Presentations of universal deformation rings

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Abstract

Let $\mathbb{F}$ be a finite field of characteristic $\ell > 0$, F a number field, G_F the absolute Galois group of F and let $\bar{\rho} : G_F \to \mathrm{GL}_N(\mathbb{F})$ be an absolutely irreducible continuous representation. Suppose S is a finite set of places containing all places above ℓ and above ∞ and all those at which $\bar{\rho}$ ramifies. Let $\mathcal{O}$ be a complete discrete valuation ring of characteristic zero with residue field $\mathbb{F}$. In such a situation one may consider all deformations of $\bar{\rho}$ to $\mathcal{O}$-algebras which are unramified outside S and satisfy certain local deformation conditions at the places in S. This was first studied by Mazur, [16], and under rather general hypotheses, the existence of a universal deformation ring was proven.

In [3] I studied, among other things, the number of generators needed for an ideal I in a presentation of such a universal deformation ring as a quotient of a power series ring over $\mathcal{O}$ by I. The present manuscript is an update of this part of [3]. The proofs have been simplified, the results slightly generalized. We also treat $\ell = 2$, more general groups than GL_N, and cases where not all relations are local. The results in [3] and hence also in the present manuscript are one of the (many) tools used in the recent attacks on Serre's conjecture by C. Khare and others.

1 Introduction

Let us consider the following simple lemma from commutative algebra:

Lemma 1.1 *Suppose a ring R has a presentation $R = W(\mathbb{F})[[T_1, \ldots, T_n]]/(f_1, \ldots, f_m)$. If $R/(\ell)$ is finite, and if $n \geq m$, then $n = m$, and R is a complete intersection and finite flat over $W(\mathbb{F})$.*

If the ring R in the lemma was a universal deformation ring for certain deformation types of a given residual representation, then the conclusion of the lemma would provide one with a lift to characteristic zero of this deformation type. This observation was first made by A.J. de Jong, [9], *(3.14)*, in 1996. Using obstruction theory and Galois cohomology, in [3] we investigated the existence of presentations of universal deformation rings of the type required in the lemma. In many cases such a presentation was found. However the finiteness of the ring $R/(\ell)$ seemed out of reach.

This was changed enormously by the ground breaking work [22] of R. Taylor where a potential version of Serre's conjecture was proved. The results of Taylor do allow one in many cases to prove the finiteness of $R/(\ell)$. I first learned about this from C. Khare soon after [22] was available. This gives a powerful tool to construct ℓ-adic Galois representations, (potentially) semistable or ordinary at ℓ and with prescribed ramification properties at primes away from ℓ. Besides the deep modularity results for such representations provided by Wiles, Taylor, Skinner et al., the results in [3] were one of the ingredients of the recent proof of Serre's conjecture for conductor $N = 1$ and arbitrary weight by Khare in [10] (cf. also [11]). This is based on previous joint work between Khare and Wintenberger [12], and a result by Dieulefait [7]. Dieulefait also has some partial results on Serre's conjecture [8].

The present manuscript is an update of those parts of [3], which study the number of generators needed for an ideal I in a presentations of a given universal deformation ring as a quotient of a power series ring over $\mathcal{O}$ by I. The proofs have been simplified, the results generalized. We also treat $\ell = 2$, more general groups than GL_N, and cases where not all relations are local. A main improvement is that the use of auxiliary primes could be avoided entirely. We hope that this will be useful for the interested reader.

Let me give a summary of the individual sections. In Section 2, we start by briefly recalling Mazur's fundamental results on universal deformations with the main emphasis on presentations of universal deformation rings. Section 3 gives a first link between the ideals of presentations of local and of global deformation rings in the setting of Mazur adapted to global number fields. The discrepancy is measured by III_S^2 of the adjoint representation of the given residual representation.

It is natural to put further local restrictions on the initial deformation problem studied by Mazur. To obtain again a representable functor the local conditions need relatively representable. In Section 4 we present a perhaps useful variant of this notion.

The core of the present article is Section 5, cf. Corollary 5.3. Here we study presentations of (uni)versal deformation rings for deformations in the sense of Mazur that moreover satisfy a number of local conditions that follow the axiomatics in Section 4. The obstruction module III_S^2 is replaced by a naturally occurring dual Selmer group. The main novelty of the present paper is that unlike in [3] we do not require that this dual Selmer group vanishes. Instead we incorporate it into the presentation of the corresponding ring.

The final three sections investigate consequences of our results. In Section 6 we make some general comments and study the case GL_2 in detail. In particular, we present the numerology for local ordinary deformation rings over arbitrary local fields (of any characteristic). In Section 7 we compare our results to those in [15, 23] of Mauger and Tilouine. The last section, Section 8, is dedicated to deriving a presentation of a global universal deformation rings as the quotient of a power series ring over the completed tensor product over all local versal deformation rings. The main result here is essentially due to M. Kisin [13]. We show how to derive it using the results of Section 5.

Notation: For the rest of this article, we fix the following notation: $\mathbb{F}$ is a finite field of characteristic ℓ. The ring of Witt vectors of $\mathbb{F}$ is denoted $W(\mathbb{F})$. For a local ring R its maximal ideal is denoted by $\mathfrak{m}_R$. By $\mathcal{O}$ we denote a complete discrete valuation ring of characteristic zero with residue field $\mathbb{F}$, so that in particular $\mathcal{O}$ is finite over $W(\mathbb{F})$. The category of complete noetherian local $\mathcal{O}$-algebras R with a fixed isomorphism $R/\mathfrak{m}_R \cong \mathbb{F}$ will be $\mathcal{C}_\mathcal{O}$. Here and in the following $\mathbb{F}[\varepsilon]/(\varepsilon^2)$ is an $\mathcal{O}$-algebra via $\mathcal{O} \to \mathbb{F} \to \mathbb{F}[\varepsilon]/(\varepsilon^2)$. For a Ring R of $\mathcal{C}_\mathcal{O}$ its mod $\mathfrak{m}_\mathcal{O}$ tangent space is defined as

$$\mathfrak{t}_R := \mathrm{Hom}_\mathcal{O}(R, \mathbb{F}[\varepsilon]/(\varepsilon^2)) \cong \mathrm{Hom}_\mathbb{F}(\mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_\mathcal{O}R), \mathbb{F}).$$

For J an ideal of a ring R in $\mathcal{C}_{\mathcal{O}}$, we define $\mathrm{gen}(J) := \dim_{\mathbb{F}} J/(\mathfrak{m}_R J)$. By Nakayama's lemma, $\mathrm{gen}(J)$ is the minimal number of generators of J as an ideal in R.

