted by $h^1_{X_K}$, or simply $h^1$.

By Theorem 2.1, and the remark thereafter, we obtain the following proposition which is due to [15] and [19] except for the presentation we give for $R_{\mathcal{O},X_K^o}$.

**Proposition 3.1** $\mathrm{Def}_{\mathcal{O},X_K}$ *has a hull, and it is representable if* $C_{\mathrm{GL}_N(k)}(Im(\bar{\rho})) = k^*$. *If we denote by* $(R_{\mathcal{O},X_K}, \rho_{\mathcal{O},X_K})$ *a hull representing* $\mathrm{Def}_{\mathcal{O},X_K}$, *then* $R_{\mathcal{O},X_K}$ *is a quotient of* $\mathcal{O}[[x_1, \ldots, x_{h^1}]]$. *In the special case* $X_K^o$, *i.e.* $\mathrm{Def}_{\mathcal{O},X_K} = \mathrm{Def}_{\mathcal{O},G_K}$, *one has a presentation*

$$0 \to J \to \mathcal{O}[[x_1, \ldots, x_{h^1}]] \to R_{\mathcal{O},X_K^o} \to 0$$

*where* $J$ *is generated by at most* $\dim_k H^2(G_K, \mathrm{ad}_{\bar{\tau}})$ *elements.*

**Remark 3.2** It would be very desirable to have a better theory of obstruction classes $H^2$, i.e. a theory that can also be applied to subfunctors of Def. If such a theory was available one could hope to have a better understanding of the number of equations necessary to describe $R_{X_K}$. In particular this should make it easier to recognize conditions $X_K$ that are unobstructed, i.e. for which $J = 0$. In the deformation theory over local fields, there are several natural situations where this happens, e.g. [19] or [9], where semi-stable deformations and generalizations thereof are considered. One of the few cases where the above is possible is the case where one looks at deformations with fixed determinant. Here the obstructions are directly described by $H^2(G_K, \mathrm{ad}^0_{\bar{\tau}})$.

[1] If $f : R \to R$ is a surjective ring homomorphism of local rings with maximal ideal $\mathfrak{m}_R$, it is easy to see that $\mathfrak{m}_R^n \subset f^{-1}(\mathfrak{m}_R^n)$ for all $n$. By comparing the lengths of the Artinian rings $R/(\mathfrak{m}_R^n)$ and $R/(f^{-1}(\mathfrak{m}_R^n))$ one finds that the inclusion must be an equality. Hence the kernel of $f$ must be zero.

Examples of properties $X_K$ satisfying properties (i) to (iii) above, are quotient properties, as defined by Mazur [15], §2.1, or the property of having a fixed determinant, or the properties as defined by Ramakrishna [19], i.e. properties of finite $R[G_K]$-modules which are closed under direct sums, subobjects and quotients, where $R \in \mathcal{C}_{\mathcal{O}}$.

In fact we shall mainly be interested in properties $X_K$ satisfying property $(*)$ described above Lemma 2.3 for the inclusion $X_K \subset X_K^o$. We have the following examples for this.

**Lemma 3.3** *The inclusion* $\mathrm{Def}_{\mathcal{O},X_K} \subset \mathrm{Def}_{\mathcal{O},X_K^o}$ *satisfies the property* $(*)$ *in the following cases.*

*(i)* $X_K^o$ *is representable and* $X_K$ *satisfies (i) to (iii) above.*

*(ii)* $X_K$ *is a quotient property of* $X_K^o$*, in the sense of Mazur [15], §2.1.*

*(iii)* $N = 2$ *and* $X_K$ *is the condition of ordinariness, where we do not necessarily assume that* $X_K^o$ *is representable.*

*(iv) In the notation of [25],* $X_K$ *is the local condition* $P_\nu$*, nearly ordinariness at* $\nu$*, and we assume it satisfies the condition (Reg) [25], §6.1.*

Proof: Part (i) is covered by Lemma 2.3. Part (iv) follows from the results quoted from Tilouine. The other cases we leave as an easy exercise. ∎

In Proposition 3.6, we shall give another condition for the case $N = 2$.

For the remainder of this section, we shall assume $N = 2$. First we shall reinterpret $R_{X_K^o}$ as a universal ring representing a functor that is a minimal smooth hull of $\mathrm{Def}_{X_K^o}$. See [23] for this terminology.

Let $L$ be the splitting field of $\bar{\tau}$, i.e. the fixed field inside $\bar{K}$ under the kernel of $\bar{\tau}$. As $K$ is local, $H := \mathrm{Gal}(L/K)$ is solvable. We consider $H$ as a subgroup of $\mathrm{GL}_2(k)$ via $\bar{\tau}$. Let $U$ denote the $l$-Sylow subgroup of $H$, and let $H'$ denote the quotient $H/U$, considered as a subgroup of $H$ by the Lemma of Schur-Zassenhaus. Let $g_i$ denote elements in $G_K$ generating $U$ as a group. $U$ is always an elementary abelian $l$-group and for $p \neq l$ it is trivial or one-generated. If $U$ is non-trivial, we assume without loss of generality that $U$ is upper-triangular, i.e. necessarily inside the set of matrices of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$. We choose a lift of $H'$ to $\mathrm{GL}_2(W(k))$. Without loss of generality we assume that this lift is inside the upper-triangular and that $\bar{\tau}(g_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ whenever $U$ is non-trivial. By $F$ we denote the fixed field corresponding to $\bar{\tau}^{-1}(U)$, and by $G_F(l)$ we denote the pro-$l$ completion of $G_F$. By the lemma of Schur-Zassenhaus, one can define an action of $H'$ on $G_F(l)$, unique up to inner automorphisms.

Now we can define a local deformation functor from $\mathcal{C}$ to *Sets*. If $H'$ is not inside the set of homotheties, we define

$$\begin{aligned} \mathrm{Equiv}_{X_K^o}(R) \ &:= \ \{\xi \in Hom_{H'}(G_F(l), \Gamma_2(R)) : \xi \pmod{\mathfrak{m}} = \bar{\tau} \\ &\qquad \text{and } \xi(g_1) \text{ has } (1,2)\text{-entry equal to } 1\} \end{aligned}$$

**Remark 3.4** If $U$ is trivial, we assume that the condition on $g_1$ is vacuous.

If $H'$ acts trivially on $U$, but $U$ is non-trivial, we impose the stronger condition that $\xi(g_1)$ is of the form $\begin{pmatrix} 1 & 1 \\ * & * \end{pmatrix}$. There are other possibilities to make the second situation more rigid.

One could require $\xi(g_1) = \begin{pmatrix} * & 1 \\ * & * \end{pmatrix}$ and for some $g_i$ with $\bar{\xi}(g_i) \neq I$, one could require its image to be $\begin{pmatrix} 1 & * \\ * & * \end{pmatrix}$, $\begin{pmatrix} a & * \\ * & a \end{pmatrix}$ or $\begin{pmatrix} * & * \\ * & 1 \end{pmatrix}$.

One has the following proposition, stated in [1], whose proof can be found essentially in [5], §§6,9.

**Proposition 3.5** $\mathrm{Equiv}_{X_K^o}$ *is always representable. The obvious morphism from* $\mathrm{Equiv}_{X_K^o}$ *to* $\mathrm{Def}_{X_K^o}$ *is smooth, i.e. for any surjection* $S \to R$ *in* $\mathcal{C}$, *the morphism*

$$\mathrm{Equiv}_{X_K^o}(S) \to \mathrm{Equiv}_{X_K^o}(R) \times_{\mathrm{Def}_{X_K^o}(R)} \mathrm{Def}_{X_K^o}(S)$$

*is surjective. It is an isomorphism if the centralizer of the image of* $H$ *in* $\mathrm{GL}_2(k)$ *is the set of scalar matrices. The induced map on tangent spaces is always an isomorphism, and in particular this implies that the universal ring representing* $\mathrm{Equiv}_{X_K^o}$ *is isomorphic to* $R_{X_K^o}$.

We can use this interpretation whenever it seems convenient. Also we note that a similar description is available for the case of fixed determinant. One simply has to require that the determinant of the homomorphisms $\xi$ is equal to the fixed one.

For a general condition $X_K$, we can consider the subfunctor $\mathrm{Equiv}_{\mathcal{O},X_K}$ of $\mathrm{Equiv}_{\mathcal{O},X_K^o}$ of such morphisms that satisfy $X_K$. Here it may be useful to choose a different normalization in the case where $U$ is non-trivial, but $H$ acts trivially on it. For example if $X$ is the condition 'co-ordinary' in the sense of [16], and $\mathrm{Im}(\bar{\tau}) = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in k\}$, it would be more appropriate for calculations to assume $g_1 \mapsto \begin{pmatrix} * & 1 \\ * & 1 \end{pmatrix}$, than our standard normalization as in Remark 3.4. One can show the following for $\mathrm{Equiv}_{\mathcal{O},X_K}$.

**Proposition 3.6** *The functor* $\mathrm{Equiv}_{\mathcal{O},X_K}$ *satisfies Schlessinger's condition* $(H_1)$ *if and only if the inclusion* $\mathrm{Def}_{\mathcal{O},X_K} \subset \mathrm{Def}_{\mathcal{O},X_K^o}$ *satisfies property* $(*)$. *If this holds, then the previous proposition remains valid for* $\mathrm{Equiv}_{\mathcal{O},X_K}$ *and* $\mathrm{Def}_{\mathcal{O},X_K}$. *The universal pair for* $\mathrm{Equiv}_{\mathcal{O},X_K}$ *will be denoted by* $(R_{\mathcal{O},X_K}, \xi_{\mathcal{O},X_K})$.

There is the following application of the above formalism.

**Proposition 3.7** *Assume we are given a representation* $\tau : G_K \to \mathrm{GL}_2(\mathcal{O})$ *for some* $\mathcal{O} \in \mathcal{C}$ *where* $\mathcal{O}$ *is Artinian. Let* $\bar{\tau} = \tau(\mathrm{mod}\ \mathfrak{m}_{\mathcal{O}})$. *If* $\mathrm{ad}_\tau^{G_K}$ *is a free* $\mathcal{O}$*-module, and hence of rank* $\dim_k \mathrm{ad}_{\bar{\tau}}^{G_K}$, *then one has*

$$Hom_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{O}) \cong H^1_{\mathcal{O},X_K}(G_K, \mathrm{ad}_\tau)$$

*where* $\mathfrak{p}$ *is the kernel of the map* $R_{\mathcal{O},X_K} \to \mathcal{O}$ *corresponding to* $\tau$, *and* $H^1_{\mathcal{O},X_K}(G_K, \mathrm{ad}_\tau)$ *is the subspace* $\mathrm{Def}_{\mathcal{O},X_K}(\mathcal{O}[\varepsilon])^*$ *inside* $H^1(G_K, \mathrm{ad}_\tau)$.

Proof: One has an isomorphism

$$Hom_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{O}) \cong Hom_{\mathcal{O}}(R_{\mathcal{O},X_K}, \mathcal{O}[\varepsilon])$$

where the $\mathcal{O}$ structure on $\mathcal{O}[\varepsilon]$ is the one where one considers the second ring as a free module over the first with basis $1, \varepsilon$. As $R_{\mathcal{O},X_K}$ represents the functor $\mathrm{Equiv}_{\mathcal{O},X_K}$, we have the isomorphism

$$Hom_{\mathcal{O}}(R_{\mathcal{O},X_K}, \mathcal{O}[\varepsilon]) \cong \mathrm{Equiv}_{\mathcal{O},X_K}(\mathcal{O}[\varepsilon]).$$

Our condition on $\mathrm{ad}_\tau^{G_K}$ implies that in fact one has

$$\mathrm{Equiv}_{\mathcal{O},X_K}(\mathcal{O}[\varepsilon]) \cong \mathrm{Def}_{\mathcal{O},X_K}(\mathcal{O}[\varepsilon])$$

This finishes the proof. We note that in general one only has a surjection in the last step, not an isomorphism. ■

If one has a deformation $\tau : G_K \to \mathrm{GL}_2(\mathcal{O})$, satisfying $X_K$, such that $\mathcal{O}$ is finite flat over $W(k)$ and $\mathrm{ad}_\tau^{G_K}$ is free over $\mathcal{O}$, one can apply the previous proposition to all the rings $\mathcal{O}/(\lambda^n)$, form a direct limit and obtain

$$Hom_{\mathcal{O}}(\mathfrak{p}/\mathfrak{p}^2, \mathcal{K}/\mathcal{O}) \cong \lim_{\rightarrow} H^1_{\mathcal{O},X_K}(G_K, \mathrm{ad}_{\tau \pmod{\lambda^n}}) \subset H^1(G_K, \mathrm{ad}_\tau \otimes \mathcal{K}/\mathcal{O})$$

where $\mathcal{K}$ is the fraction field of $\mathcal{O}$.

We now describe what is known for $X_K = X_K^o$ in the case $N = 2$.

**Theorem 3.8** *Let $N = 2$, then for all $K$ and $\bar\tau$, the hull $R_{X_K^o}$ is flat over $W(k)$ of relative dimension $\dim_k H^1(G_K, \mathrm{ad}_{\bar\tau}) - \dim_k H^2(G_K, \mathrm{ad}_{\bar\tau})$, and a complete intersection. In particular the ideal $J$ in Proposition 3.1 is generated by precisely $\dim_k H^2(G_K, \mathrm{ad}_{\bar\tau})$ many analytically independent elements of $W(k)[[x_1, \ldots, x_{h^1}]]$. The analogous statement holds over any base ring $\mathcal{O}$, by flatness of $R_{X_K^o}$ over $W(k)$.*

**Remark 3.9** Apart from the results above, for $N = 2$ the following universal deformation spaces are known to be flat over $\mathbb{Z}_p$ and complete intersections by [2, 9, 19].

- For $K = \mathbb{Q}_p$ the universal deformation ring of semistable or potentially semistable deformations (that become semistable in an extension of ramification degree at most $p$ and satisfy some further conditions) - if furthermore the determinant is fixed, the ring is isomorphic to $W(k)[[T]]$.
- For any $K$ the universal deformation space of ordinary deformations.

Proof of Theorem 3.8: The case $l = p$ is completely treated in [2]. So from now on we may assume that $l \neq p$. Furthermore as explained in [3], one can decompose $R_{K,\bar\tau}$ into the complete tensor product of the universal ring that arises from deformations of $\det(\bar\tau)$ and the hull that arises from considering deformations with a fixed determinant. Note that locally one can always find lifts to $W(k)$. The case of deforming the determinant has been considered in [15], so we only need to consider the case of fixed determinant, i.e. where the cohomological obstruction is given by $H^2(G_K, \mathrm{ad}^0_{\bar\tau})$. Throughout this proof only, we shall use $h^2$ for the $k$-dimension of this vector space. By Tate local duality $h^2 = \dim_k H^0(G_K, \mathrm{ad}^{0\,*}_{\bar\tau}(1))$.

If $h^2 = 3$, then $K$ must contain $l$-th roots of unity and $\bar\tau$ must have image in the homotheties. By the above proposition we are reduced to analyzing the functor assigning to $R \in \mathcal{C}$ the set of homomorphisms from $G_K(l)$ to $\mathrm{GL}_2(R)$ with no equivariance condition, because of the shape of $\bar\tau$. $G_K(l)$, the maximal pro-$l$ quotient of $G_K$, is isomorphic to $\mathbb{Z}_l \rtimes \mathbb{Z}_l$ where the action on a generator $t$ of the first component by a generator $s$ of the second is $s^{-1}ts = t^q$, where $q$ is the number of elements of the residue field of $K$. The calculations to obtain the result are as in [2], §8. In fact, in all cases discussed below we will have to consider maps from a group $G_K(l) \cong \mathbb{Z}_l \rtimes \mathbb{Z}_l$ to $\mathrm{GL}_2(R)$ that are however usually equivariant for a non-trivial operation of $H'$.

The case $h^2 = 2$ occurs precisely when $F$ contains $l$-th roots of unity, and $H'$ modulo homotheties is a group of order two acting non-trivially on the $l$-th roots of unity in $F$. This case was described in [6], Theorem 1, and in [1], Theorem 4.7. The explicit equations given there form a regular sequence and show that $R_{X_K^o}$ is flat over $W(k)$.

We now discuss the cases where $U$ is trivial and $h^2 = 1$. The other cases are listed in the lemma below and are obtained by the methods in [2]. Thus after discussing the cases here, we will have shown the theorem.