By F we denote a number field and by S a finite set of places of F. We always assume that S contains all places of F above ℓ and ∞. The maximal outside S unramified extension of F inside a fixed algebraic closure F^{alg} of F is denoted F_S. It is a Galois extension of F whose corresponding Galois is $G_{F,S} := \mathrm{Gal}(F_S/F)$.

For each place ν of F let F_ν be the completion of F at ν, let G_ν be the absolute Galois group of F_ν, and $I_\nu \subset G_\nu$ the inertia subgroup. Choosing for each such ν a field homomorphism $F_S \hookrightarrow F_\nu^{\mathrm{alg}}$, we obtain induced group homomorphisms $G_\nu \to G_{F,S}$.

Acknowledgments: This article owes many ideas and much inspiration to the work of Mazur, Wiles, Taylor, de Jong, and many others. Many thanks go to C. Khare for constantly reminding me to write un 'update' of the article [3] and for many comments. Many thanks also to Mark Kisin for having made available [13] and for some interesting related discussions.

2   A simple deformation problem

In this section we recall various basic notions and concepts from [16]. In terms of generality, we follow [23], and so we fix a smooth linear algebraic group $\mathcal{G}$ over $\mathcal{O}$. By $\mathcal{Z}_{\mathcal{G}}$ we denote the center of $\mathcal{G}$, by $\mathcal{T}$ we denote a smooth affine algebraic group over $\mathcal{O}$ that is a quotient of $\mathcal{G}$ via some surjective homomorphism $d\colon \mathcal{G} \to \mathcal{T}$ of algebraic groups over $\mathcal{O}$. The kernel of d is denoted $\mathcal{G}_0$. The Lie algebras over $\mathcal{O}$ corresponding to $\mathcal{G}$ and $\mathcal{G}_0$ will be $\mathfrak{g}$ and $\mathfrak{g}_0$, respectively.

Examples 2.1   (a) $d := \det\colon \mathcal{G} := \mathrm{GL}_N \to \mathcal{T} := \mathrm{GL}_1$. Then $\mathfrak{g} = M_N(\mathbb{F})$ and $\mathfrak{g}_0 \subset \mathfrak{g}$ is the subset of trace zero matrices.

(b) $\mathcal{G}$ is the Borel subgroup of GL_N formed by the set of upper triangular matrices, $\mathcal{T} := \mathrm{GL}_1^N$, and $d\colon \mathcal{G} \to \mathrm{GL}_1^N$ is the the quotient homomorphism of $\mathcal{G}$ modulo its unipotent radical. The corresponding Lie algebras are the obvious ones.

Throughout this section let Π be a profinite group such that the pro-ℓ completion of every open subgroup is topologically finitely generated. (This is the finiteness condition Φ_ℓ of [16], Def. 1.1.) Let us fix a continuous (residual) representation

$$\bar{\rho} : \Pi \to \mathcal{G}(\mathbb{F}).$$

The **adjoint representation of Π on** $\mathfrak{g}(\mathbb{F})$ is denoted by $\mathrm{ad}_{\bar{\rho}}$, **its subrepresentation on** $\mathfrak{g}_0(\mathbb{F}) \subset \mathfrak{g}(\mathbb{F})$ by $\mathrm{ad}_{\bar{\rho}}^0$. For M an $\mathbb{F}[\Pi]$-module, we define its **dimension** as $h^i(\Pi, M) := \dim_{\mathbb{F}} H^i(\Pi, M)$.

Following Mazur we first consider the following simple deformation problem: A **lifting** of $\bar{\rho}$ to $R \in \mathcal{C}_{\mathcal{O}}$ is a continuous representation $\rho : \Pi \to \mathcal{G}(R)$, such that $\rho \pmod{\mathfrak{m}_R} = \bar{\rho}$. A **deformation** of $\bar{\rho}$ to R is a strict equivalence class $[\rho]$ of liftings ρ of $\bar{\rho}$ to R, where two liftings ρ_1 and ρ_2 from Π to $\mathcal{G}(R)$ are strictly equivalent, if there exists an element in the kernel of $\mathcal{G}(R) \to \mathcal{G}(\mathbb{F})$ which conjugates one into the other. We consider the functor

$$\mathrm{Def}_{\mathcal{O},\Pi} : \mathcal{C}_{\mathcal{O}} \to \mathbf{Sets} : R \mapsto \{[\rho] \mid [\rho] \text{ is a deformation of } \bar{\rho} \text{ to } R\}.$$

Theorem 2.2 ([16]) *Suppose Π and $\bar{\rho}$ are as above. Then*

(a) The functor $\mathrm{Def}_{\mathcal{O},\Pi}$ has a versal hull, which we denote by $\rho_{\bar{\rho},\mathcal{O}} \colon \Pi \to \mathcal{G}(R_{\bar{\rho},\mathcal{O}})$.

(b) If furthermore the centralizer of $\mathrm{Im}(\bar{\rho})$ in $\mathcal{G}(\mathbb{F})$ is contained in $\mathcal{Z}_{\mathcal{G}}(\mathbb{F})$, then $\mathrm{Def}_{\mathcal{O},\Pi}$ is representable by the above pair $(R_{\bar{\rho},\mathcal{O}}, \rho_{\bar{\rho},\mathcal{O}})$.

(c) $\mathfrak{t}_{R_{\bar{\rho},\mathcal{O}}} \cong H^1(\Pi, \mathrm{ad}_{\bar{\rho}})$

(d) $R_{\bar{\rho},\mathcal{O}}$ has a presentation $R_{\bar{\rho},\mathcal{O}} \cong \mathcal{O}[[T_1, \ldots, T_h]]/J$ for some ideal $J \subset \mathcal{O}[[T_1, \ldots, T_h]]$, where $h = h^1(\Pi, \mathrm{ad}_{\bar{\rho}})$ and $\mathrm{gen}(J) \leq h^2(\Pi, \mathrm{ad}_{\bar{\rho}})$.

PROOF: The proof is essentially contained in [16] §1.2, §1.6, where a criterion of Schlessinger is verified. For (b) Mazur originally assumed that $\bar{\rho}$ was absolutely irreducible. It was later observed by Ramakrishna, [20], that this could be weakened to the condition given.
A proof for GL_N instead of a general group $\mathcal{G}$ in the precise form above can be found in [3], Thm. 2.4. The adaption to general $\mathcal{G}$ is obvious, and so we omit details.
Since this will be of importance later, we remark that the proofs in [16] or [3] show that there is a canonical surjective homomorphism

(1) $$H^2(\Pi, \mathrm{ad}_{\bar{\rho}})^* \longrightarrow\!\!\!\!\rightarrow J/\mathfrak{m}_{\mathcal{O}[[T_1, \ldots, T_h]]} J.$$

of vector spaces over $\mathbb{F}$ ∎

Remark 2.3 If $R_{\bar{\rho},\mathcal{O}}/(\ell)$ is known to have Krull dimension $h^1(\Pi, \mathrm{ad}_{\bar{\rho}}) - h^2(\Pi, \mathrm{ad}_{\bar{\rho}})$, then Theorem 2.2, which is obtained entirely by the use of obstruction theory, implies that $R_{\bar{\rho},\mathcal{O}}$ is flat over $\mathcal{O}$ of relative dimension $h^1(\Pi, \mathrm{ad}_{\bar{\rho}}) - h^2(\Pi, \mathrm{ad}_{\bar{\rho}})$, and a complete intersection. So in this situation Theorem 2.2 has some strong ring-theoretic consequences for $R_{\bar{\rho},\mathcal{O}}$.
In the generality of the present section, it can not be expected that the Krull dimension of $R_{\bar{\rho},\mathcal{O}}/(\ell)$ is always equal to $h^1(\Pi, \mathrm{ad}_{\bar{\rho}}) - h^2(\Pi, \mathrm{ad}_{\bar{\rho}}) + 1$. Recent work [4] by Bleher and Chinburg shows that this fails for finite groups Π and also for Galois groups $\Pi = G_{F,S}$ in case S does not contain all the primes above ℓ.
If $\mathcal{G} = \mathrm{GL}_n$ and $\bar{\rho}$ is absolutely irreducible, and if $\Pi = G_{F,S}$ and S contains all primes above ℓ, or if $\Pi = G_{F_\nu}$ (cf. Remark 6.2), all evidence suggests that $R_{\bar{\rho},\mathcal{O}}$ is a complete intersection, flat over $\mathcal{O}$ and of relative dimension $h^1(\Pi, \mathrm{ad}_{\bar{\rho}}) - h^2(\Pi, \mathrm{ad}_{\bar{\rho}})$. But the amount of evidence is small. On the one hand, there is Leopoldt's conjecture, cf. [16], 1.10. On the other, if one is bootstrapping the content of [1] in light of the recent modularity results of Taylor et al., e.g. [22], there is some evidence for odd two-dimensional representations $\bar{\rho}$ of $G_{F,S}$ where F is a totally real field.

One often considers the following subfunctor of $\mathrm{Def}_{\Pi,\mathcal{O}}$: Let $\eta \colon \Pi \to \mathcal{T}(\mathcal{O})$ be a fixed lift of the residual representation $d \circ \bar{\rho} \colon \Pi \to \mathcal{T}(\mathbb{F})$. Then one defines the subfunctor $\mathrm{Def}^{\eta}_{\Pi,\mathcal{O}}$ of $\mathrm{Def}_{\Pi,\mathcal{O}}$ as the functor which to $R \in \mathcal{C}_{\mathcal{O}}$ assigns the set of all deformations $[\rho]$ of $\bar{\rho}$ to R for which the composite $d \circ \rho \colon \Pi \to \mathcal{T}(R)$ and the composite $\tau \circ \eta$ of η with the canonical homomorphism $\tau \colon \mathcal{T}(\mathcal{O}) \to \mathcal{T}(R)$ only differ by conjugation inside $\mathcal{T}(R)$. One obtains the following analog of Theorem 2.2:

Theorem 2.4 ([16]) *Suppose* Π *and* $\bar{\rho}$ *are as above. Then:*

(a) *The functor* $\mathrm{Def}^{\eta}_{\Pi,\mathcal{O}}$ *has a versal hull, which we denote by* $\rho^{\eta}_{\bar{\rho},\mathcal{O}} \colon \Pi \to \mathcal{G}(R^{\eta}_{\bar{\rho},\mathcal{O}})$.

(b) *If furthermore the centralizer of* $\mathrm{Im}(\bar{\rho})$ *in* $\mathcal{G}(\mathbb{F})$ *is contained in* $\mathcal{Z}_{\mathcal{G}}(\mathbb{F})$, *then* $\mathrm{Def}_{\mathcal{O},\Pi}^{\eta}$ *is representable by the above pair* $(R_{\bar{\rho},\mathcal{O}}^{\eta}, \rho_{\bar{\rho},\mathcal{O}})$.