The cases where $L_0$, the splitting field of $\mathrm{ad}^0_{\bar\tau}$, is unramified over $K$, and where $h^2 = 1$, were discussed either in [6], or in [1], Theorem 4.7. So we only need to discuss the cases where $L_0$ is ramified over $K$ and $h^2 = 1$.

Then we can pick a non-trivial element in the ramification subgroup of $\mathrm{Gal}(L_0/K)$. This element, considered inside $\mathrm{GL}_2(W(k))$, must act trivially on the image of $G_L(l) \cong \mathbb{Z}_l \rtimes \mathbb{Z}_l = \langle s,t | s^{-1}tst^{-q}\rangle$. As the centralizer in $\mathrm{GL}_2(k)$ of $\bar\tau(G_K)$ is abelian, we may assume, after possibly enlarging $k$ to a quadratic extension, that $\xi_{X_K^o}(s)$, and $\xi_{X_K^o}(t)$ are matrices of the form $\begin{pmatrix} 1+a & 0 \\ 0 & (1+a)^{-1} \end{pmatrix}$ with different $a = S, T$, respectively. - We assumed that the determinant be one. - Hence $R_{X_K^o} = W(k)[[S,T]]/((1+T)^{q-1}-1)$, at least after replacing $k$ by its unique quadratic extension, which doesn't change the properties in question. ∎

**Lemma 3.10** *Let $\bar\tau$ be as above. Let $\bar H$ be the image of $H$ inside $\mathrm{PGL}_2(k)$ and note that $U$ can be considered as a subgroup of $\mathrm{PGL}_2(k)$. We assume that $U$ is non-trivial, and $p \neq l$. Let $K'$ be the extension of $K$ corresponding to $\bar H/U$, and let $l^n$ be the $l$-part of the order of the set of roots of unity in $K'$. Then there are the following cases for the universal pair for $\mathrm{Equiv}_{X_K^o}$, where we assume the determinant to be fixed, equal to one, for the equivariant homomorphism. (Throughout, we use the condition $g_1 \mapsto \begin{pmatrix} a & 1 \\ c & a \end{pmatrix}$.)*

*(i) $\bar H/U$ has order greater than two. Then $h^2 = 0$, $R_{K,\bar\tau} = W(k)$ and the unique universal morphism $\xi_{K,\bar\tau}$ is given by*

$$s \mapsto \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix} \quad t \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

*(ii) $\bar H/U$ has order equal to two. Then $h^2 = 1$, $R_{K,\bar\tau} = W(k)[[T]]/(Tg_{l^n}(T))$ and the unique universal morphism $\xi_{K,\bar\tau}$ is given by*

$$s \mapsto \begin{pmatrix} \sqrt{1+d} & 0 \\ 0 & \sqrt{1+d}^{-1} \end{pmatrix} \quad t \mapsto \begin{pmatrix} \sqrt{1+T} & \frac{1}{\sqrt{1+T}} \\ T & \sqrt{1+T} \end{pmatrix}$$

*where $d = g_q(T) - 1$ and the polynomials $f_n, g_n$ are defined as in in [2], Remark 5.4. They are the polynomials $g_n, h_n$ of [6], Theorem 1. Further $g_{l^n}(T) \equiv T^{(l^n-1)/2} (\mathrm{mod}\ l)$.*

*(iii) $\bar H = U$ and the extension is ramified. Then $h^2 = 1$,*

$$R_{K,\bar\tau} = W(k)[[S,T]]/(Tg_{l^n}(T))$$

*and the unique universal morphism $\xi_{K,\bar\tau}$ is given by*

$$s \mapsto \begin{pmatrix} 1+a & b+S \\ c & 1+d \end{pmatrix} \quad t \mapsto \begin{pmatrix} \sqrt{1+T} & \frac{1}{\sqrt{1+T}} \\ T & \sqrt{1+T} \end{pmatrix}$$

*where $c = ST$, $d, a = \pm g_{q-1}(T)/2 + \sqrt{1 - TS^2 + g^2_{q-1}(T)/4}$, and $b$ is the Teichmüller lift (or zero) of the $(1,2)$ entry of $\bar\tau(s)$.*

*(iv)* $\bar{H} = U$ *and the extension is unramified. Then* $h^2 = 1$,

$$R_{K,\bar{\tau}} = W(k)[[S,T]]/(Th(S,T)g_{l^n}(T^2h(S,T)))$$

*and the unique universal morphism* $\xi_{K,\bar{\tau}}$ *is given by*

$$s \mapsto \begin{pmatrix} 1+S & 1 \\ 2S+S^2 & 1+S \end{pmatrix} \quad t \mapsto \begin{pmatrix} a & T \\ T\Big(h(S,T)-(q-1)\frac{(1+S)^2}{(1+q)^2}g_{q-1}(T^2h(S,T)\Big) & a \end{pmatrix}$$

*where* $a = \sqrt{1+T^2\Big(h(S,T)-(q-1)\frac{(1+S)^2}{(1+q)^2}g_{q-1}(T^2h(S,T))\Big)}$. *The function* $h(S,T)$ *is constructed by iteratively replacing* $c$ *by itself in the following expression of formal power series*

$$c = 2S + S^2 - (q-1)\frac{(1+S)^2}{(1+q)^2}g_{q-1}(T^2c)$$

*The polynomial in the limit, expressing* $c$ *in terms of* $S, T$, *will be denoted by* $h(S,T)$. *One can show that* $h(S,T) = \Big(S+1-\sqrt{1+\frac{(q-1)^2}{4q}}\Big)\cdot$ *unit.*

Proof: We shall only comment on part (iv). Starting with general images

$$s \mapsto \begin{pmatrix} 1+U & 1 \\ U+X+UX & 1+X \end{pmatrix} \quad t \mapsto \begin{pmatrix} \sqrt{1+TW} & T \\ W & \sqrt{1+TW} \end{pmatrix}$$

of determinant one, where we took the normalization $s \mapsto \begin{pmatrix} * & 1 \\ * & * \end{pmatrix}$ and $t \mapsto \begin{pmatrix} a & * \\ * & a \end{pmatrix}$, one can derive three equations from the relation $s^{-1}ts = t^q$, namely

$$W = T(U+X+UX) \quad W(U-X) = 0 \quad T(U-X-(1+X)g_{q-1}(TW)) = 0.$$

Using the above images for $s,t$, we could derive equations among the variables. However the images we took are not acceptable for a description of $\mathrm{Equiv}_{X_K^o}$ as the case $T = W = 0$ is not rigid. To correct this, we conjugate the matrices representing the images of $s,t$ by $\begin{pmatrix} 1 & 0 \\ (U-X)/2 & 1 \end{pmatrix}$ and introduces new variables $S = (U+X)/2, Y = (U-X)/2$. After some further calculations one obtains for $W$ the recursion

$$W = T(2S+S^2-Y^2) = T\Big(2S+S^2-(q-1)\frac{(1+S)^2}{(1+q)^2}g_{q-1}(TW)\Big).$$

This explains the definition of $h(S,T)$. Its well-definedness has to be checked. Once this is done, it is easy to finish the proof of part (iv). ∎

**Remark 3.11** We shall use these explicit descriptions in Section 7, when comparing minimal universal deformations to slightly larger ones, and when discussing recent results of Ramakrishna.

In cases (ii) and (iii), we note that substituting $T = 0$ implies that any lift to characteristic zero must have infinite ramification, while substituting any solution of $g_{l^n}(T) = 0$ provides a finite image of the ramification group for lifts to characteristic zero. The local condition $T = 0$ is the one used in [26] to define minimal deformation. Moreover, cases (iii) and (iv) are the cases used in [20, 21].

## 4 Global deformation functors

Let $E$ be any number field. We assume that we are given a representation $\bar{\rho}: G_E \to \mathrm{GL}_N(k)$, such that the centralizer of $H = \mathrm{Im}(\bar{\rho})$ is the set of homotheties. $M$ will denote the splitting field of $\bar{\rho}$. By $\bar{H}$ we denote the image of $\bar{\rho}$ inside $\mathrm{PGL}_N(k)$, by $E_0$ the corresponding Galois extension of $E$. There will be no confusion with the notation of the previous section, as from now on if we talk about the local situation at a prime $\mathfrak{p}$ of $E$, we will use the subscript $\mathfrak{p}$.

We now consider the following deformation problem. We fix a ring $\mathcal{O}$ in $\mathcal{C}$ and let $Q$ be a finite set of places containing all places above $l$ and $\infty$. For each place $\mathfrak{p}$, we are given a property $X_{E_\mathfrak{p}}$, in short $X_\mathfrak{p}$, of Galois representations $\rho: G_\mathfrak{p} \to \mathrm{GL}_N(R)$, $R \in \mathcal{C}_\mathcal{O}$, where $G_\mathfrak{p}$ is a fixed decomposition group inside $G_E$. We have an isomorphism $G_\mathfrak{p} \cong G_{E_\mathfrak{p}}$, that corresponds to an embedding $\bar{E} \to \bar{E}_\mathfrak{p}$. We shall assume that all the inclusions $\mathrm{Def}_{\mathcal{O},X_\mathfrak{p}} \subset \mathrm{Def}_{\mathcal{O},X^o_{E_\mathfrak{p}}}$ satisfy the property $(*)$. (For conditions under which this holds, see Lemma 3.3.) If $\mathcal{O} = W(k)$, we shall omit it in the notation.

By $I_\mathfrak{p}$ we shall denote the inertia group of $G_\mathfrak{p}$. For all places $\mathfrak{p} \notin Q$, the property $X_\mathfrak{p}$ will be that the representation is unramified at $\mathfrak{p}$. This implies that we consider only representations of $\Pi = G_{E,Q}$, the maximal outside $Q$ unramified extension of $E$. This extension satisfies the condition stated for $\Pi$ at the beginning of Section 2. Let $X$ be the set of all local deformation conditions $X_\mathfrak{p}$. $X_Q$ shall denote the set of conditions defined by $X_\mathfrak{p} = \emptyset$ for $\mathfrak{p} \in Q$, and $X_\mathfrak{p} = X^o_{E_\mathfrak{p}}$ for $\mathfrak{p} \notin Q$.

These assumptions at all places $\mathfrak{p}$ together with the observations in Remark 2.2 and Theorem 2.1 provide us with the following theorem, essentially due to Mazur and Ramakrishna, on the functor $\mathrm{Def}_{\mathcal{O},X}$ from $\mathcal{C}_\mathcal{O}$ to Sets

$$\mathrm{Def}_{\mathcal{O},X}(R) = \{\text{deformations } [\rho] \text{ of } [\bar{\rho}] \text{ to } R \text{ satisfying } X_\mathfrak{p} \text{ for each place } \mathfrak{p}\}$$

**Theorem 4.1** $\mathrm{Def}_{\mathcal{O},X}$ *is representable. The corresponding universal object is denoted by* $(R_{\mathcal{O},X}, \rho_{\mathcal{O},X})$. *Furthermore, if* $X = X_Q$, *and* $h_i := \dim_k H^i(G_K, \mathrm{ad}_{\bar{\rho}})$ *for* $i = 1, 2$, *then* $R_{X_Q}$ *has a presentation*

$$0 \to J \to W(k)[[x_1, \ldots, x_{h_1}]] \to R_{X_Q} \to 0$$

*where* $J$ *is generated by at most* $h_2$ *elements.*

All of our conditions so far are local. However we want to include one global condition, namely the condition of a fixed determinant. Suppose we are given a condition $X$ as above and a deformation $[\rho_\mathcal{O}] \in \mathrm{Def}_{\mathcal{O},X}(\mathcal{O})$. Let $\eta = \det(\rho_\mathcal{O})$. Then we define $X^\eta$ to be the deformation problem as above with the additional condition that $\det(\rho) = \eta$ for all deformations $[\rho]$. One obtains the analogue of Theorem 4.1 for $X^\eta$ with the only modification that $\mathrm{ad}_{\bar{\rho}}$ has to be replaced by $\mathrm{ad}^0_{\bar{\rho}}$.

One of the advantages of working with fixed determinant problems is that Leopoldt's conjecture for $E$ at the prime $l$ does not necessarily intervene when calculating dimensions of deformation rings, while it has clearly an effect on the dimension of universal deformation spaces of one-dimensional representations, c.f. [15], p. 405. Also for $X_Q$, for example, it is rather easy to write $R_X$ as a complete tensor product of $R_{X^\eta}$ and the deformation space describing deformations of the one-dimensional representation $\det(\bar{\rho})$, provided $l \nmid N$.

We shall now follow [10], which in turn follows [26], and make the following definition.

**Definition 4.2** A *collection of infinitesimal conditions* $\mathcal{L}$, or $\mathcal{L}_X$, associated to a set $X$ of deformation conditions is defined as the collection of the subspaces

$$\mathcal{L}_\mathfrak{p} = t^*_{R_{\mathcal{O},X_\mathfrak{p}},\mathcal{O}} \subset H^1(G_\mathfrak{p}, \mathrm{ad}_{\bar\rho}),$$

the duals of the mod $\mathfrak{m}_\mathcal{O}$ tangent spaces of the rings $R_{\mathcal{O},X_\mathfrak{p}}$, in agreement with the definition at the beginning of Section 3.

In particular this implies that for all $\mathfrak{p} \notin Q$ we have

$$\mathcal{L}_\mathfrak{p} \cong H^1(G_\mathfrak{p}/I_\mathfrak{p}, (\mathrm{ad}_{\bar\rho})^{I_\mathfrak{p}})$$

Furthermore using Tate local duality, as in [10], §2, one can define $\mathcal{L}^\perp$ by defining $\mathcal{L}_\mathfrak{p}^\perp$ to be the dual of $\mathcal{L}_\mathfrak{p}$ under the prefect pairing

$$H^1(G_\mathfrak{p}, \mathrm{ad}_{\bar\rho}) \times H^1(G_\mathfrak{p}, \mathrm{ad}^*_{\bar\rho}(1)) \to \mathbb{Q}/\mathbb{Z}$$

The same definitions can be made for deformation problems $X^\eta$ for fixed determinant deformations where one has to replace throughout the module $\mathrm{ad}_{\bar\rho}$ by $\mathrm{ad}^0_{\bar\rho}$.

Following [10] or [26], we define $H^1_{\mathcal{L}_X}(G_E, \mathrm{ad}_{\bar\rho})$ by

$$0 \to H^1_{\mathcal{L}_X}(G_E, \mathrm{ad}_{\bar\rho}) \to H^1(G_{E,Q}, \mathrm{ad}_{\bar\rho}) \to \coprod_{\mathfrak{p}\in Q} H^1(G_\mathfrak{p}, \mathrm{ad}_{\bar\rho})/\mathcal{L}_{X_\mathfrak{p}},$$

and similarly for $\mathrm{ad}^0_{\bar\rho}$. We note that for $X = X_Q$, one has $\text{Ш}^1_Q(E, \mathrm{ad}_{\bar\rho}) = H^1_{\mathcal{L}_{X_Q}}(G_E, \mathrm{ad}_{\bar\rho})$, and by Poitou-Tate duality, in the same situation $\text{Ш}^2_Q(E, \mathrm{ad}_{\bar\rho}) = H^1_{\mathcal{L}^\perp_{X_Q}}(G_E, \mathrm{ad}^*_{\bar\rho}(1))^*$. This implies in particular for general $X$, that $\text{Ш}^2_Q(E, \mathrm{ad}_{\bar\rho}) = 0$ whenever $H^1_{\mathcal{L}^\perp_X}(G_E, \mathrm{ad}^*_{\bar\rho}(1)) = 0$.

The essential point of having a deformation problem described by an inifinitesimal set of conditions is that one can then appeal as in [26] to the cohomological methods developed by Poitou and Tate.

## 5 Local-to-global principles

In the previous two sections we defined local and global deformation problems. Given a representation $\bar\rho : G_E \to \mathrm{GL}_N(k)$, a set $X$ of deformation conditions and a set $Q$ of primes, as in the previous sections, there are obvious maps

$$\mathrm{Def}_{\mathcal{O},X} \to \mathrm{Def}_{\mathcal{O},X_\mathfrak{p}}$$

for all primes $\mathfrak{p}$ of $E$. Hence there are maps between the respective versal or universal deformation spaces $R_{\mathcal{O},X_\mathfrak{p}} \to R_{\mathcal{O},X}$. Those maps are unique precisely when the restriction of $\bar\rho$ to $G_\mathfrak{p}$ has a universal deformation. In the case $N = 2$ one can use the description in Section 3 of the equivariant mapping functors as smooth minimal covers of the local deformation functors, to obtain unique maps between the global and versal local deformation spaces.