(c) $\mathfrak{t}_{R_{\bar{\rho},\mathcal{O}}^{\eta}} \cong H^1(\Pi, \mathrm{ad}_{\bar{\rho}})^{\eta} := \mathrm{Im}(H^1(\Pi, \mathrm{ad}_{\bar{\rho}}^0) \to H^1(\Pi, \mathrm{ad}_{\bar{\rho}}))$.

(d) $R_{\bar{\rho},\mathcal{O}}^{\eta}$ *has a presentation* $R_{\bar{\rho},\mathcal{O}}^{\eta} \cong \mathcal{O}[[T_1, \ldots, T_{h^{\eta}}]]/J^{\eta}$ *for some ideal* $J^{\eta} \subset \mathcal{O}[[T_1, \ldots, T_{h^{\eta}}]]$, *where* $h^{\eta} = \dim_{\mathbb{F}} H^1(\Pi, \mathrm{ad}_{\bar{\rho}}^0)^{\eta}$ *and* $\mathrm{gen}(J^{\eta}) \leq \dim_{\mathbb{F}} H^2(\Pi, \mathrm{ad}_{\bar{\rho}}^0)$.

Remark 2.5 From the long exact cohomology sequence for

$$0 \longrightarrow \mathfrak{g}_0 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{g}_0 \longrightarrow 0$$

it follows that $H^1(\Pi, \mathrm{ad}_{\bar{\rho}})^{\eta} \cong H^1(\Pi, \mathrm{ad}_{\bar{\rho}}^0)$ is an isomorphism if and only if

$$H^0(\Pi, \mathrm{ad}_{\bar{\rho}}) \longrightarrow\!\!\!\!\to H^0(\Pi, \mathrm{ad}_{\bar{\rho}}/\mathrm{ad}_{\bar{\rho}}^0)$$

is surjective. To measure the discrepancy between h^{η} and $h^1(\Pi, \mathrm{ad}_{\bar{\rho}}^0)$, we define $\delta(\Pi, \mathrm{ad}_{\bar{\rho}}) = 0$ and

(2) $\delta(\Pi, \mathrm{ad}_{\bar{\rho}})^{\eta} := \dim_{\mathbb{F}} \mathrm{Coker}(H^0(\Pi, \mathrm{ad}_{\bar{\rho}}) \longrightarrow H^0(\Pi, \mathrm{ad}_{\bar{\rho}}/\mathrm{ad}_{\bar{\rho}}^0)) = h^1(\Pi, \mathrm{ad}_{\bar{\rho}}^0) - h^{\eta}$.

As an example consider the case $d = \det \colon \mathcal{G} = \mathrm{GL}_N \to \mathcal{T} = \mathrm{GL}_1$. If ℓ does not divide N, then $\mathrm{ad}_{\bar{\rho}} = \mathrm{ad}_{\bar{\rho}}^0 \oplus \mathbb{F}$, where here $\mathbb{F}$ denotes the trivial representation of Π, and so $\delta(\Pi, \mathrm{ad}_{\bar{\rho}})^{\eta} = 0$. However for $\ell | N$ and absolutely irreducible $\bar{\rho}$, one finds $\delta(\Pi, \mathrm{ad}_{\bar{\rho}})^{\eta} = 1$.
If $\eta = 1$, one can in fact consider two deformation functors: (i) the functor that arises from considering deformations into $\mathcal{G}_0$ instead of $\mathcal{G}$, and (ii) the functor $\mathrm{Def}_{\mathcal{O},\Pi}^{\eta}$ considered above. If $\delta(\Pi, \mathrm{ad}_{\bar{\rho}})^{\eta} = 0$, the two agree. Otherwise, the functor for $\mathcal{G}_0$ is less rigid, and in fact its mod $\mathfrak{m}_{\mathcal{O}}$ tangent space has a larger dimension (the difference being given by $\delta(\Pi, \mathrm{ad}_{\bar{\rho}})^{\eta}$).

Note also that the bound for $\mathrm{gen}(J^{\eta})$ in part (d) is solely described in terms of $\mathrm{ad}_{\bar{\rho}}^0$.

3   A first local to global principle

For the remainder of this article we fix a residual representation

$$\bar{\rho} \colon G_{F,S} \to \mathcal{G}(\mathbb{F}).$$

Whenever it makes sense, we fix a lift $\eta \colon G_{F,S} \to \mathcal{T}(\mathcal{O})$ of $d \circ \bar{\rho}$. (If $\mathcal{T}(\mathbb{F})$ is of order prime to ℓ, such a lift always exists.) As in the previous section, the adjoint representation of $G_{F,S}$ on $\mathfrak{g}(\mathbb{F})$ is denoted by $\mathrm{ad}_{\bar{\rho}}$, its subrepresentation on $\mathfrak{g}_0(\mathbb{F}) \subset \mathfrak{g}(\mathbb{F})$ by $\mathrm{ad}_{\bar{\rho}}^0$.
To $\bar{\rho}$ we can attach the following canonical deformation functors: First, we define

$$\mathrm{Def}_{S,\mathcal{O}} := \mathrm{Def}_{G_{F,S},\mathcal{O}}, \quad \text{and} \quad \mathrm{Def}_{S,\mathcal{O}}^{\eta} := \mathrm{Def}_{G_{F,S},\mathcal{O}}^{\eta}.$$