Our goal will be a local-to-global principle. We shall give two versions. The first applies only to the special situation $X = X_Q$. But it shall be used to derive the second very general one. For the Poitou-Tate formalism and Tate local duality we refer the reader to [17].

**Lemma 5.1** *Suppose we are given $R_1, R_2 \in \mathcal{C}_{\mathcal{O}}$ and a map $f : R_1 \to R_2$ in $\mathcal{C}_{\mathcal{O}}$. Let $S_i$, $i = 1, 2$, be power series rings over $\mathcal{O}$ such that the mod $\mathfrak{m}_{\mathcal{O}}$ tangent spaces $t_{S_i,\mathcal{O}}$ and $t_{R_i,\mathcal{O}}$ are isomorphic as $k$ vector spaces. Then for any diagram* $\mathbf{D}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_1 & \longrightarrow & S_1 & \xrightarrow{f_1} & R_1 & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & J_2 & \longrightarrow & S_2 & \xrightarrow{f_2} & R_2 & \longrightarrow & 0 \end{array}$$

*and any lift $\hat{f} : S_1 \to S_2$ of $f$ in $\mathcal{C}_{\mathcal{O}}$ that makes the diagram commutative, the $k$-dimension* $\dim_k J_2/(\mathfrak{m}_2 J_2, \hat{f}(J_1))$ *is independent of these data. In particular the property of $J_1$ mapping onto $J_2$ under such lifts is independent of the chosen lift.*

Proof: To show independence of $f_1, f_2$, we first consider the special case that $f = \mathrm{Id}$. Let $f_i : S_i \to R_i$ be surjections with $\dim_k t_{S_i,\mathcal{O}} = \dim_k t_{R_i,\mathcal{O}}$. Given $\mathbf{D}$ as above and any $\hat{f} : S_1 \to S_2$ such that

$$\begin{array}{ccc} S_1 & \xrightarrow{f_1} & R_1 \\ \downarrow{\scriptstyle \hat{f}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ S_2 & \xrightarrow{f_2} & R_2 \end{array}$$

commutes. By considering mod $\mathfrak{m}_{\mathcal{O}}$ tangent spaces, it follows that $\hat{f}$ must be isomorphisms, and so the lemma is obvious. Thus from now on, we may fix $f_1, f_2$ in $\mathbf{D}$ and only vary the lift $\hat{f}$.

Let $\hat{f}, \hat{f}'$ be any two lifts of $f$ that make $\mathbf{D}$ commutative. It follows that $\delta := \hat{f}' - \hat{f} \in \mathrm{Der}_{\mathcal{O}}(S_1, J_2/(\mathfrak{m}_2 J_2))$ if the lifts are considered as maps to $S_2/(\mathfrak{m}_2 J_2)$. Clearly $\delta(\mathfrak{m}_1^2) = 0$ as $J_2/(\mathfrak{m}_2 J_2)$ is a $k$ vector space. As $J_1 \subset \mathfrak{m}_1^2 + \mathfrak{m}_{\mathcal{O}} S_1$, we have $\delta(J_1) = 0$, and hence the lifts modulo $\mathfrak{m}_2 J_2$ agree on $J_1/(\mathfrak{m}_1 J_1)$. This shows the invariance of the dimension in question. ■

We now present the set-up for our local-to-global principles. Suppose we are given a set of conditions $X$ and a corresponding set of place $Q$. Let $Q'$ be a finite set containing $Q$ whose relevance shall be clarified below. We assume that $\mathrm{Def}_{\mathcal{O},X}$ is a functor defined on $\mathcal{C}_{\mathcal{O}}$. $X'$ shall denote the set of conditions such that $X'_{\mathfrak{p}} = X_{\mathfrak{p}}$ for the primes outside $Q' - Q$, and it is the empty condition at the primes of $Q' - Q$. Let $n'_{\mathfrak{p}} = \dim_k t_{R_{\mathcal{O},X'_{\mathfrak{p}}},\mathcal{O}}$, and let $n' = \dim_k t_{R_{\mathcal{O},X'},\mathcal{O}}$. Consider the following diagram $\mathcal{D}_{X',Q'}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \hat{\otimes} J'_{\mathfrak{p}} & \longrightarrow & \hat{\otimes} S'_{\mathfrak{p}} := \hat{\otimes}\mathcal{O}[[x_1, \ldots, x_{n'_{\mathfrak{p}}}]] & \longrightarrow & \hat{\otimes} R_{\mathcal{O},X'_{\mathfrak{p}}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \tilde{\alpha}} & & \\ 0 & \longrightarrow & J_{X'} & \longrightarrow & S_{X'} := \mathcal{O}[[x_1, \ldots, x_{n'}]] & \longrightarrow & R_{\mathcal{O},X'} & \longrightarrow & 0 \end{array}$$

where the horizontal maps are minimal presentations of the respective rings on the right, and the tensor products are formed over the ring $\mathcal{O}$, and with the primes of $Q'$ as the index set.

The map $\tilde{\alpha}$ is any map from the local deformation spaces to the global one as described above Lemma 5.1, the map $\alpha$ is any lift that exist by $\mathcal{O}$-smoothness of $S_{X'}$, and the left vertical map is the restriction of $\alpha$ to the kernel. One can think of $\tilde{\alpha}$ and $\alpha$ as the tensor product of maps $\tilde{\alpha}_{\mathfrak{p}}$ and $\alpha_{\mathfrak{p}}$ for the individual places $\mathfrak{p}$. $\hat{\otimes} J'_{\mathfrak{p}}$ is defined to be the kernel of the top horizontal sequence. It is not meant to be the completed tensor product of the $J'_{\mathfrak{p}}$. We use this notation to remind us that this kernel is generated by the $J'_{\mathfrak{p}}$.

For the places $\mathfrak{p} \in Q' - Q$ we define the ideals $J_{\mathfrak{p}}$ inside $S'_{\mathfrak{p}} = \mathcal{O}[[x_1, \ldots, x_{n'_{\mathfrak{p}}}]]$ so that $(R_{\mathcal{O},X'_{\mathfrak{p}}}, \rho_{\mathcal{O},X'_{\mathfrak{p}}})$ mod $J_{\mathfrak{p}}$ is isomorphic to the versal deformation for unramified deformations of $\bar{\rho}_{|G_{\mathfrak{p}}}$.

**Theorem 5.2** *If $X = X_Q$ and $Ш^2_{Q'}(E, \mathrm{ad}_{\bar{\rho}}) = 0$, and so in particular $X' = X_{Q'}$, and one has the explicit expressions $n' = \dim_k H^1(G_{E,Q'}, \mathrm{ad}_{\bar{\rho}})$ and $n'_{\mathfrak{p}} = \dim_k H^1(G_{\mathfrak{p}}, \mathrm{ad}_{\bar{\rho}})$ for $\mathfrak{p}$ in $Q'$, then the ideal $J_{X'}$ is the span of $\alpha_{\mathfrak{p}}(J'_{\mathfrak{p}}) : \mathfrak{p} \in Q'$.*

*If $J$ denotes the ideal of $\mathcal{O}[[x_1, \ldots, x_{n'}]]$, whose quotient is $R_{\mathcal{O},X}$, then $J$ is the span of*

$$\{\alpha_{\mathfrak{p}}(J_{\mathfrak{p}}) : \mathfrak{p} \in Q'\}$$

*If a lift to $\mathcal{O}$ of determinant $\eta$ is fixed, the result holds for $X^{\eta}_Q$, too, with the only modification that one has to replace throughout $\mathrm{ad}_{\bar{\rho}}$ by $\mathrm{ad}^0_{\bar{\rho}}$.*

Proof: The proof is similar to the proof of Theorem 2.4. We assume that the composite of the $J'_{\mathfrak{p}}$ does not generate all of $J_{X'}$. Thus the quotient

$$J_{X'}/(\mathfrak{m}_{S_{X'}} J_{X'}, \alpha(J'_{\mathfrak{p}}) : \mathfrak{p} \in Q')$$

is a non-trivial vector space over $k$. Let $J''$ be a one-dimensional quotient by a submodule $J_0$ of $J_{X'}$, and consider

$$0 \to J'' \to R'' := S_{X'}/J_0 \to R_{\mathcal{O},X} \to 0.$$

As in the proof of Theorem 2.4, the ideal $J''$ produces a non-zero class in $H^2(G_E, \mathrm{ad}_{\bar{\rho}}) \otimes J''$ By our construction this class becomes zero in all the local Galois groups $H^2(G_{\mathfrak{p}}, \mathrm{ad}_{\bar{\rho}}) \otimes J''$, because we have an explicit lift from $\hat{\otimes} R_{\mathcal{O},X_{\mathfrak{p}}}$ to $R''$, as $\hat{\otimes} J'_{\mathfrak{p}}$ maps to zero in $J''$. Thus this class must be in the kernel of

$$H^2(G_{E,Q'}, \mathrm{ad}_{\bar{\rho}}) \otimes J'' \to \coprod H^2(G_{\mathfrak{p}}, \mathrm{ad}_{\bar{\rho}}) \otimes J''$$

which is $Ш^2_{Q'}(E, \mathrm{ad}_{\bar{\rho}}) \otimes J'' = 0$. This contradicts our assumption and completes the proof for $R_{\mathcal{O},X'}$.

For the statement about $R_{\mathcal{O},X}$, one observes that there is clearly a surjection $R := R_{\mathcal{O},X'}/(\alpha(J_{\mathfrak{p}}) : \mathfrak{p} \in Q' - Q) \to R_{\mathcal{O},X}$. One can directly verify that the dimensions of the tangent spaces (modulo $\mathfrak{m}_{\mathcal{O}}$) are equal, and one has a section as $R$ satisfies the conditions $X$. As in the proof of Theorem 2.4, it follows that $R \to R_{\mathcal{O},X}$ is an isomorphism. ∎

**Remarks 5.3** (i) Conditions under which one can enlarge $Q$ to a finite set $Q'$ such that $Ш^2_{Q'}(E, \mathrm{ad}_{\bar{\rho}}) = 0$ shall be considered in the following section.

(ii) By inspecting the above proof, it is clear that $J_{X'}$ is generated by any set

$$\{\alpha(J'_{\mathfrak{p}}) : \mathfrak{p} \in \tilde{Q}\},$$

where $\tilde{Q} \subset Q'$, such that the map

$$H^2(G_E, \mathrm{ad}_{\bar{\rho}}) \to \coprod_{\mathfrak{p}\in\tilde{Q}} H^2(G_{\mathfrak{p}}, \mathrm{ad}_{\bar{\rho}})$$

is injective. (In case of fixed determinant one has to replace $\mathrm{ad}_{\bar{\rho}}$ by $\mathrm{ad}^0_{\bar{\rho}}$.)

This can be relevant for example in the case $N = 2$ when one considers $\bar{\rho}$ of Borel type, see [4]. In loc. cit., natural conditions are given under which $\text{Ш}^2_{\tilde{Q}}(E, \mathrm{ad}_{\bar{\rho}}) = 0$. This allows the explicit calculation of some interesting deformation spaces.

**Lemma 5.4** *Suppose we are given $S_1 \leftarrow S_0 \to S_2$ in $\mathcal{C}_{\mathcal{O}}$ where the $S_i$ are power series rings over $\mathcal{O}$. Let $\bar{S}_i = S_i/(\mathfrak{m}^2_{S_i}, \mathfrak{m}_{\mathcal{O}})$, and suppose we know that in the following pushout diagram, the horizontal maps are surjections with isomorphic kernels*

$$\begin{array}{ccc} \bar{S}_0 & \longrightarrow & \bar{S}_1 \\ \downarrow & & \downarrow \\ \bar{S}_2 & \longrightarrow & \bar{S}_1 \otimes_{\bar{S}_0} \bar{S}_2 \end{array}$$

*Then $S_3 = S_1 \otimes_{S_0} S_2$ is a power series ring over $\mathcal{O}$ on $n_3 = n_1 + n_2 - n_0$ variables where $n_i := \dim_k t_{S_i,\mathcal{O}}$, the number of variables of the ring $S_i$.*

Proof: The diagram for the $S_i$ implies that the kernel of $S_2 \to S_3$ is generated by at most $n_1 - n_0$ elements, the diagram for the $\bar{S}_i$ that, at the same time, the difference of the mod $\mathfrak{m}_{\mathcal{O}}$ tangent spaces for $S_2$ and $S_3$ is exactly this number. Hence $S_3$ must be an $\mathcal{O}$-smooth quotient of $S_2$. ∎

From [10], Theorems 2.13, 2.14, we quote the following lemma which holds for general $N$, not just $N = 2$.

**Lemma 5.5** *Let $X$ be a set of deformation conditions and $\mathcal{L} = \mathcal{L}_X$ the corresponding collection of infinitesimal conditions.*

*(i) If $\mathfrak{p}$ is a prime above some rational prime $p$, then for any finite $G_{\mathfrak{p}}$-module $M$ one has $|H^1(G_{\mathfrak{p}}/I_{\mathfrak{p}}, M^{I_{\mathfrak{p}}})| = |H^0(G_{\mathfrak{p}}, M)|$ and*

$$|H^1(G_{\mathfrak{p}}, M)| = |H^0(G_{\mathfrak{p}}, M)||H^0(G_{\mathfrak{p}}, M^*(1))||M \otimes \mathcal{O}_{E_{\mathfrak{p}}}|.$$

*(ii) The Selmer group $H^1_{\mathcal{L}}(G_E, \mathrm{ad}_{\bar{\rho}})$ is finite and*

$$\frac{|H^1_{\mathcal{L}}(G_E, \mathrm{ad}_{\bar{\rho}})|}{|H^1_{\mathcal{L}^\perp}(G_E, \mathrm{ad}^*_{\bar{\rho}}(1))|} = \frac{|H^0(G_E, \mathrm{ad}_{\bar{\rho}})|}{|H^0(G_E, \mathrm{ad}^*_{\bar{\rho}}(1))|S} \prod_{\mathfrak{p}\in Q} \frac{|\mathcal{L}_{X,\mathfrak{p}}|}{|H^0(G_{\mathfrak{p}}, \mathrm{ad}_{\bar{\rho}})|} \tag{2}$$

*In particular this gives a formula to calculate the number of topological generators of $R_{\mathcal{O},X}$, i.e. $\dim_k t_{R_{\mathcal{O},X},\mathcal{O}}$, from local data and $H^1_{\mathcal{L}^\perp_X}(G_E, \mathrm{ad}^*_{\bar{\rho}}(1))$, or entirely from local data, provided $H^1_{\mathcal{L}^\perp_X}(G_E, \mathrm{ad}^*_{\bar{\rho}}(1)) = 0$, and similarly for $R_{\mathcal{O},X^\eta}$ if one replaces $\mathrm{ad}_{\bar{\rho}}$ by $\mathrm{ad}^0_{\bar{\rho}}$.*

**Theorem 5.6** *We use the notation introduced before Theorem 5.2. Let $X$ be arbitrary, and so $n_X = \dim_k H^1_{\mathcal{L}_X}(G_E, M)$ and $n_\mathfrak{p} = \dim_k \mathcal{L}_{X_\mathfrak{p}}$. $n'_X$ and $n'_\mathfrak{p}$ are similarly defined. If $H^1_{\mathcal{L}^\perp_{X'}}(G_E, \mathrm{ad}^*_{\bar\rho}(1)) = 0$, then*

$$J_{X'} = \langle \alpha_\mathfrak{p}(J'_\mathfrak{p}) : \mathfrak{p} \in Q' \rangle,$$

*and the ideal $J_X$ for a presentation of $R_{\mathcal{O},X}$ as a quotient of $S'_X$ is spanned by the sets*

$$\alpha_\mathfrak{p}(J_\mathfrak{p})\,, \mathfrak{p} \in Q'.$$

*The theorem holds for problems $X^\eta$ of fixed determinant, too, if one uses $\mathrm{ad}^0_{\bar\rho}$ instead of $\mathrm{ad}_{\bar\rho}$, throughout.*

Proof: By Lemma 5.1, our statements are independent of the respective presentations we use. We fix a diagram $\mathcal{D}_{X_{Q'},Q'}$. (In general, $X_{Q'}$ is not the same as $X'$. It only agrees with it in the situation of Theorem 5.2.) We shall refer to it as the top diagram. We shall think of the local row being the back and the global one being the front. By the previous theorem, this diagram satisfies the local-to-global principle we try to establish as $Ш^2_{Q'}(E, \mathrm{ad}_{\bar\rho}) = 0$ whenever $H^1_{\mathcal{L}^\perp_{X'}}(G_E, \mathrm{ad}_{\bar\rho})$ vanishes. The goal will be to obtain a bottom diagram as in the statement of Theorem 5.6. We start by putting in the back of it, i.e. the first row of the diagram $\mathbf{D}_{X',Q'}$. To do this, we choose any first row of it, we pick a versal morphism from $\hat\otimes R_{\mathcal{O},X^o_\mathfrak{p}}$ to $\hat\otimes R_{\mathcal{O},X'_\mathfrak{p}}$, where the tensor products are over all places in $Q'$. Then we choose lifts on the middle terms using $\mathcal{O}$-smoothness, and complete the back of the diagram so that everything commutes. We observe that $R_{\mathcal{O},X'}$ is the pushout of the three rings on the right, i.e. of

$$R_{\mathcal{O},X} \leftarrow \hat\otimes R_{\mathcal{O},X^o_\mathfrak{p}} \to \hat\otimes R_{\mathcal{O},X'_\mathfrak{p}},$$

as the corresponding diagram of local and global functors is a pullback.