The first functor parameterizes all deformations of $\bar{\rho}$ which are unramified outside S, the second (sub)functor moreover fixes the chosen determinant η.
Let ν be any place of F. The restriction of ρ to G_{ν} defines a residual representation $G_{\nu} \to \mathcal{G}(\mathbb{F})$, the restriction of η to G_{ν} a lift $G_{\nu} \to \mathcal{T}(\mathcal{O})$ of $d \circ \bar{\rho}$ restricted to G_{ν}. Thus we may define local deformation functors by

$$\mathrm{Def}_{\nu,\mathcal{O}} := \mathrm{Def}_{G_{\nu},\mathcal{O}}, \quad \text{and} \quad \mathrm{Def}_{\nu,\mathcal{O}}^{\eta} := \mathrm{Def}_{G_{\nu},\mathcal{O}}^{\eta}.$$

Notational convention: In the sequel we often write $?^{(\eta)}$ in formulas. This expresses two assertions at once: First, the formula is true if the round brackets are missing throughout. Second, the formula is also true if (η) is entirely omitted throughout the formula. Corresponding to the above cases the usage of $\mathrm{ad}_{\bar{\rho}}^{(0)}$ has to be interpreted as follows: If brackets around (η) are omitted, then they are to be omitted around (0) in $\mathrm{ad}_{\bar{\rho}}^{(0)}$, too; if (η) is omitted, then (0) in $\mathrm{ad}_{\bar{\rho}}^{(0)}$ is to be omitted, as well.

By global, respectively local class field theory, the groups $G_{F,S}$ and G_ν satisfy the conditions imposed on the abstract profinite group Π in Section 2. Therefore Theorems 2.2 and 2.4 are applicable to $\bar{\rho}$ and its restriction to the groups G_ν. The resulting (uni)versal global deformations are denoted by $\rho_{S,\mathcal{O}}^{(\eta)} : G_{F,S} \to R_{S,\mathcal{O}}^{(\eta)}$, and the local ones by $\rho_{\nu,\mathcal{O}}^{(\eta)} : G_\nu \to R_{\nu,\mathcal{O}}^{(\eta)}$. We also set

$$h := h^1(G_{F,S}, \mathrm{ad}_{\bar{\rho}}), h^\eta := h^1(G_{F,S}, \mathrm{ad}_{\bar{\rho}})^\eta, h_\nu := h^1(G_\nu, \mathrm{ad}_{\bar{\rho}}), h_\nu^\eta := h^1(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta.$$

With the above notation, Theorem 2.2 shows that there exist presentations

(3) $$0 \longrightarrow J_\nu^{(\eta)} \longrightarrow \mathcal{O}[[T_{\nu,1}, \ldots, T_{\nu,h_\nu^{(\eta)}}]] \longrightarrow R_{\nu,\mathcal{O}}^{(\eta)} \longrightarrow 0.$$

(4) $$0 \longrightarrow J^{(\eta)} \longrightarrow \mathcal{O}[[T_1, \ldots, T_{h^{(\eta)}}]] \longrightarrow R_{S,\mathcal{O}}^{(\eta)} \longrightarrow 0,$$

The restriction $G_\nu \to G_{F,S}$ applied to deformations, induces a natural transformation of functors

$$\mathrm{Def}_{S,\mathcal{O}} \to \prod_{\nu \in S} \mathrm{Def}_{\nu,\mathcal{O}}.$$

This yields a ring homomorphism

$$\widehat{\bigotimes_{\nu \in S}} R_\nu^{(\eta)} \longrightarrow R_S^{(\eta)},$$

where by $\widehat{\otimes}$, we denote the completed tensor product over the ring $\mathcal{O}$. Using the smoothness of $\mathcal{O}[[T_1, \ldots, T_{h^{(\eta)}}]]$, and the above presentations, we obtain a commutative diagram with inserted dashed arrows, where α is a product of local maps $\alpha_\nu : \mathcal{O}[[T_{\nu,1}, \ldots, T_{\nu,h_\nu^{(\eta)}}]] \to \mathcal{O}[[T_1, \ldots, T_{h^{(\eta)}}]]$ and where $\langle J_\nu^{(\eta)} \rangle$ denotes the ideal generated by the $J_\nu^{(\eta)}$ in

(5)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \langle J_\nu^{(\eta)} : \nu \in S \rangle & \longrightarrow & \mathcal{O}[[T_{\nu,1}, \ldots, T_{\nu,h_\nu^{(\eta)}} \mid \nu \in S]] & \longrightarrow & \hat{\otimes}_{\nu \in S} R_\nu^{(\eta)} & \longrightarrow & 0 \\
& & \Big\downarrow & & \Big\downarrow {\scriptstyle \alpha = \prod \alpha_\nu} & & \Big\downarrow & & \\
0 & \longrightarrow & J^{(\eta)} & \longrightarrow & \mathcal{O}[[T_1, \ldots, T_{h^{(\eta)}}]] & \longrightarrow & R_{S,\mathcal{O}}^{(\eta)} & \longrightarrow & 0
\end{array}
$$

For any $\mathbb{F}[G_{F,S}]$ module M, define $\text{Ш}_S^2(M) = \mathrm{Ker}(H^2(G_{F,S}, M) \longrightarrow \oplus_{\nu \in S} H^2(G_\nu, M))$. Our first result on a local to global relation is the following simple consequence of Theorems 2.2 and 2.4:

Theorem 3.1 *The ideal $J^{(\eta)}$ is generated by the images of the ideals $J_\nu^{(\eta)}$, $\nu \in S$, together with at most $\dim_{\mathbb{F}} \text{Ш}_S^2(\mathrm{ad}_{\bar{\rho}}^{(0)})$ further elements.*
In particular, if the corresponding $\text{Ш}_S^2(\ldots)$ vanishes, then all relations in $J^{(\eta)}$ are local.