Now we use Lemma 5.5 (ii). It says precisely that the differences of the mod $\mathfrak{m}_\mathcal{O}$ tangent space dimensions on the global side, i.e. between $R_{\mathcal{O},X}$ and $R_{\mathcal{O},X'}$, and on the local side, i.e. between $\hat\otimes R_{\mathcal{O},X^o_\mathfrak{p}}$ and $\hat\otimes R_{\mathcal{O},X'_\mathfrak{p}}$ are the same. Lemma 5.4 shows that the pushout of the three middle rings, which are power series rings over $\mathcal{O}$, must be a power series ring over $\mathcal{O}$ whose number of variables is equal to $n'_X$. By the pushout property it is clear that we can now complete the diagram to a diagram $\mathbf{D}_{X_{Q'},Q'}$ on the top and $\mathbf{D}'_{X',Q'}$ at the bottom with arrows between them that make all squares commutative. We note that we put a prime at $\mathbf{D}'_{X',Q'}$ to indicate that it is not necessarily the diagram we constructed above Theorem 5.2.

We are now essentially done. Let $\kappa_\mathfrak{p}$ be the kernel of

$$S_{X^o_\mathfrak{p}} \to S_{X'_\mathfrak{p}}$$

where, as above, $S_?$ denotes the power series ring that appears in the minimal presentation for $R_?$. Clearly $\kappa_\mathfrak{p}$ is generated by $h^1_{X^o_\mathfrak{p}} - h^1_{X'_\mathfrak{p}}$ elements, and any minimal set of generators of $\kappa_\mathfrak{p}$ forms a part of a basis of $t_{S_{X^o_\mathfrak{p}},\mathcal{O}}$. We define $J''_\mathfrak{p}$ to be the kernel of the composite

$$S_{X^o_\mathfrak{p}} \to S_{X'_\mathfrak{p}} \to R_{\mathcal{O},X'_\mathfrak{p}}.$$

Then $J'_\mathfrak{p} \cong J''_\mathfrak{p}/\kappa_\mathfrak{p}$, and

$$R_{\mathcal{O},X'} \cong S_{X_{Q'}}/(J''_\mathfrak{p} : \mathfrak{p} \in Q') \cong (S_{X_{Q'}}/(\kappa_\mathfrak{p} : \mathfrak{p} \in Q'))/(\alpha(J'_\mathfrak{p}) : \mathfrak{p} \in Q').$$

The isomorphism on the left follows by arguments similar to those at the end of the proof of Theorem 2.4. One uses that $S_{X_{Q'}}/(J''_{\mathfrak{p}} : \mathfrak{p} \in Q')$ surjects onto $R_{\mathcal{O},X'}$, that this surjection induces an isomorphism of tangent spaces and that $S_{X_{Q'}}/(J''_{\mathfrak{p}} : \mathfrak{p} \in Q')$ satisfies $X'$. By what we remarked before, $S_{X_{Q'}}/(\kappa_{\mathfrak{p}})$ is the pushout of the three middle rings, and hence a power series ring isomorphic to $S_{X'}$. So we have the local-to-global principle for $X'$. The description for $X$ follows as in the proof of the previous theorem. ∎

It remains to see under what conditions one can construct auxiliary primes, where we call a finite set of primes $Q_{aux}$ *auxiliary* for a given set of data $Q, X$, if $Q' = Q \cup Q_{aux}$ satisfies the condition in Theorem 5.6.

## 6 Existence of auxiliary primes

First we consider the case $N = 2$, and adapt the construction in [24] to obtain a criterion for the existence of auxiliary primes. A generalization of this construction already appeared in [13]. We shall follow closely [10], §2.6, 2.7, 2.8. After this, we shall give a criterion for the existence of auxiliary primes for general $N$. We shall obtain a purely group-theoretical condition for the existence of such sets, that only depends on $\mathrm{Im}(\bar\rho)$ and its cohomology with $\mathrm{ad}_{\bar\rho}$ coefficient. We keep the notation from the previous section. In particular, $\bar\rho$, $X$ and $\mathcal{L} = \mathcal{L}_X$ will have the same meaning.

For the case $N = 2$ we need the following classification of subgroups of $\mathrm{PGL}_2(k)$, see [12], §§255, 260.

**Theorem 6.1** *Let $H$ be a subgroup of $\mathrm{PGL}_2(k)$, $k$ a finite field. Then either $H$ is cyclic, dihedral, isomorphic to $A_4$, $S_4$, $A_5$, or it lies inside a Borel subgroup, or it is isomorphic to $\mathrm{PGL}_2(k')$ or $\mathrm{PSL}_2(k')$ for some subfield $k'$ of $k$.*

In the $N = 2$ case, we call $\bar\rho$ dihedral, if $\bar H$, the image of $\mathrm{GL}_2(k) \to \mathrm{PGL}_2(k)$ composed with $\bar\rho$, is dihedral, we call it of $A_4$ type if $\bar H$ is isomorphic to $A_4$, etc. We also note that one always has an isomorphism $\mathrm{ad}^*_{\bar\rho} \cong \mathrm{ad}_{\bar\rho}$, as $\mathrm{ad}_{\bar\rho} \cong \bar\rho^* \otimes_k \bar\rho$. One also has such an isomorphism for $\mathrm{ad}^0_{\bar\rho}$.

Now we sketch the proof of the following lemma that is analogous to [10], Theorem 2.39.

**Lemma 6.2** *Let $\bar\rho : G_E \to \mathrm{GL}_2(k)$ be an absolutely irreducible Galois representation, where $k$ is a finite field of characteristic $l$, and $E$ is a number field. Let $[\psi] \in H^1_{\mathcal{L}^\perp}(G_E, \mathrm{ad}^0_{\bar\rho}(1))$ be a non-zero cohomology class. Assume that:*

*(i) If $\bar\rho$ is dihedral, then $\bar\rho$ restricted to $E(\zeta_l)$ stays absolutely irreducible.*

*(ii) If $l = 5$ and $E_0 \supset \mathbb{Q}(\zeta_l)$, then $\mathrm{Gal}(E_0/E(\zeta_l))$ is not isomorphic to $\mathrm{PSL}_2(\mathbf{F}_5)$.*

*Then given any $n \in \mathbb{N}$, there exists a set of primes $\mathfrak{p}$ of $E$ of positive density satisfying*

*(i) $N\mathfrak{p} \equiv 1 \,(\mathrm{mod}\ l^n)$*

*(ii) $\bar\rho$ is unramified at $\mathfrak{p}$*

*(iii) $\mathrm{Frob}_{\mathfrak{p}}$ has distinct eigenvalues for its action on $\mathrm{ad}^0_{\bar\rho}$*

*(iv) The class* $\mathrm{res}_{\mathfrak{p}}[\psi]$ *under the restriction homomorphism* $G_{\mathfrak{p}} \to G_E$ *in*

$$\ker(H^1(G_{\mathfrak{p}}, \mathrm{ad}^0_{\bar\rho}(1)) \to H^1(I_{\mathfrak{p}}, \mathrm{ad}^0_{\bar\rho}(1))) \cong H^1(G_{\mathfrak{p}}/I_{\mathfrak{p}}, \mathrm{ad}^0_{\bar\rho}(1))$$

*is non-zero.*

Proof: As in [10], using the Cebotarev density theorem it suffices to find an element $\sigma \in G_E$ such that

(i) $\sigma|_{E(\zeta_{l^n})} = 1$,

(ii) $\mathrm{ad}^0_{\bar\rho}(\sigma)$ has an eigenvalue unequal to one,

(iii) $\psi(\sigma) \notin (\sigma - 1)\mathrm{ad}^0_{\bar\rho}(1)$.

We let $E_n$ be the fixed field of $\mathrm{ad}^0_{\bar\rho}|_{E(\zeta_{l^n})}$. As in the proof of [10], one first shows that $H^1(\mathrm{Gal}(E_n/E), \mathrm{ad}^0_{\bar\rho}(1)) = 0$ by considering the inflation-restriction sequence

$$\begin{aligned} 0 &\to H^1(\mathrm{Gal}(E_0/E), (\mathrm{ad}^0_{\bar\rho}(1))^{G_{E_0}}) \\ &\to H^1(\mathrm{Gal}(E_n/E), \mathrm{ad}^0_{\bar\rho}(1)) \to H^1(\mathrm{Gal}(E_n/E_0), \mathrm{ad}^0_{\bar\rho}(1))^{G_E} \end{aligned}$$

The hypothesis (ii) guarantees that the term on the left is zero. The term on the right can shown to be isomorphic to

$$Hom(\mathrm{Gal}(E_n/E_1), (\mathrm{ad}^0_{\bar\rho}(1))^{G_E})$$

If $\mathrm{Im}(\bar\rho)$ is not of dihedral type, the latter set is clearly zero. If it is of dihedral type, it is zero precisely when condition (i) is satisfied.

The next step in the argument of loc. cit. is to analyze the $\mathrm{Gal}(E_n/E)$ submodule $0 \neq \psi(G_{E_n}) \subset \mathrm{ad}^0_{\bar\rho}(1)$ after noticing, using the inflation-restriction sequence, that

$$[\psi] \in H^1(G_{E_n}, \mathrm{ad}^0_{\bar\rho}(1))^{\mathrm{Gal}(E_n/E)} \cong Hom_{\mathrm{Gal}(E_n/E)}(G_{E_n}, \mathrm{ad}^0_{\bar\rho}(1))$$

By considering the $A_4$ and dihedral cases separately, one can find an element $g$ of order not dividing $l$ in $\mathrm{Gal}(E_n/E(\zeta_{l^n}))$, that fixes a non-zero element of $\psi(G_{E_n})$. To find $g$ one needs again hypothesis (i).

This $g$ one lifts arbitrarily to $\sigma_0 \in G_E$. By a case by case analysis, one finds that $\psi(G_{E_n}) \not\subset (g-1)\mathrm{ad}^0_{\bar\rho}(1)$. Here one needs that the order of $g$ is prime to $l$ and one regards $\mathrm{ad}^0_{\bar\rho}$ as a $\langle g\rangle$-module, $\langle g\rangle \subset \mathrm{Gal}(E_n/E)$. This allows one to replace $\sigma_0$ by $\sigma = \tau\sigma_0$ where $\tau \in G_{E_n}$ is chosen so that $\psi(\tau\sigma_0) = \psi(\tau) + \psi(\sigma_0) \notin (\sigma_0 - 1)\mathrm{ad}^0_{\bar\rho}(1)$. The so obtained $\sigma$ satisfies the requirements of the lemma. ∎

**Remark 6.3** The condition that $\bar\rho|_{G_{E(\zeta_l)}}$ be absolutely irreducible if $\bar\rho$ is dihedral, excludes the case that $\mathrm{Gal}(E(\zeta_l)/E)$ is cyclic of order two and that $E(\zeta_l)$ is the fixed field of the kernel of map $G_E \to C_2$ induced from the canonical map of the dihedral group onto $C_2$.

The assumptions on $\mathfrak{p}$ in the previous lemma imply that $\dim_k H^0(G_{\mathfrak{p}}, \mathrm{ad}^0_{\bar\rho}(1)) = 1$, and hence that $\dim_k H^1(G_{\mathfrak{p}}/I_{\mathfrak{p}}, \mathrm{ad}^0_{\bar\rho}(1)) = 1$, as the respective groups are isomorphic.

As a corollary, we obtain.

**Corollary 6.4** *Let $\bar\rho : G_E \to \mathrm{GL}_2(k)$ be as in Lemma 6.2, let $Q$ be a set of places containing all places where $\bar\rho$ is ramified, and all places above $l$ and $\infty$, and let $X$ be a set of deformation conditions. We assume that there exists a lift $\rho$ of $\bar\rho$ of type $X$ to some $\mathcal{O} \in \mathcal{C}$ and define $\eta = \det(\rho)$. Let $d = \dim_k H^1_{\mathcal{L}^\perp}(G_E, \mathrm{ad}^0_{\bar\rho}(1))$. We consider the deformation problem $X^\eta$ of fixed determinant $\eta$, as a problem defined over $\mathcal{C}_{\mathcal{O}}$. Then one can find a set $Q_{aux}$ of $d$ many primes as in the previous lemma such that for $Q' = Q \cup Q_{aux}$ the assumptions of Theorem 5.6 are satisfied, and such that the number of topological generators of $R_{\mathcal{O},X'^\eta}$ and $R_{\mathcal{O},X^\eta}$, in the notation of this theorem, agrees. Such a set $Q_{aux}$ is called optimal.*

*Without the restriction of the determinant, a set $Q'$ as in Theorem 5.6 always exists, but it is not necessarily optimal.*

Proof: The first half is clear by the previous lemma and Remark 6.3. That the number of topological generators does not change if one enlarges $Q$ by $Q_{aux}$, follows from formula (2) in Lemma 5.5, as the increment of the dimensions of the local terms is precisely cancelled by the decrement of the dimension of $\dim_k H^1_{\mathcal{L}^\perp}(G_E, \mathrm{ad}^0_{\bar\rho}(1))$. It remains to remark that the lemma works also for $k^{triv}$ as a $G_E$-module. In fact there it is rather trivial. Then one can use the combined set of auxiliary primes for $\mathrm{ad}^0_{\bar\rho}$ and $k^{triv}$, as $\mathrm{ad}_{\bar\rho} = \mathrm{ad}^0_{\bar\rho} \oplus k^{triv}$ ∎

We shall now discuss the existence of auxiliary primes for arbitrary $N$ for the situation $X = X_Q$ in Theorem 5.2. For this we may decompose $\mathrm{ad}_{\bar\rho}$ into a direct sum of indecomposable submodules, and we can consider each of them separately. So let $V$ be such a $G_{E,Q}$-module. We shall investigate under what conditions one can replace $Q$ by a finite extension $Q'$ such that $Ш^2_{Q'}(G_E, V) = 0$.

From the definition of $Ш^i$ and Poitou-Tate, it follows that

$$Ш^2_Q(G_E, V) \cong Ш^1_Q(G_E, V^*(1))^*.$$

Let $\tilde E$ be the splitting field of the restriction of $V$ to $G_{E(\zeta_l)}$. Let $H = \mathrm{Gal}(\tilde E/E)$, $H_{\mathfrak p}$ a decomposition group of $H$ at $\mathfrak p$, and $Q_{\tilde E}$ the places of $\tilde E$ above those in $Q$. We consider the following diagram, where the middle and right columns are inflation-restriction sequences.

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & 0\\
 & & \downarrow & & \downarrow & & \downarrow\\
0 & \to & K & \to & H^1(H, V^*(1)) & \to & \coprod_{\mathfrak p \in Q} H^1(H_{\mathfrak p}, V^*(1))\\
 & & \downarrow & & \downarrow & & \downarrow\\
0 & \to & Ш^1_Q(G_E, V^*(1)) & \to & H^1(G_{E,Q}, V^*(1)) & \to & \coprod_{\mathfrak p \in Q} H^1(G_{E_{\mathfrak p}}, V^*(1))\\
 & & \downarrow & & \downarrow & & \downarrow\\
0 & \to & Ш^1_Q(G_{\tilde E}, V^*(1))^H & \to & H^1(G_{\tilde E, Q_{\tilde E}}, V^*(1))^H & \to & \Big(\coprod_{\mathfrak P \in Q_{\tilde E}} H^1(G_{\tilde E_{\mathfrak P}}, V^*(1))\Big)^H
\end{array}
\tag{3}
$$

One easily checks the isomorphism of the kernels of the maps

$$H^1(H, V^*(1)) \to \coprod_{\mathfrak p \in Q} H^1(H_{\mathfrak p}, V^*(1)) \quad \text{and} \quad Ш^1_Q(G_E, V^*(1)) \to Ш^1_Q(G_{\tilde E}, V^*(1))^H$$

The action on $V^*(1)$ in the bottom row is trivial. Hence, in the notation of [14], the first term simply is $Hom(Б_{Q_{\tilde{E}}}, V^*)^H$. As explained in Satz 12.3 of loc. cit., it is easy to achieve that $Б_{Q_{\tilde{E}}} = 0$, by simply adding enough primes to $Q_{\tilde{E}}$ so that the $Q'$-class group of $\tilde{E}$ is zero. (A refinement of this will be discussed below, when considering general $X$.) Hence one is left with a purely group theoretical problem, namely to give conditions under which $K$ becomes trivial under enlarging $Q$.