PROOF: By (1), there is a surjection

$$H^2(G_{F,S}, \mathrm{ad}_{\bar{\rho}}^{(0)})^* \longrightarrow J^{(\eta)}/\mathfrak{m}_{\mathcal{O}[[T_1,\dots,T_{h(\eta)}]]}J^{(\eta)},$$

and similarly for the local terms. Comparing local and global terms yields the commutative diagram

$$
\begin{array}{ccccccc}
\bigoplus_{\nu\in S} H^2(G_\nu, \mathrm{ad}_{\bar{\rho}}^{(0)})^* & \longrightarrow & H^2(G_{F,S}, \mathrm{ad}_{\bar{\rho}}^{(0)})^* & \longrightarrow & \mathrm{III}_S^2(\mathrm{ad}_{\bar{\rho}}^{(0)})^* & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
\bigoplus_{\nu\in S} J_\nu^{(\eta)}/\mathfrak{m}_{\mathcal{O}[[T_1,\dots,T_{h_\nu(\eta)}]]}J_\nu^{(\eta)} & \longrightarrow & J^{(\eta)}/\mathfrak{m}_{\mathcal{O}[[T_1,\dots,T_{h(\eta)}]]}J^{(\eta)}, & & & &
\end{array}
$$

where the lower horizontal homomorphism is induced from the ring homomorphism α of the previous diagram, and where the vertical homomorphisms are surjective.

By Nakayama's Lemma, any subset of $J^{(\eta)}$ whose image generates $J^{(\eta)}/\mathfrak{m}_{\mathcal{O}[[T_1,\dots,T_{h(\eta)}]]}J^{(\eta)}$ forms a generating system for $J^{(\eta)}$. Therefore the assertion of the theorem follows immediately from the above diagram. ∎

Remark 3.2 An obvious consequence of Theorem 3.1 is the inequality

$$\mathrm{gen}(J^{(\eta)}) \leq \dim_{\mathbb{F}} \mathrm{III}_S^2(\mathrm{ad}_{\bar{\rho}}^{(0)}) + \sum_{\nu\in S} \mathrm{gen}(J_\nu^{(\eta)}).$$

In general, this inequality is not best possible, since one has the exact sequence

$$0 \longrightarrow \mathrm{III}_S^2(\mathrm{ad}_{\bar{\rho}}^{(0)}) \longrightarrow H^2(G_{F,S}, \mathrm{ad}_{\bar{\rho}}^{(0)}) \longrightarrow \oplus_{\nu\in S} H^2(G_\nu, \mathrm{ad}_{\bar{\rho}}^{(0)}) \longrightarrow H^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^{(0)})^\vee)^* \longrightarrow 0.$$

4   Local conditions

For the applications to modularity questions, the functors considered in the previous section are too general. At places ν above the prime ℓ modular Galois representations are potentially semistable; at places ν away from ℓ, one often wants to prescribe a certain behavior of the local Galois representations in question. This leads one to consider subfunctors $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{(\eta)}$ of the functors $\mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)}$ that describe a certain type of local deformation.

An important requirement on these subfunctors is that the resulting global deformation problems should have a versal hull. There are various approaches to achieve this. We find it most convenient to work with the notion of relative representability, which is basically described in [17], § 19.

Let us recall from [3], § 2, the relevant notion of relative representability: Following Schlessinger a homomorphism $\pi\colon A \to C$ of Artin rings in $\mathcal{C}_{\mathcal{O}}$ is called a **small extension** if π is surjective and if the kernel of π is isomorphic to the A-module $\mathbb{F}$.

In [17], p. 277, in the definition of *small*, the requirement of surjectivity is left out. Therefore the statement of Schlessinger's Theorem as given there is weaker than that given in [21]. The statement in [17], p. 277, is also true if small morphisms are assumed to be surjective.

A covariant functor $F\colon \mathcal{C}_{\mathcal{O}} \to \mathbf{Sets}$ is called **continuous**, if for any directed inverse system $(A_i)_{i\in I}$ of Artin rings in $\mathcal{C}_{\mathcal{O}}$ with limit $A := \varprojlim A_i$ in $\mathcal{C}_{\mathcal{O}}$, one has

$$F(A) = \varprojlim F(A_i).$$

Definition 4.1 *Given two covariant continuous functors $F, G \colon \mathcal{C}_{\mathcal{O}} \to \mathbf{Sets}$ such that G is a subfunctor of F, we say that G is **relatively representable** if*

(a) *$G(k) \neq \varnothing$, and*

(b) *for all small surjections $f_1 \colon A_1 \to A_0$ and maps $f_2 \colon A_2 \to A_0$ of artinian rings $\mathcal{C}_{\mathcal{O}}$, the following is a pullback diagram:*

$$
\begin{array}{ccc}
G(A_1 \times_{A_0} A_2) & \longrightarrow & G(A_1) \times_{G(A_0)} G(A_2) \\
\downarrow & & \downarrow \\
F(A_1 \times_{A_0} A_2) & \longrightarrow & F(A_1) \times_{F(A_0)} F(A_2)
\end{array}
$$

Remark 4.2 The definition of relative representability given in [17] seems at the outset more restrictive. However, by a reduction procedure similar to that of Schlessinger in [21], our definition might be equivalent to the the one given in [17].

The property from [17] is the one that is satisfied for essentially all subfunctors $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{(\eta)} \subset \mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)}$ that have been considered in deformation problems for Galois representations. Hence in all this cases, the local deformation problems are relatively representable in the above sense.

Proposition 4.3 *Suppose $F, F_i, G_i \colon \mathcal{C}_{\mathcal{O}} \to \mathbf{Sets}$, $i \in I$, I a finite set, are covariant continuous functors. Suppose for each $i \in I$ that G_i is a relatively representable subfunctor of F_i. Then the following holds:*

(a) *If F_i has a hull, i.e., F_i satisfies conditions $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3})$ of Schlessinger, [21], Thm. 2.11, or [17], § 18, then so does G_i. If F_i is representable, then so is G_i.*

(b) *The product $\prod_{i \in I} G_i$ is a continuous subfunctor of $\prod_i F_i$ which is relatively representable.*

(c) *Suppose the F_i have a versal hull. Let $\alpha \colon F \to \prod_i F_i$ be a natural transformation, and let G be defined as the pullback of*

$$
\begin{array}{ccc}
G & \dashrightarrow & \prod G_i \\
\vdots & & \downarrow \\
F & \longrightarrow & \prod F_i.
\end{array}
$$

Then, if F has a versal hull, then so does G, and if F is representable, then so is G.

The proof exploits the representability criterion of Schlessinger. It is a simple exercise in diagram chasing, and left to the reader.

After the above detour on general representability criteria, let us come back to the deformation functors we introduced in the previous section. The functors $\mathrm{Def}_{S,\mathcal{O}}^{(\eta)}$ and $\mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)}$ are continuous. To work with finer local conditions, for each place ν in S we fix relatively representable subfunctors

$$
\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{(\eta)} \subset \mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)}.
$$

We also define $\widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{(\eta)}$ as the pullback of functors in the diagram

$$
\begin{array}{ccc}
\widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{(\eta)} & \dashrightarrow & \prod_{\nu \in S} \widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{(\eta)} \\
\big\downarrow & & \big\downarrow \\
\mathrm{Def}_{S,\mathcal{O}}^{(\eta)} & \longrightarrow & \prod \mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)}.
\end{array}
$$

By Proposition 4.3, we obtain:

Proposition 4.4 *The functors $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{(\eta)}$ have a versal hull $\widetilde{\rho}_{\nu,\mathcal{O}}^{(\eta)} \colon G_\nu \to \mathcal{G}(\widetilde{R}_{\nu,\mathcal{O}}^{(\eta)})$. The functor $\widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{(\eta)}$ is representable, say by, $\widetilde{\rho}_{S,\mathcal{O}}^{(\eta)} \colon G_\nu \to \mathcal{G}(\widetilde{R}_{S,\mathcal{O}}^{(\eta)})$. The induced ring homomorphisms $R_{\nu,\mathcal{O}}^{(\eta)} \to \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}$ and $R_{S,\mathcal{O}}^{(\eta)} \to \widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ are surjective.*

5   A refined local to global principle

We keep the hypotheses of the previous sections that the subfunctors $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{(\eta)} \subset \mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)}$ are relatively representable. In this section, we want to derive an analog of Theorem 3.1, i.e., some kind of local to global principle for the refined deformation problem $\widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{(\eta)}$. The needed substitute for $\mathrm{III}_S^2(\mathrm{ad}_{\bar{\rho}})$ is a certain dual Selmer group. In our exposition of generalized Selmer groups, we follow Wiles, cf. also [18], (8.6.19) and (8.6.20).

Let us consider a place ν of S. Since $R_{\nu,\mathcal{O}}^{(\eta)} \to \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}$ is an epimorphism, there is an inclusion of mod ℓ tangent spaces $\mathfrak{t}_{\widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}} \hookrightarrow \mathfrak{t}_{R_{\nu,\mathcal{O}}^{(\eta)}}$. Via the isomorphism $H^1(G_\nu, \mathrm{ad}_{\bar{\rho}})^{(\eta)} \cong \mathfrak{t}_{R_{\nu,\mathcal{O}}^{(\eta)}}$ this yields a subspace $L_\nu^{(\eta)} \subset H^1(G_\nu, \mathrm{ad}_{\bar{\rho}})^{(\eta)}$ canonically attached to $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{(\eta)}$. Its dimension will be denoted $\widetilde{h}_\nu^{(\eta)}$. From the interpretation of $L_\nu^{(\eta)}$ as a mod $\mathfrak{m}_\mathcal{O}$ tangent space, we deduce the existence of a presentation

$$(6) \qquad 0 \longrightarrow \widetilde{J}_\nu^{(\eta)} \longrightarrow \mathcal{O}[[T_{\nu,1}, \ldots, T_{\nu,\widetilde{h}_\nu^{(\eta)}}]] \longrightarrow \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)} \longrightarrow 0.$$

The collection $(L_\nu^{(\eta)})_{\nu \in S}$ is often abbreviated by $\mathcal{L}^{(\eta)}$. Let us also denote by $L_\nu^0 \subset H^1(G_\nu, \mathrm{ad}_{\bar{\rho}}^0)$ the inverse image of $L_\nu^\eta \subset H^1(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta$ under the surjection $H^1(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) \longrightarrow H^1(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta$.
Convention on notation: For the refined deformation problems, the universal ring and the ideals in a presentation, and the dimensions of the mod $\mathfrak{m}_\mathcal{O}$ tangent spaces are given a tilde. For the corresponding subspaces of $H^1(\ldots)^{(\eta)}$ we stick to the commonly used notation $L_\nu^?$.

We denote by $\bar{\chi}_{\mathrm{cyc}}$ the mod ℓ cyclotomic character. For any finite $\mathbb{F}[G_{F,S}]$-module M, we define $M(i) := M \otimes \bar{\chi}_{\mathrm{cyc}}^i$ and denote by $M^\vee$ the Cartier dual of M as an $\mathbb{F}[G_{F,S}]$-module, i.e., $M^\vee = \mathrm{Hom}_\mathbb{F}(M, \mathbb{F})(1)$.