Clearly whenever $H^1(H, V^*(1)) = 0$, then $K = 0$. If an $l$-Sylow subgroup of $H$ is cyclic, then one can add a prime $\mathfrak{p} \in Q$ whose local decomposition group is this local $l$-Sylow subgroup $P$. By [8], Proposition 10.4, it follows that the map

$$H^1(H, V^*(1)) \to H^1(P, V^*(1))$$

is injective, and hence $K$ is zero after adding $\mathfrak{p}$ to $Q$. This can be applied, for example, to the exceptional case $H = \mathrm{PSL}_2(\mathbf{F}_5)$ in Lemma 6.2, to obtain a set of auxiliary primes, which is even optimal in the sense of Corollary 6.4 for deformations with fixed determinant.

It is not clear to us, whether one can always remove $K$. We are sceptical, as for general $\mathbf{F}_l[H]$-modules $M$ and general finite groups $H$ it is not true that the map

$$H^1(H, M) \to \coprod H^1(H_i, M)$$

is injective, where $H_i$ runs through all cyclic subgroups of $H$. One can in fact construct a counterexample for $H \cong \mathbb{Z}/(l) \times \mathbb{Z}/(l)$ and $M$ a $k[H]$-module with $\dim_k M = 5$. This dependence on $H$ and $M$ should not come as a surprise. If one looks back at the proof of Lemma 6.2, one finds that the vanishing of $H^1(H, \mathrm{ad}^0_{\bar{\rho}})$ was not a result of a general method, but of the known classification of the subgroups of $\mathrm{GL}_2(k)$. So the group theoretical problem has to be solved for each case under consideration. We summarize this discussion.

**Proposition 6.5** *The set of deformation conditions $X_Q$ admits an auxiliary set of primes if and only if for each indecomposable summand $V$ of $\mathrm{ad}_{\bar{\rho}}(1)$ there exist cyclic subgroups $H'_i$ of $H' = \mathrm{Gal}(E'/E)$, where $E'$ is the splitting field of $V$, such that the kernel of*

$$H^1(H', V) \to \coprod_{\mathfrak{p} \in Q} H^1(H'_{\mathfrak{p}}, V) \sqcup \coprod_i H^1(H'_i, V)$$

*is zero. A sufficient condition for this is that all $H^1(H', \mathrm{ad}_{\bar{\rho}}(1)) = 0$, or that any maximal $l$-Sylow subgroup of any $H'$ is cyclic. For $N \leq l$, the latter means that such a group is isomorphic to $\mathbb{Z}/(l)$.*

Proof: The only thing to show is that we can replace $H$ by $H'$ and similarly for the decomposition groups. We consider

$$1 \to \mathrm{Gal}(\tilde{E}/E') \to H \to H' \to 1.$$

As the order of $\mathrm{Gal}(\tilde{E}/E')$ divides that of $\mathrm{Gal}(E(\zeta_l)/E)$ and is thus prime to $l$, applying the inflation-restriction sequence yields the result. ∎

When trying to generalize this to arbitrary $X, \mathcal{L}$, the problem arises that the local conditions $\mathcal{L}_{\mathfrak{p}}^{\perp}$ are not necessarily compatible with a direct sum decomposition of $\mathrm{ad}_{\bar\rho}$ into indecomposable summands. So the analogue of diagram (3), where the third row incorporates the infinitesimal conditions $\mathcal{L}_{\mathfrak{p}}^{\perp}$, only makes sense for $V = \mathrm{ad}_{\bar\rho}$. To be precise, we define $\mathcal{L}_{\mathfrak{p}}^{\perp,0}$ to be the contraction of $\mathcal{L}_{\mathfrak{p}}^{\perp}$ to $H^1(H_{\mathfrak{p}}, \mathrm{ad}_{\bar\rho}(1))$ using the right column of diagram (3), and $\tilde{\mathcal{L}}_{\mathfrak{p}}^{\perp}$ its image inside $H^1(G_{\tilde{E}_{\mathfrak{P}}}, \mathrm{ad}_{\bar\rho}(1))$, where for each $\mathfrak{p} \in Q$ a place $\mathfrak{P} \in Q_{\tilde{E}}$ is chosen above it. One obtains a diagram analogous to diagram (3) whose first and third rows are

$$0 \to K_{\mathcal{L}^\perp} \to H^1(H, \mathrm{ad}_{\bar\rho}(1)) \to \coprod_{\mathfrak{p}\in Q} H^1(H_{\mathfrak{p}}, \mathrm{ad}_{\bar\rho}(1))/\tilde{\mathcal{L}}_{\mathfrak{p}}^{\perp,0}$$

$$0 \to H^1_{\tilde{\mathcal{L}}^\perp}(G_{\tilde{E}}, \mathrm{ad}_{\bar\rho}(1))^H \to H^1(G_{\tilde{E},Q_{\tilde{E}}}, \mathrm{ad}_{\bar\rho}(1))^H \to \coprod_{\mathfrak{p}\in Q} H^1(G_{\tilde{E}_{\mathfrak{P}}}, \mathrm{ad}_{\bar\rho}(1))^{H_{\mathfrak{p}}}/\tilde{\mathcal{L}}_{\mathfrak{p}}^{\perp},$$

where the objects on the left are defined to be the kernels of the morphism on the right.

$K_{\mathcal{L}^\perp}$ can be analyzed as above. However this analysis can be quite difficult, if the subgroups $\tilde{\mathcal{L}}_{\mathfrak{p}}^{\perp,0}$ are needed explicitly to see if this kernel is zero, or if it can be made so by adding some primes to $Q$. We shall now show that for any given class in $Hom_H(G_{\tilde{E},Q_{\tilde{E}}}, \mathrm{ad}_{\bar\rho}(1)) \cong H^1(G_{\tilde{E},Q_{\tilde{E}}}, \mathrm{ad}_{\bar\rho}(1))^H$ one can choose a prime $\mathfrak{P}$ such that this class maps under the restriction map to a non-zero class of $H^1(G_{\tilde{E}_{\mathfrak{P}}}, \mathrm{ad}_{\bar\rho})$, and so, if $K_{\mathcal{L}^\perp}$ can be made zero by enlarging $Q$, there always exist an auxiliary set of primes. For this we can consider the indecomposable summands of $\mathrm{ad}_{\bar\rho}$ separately. So let $V$ be such a summand. We follow the argument in the proof of Lemma 6.2.

Given any $\psi \in Hom_H(G_{\tilde{E},Q_{\tilde{E}}}, V)$, let $\tilde{E}_\psi$ be the splitting field of $\psi$ above $\tilde{E}$ and $E_\psi$ the Galois closure above $E$ of $\tilde{E}_\psi$. Then there is a $\bar\sigma \in \mathrm{Gal}(\tilde{E}_\psi/\tilde{E})$ such that $\psi(\bar\sigma) \neq 0$. Let $\sigma$ be a lift in $\mathrm{Gal}(E_\psi/\tilde{E}) \subset \mathrm{Gal}(E_\psi/E)$. By the Cebotarev density theorem, there is a set of primes of positive density of $E$, whose decomposition group in $\mathrm{Gal}(E_\psi/E)$ is conjugate to the cyclic group $\langle\sigma\rangle$. Enlarging $Q$ by such primes, after a finite number of steps, one obtains $H^1_{\tilde{\mathcal{L}}^\perp_{X'}}(G_{\tilde{E}}, V)^H = 0$.

Furthermore, under the following hypothesis (**A**), it is possible for problems with fixed determinant $\eta$ and for which $K_{\mathcal{L}^\perp}(\mathrm{ad}^0_{\bar\rho}(1)) = 0$, to find an optimal set of auxiliary primes, i.e. one such that

$$H^1_{\mathcal{L}_{X^\eta}}(G_E, \mathrm{ad}^0_{\bar\rho}) \cong H^1_{\mathcal{L}_{X^{\eta\prime}}}(G_E, \mathrm{ad}^0_{\bar\rho}),$$

where $X^{\eta\prime}$ is obtained from $X^\eta$ as above Theorem 5.2. The hypothesis (**A**) is:

(**A**) Given any indecomposable submodule $V$ of $\mathrm{ad}^0_{\bar\rho}(1)$, there is a non-zero element $g \in G_E$ such that $V$ is not a submodule of $(g-1)\mathrm{ad}^0_{\bar\rho}(1)$ and such that $\dim_k(\mathrm{ad}^0_{\bar\rho}(1))^{\langle g\rangle} = 1$.

The latter condition is for example satisfied, if one can always choose a $g \in G_{E(\zeta_l)}$ such that $\bar\rho(g)$ is diagonalizable with distinct eigenvalues. We obtain.

**Proposition 6.6** *A general $X$ admits an auxiliary set of primes if there exist cyclic subgroups $H'_i$ of $H' = \mathrm{Gal}(E'/E)$, where $E'$ is the splitting field of $\mathrm{ad}_{\bar\rho}(1)$, such that the kernel of*

$$H^1(H', \mathrm{ad}_{\bar\rho}(1)) \to \coprod_{\mathfrak{p}\in Q} H^1(H'_{\mathfrak{p}}, \mathrm{ad}_{\bar\rho}(1))/\tilde{\mathcal{L}}_{\mathfrak{p}}^{\perp,0} \sqcup \coprod_i H^1(H'_i, \mathrm{ad}_{\bar\rho}(1))$$

*is zero. If $K_{\mathcal{L}_X^\perp}(\mathrm{ad}^0_{\bar\rho}) = 0$ and if hypothesis (**A**) holds, then the deformation problem $X^\eta$, that includes a fixed choice $\eta$ of the determinant, admits an optimal set of auxiliary primes.*

**Example 6.7** Let $N = 2$, $E = \mathbb{Q}$ and $\bar{\rho} = \begin{pmatrix} \phi_1 & * \\ 0 & \phi_2 \end{pmatrix}$ be of Borel type, such that $\phi = \phi_1\phi_2^{-1}$ is non-trivial, and so that $\mathrm{Cent}_{\mathrm{GL}_2(k)}(\mathrm{Im}(\bar{\rho})) \cong k^*$. Then condition (**A**) holds if and only if there exists $g \in G_E$, such that $\chi(g)^2 \neq 1$ and $\phi(g)\chi(g) = 1$. From this one can derive precise conditions when condition (**A**) is satisfied.

If one wants to imitate Lemma 6.2, one simply needs to work with a condition $p \equiv a \pmod{l^n b}$ for some fixed integers $a, b$ that are relatively prime to $l$ and depend on $\bar{\rho}$.

Finally, if no such $g$ exists, one could try to use subfunctors of $\mathrm{Equiv}_{X^0_{\mathfrak{p}}}$. The problem with this approach is that it seems rather unlikely that one can find a modular interpretation that describes such a functor.

# 7 Applications

Throughout we assume that $\bar{\rho} : G_E \to \mathrm{GL}_N(k)$ is an absolutely irreducible representation of $G_E$. Our first application concerns the results in [6]. There the shape of local relations for certain unramified places is calculated for $N = 2$, and the consequences are discussed for the presentation of $R_X$ for a given $X$, if one replaces the deformation condition $X$ by the condition $X'$ that allows ramification at additional primes. A presentation for $R_{X'}$ is obtained involving the shape of the new local relations.

Theorems 5.2 and 5.6 generalize this immediately, to arbitrary $N, E$, provided the local situation is well understood. The latter seems to be only the case for $N = 2$ – in Section 3 we complete the knowledge of the local situations by discussing all the local situations that had not been analyzed previously. We state the following Corollary of Theorem 5.6 which in combination with the results of Section 3 gives a generalization of Theorems 1 and 2 in [6].

**Corollary 7.1** *Let $X$ be a set of deformation conditions for $\bar{\rho}$ which admits an auxiliary set of primes $Q_{aux}$. Suppose $Q' \supset Q$ is finite and disjoint from $Q_{aux}$. Define $X'$ corresponding to $Q'$ as above Theorem 5.2. Then one has a presentation*

$$0 \to (f_1, \ldots, f_{r+s}) \to \mathcal{O}[[x_1, \ldots, x_{t+s}]] \to R_{\mathcal{O},X'} \to 0$$

*such that $s = \sum_{\mathfrak{p} \in Q'-Q} \dim_k H^2(G_{\mathfrak{p}}, \mathrm{ad}_{\bar{\rho}})$, $f_{r+i} \in (f_1, \ldots, f_r, x_{t+1}, \ldots, x_{t+r})$ for $i = 1, \ldots, s$, and such that the above presentation modulo $J = (x_{t+1}, \ldots . x_{t+r})$ gives a presentation of $R_{\mathcal{O},X}$, i.e.*

$$0 \to (\bar{f}_1, \ldots, \bar{f}_{r+s}) \to \mathcal{O}[[x_1, \ldots, x_t]] \to R_{\mathcal{O},X} \to 0.$$

*Here $\bar{f}_i = f \pmod{J}$. Furthermore the functions $f_{r+1}, \ldots, f_{r+s}$ are the images under a map from local to global presentations of a complete (minimal) set of relations for all the local places in $Q' - Q$.*

*Any one of the following conditions is sufficient so that none of the $f_i$ is redundant.*

*(i) $N = 2$, $\bar{\rho}$ is tame, i.e. $l \not| \#Im(\bar{\rho})$, $E$ satisfies the Leopoldt conjecture, $\zeta_p \notin E$, and $X = X_Q$.*

*(ii) $N = 2$, $\bar{\rho}$ is modular and absolutely irreducible, and at the places $\mathfrak{l}$ above $l$ either $\bar{\rho}$ is semistable, or it is ordinary and the character $\det(\bar{\rho}_{|I_{\mathfrak{l}}})$ is not the local cyclotomic character, and $X = X_Q$ or one imposes the above mentioned conditions (i.e. semistability, ordinariness, etc.) at the places above $l$.*

Proof: The only part that is not obvious is that conditions (i) and (ii) are sufficient so that none of the $f_i$ is redundant. Under condition (i), this is shown in [3], Corollary 1.2, under condition (ii), in [2], Corollary 9.5 and Remark 9.6. ∎

**Remark 7.2** The question of redundancy of local equations in a presentation is a rather difficult one. If $\bar\rho$ is not tame, the only general way to approach this is to assume that $\bar\rho$ is modular, and to use Ribet's results on raising the level (and generalization thereof). This was first exploited in [6].

When studying redundancy of equations $f_i$, the main obstacle is that in general one has no control over the images in the global deformation ring of the variables of local deformation rings at unramified primes. After enlarging ramification, the images of the corresponding lifts remain mysterious, while usually the new local variables that arise from admitting more ramification can be directly seen in the global presentation, as they give rise to new parameters. In most cases, the local equations involve both types of variables. Thus there is no reason why such an equation shouldn't disappear completely when considered globally, unless one has some information on the images of the above elements.

As a second application we consider the following situation, which for example appears in [11], [13] or [26], when ascending from a minimal deformation problem to a slightly larger one.

**Corollary 7.3** *Let $N=2$. We assume that we have a lift $\rho : G_E \to \mathrm{GL}_2(\mathcal{O})$ in $\mathrm{Def}_{\mathcal{O},X^\eta}(\mathcal{O})$ of $\bar\rho$ where $\mathcal{O}\in\mathcal{C}$ is a discrete valuation ring, finite flat over $W(k)$. Let $\eta=\det(\rho)$, and $X^\eta$ be a deformation problem that fixes the determinant to be $\eta$. Let $\Delta Q$ be a finite set of places $\mathfrak{p}$ such that the following holds. Either $\mathrm{ad}_{\bar\rho}$ is ramified at $\mathfrak{p}$ and $\mathcal{L}_{X^\eta_\mathfrak{p}}=0$, this means that $X^\eta$ is minimally ramified in the sense of Remark 3.11, or $\mathfrak{p}$ is a place where $\bar\rho$ is unramified, $E_\mathfrak{p}$ contains $\zeta_l$, and the local decomposition group acts non-trivially on $\mathrm{ad}^0_{\bar\rho}$. Those are exactly the primes used in Lemma 6.2. Let $X'^\eta$ be the problem obtained from $X^\eta$ by allowing arbitrary ramification at the places in $\Delta Q$. We assume that all the $R_{\mathcal{O},X'^\eta_\mathfrak{p}}$ are complete intersections, flat over $\mathcal{O}$. By $r_\mathfrak{p}$ we denote the minimal number of generators of the local relations $J_\mathfrak{p}$, and we define $d=\dim_k H^1_{\mathcal{L}_X}(G_E,\mathrm{ad}^0_{\bar\rho})-\sum_{\mathfrak{p}\in Q} r_\mathfrak{p}$. Then $R_{\mathcal{O},X^\eta}$ is a complete intersection and flat over $\mathcal{O}$ of dimension $d$ if and only if $R_{\mathcal{O},X'^\eta}$ has this property.*

Proof: We shall suppress the subscript $\mathcal{O}$ in the proof. We choose any set of auxiliary primes for the problem $X'^\eta$ and consider the presentation for $R_{X'^\eta}$ given in Theorem 5.6. A presentation for $R_{X^\eta}$ is obtained by replacing the ideals $J_\mathfrak{p}$ for places in $\Delta Q$ by appropriate ideals $J'_\mathfrak{p}$. For all primes in $\Delta Q$, modulo $\mathfrak{m}_\mathcal{O}$, the local equation for $J'_\mathfrak{p}$ is given by $T_\mathfrak{p}$, while that of $J_\mathfrak{p}$ is given by $T_\mathfrak{p}^{n_\mathfrak{p}}$. For ramified primes this follows from Lemma 3.10 and from the discussion in the proof of Proposition 3.7, for the other ones from [10], Lemma 2.36.

We start by showing that the property for $R_{X^\eta}$ implies that for $R_{X'^\eta}$. By our assumptions and counting relations, the number of generators of the ideal in our presentation for $R_{X^\eta}$, we call it $J_{X^\eta}$, is exactly the sum of the number of generators of the ideals $J_\mathfrak{p}$, resp. $J'_\mathfrak{p}$ for all local problems. Let $f_1,\ldots,f_m$ be a list of the images of the generators of the local ideals considered inside $J_{X^\eta}$. As $R_{X^\eta}$ is a complete intersection of dimension $d$, they must form a regular sequence. By flatness over $W(k)$, adding $\lambda$, the uniformizing parameter of $\mathcal{O}$, to this list, we still obtain a regular sequence. As the rings we consider are local, replacing any equation by a power of itself, or reordering the sequence doesn't change the regularity property of the sequence. So we may assume that $\lambda$ is the first element of the sequence. Then we can replace all the equations $T_{\mathfrak{p}}$ by $T_{\mathfrak{p}}^{n_{\mathfrak{p}}}$. By the above remark on the equations for the primes in $\Delta Q$, this is exactly the sequence one obtains from the equations for $R_{X'^\eta}$ after adding $\lambda$ and considering it as the first equation. Hence $R_{X'^\eta}$ is a complete intersection, flat over $\mathcal{O}$ of the same dimension as $R_{X^\eta}$. The converse follows by reversing the argument. ∎

**Remark 7.4** If $\bar\rho$ is ramified at $\mathfrak{p}$, but not $\mathrm{ad}_{\bar\rho}$, then this means that the image of $I_{\mathfrak{p}}$ under $\bar\rho$ is inside the set of homotheties of $\mathrm{GL}_2(k)$. Hence by twisting by a character $\chi$ that is unramified away from $\mathfrak{p}$, one can remove this ramification. As $\chi$ can be viewed as a $W(k)^*$ valued character, this can also be done for deformations with fixed determinant. After doing this, one can study deformations of $\bar\rho' = \bar\rho \otimes \chi$. This reduces the study of deformation rings to situations where the primes that ramify for $\bar\rho$ are the same as those for $\mathrm{ad}_{\bar\rho}$. Removing such ramification was already used in [11], Remark 2.1.

Another application of the local-to-global principle is the construction of lifts to $W(k)/(l^2)$ under rather general conditions on the given $\bar\rho$, where in particular $N$ can be larger than 2. This is the subject of current joint work with C. Khare which shall be discussed elsewhere.

In Remark 7.2, we mentioned the problem that arises from the fact that in general one has no control over the images of the local variables corresponding to a local Frobenius element. We shall now investigate some consequences if these images are controlled by the local-to-global map of mod $\mathfrak{m}_{\mathcal{O}}$ tangent spaces. This will then provide us with a reinterpretation of some recent results of Ramakrishna. We do not strive for full generality, we simply want to indicate the underlying principle of the approach.

Let $\rho_0 : G_{E,Q} \to \mathrm{SL}_2(\mathcal{O})$ be a representation to a discrete valuation ring $\mathcal{O}$, that is finite flat over $W(k)$, with uniformizing parameter $\lambda$. We assume that $\bar\rho = (\rho_0(\mathrm{mod}\ \lambda))$ is surjective onto $\mathrm{SL}_2(k)$. Further we assume that we consider a set $X^\eta$ of deformation conditions satsifying the following.

- $\eta$ is trivial, $X^\eta : \mathcal{C}_{\mathcal{O}} \to \mathit{Sets}$, $\rho_0 \in \mathrm{Def}_{\mathcal{O},X^\eta}(\mathcal{O})$.
- $H^1_{\mathcal{L}^\perp_{X^\eta}}(G_E, \mathrm{ad}^0_{\bar\rho}(1)) = 0$.
- All local deformation problems corresponding to $X^\eta$ satisfy $J_{\mathfrak{p}} = (0)$, so in particular $R := R^\eta_{\mathcal{O},X}$ is smooth over $\mathcal{O}$ of dimension $d = \dim_k H^1_{\mathcal{L}_X}(G_E, \mathrm{ad}^0_{\bar\rho}(1))$.

Let $\Delta Q$ be a set of places of $E$ such that $\mathcal{L}_{\mathfrak{p}} \cong H^1(G_{\mathfrak{p}}/I_{\mathfrak{p}}, (\mathrm{ad}^0_{\bar\rho})^{I_{\mathfrak{p}}})$ and such that $\bar\rho(G_{\mathfrak{p}})$ has order $l$. Such places could be ramified or not. This corresponds to the conditions described in Lemma 3.10 (iii) and (iv). We let $X'^\eta$ by as usual the same as $X^\eta$ for places outside $\Delta Q$, and no condition but trivial determinant at places of $\Delta Q$, and $R' := R_{\mathcal{O},X'^\eta}$. By $Q_1$ we denote the set of primes of $\Delta Q$ that ramify in $\bar\rho$, and by $Q_2$ its complement in $\Delta Q$. We let $R_{\mathfrak{p}} = R_{\mathcal{O},X^o_{E_{\mathfrak{p}}}}$. So $R_{\mathfrak{p}} \cong \mathcal{O}[[S_{\mathfrak{p}}, T_{\mathfrak{p}}]]/(g_{\mathfrak{p}}(S_{\mathfrak{p}}, T_{\mathfrak{p}}))$ for an unramified place $\mathfrak{p}$ in $\Delta Q$, where

$$g_{\mathfrak{p}}(S,T) = T h_{\mathfrak{p}}(S,T) g_{N(\mathfrak{q})-1}(T^2 h_{\mathfrak{p}}(S,T))$$

and $h_{\mathfrak{p}}$ is the function $h$ in the notation of Lemma 3.10 (iv), and $R_{\mathfrak{p}} \cong \mathcal{O}[[T_{\mathfrak{p}}]]/(g_{\mathfrak{p}}(T_{\mathfrak{p}}))$ where $g_{\mathfrak{p}}(T) = T g_{N(\mathfrak{q})-1}(T)$ in the notation of Lemma 3.10 (iii).

We now describe the linearity condition we want. Let $\bar{S}_\mathfrak{p}$ and $\bar{T}_\mathfrak{p}$ by the images of $S_\mathfrak{p}, T_\mathfrak{p}$ in $t_{R_\mathfrak{p},\mathcal{O}}$, and $\sigma_\mathfrak{p}, \tau_\mathfrak{p}$ their images in $t_{R'}$. As $H^1_{\mathcal{L}_X^\perp}(G_E, \mathrm{ad}^0_{\bar\rho}(1)) = 0$, it follows that

$$t_{R',\mathcal{O}} \cong t_{R,\mathcal{O}} \oplus \bigoplus_{\mathfrak{p}\in\Delta Q} k\tau_\mathfrak{p}$$

Thus we can write $\sigma_\mathfrak{p} = a_\mathfrak{p} + \sum_{\mathfrak{p}\in\Delta Q} \gamma_{\mathfrak{p}\mathfrak{p}'}\tau_{\mathfrak{p}'}$, $a_\mathfrak{p} \in t_{R,\mathcal{O}}$, $\gamma_{\mathfrak{p}\mathfrak{p}'} \in k$. We let $A$ be the matrix $A = (\gamma_{\mathfrak{p}\mathfrak{p}'})_{\mathfrak{p},\mathfrak{p}'\in Q_2}$.

**Theorem 7.5** *We assume that we are in the above set-up.*

*(i) If for all subsets $Q_0$ of $Q_2$,*

$$\det((\gamma_{\mathfrak{p}\mathfrak{p}'})_{\mathfrak{p},\mathfrak{p}'\in Q_0}) \neq 0$$

*and if $J_\mathfrak{p} = (0)$ for all places $\mathfrak{p} \in Q - \Delta Q$, then $R'$ is flat over $W(k)$ and a complete intersection of Krull dimension $d$. In particular all deformations lift to characteristic zero.*

*(ii) If in addition to the assumptions in part (i), we assume that for $\mathfrak{p}' \in Q_1$ all $\gamma_{\mathfrak{p}\mathfrak{p}'} = 0$, and that all entries of the vector*

$$\frac{1}{\lambda} A^{-1}(\mathrm{trace}(\rho_0(\mathrm{Frob}_\mathfrak{p})) - 2)_{\mathfrak{p}\in Q_2}$$

*are units in $\mathcal{O}$, then there exists a lift of $\bar\rho$ to $\mathrm{GL}_2(\mathcal{O})$ infinitely ramified at all places in $\Delta Q$.*

**Remark 7.6** The conditions of the second part can be realized in the following two situations. The image of $\rho_0$ is finite, all $\rho_0(\mathrm{Frob}_\mathfrak{p})$ have order $l$ for $\mathfrak{p} \in Q_2$, $\mathcal{O} = W(k)[\zeta_l + \zeta_l^{-1} - 2]$, and $A$ is a diagonal matrix. Then the conditions on the vector entries are satisfied. However the only situations where one can hope for this are $k = \mathbf{F}_3$ and $k = \mathbf{F}_5$, see [15], §1.9. We shall remark below on how to calculate $A$.

The other situation is where $\rho_0$ is surjective. Then by the Cebotarev density theorem, one has a set of places $\mathfrak{p}$ of a positive density for which $\mathrm{trace}(\rho_0(\mathrm{Frob}_\mathfrak{p})) - 2$ has $\lambda$-adic valuation 1. For instance for representations associated to a modular form $f$, this condition can be checked by looking at the coefficients of the Fourier expansion of $f$. Properties of $A$ however have to be checked differently.

The condition on the determinants in (i) means that one can freely replace any set of parameters $(\alpha_\mathfrak{p}(T_\mathfrak{p}))_{\mathfrak{p}\in Q_2 - Q_0}$ for $R'$ by the elements $(\alpha_\mathfrak{p}(S_\mathfrak{p}))_{\mathfrak{p}\in Q_2-Q_0}$ in a minimal presentation of $R'$, where $\alpha_\mathfrak{p} : R_\mathfrak{p} \to R'$ is any local-to-global map.

Proof: We shall first prove part (i). We consider $R'/(\lambda)$. As topological generators we take the images of the local $T_\mathfrak{p}$, which we denote by $(\hat\tau_\mathfrak{p})_{\mathfrak{p}\in\Delta Q}$, together with lifts $\{x_1, \ldots, x_d\}$ of a any set of elements of $\mathfrak{m}_R$ that forms a basis of $t_{R,\mathcal{O}}$. The equations for the primes $\mathfrak{p} \in Q_1$ modulo $\lambda$ are of the form $T_\mathfrak{p}^{n_\mathfrak{p}}$, those for the primes in $Q_2$ of the form $S_\mathfrak{p}^{m_\mathfrak{p}} T_\mathfrak{p}^{n_\mathfrak{p}}$ for some positive integers $m_\mathfrak{p}, n_\mathfrak{p}$.

We shall now calculate the Krull dimension of $R'/(\lambda)$. For this we may consider the reduced ring

$$(R'/(\lambda))_{red} = \Big(k[[x_1, \ldots, x_d, \hat\tau_\mathfrak{p} : \mathfrak{p} \in \Delta Q]]/(\hat\tau_\mathfrak{p}^{n_\mathfrak{p}} : \mathfrak{p} \in Q_1, \hat\sigma_\mathfrak{p}^{m_\mathfrak{p}} \hat\tau_\mathfrak{p}^{n_\mathfrak{p}} : \mathfrak{p} \in Q_2)\Big)_{red}.$$

Let $Q_0$ be any subset of $Q_2$. Then clearly it suffices to calculate the Krull dimension of all rings

$$\tilde{R} = \Big(k[[x_1,\ldots,x_d,\hat{\tau}_\mathfrak{p}:\mathfrak{p}\in Q_0]]/(\hat{\sigma}_\mathfrak{p}:\mathfrak{p}\in Q_0)\Big)_{red}$$

Our assumptions on the matrix $A$ imply that the equations for the $\hat{\sigma}_\mathfrak{p}$ in terms of the $\hat{\tau}_{\mathfrak{p}'}$ and the $x_i$ are such that we can form linear combinations that say that each of the $\hat{\tau}_\mathfrak{p}$, $\mathfrak{p}\in Q_0$, can be expressed in the variables $x_i$ and $\hat{\tau}_{\mathfrak{p}'}$ by power series with no linear terms in the variables $\hat{\tau}_\mathfrak{p}$, $\mathfrak{p}\in Q_0$. Hence these equations can be used to eliminate all the $\hat{\tau}_\mathfrak{p}$, $\mathfrak{p}\in Q_0$. It follows that $\tilde{R}\cong k[[x_1,\ldots,x_d]]$. This shows that all components of $R'/(\lambda)$ have Krull dimension at most $d$. From the explicit presentation of $R'$ it follows that the equations involved together with the element $\lambda$ must form a regular sequence. Hence $R'$ has the properties claimed.

Now we come to the proof of (ii). We assume now that the choices of the $x_i$ and $\hat{\tau}_\mathfrak{p}$ are made in such a way that in the presentation of $R'$ above, specializing them to zero corresponds to $\rho_0$. For $\mathfrak{p}\in Q_2$, let $\hat{\sigma}_{\mathfrak{p},0}$ denote the image of $S_\mathfrak{p}$ under the map to $\mathcal{O}$ corresponding to $\rho_0$. From the shape of the matrices given in Lemma 3.10 part (iv), this means that $\hat{\sigma}_{\mathfrak{p},0}=\frac{1}{2}\mathrm{trace}(\rho_0(\mathrm{Frob}_\mathfrak{p}))-1$. This implies that

$$\hat{\sigma}_\mathfrak{p}=\hat{\sigma}_{\mathfrak{p},0}+\sum_{\mathfrak{p}'\in\Delta Q}\hat{\gamma}_{\mathfrak{p}\mathfrak{p}'}\hat{\tau}_\mathfrak{p}+\sum_i\beta_{\mathfrak{p}i}x_i+r_\mathfrak{p}$$

where $r_\mathfrak{p}\in(x_i,\hat{\tau}_\mathfrak{p},\lambda)^2$. The $\hat{\gamma}_{\mathfrak{p}\mathfrak{p}'}$ are any lifts of the $\gamma_{\mathfrak{p}\mathfrak{p}'}$ to $\mathcal{O}$. By our assumptions, we may assume that $\hat{\gamma}_{\mathfrak{p}\mathfrak{p}'}=0$ for $\mathfrak{p}'\in Q_1$.

We specialize the $x_i$ to zero, the $\hat{\tau}_\mathfrak{p}$ for $\mathfrak{p}\in Q_1$ to the value $\hat{\tau}_{\mathfrak{p},0}$ that corresponds locally to infinity ramification, i.e. locally to $T_\mathfrak{p}=0$, and we then form the quotient by the equations $h_\mathfrak{p}(\hat{\sigma}_\mathfrak{p},\hat{\tau}_\mathfrak{p})$ for $\mathfrak{p}\in Q_2$. We obtain the ring

$$R''=\mathcal{O}[[\hat{\tau}_\mathfrak{p}:\mathfrak{p}\in\Delta Q]]/(\hat{\tau}_\mathfrak{p}-\hat{\tau}_{\mathfrak{p},0}:\mathfrak{p}\in Q_1,h_\mathfrak{p}(\hat{\sigma}_\mathfrak{p},\hat{\tau}_\mathfrak{p}):\mathfrak{p}\in Q_2).$$

By the assumption for part (i), in particular that $A$ is invertible, and as $h_\mathfrak{p}(\hat{\sigma}_\mathfrak{p},\hat{\tau}_\mathfrak{p})\equiv\hat{\sigma}_\mathfrak{p}$ (mod $\lambda$), it follows that $R''/(\lambda)\cong k$, and from there that $\mathcal{O}\stackrel{\cong}{\to}R''$.

It remains to show that the images $\hat{\tau}_\mathfrak{p}$ of the $T_\mathfrak{p}$ are non-zero for $\mathfrak{p}\in Q_2$. Then the image of the generator of the local inertia group maps to a conjugate of $\left(\begin{smallmatrix}1&\hat{\tau}_\mathfrak{p}\\0&1\end{smallmatrix}\right)$ and is therefore of infinite order. The $(2,1)$ entry must be zero by our condition that $h_\mathfrak{p}(\hat{\sigma}_\mathfrak{p},\hat{\tau}_\mathfrak{p})=0$.

As we remarked in Lemma 3.10 (iv), the equation $h_\mathfrak{p}(\hat{\sigma}_\mathfrak{p},\hat{\tau}_\mathfrak{p})=0$ implies that $\hat{\sigma}_\mathfrak{p}\in(l)^4$. Thus modulo $(\lambda^2)$ we have

$$0\equiv\hat{\sigma}_{\mathfrak{p},0}+\sum_{\mathfrak{p}'\in\Delta Q}\gamma_{\mathfrak{p}\mathfrak{p}'}\hat{\tau}_\mathfrak{p},$$

i.e. $(\hat{\tau}_\mathfrak{p})_{\mathfrak{p}\in Q_2}\equiv -A^{-1}(\hat{\sigma}_{\mathfrak{p},0})_{\mathfrak{p}\in Q_2}\,(\mathrm{mod}\ \lambda^2)$. The claim follows from our assumption. ∎

**Remark 7.7** As shown by [1], Example 4.11, one cannot always expect the linearity condition to hold which we imposed above. In loc. cit., a situation was described where a local variable $S$, considered in the global deformation ring, is a multiple of $T^2$, where $T$ is the image of the local variable describing ramification.

Also we note that if $Q_2=\emptyset$, then the theorem holds unconditionally - provided we are in the set-up described above it.

We explain now how to calculate the matrix $A$ above under the assumption that $\mathcal{L}_{X_\mathfrak{p}} = H^1(G_\mathfrak{p}/I_\mathfrak{p}, (\mathrm{ad}^0_{\bar\rho})^{I_\mathfrak{p}})$ for all places in $\Delta_Q$. We assume without loss of generality that $Q$ contains all the places of $\Delta Q$. This doesn't force them to be ramified for $\bar\rho$.

The maps $t_{R_\mathfrak{p}} \to t_{R'}$ are dual to the map

$$H^1(G_{E,Q}, \mathrm{ad}^0_{\bar\rho}) \stackrel{\mathrm{res}_\mathfrak{p}}{\to} H^1(G_\mathfrak{p}, \mathrm{ad}^0_{\bar\rho})$$

A local dual basis of $\bar S_\mathfrak{p}, \bar T_\mathfrak{p}$ is obtained by fixing a generator $c_\mathfrak{p}$ of $H^1(G_\mathfrak{p}/I_\mathfrak{p}, (\mathrm{ad}^0_{\bar\rho})^{I_\mathfrak{p}})$ and the lift $d_\mathfrak{p}$ of a generator of $H^1(I_\mathfrak{p}, \mathrm{ad}^0_{\bar\rho})^{G_\mathfrak{p}}$ such that $(d_\mathfrak{p}, S_\mathfrak{p}) = 0$, where $(,)$ denotes the pairing between a vector space and its dual.

We choose cycles $s_\mathfrak{p} \in H^1(G_{E,Q}, \mathrm{ad}^0_{\bar\rho})$ that are ramified at $\mathfrak{p}$ but unramified at all places in $\Delta Q - \{\mathfrak{p}\}$. We may assume that $(\mathrm{res}_{\mathfrak{p}'}(s_\mathfrak{p}), T_{\mathfrak{p}'}) = \delta_{\mathfrak{p},\mathfrak{p}'}$, where $\delta_{\dots}$ denotes the Kronecker $\delta$-function. Then the value $\gamma_{\mathfrak{p},\mathfrak{p}'}$ can be either calculated as $(\mathrm{res}_{\mathfrak{p}'}(s_\mathfrak{p}), T_{\mathfrak{p}'})$, or equivalently as the coefficient of $c_{\mathfrak{p}'}$ when writing $\mathrm{res}_{\mathfrak{p}'}$ in the basis $c_{\mathfrak{p}'}, d_{\mathfrak{p}'}$. It is by no means easy to perform such calculations, c.f. [21].

Finally, we give an application of the above theorem to the universal deformation rings and the deformations calculated in [20] and [21].

For this we let $l = 3$, $E = \mathbb{Q}$, the prime called $l$ in loc. cit. we shall denote by $p_0$. We shall use Roman letters for the primes, as we work over $\mathbb{Q}$. For the sets $Q$ and $\Delta Q$ we take $\{3, p_0, p_1, \dots, p_r\}$ and $\{p_0, p_1, \dots, p_r\}$, respectively, in the notation from loc. cit. (so $p_0$ would be the $l$ in that notation). In loc. cit. a representation $G_\mathbb{Q} \to \mathrm{SL}_2(\mathbf{F}_3)$ is fixed that is unramified outside $l, p_0$. We let $X$ be the set of conditions such that $X_l$ is empty, $X_p$ is unramified for $p \neq l$, and such that $\#\rho(I_{p_0}) = 3$ for all deformations $[\rho]$. In loc. cit. it is checked that $R_X \cong \mathbb{Z}_p$ and it is not hard to check that all the conditions we required in the above set-up are satisfied. One has $Q_1 = \{p_0\}$ and $Q_2 = \{p_1, \dots, p_r\}$. We obtain the following corollary, which is, apart from the explicit shape of the universal deformation ring, the central result in [20].

**Corollary 7.8** *The universal deformation ring in [20], Theorem 3, is isomorphic to the ring* $\mathbb{Z}_3[[x]]/(x(3+4x))$, *and setting* $x = 0$ *in this description gives a surjective deformation to* $\mathrm{SL}_2(\mathbb{Z}_3)$ *that is infinitely ramified at* $p_0$.

Proof: We take the case $r = 0$. Then all assumptions of the previous theorem are trivially satisfied, and from its proof it is clear that the universal ring is isomorphic to $\mathbb{Z}_3[[x]]/(g(x))$, where $g$ is the local equation for the place $p_0$. This relation can easily be calculated using the recursion formula in [6] for the local polynomial $g_{l^n}(x)$ or from the explicit formula in [2], Remark 5.4.

If one substitutes $x = 0$ in the local equation, it is clear that the image of the local deformation at $p_0$ is $\{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{Z}_3\}$. Furthermore, $\bar\rho(G_\mathbb{Q}) = \mathrm{SL}_2(\mathbf{F}_3)$. Form those two facts it is clear how to derive surjectivity. ∎

The previous theorem also applies to the situation described in [21]. If $\Delta Q$ is a set of primes satisfying the conditions of Propositions 6 in loc. cit., then from the description given there, following the above sketch how to calculate $A$, one can show that $A$ is a diagonal matrix with non-zero entries in $\mathbf{F}_3$ along the diagonal. Hence both parts of the above Theorem apply. From the second part we obtain the results of [21], Theorems 1 to 3. In loc. cit. the conditions of Theorem 7.5 are established for explicit sets $\Delta Q$ with $r = 1, 2, 3$. The first part of Theorem 7.5 shows the following.

**Corollary 7.9** *The universal deformation rings for deformations with determinant 1 of the residual representation $\bar{\rho}$ of [20] unramified outside $Q$, where $Q$ is as in [21], Theorems 1 to 3, are finite flat over $\mathbb{Z}_3$.*

**Remark 7.10** One could also use our approach to construct infinitely ramified deformations as in [22]. Yet the construction would be a very similar induction procedure, and so we shall not carry it out. In this case our approach shows, that if at each step in the construction of loc. cit. it suffices to add one further prime, then at step $n$ the universal deformation space $R_n$ is isomorphic to $\mathbb{Z}_p[[x_1,\ldots,x_n]]/(x_i(x_i-a_i): i=1,\ldots n)$ where the $a_i$ converge $p$-adically to zero as $n\to\infty$. If one has to add two primes from a certain step on, we can no longer show this as there are components that are no more controlled by linear conditions.

# 8 An interpretation in the tame case

Our goal here is on the one hand to improve the prime-to-adjoint principle from [5], §2, so that it also applies to relations of presentations of pro-$l$ groups, and not just to generators, and then to show how this gives in the tame case an alternative proof of Theorem 5.2 in the case of fixed determinant. Throughout this section we assume that we are given a tame absolutely irreducible representation $\bar{\rho}: G_E\to \mathrm{GL}_2(k)$, i.e. the order of $\mathrm{Im}(\bar{\rho})$ is prime to $l$.

We follow the construction in [5], §6. By the lemma of Schur-Zassenhaus one can find a lift $\rho_0: G_E\to \mathrm{GL}_N(W(k))$ of $\bar{\rho}$ such that $\mathrm{Im}(\rho_0)\cong \mathrm{Im}(\bar{\rho})$. This is unique up to strict equivalence and we shall call it the trivial lift. We give ourselves a set of primes $Q$, containing as usual all primes above $l$ and $\infty$ and all primes where $\bar{\rho}$ ramifies. The deformation problem we shall consider is the problem $X_Q^\eta$, where $\eta=\det(\rho_0)$. We simply call it $X$.

Let $L$ be the splitting field of $\bar{\rho}$ and $H=\mathrm{Gal}(L/E)$. Let $P_Q$ be the Galois group of the maximal pro-$l$ extension $L_Q(l)$ of $L$ unramified outside the primes above $Q$. $L_Q(l)$ is Galois over $E$, and we shall denote the corresponding profinite Galois group by $G_{\bar{\rho}}(l)$. One has a short exact sequence

$$1\to P_Q\to G_{\bar{\rho}}(l)\to H\to 1$$

which is split by the lemma of Schur-Zassenhaus. We shall fix such a splitting. Thus we have an action of $H$ on $P_Q$ and one on $\Gamma_N(W(k))$ and hence a canonical action on all $\Gamma_N(R)$, $R\in\mathcal{C}$, via $\mathrm{GL}_N(W(k))\to \mathrm{GL}_N(R)$ and the conjugation action of $\mathrm{GL}_N(R)$ on $\Gamma_N^0(R)$. Let $\Gamma_N^0(R)$ be the subgroup of $\Gamma_N(R)$ consisting of matrices of determinant one. From [5] it follows that one has a natural isomorphism

$$\mathrm{Def}_X(R)\cong \mathrm{Equiv}_X(R)=\{H \text{ equivariant morphisms } P_Q\to\Gamma_N^0(R)\}$$

From now on we fix two pro-$l$ groups $\Pi, P$ that carry the action of a finite group $H$ that is of order prime to $l$. We want to replace $\Pi$ by a group $\Pi'$ in a way related to the action of $H$ on $P$ such that

$$Hom(\Pi,P)=Hom(\Pi',P)$$

Our motivation is the prime-to-adjoint principle by Boston, that he uses to reduce the number of generators in [5], §6. We shall fix a filtration $\{P_n\}$ of $P$ such that all the subquotients are $\mathbf{F}_l[H]$-modules. By $\mathcal{V}=\{V_1,\ldots,V_t\}$ we shall denote a full list of irreducible $\mathbf{F}_l[H]$-modules that occur as irreducible summands in those subquotients. It is not hard to see that this list is independent of the filtration chosen. We also fix an $H$-equivariant presentation

$$1\to\mathcal{R}\to\mathcal{F}\to\Pi\to 1$$

As $\mathcal{R}$ is usually infinitely generated as an abstract pro-$l$ group, we have to be careful when applying a version of [5], Lemma 2.5. Thus we first establish the following lemma.

**Lemma 8.1** *Let $A_0$ be an irreducible submodule of the $\mathbf{F}_l[H]$-module $\mathcal{R}/[\mathcal{R},\mathcal{F}]\mathcal{R}^p$. Then there exists a finitely generated, $H$-invariant, free pro-$l$ subgroup $\mathcal{R}_0$ of $\mathcal{R}$ whose Frattini quotient maps isomorphically onto $A_0$ under the map induced from $\mathcal{R}_0 \to \mathcal{R}/[\mathcal{R},\mathcal{F}]\mathcal{R}^p$.*

Proof: Let $\bar{r} \neq 0$ be in $A_0$. Take any lift $r$ to $\mathcal{R}$. Let $\mathcal{R}_1$ be the topologically closed subgroup of $\mathcal{R}$ that is generated by the set $Hr$. This is clearly a free pro-$l$ group, as any closed subgroup of a free pro-$l$ group is a free pro-$l$ group.

By construction, the Frattini quotient of $\mathcal{R}_1$, we call it $A_1$, is generated as an $\mathbf{F}_l[H]$-module by the image of $r$, and so it is finite. Under the map $A_1 \to \mathcal{R}/[\mathcal{R},\mathcal{F}]\mathcal{R}^p$ induced from $\mathcal{R}_1 \to \mathcal{R}/[\mathcal{R},\mathcal{F}]\mathcal{R}^p$, $A_1$ maps onto $A_0$. Let $B$ be the kernel of this map. By the proof of [5], Lemma 2.3, we can find pro-$l$ subgroups $\mathcal{R}_0$ and $\mathcal{R}_2$ of $\mathcal{R}_1$ with Frattini quotients $A_0$, $B$, respectively, such that $\mathcal{R}_1$ is isomorphic to the free pro-$l$ product of $\mathcal{R}_0$ and $\mathcal{R}_2$, compatibly with the $H$ action. The submodule $\mathcal{R}_0$ we constructed, satisfies the claim of the lemma. ∎

We can immediately derive some consequences about possible replacements of $\Pi$ by other groups $\Pi'$. For this we shall make the following definitions. Given a finite $\mathbf{F}_l[H]$-module $M$, by $M^{\mathcal{V}}$ we shall denote the direct sum of all $V$-isotypical components of $M$ for all $V \in \mathcal{V}$. We say that $M$ is *prime to* $\mathcal{V}$, if $M^{\mathcal{V}} = 0$.

**Proposition 8.2** *We fix $P$, $\mathcal{V}$ and a presentation of $\Pi$ as above. Suppose we are given a chain of maps*

$$\Pi = \Pi_0 \leftarrow \Pi_1 \to \Pi_2 \leftarrow \Pi_3 \to \ldots \leftarrow \Pi_{2k-1} \to \Pi_{2k} = \Pi'$$

*such that at each step the $\mathbf{F}_l[H]$-module*

$$(H^1(\ker(\Pi_i \to \Pi_j), \mathbf{F}_l)^{\Pi_j})^* \quad \textit{is prime to } \mathcal{V}, \tag{4}$$

*where $j = i+1$ or $i-1$ so that $\Pi_i$ surjects onto $\Pi_j$. ($^*$ denotes the Pontryagin dual of finite groups.) By the inflation-restriction sequence for the surjection $\Pi_i \to \Pi_j$, condition (4) above is equivalent to the condition that the modules*

$$\ker(H^1(\Pi_i, \mathbf{F}_l)^* \to H^1(\Pi_j, \mathbf{F}_l)^*) \textit{ and } Coker(H^2(\Pi_i, \mathbf{F}_l)^* \to H^2(\Pi_j, \mathbf{F}_l)^*)$$

*are prime to $\mathcal{V}$. Then $Hom_H(\Pi, P) \cong Hom_H(\Pi', P)$ where the isomorphism is induced from the given chain of homomorphisms between $\Pi$ and $\Pi'$.*

*A group $\Pi'$ constructed as above will be called a $\mathcal{V}$-modification of $\Pi$. It only depends on $\mathcal{V}$ and not directly on $P$.*

Proof: We first make the following observation. Let $W$ be an irreducible $\mathbf{F}_l[H]$-module. We assume that $W$ is not isomorphic to any of the $V_i$. Let $\mathcal{G}$ be a closed, finitely generated, $H$-invariant subgroup of $\mathcal{F}$ whose Frattini quotient is $W$. Then by [5], Lemma 2.5, for any $H$-equivariant homomorphism $\xi$ from $\mathcal{F}$ to $P$, the restriction of $\xi$ to $\mathcal{G}$ is trivial.

We shall give the proof for the situation where $\Pi$ surjects onto $\Pi'$ and where $(H^1(\ker(\Pi \to \Pi'), \mathbf{F}_l)^{\Pi'})^*$ is prime to $\mathcal{V}$. We consider the diagram

$$(5)\qquad \begin{array}{ccccc} \mathcal{R} & \hookrightarrow & \mathcal{F} & \twoheadrightarrow & \Pi \\ \downarrow & & \downarrow \cong & & \downarrow \\ \mathcal{R}' & \hookrightarrow & \mathcal{F} & \twoheadrightarrow & \Pi' \\ \downarrow & & \downarrow & & \cong \downarrow \\ \tilde{\mathcal{R}} & \hookrightarrow & \mathcal{F}' & \twoheadrightarrow & \Pi' \end{array}$$

where the first line in (5) is our given presentation of $\Pi$, $\mathcal{R}'$ is the kernel from $\mathcal{F}$ to $\Pi'$, and the third line of the diagram denotes a minimal ($H$-equivariant) presentation of $\Pi'$. We shall further consider the following diagram

$$(6)\qquad \begin{array}{ccccc} & & \mathcal{R}/[\mathcal{F},\mathcal{R}]\mathcal{R}^p & & \\ & & \downarrow & \searrow^{f} & \\ N & \longrightarrow & \mathcal{R}'/[\mathcal{F},\mathcal{R}']\mathcal{R}'^p & \xrightarrow{g} & \tilde{\mathcal{R}}/[\mathcal{F}',\tilde{\mathcal{R}}]\tilde{\mathcal{R}}^p \end{array}$$

By the inflation-restriction sequence applied to the middle and lower three term sequence in (5), one can identify the kernel $N$ of the bottom row of (6) with the kernel of $H^1(\Pi, \mathbf{F}_l)^* \to H^1(\Pi', \mathbf{F}_l)^*$. Our assumption means precisely that this kernel, and the cokernel of $f$ are both prime to $\mathcal{V}$.

We now choose irreducible $\mathbf{F}_l[H]$-submodules $W_i$, $i = 1, \ldots, n$ of $W = \mathcal{R}'/[\mathcal{F}, \mathcal{R}']\mathcal{R}'^p$ and corresponding subgroups $\mathcal{R}_i$ according to Lemma 8.1, such that $W$ is the direct sum of the $W_i$, the modules $W_1, \ldots, W_k$ generate the kernel $N$ of $g$, $g(W_{k+1}), \ldots, g(W_m)$ generate the cokernel of $f$, and $g(W_{m+1}), \ldots, g(W_n)$ generate the image of $f$. In particular, using Lemma 8.1, we shall assume that the subgroups $\mathcal{R}_{m+1}, \ldots, \mathcal{R}_n$ are inside $\mathcal{R}$. So now, if $\xi$ is in $Hom_H(\Pi, P)$ let $\hat{\xi}$ be the corresponding element in $Hom_H(\mathcal{F}, P)$. As explained above, the restriction of $\hat{\xi}$ to the $\mathcal{R}_i$ for $i = 1, \ldots, m$ is trivial. As it is induced from a homomorphism from $\Pi$ to $P$, it is trivial on the remaining $\mathcal{R}_i$, too, and thus on $\mathcal{R}'$ which is, as a closed normal subgroup, generated by the $\mathcal{R}_i$. ∎

**Remark 8.3** Typical conditions under which $\Pi'$ is a modification of $\Pi \cong \mathcal{F}/\mathcal{R}$ are the following.

(i) There are finitely generated subgroups $\mathcal{F}_i$ of $\mathcal{F}$ that are $H$-stable and such that the Frattini quotients $\bar{\mathcal{F}}_i$ are prime to $\mathcal{V}$. Let $\mathcal{R}'$ be the closed normal subgroup generated by the $\mathcal{F}_i$ and $\mathcal{R}$, and define $\Pi' = \mathcal{F}/\mathcal{R}'$.

(ii) $\mathcal{R}_0$ is a closed finitely generated subgroup of $\mathcal{R}$ such that its Frattini quotient $\bar{\mathcal{R}}_0$ surjects onto $(\mathcal{R}/\mathcal{R}^p[\mathcal{R}, \mathcal{F}])^{\mathcal{V}}$. Let $\mathcal{R}'$ be the closed, normal, $H$-invariant hull of $\mathcal{R}_0$, and let $\Pi' = \mathcal{F}/\mathcal{R}'$.

**Corollary 8.4** *There is a modification* $\Pi'$ *of* $\Pi$ *such that* $H^i(\Pi', \mathbf{F}_l)^* \cong (H^i(\Pi, \mathbf{F}_l)^*)^{\mathcal{V}}$ *for* $i = 1, 2$.

We shall now explain how one can generalize the prime-to-adjoint principle, that is formulated in [5], §2, completely. We shall consider the following situation

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi & \longrightarrow & \Pi^v & \longrightarrow & \tilde{\Pi} & \longrightarrow & 1 \\
 & & & & & & \big\downarrow \tilde{\xi} & & \\
1 & \longrightarrow & P & \longrightarrow & P^v & \longrightarrow & \tilde{P} & \longrightarrow & 1
\end{array}
\tag{7}
$$

where $\tilde{\Pi}$ and $\tilde{P}$ are finite, $\Pi^v$ and $P^v$ are profinite and $\Pi$ and $P$ are pro-$l$ groups. By $\mathrm{Def}(\Pi^v, P^v)$ we shall denote the set of homomorphisms $\xi : \Pi^v \to P^v$ that are lifts of the residual homomorphism $\tilde{\xi} : \tilde{\Pi} \to \tilde{P}$, modulo the conjugation operation by elements of $P$.

For any subgroup $H$ of $\tilde{\Pi}$ that is of order prime to $l$, by the Lemma of Schur-Zassenhaus, there is an action of $H$ on $P$. Hence one can define the set $\mathcal{V}_H$ of $P$ with respect to the $H$-action as above. A filtration $\{P_n\}$ as above always exists. In fact one can take the Zassenhaus filtration, independently of $H$.

If one also fixes a lift of $H$ to $\mathrm{Aut}(\Pi)$, then it is clear that one has an inclusion

$$\mathrm{Def}(\Pi^v, P^v) \subset Hom_H(\Pi, P)$$

One can now formulate Proposition 8.2 in this context.

**Proposition 8.5** *Suppose there is a chain of maps*

$$
\begin{array}{ccccccccc}
\Pi^v = \Pi_0^v & \longleftarrow & \Pi_1^v & \longrightarrow & \Pi_2^v & \longleftarrow & \cdots & \longrightarrow & \Pi_{2k}^v = (\Pi^v)' \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & & & \big\downarrow \\
\tilde{\Pi} = \tilde{\Pi}_0 & \xleftarrow{\cong} & \tilde{\Pi}_1 & \xrightarrow{\cong} & \tilde{\Pi}_2 & \xleftarrow{\cong} & \cdots & \xrightarrow{\cong} & \tilde{\Pi}_{2k} = \tilde{\Pi}'
\end{array}
$$

*such that for each step $\Pi_i^v \to \Pi_j^v$, where $j = i-1$ or $i+1$ depending on $i$, this map is a surjection, and there exists a subgroup $H$ of $\tilde{\Pi}_i \cong \tilde{\Pi}$ of order prime to $l$ such that*

$$(H^1(\ker(\Pi_i \to \Pi_j), \mathbf{F}_l)^{\Pi_j})^* \quad \textit{is prime to } \mathcal{V}_H.$$

*Then the maps between the various pairs $\Pi_i^v$ and $\Pi_j^v$, $j \in \{i-1, i+1\}$ induce an isomorphism* $\mathrm{Def}(\Pi^v, P^v) \to \mathrm{Def}(\Pi'^v, P^v)$. *As in Proposition 8.2, the construction of $(\Pi^v)'$ depends only on the sets $\mathcal{V}_H$ and not on $P^v$.*

The following is a supplement to Proposition 8.2 needed in the discussion below. Its proof is obvious.

**Lemma 8.6** *We take the assumptions from Proposition 8.2. We assume $\mathcal{R}$ is generated as a closed normal subgroup by elements $r_1, \ldots, r_n$. Assume that $r_1, \ldots, r_m$ are of the form $[s_i, t_i]t_i^{n_i}$ for some $s_i, t_i \in \mathcal{F}$ such that there exist subgroups $H_i$ of $H$ that act trivially on $s_i, t_i$, and where the $H_i$ satisfy the condition that $P^{H_i}$ is abelian. If one replaces $r_i$ by $r_i' = t_i^{n_i}$ for $i = 1, \ldots, m$, we call the corresponding groups $\mathcal{R}'$ and $\Pi'$, then one has the isomorphism*

$$Hom_H(\Pi, P) \cong Hom_H(\Pi', P)$$

We shall now specialize to the situation described at the beginning of this section, i.e. $\Pi = P_Q$, $\Omega = H$, $P = \Gamma^0_N(R)$. We also fix $N = 2$ (else we would need explicit results on auxiliary primes for $N > 2$). We note that the filtration of $\Gamma^0_2(R)$, induced by powers of the maximal ideal, produces subquotients that are all isomorphic to a direct sum of copies of $\mathrm{ad}^0_{\bar\rho}$ - as $k[H]$-modules. We define $\mathcal{V}$ to be the set of the distinct irreducible summands of $\mathrm{ad}^0_{\bar\rho}$ considered as an $\mathbf{F}_l[H]$-module. If $\mathrm{Im}(\bar\rho)$ is not of dihedral type, $\mathrm{ad}^0_{\bar\rho}$ is irreducible as a $k[H]$-module, if it is of $D_2$ type, it is the direct sum of three one-dimensional $k[H]$-modules - all defined over $\mathbf{F}_l$ -, and else the sum of two irreducible ones - the one-dimensional one defined over $\mathbf{F}_l$. The set $\mathcal{V}$, too, contains one, three or two elements, corresponding to the above cases.

By [14], §11, and [18] one has the following description of $P_Q$, or more precisely of the groups $H^i(P_Q, \mathbb{Z}/(l))$.

$$0 \to H^1(P_\emptyset, \mathbb{Z}/(l)) \to H^1(P_Q, \mathbb{Z}/(l)) \to \coprod_{\mathfrak{P}|\mathfrak{p}\in Q} H^1(I_{L_\mathfrak{P}}, \mathbb{Z}/(l))^{G_{L_\mathfrak{P}}} \to \text{Б}_\emptyset \to \text{Б}_Q \to 0 \tag{8}$$

and

$$0 \to \text{Б}_Q \to H^2(P_Q, \mathbb{Z}/(l)) \to \coprod_{\mathfrak{P}|\mathfrak{p}\in Q} H^2(G_{L_\mathfrak{P}}, \mathbb{Z}/(l)) \to \mu_l(L) \to 0 \tag{9}$$

Here $\text{Б}_Q$ is as in Section 6. An explicit definition may be found in [14], §11.

Let $\mathcal{L}$ be the set of local conditions corresponding to our problem $X$. From Section 6, we have

$$H^1_{\mathcal{L}^\perp}(E, \mathrm{ad}^0_{\bar\rho}(1)) \cong \text{Ш}^2_Q(E, \mathrm{ad}^0_{\bar\rho}) \cong Hom_H(\text{Б}^*_Q, \mathrm{ad}^0_{\bar\rho}) \tag{10}$$

Using Lemma 6.2, we pick a set of auxiliary primes $Q_{aux}$ for $X$, and denote the problem for $Q' = Q \cup Q_{aux}$ (with determinant equal to $\det(\rho_0)$) by $X'$. But we can also give a simple direct argument. We decompose the $Q$-class group of $L$, which is a $\mathbf{F}_p[H]$-module, into irreducible summands. In each summand that, after tensoring with $k$, contains a copy of $\mathrm{ad}^0_{\bar\rho}(1)$, we choose a prime ideal. The contraction of those primes to $E$ will form a set $Q_{aux}$, if we make sure that all primes contract to different primes. With a bit more effort, one can find an optimal set of auxiliary primes.

For the $H_i$ in Lemma 8.6, we take a choice of a local decomposition group for the places $\mathfrak{p} \in Q_{aux}$. By Proposition 8.2 or the corollary thereafter, we can eliminate many relations and generators in a presentation of $P_{Q'}$ if we are interested in morphisms to groups $\Gamma^0_2(R)$ only, and we can control this by keeping only those local relations that are not prime to $\mathcal{V}$ with respect to the subgroup $H_\mathfrak{p}$. For the newly chosen places in $Q_{aux}$, we can simplify the local relations by Lemma 8.6. If we start with a presentation of $P_{Q'}$ as in [14], Seite 110, Bemerkung, we obtain.

**Corollary 8.7** *If one considers $H$-equivariant maps from $P_{Q'}$ to some pro-l group $P$ carrying an action of $H$, where $\mathcal{V}$ is the set of irreducible $\mathbf{F}_l[H]$-submodules of $\mathrm{ad}^0_{\bar\rho}$, and such that the centralizers in $P$ of the $H_i$, defined above, are abelian, then one can find a modification $P'_{Q'}$ of $P_{Q'}$ whose Frattini quotient is isomorphic to $\bar P^{\mathcal{V}}_{Q'}$, and one has maps*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{R}_\mathfrak{P} & \longrightarrow & \mathcal{F}_\mathfrak{P} & \longrightarrow & P_\mathfrak{P} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{R} & \longrightarrow & \mathcal{F} & \longrightarrow & P'_{Q'} & \longrightarrow & 1 \end{array}$$

*where the rows are minimal presentations of the groups on the right, and $P_{\mathfrak{P}}$ is a modification of the local Galois group at $\mathfrak{P}$, such that the number of generators and relations to describe $P_{\mathfrak{P}}$ is given by $\dim_{\mathbf{F}_l}(H^i(G_{L_\mathfrak{p}}, \mathbf{F}_l)^*)^{\mathcal{V}_{H_\mathfrak{p}}}$, $i = 1, 2$, and for the primes $\mathfrak{P}$ above primes in $\mathfrak{p} \in Q_{aux}$, the relation is given by $t_\mathfrak{p}^{n_\mathfrak{P}}$ where $n_\mathfrak{P}$ is the $l$-part of the order of $(O_L/\mathfrak{P})^*$. Then $\mathcal{R}$ is generated in the following way. For each $\mathfrak{p}$ in $Q'$ we choose a $\mathfrak{P}_0$ above it and a set of relations $\mathcal{R}_{\mathfrak{P}_0}$. Then $\mathcal{R}$ is the smallest closed normal $H$-invariant subgroup of $\mathcal{F}$ generated by the images of all $\mathcal{R}_{\mathfrak{P}_0}$. In particular, if one is given a homomorphism $\mathcal{F} \to P$ that is $H$-equivariant, it descends to $P'_{Q'}$, and hence to $P'_Q$, if all the relations in all the sets $\mathcal{R}_{\mathfrak{P}_0}$ hold for it.*

*Furthermore a modification of $P_Q$ is obtained by replacing the relations $t_{\mathfrak{P}_0}^{n_{\mathfrak{P}_0}}$ for $\mathfrak{p} \in Q_{aux}$ by the relations $t_{\mathfrak{P}_0}$.*

This is now the analogue of Theorem 5.2, in the tame case, as it reduces the calculations for a presentation of $R_X$ to local calculations, and so the relations describing the universal deformation space can be calculated by local relations, to the same extend as in Theorem 5.2. Hence for $N = 2$ and $\bar{\rho}$ tame, Theorem 5.2 is implied by Corollary 8.7.

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