Example 5.1 Any simple Lie algebra is self-dual via the Killing form. This often proves $\mathrm{ad}_{\bar{\rho}}^{(0)} \cong (\mathrm{ad}_{\bar{\rho}}^{(0)})^\vee$. For instance consider $d = \det \colon \mathcal{G} = \mathrm{GL}_N \to \mathrm{GL}_1$. If $\ell \nmid N$, then $\mathfrak{g}^0$ is simple, and so $(\mathrm{ad}_{\bar{\rho}}^0)^\vee \cong \mathrm{ad}_{\bar{\rho}}^0$. This self-duality can be realized quite explicitly by the perfect trace pairing $(A, B) \mapsto \mathrm{Tr}(AB)$ on $M_N(\mathbb{F})$ (which also shows that $\mathrm{ad}_{\bar{\rho}}$ is self-dual for $\mathcal{G} = \mathrm{GL}_N$). If $\ell \nmid N$ this pairing restricts to a non-degenerate pairing on $\mathfrak{g}^0(\mathbb{F})$. For $\ell | N$, the pairing pairing is degenerate on the traceless matrices $M_N^0(\mathbb{F})$, but induces a non-degenerate pairing on $M_N^0(\mathbb{F})$ modulo the subrepresentation of scalar matrices.

The obvious pairing $M \times M^\vee \to \mathbb{F}(1)$ yields the perfect Tate duality pairing

$$H^{2-i}(G_\nu, M) \times H^i(G_\nu, M^\vee) \to H^2(G_\nu, \mathbb{F}(1)) \cong \mathbb{F},$$

$i \in \{0,1,2\}$. Applied to $M = \mathrm{ad}_{\bar\rho}$, one defines $L_\nu^\perp \subset H^1(G_\nu, \mathrm{ad}_{\bar\rho}^\vee)$ as the annihilator of $L_\nu \subset H^1(G_\nu, \mathrm{ad}_{\bar\rho})$ under this pairing for $M = \mathrm{ad}_{\bar\rho}$, and one sets $\mathcal{L}^\perp := (L_\nu^\perp)_{\nu \in S}$. For $M = \mathrm{ad}_{\bar\rho}^0$, one defines $L_\nu^{\eta,\perp}$ as the annihilator of L_ν^0 under this pairing, and one sets $\mathcal{L}^{\eta,\perp} := (L_\nu^{\eta,\perp})_{\nu \in S}$. It is now standard to define the Selmer group $H^1_{\mathcal{L}}(G_{F,S}, \mathrm{ad}_{\bar\rho})$ as the pullback of the diagram

$$
\begin{array}{ccc}
H^1_{\mathcal{L}}(G_{F,S}, \mathrm{ad}_{\bar\rho}) & - - - - \rightarrow & \bigoplus_{\nu \in S} L_\nu \\
\cap \downarrow\uparrow & & \uparrow \\
H^1(G_{F,S}, \mathrm{ad}_{\bar\rho}) & \xrightarrow{\mathrm{res}} & \bigoplus_{\nu \in S} H^1(G_\nu, \mathrm{ad}_{\bar\rho}),
\end{array}
$$

where the lower horizontal map is the restriction on cohomology. The analogous diagram with $\mathrm{ad}_{\bar\rho}^\vee$ in place of $\mathrm{ad}_{\bar\rho}$ and $L_\nu^\perp$ in place of L_ν defines the dual Selmer group $H^1_{\mathcal{L}^\perp}(G_{F,S}, \mathrm{ad}_{\bar\rho}^\vee)$. By analogy, we define $H^1_{\mathcal{L}}(G_{F,S}, \mathrm{ad}_{\bar\rho})^\eta$ as the pullback of the diagram

$$
\begin{array}{ccc}
H^1_{\mathcal{L}}(G_{F,S}, \mathrm{ad}_{\bar\rho})^\eta & - - - - \rightarrow & \bigoplus_{\nu \in S} L_\nu^\eta \\
\cap \downarrow\uparrow & & \uparrow \\
H^1(G_{F,S}, \mathrm{ad}_{\bar\rho})^\eta & \xrightarrow{\mathrm{res}} & \bigoplus_{\nu \in S} H^1(G_\nu, \mathrm{ad}_{\bar\rho})^\eta.
\end{array}
$$

The space $H^1_{\mathcal{L}}(G_{F,S}, \mathrm{ad}_{\bar\rho})^{(\eta)}$ is readily identified with the tangent space of $\widetilde{R}^{(\eta)}_{S,\mathcal{O}}$. For its dimension we write $\widetilde{h}^{(\eta)}$. Thus we have presentations:

$$(7) \qquad\qquad 0 \longrightarrow \widetilde{J}^{(\eta)} \longrightarrow \mathcal{O}[[T_1, \ldots, T_{\widetilde{h}^{(\eta)}}]] \longrightarrow \widetilde{R}^{(\eta)}_{S,\mathcal{O}} \longrightarrow 0.$$

Note that $\mathrm{Im}(H^0(G_{F,S}, \mathrm{ad}_{\bar\rho}/\mathrm{ad}_{\bar\rho}^0) \to H^1(G_{F,S}, \mathrm{ad}_{\bar\rho}^0))$ injects under the canonical restriction homomorphism into each of the $H^1(G_\nu, \mathrm{ad}_{\bar\rho}^0)$. From this and our definition of the L_ν^0, one deduces that there is a short exact sequence

$$(8) \quad 0 \to \mathrm{Im}(H^0(G_{F,S}, \mathrm{ad}_{\bar\rho}/\mathrm{ad}_{\bar\rho}^0) \to H^1(G_{F,S}, \mathrm{ad}_{\bar\rho}^0)) \to H^1_{\mathcal{L}^0}(G_{F,S}, \mathrm{ad}_{\bar\rho}^0) \to H^1_{\mathcal{L}}(G_{F,S}, \mathrm{ad}_{\bar\rho})^\eta \to 0.$$

For the proof of Theorem 5.2 below, we recall the following consequence of Poitou-Tate global duality, [18], (8.6.20): For $M \in \{\mathrm{ad}_{\bar\rho}, \mathrm{ad}_{\bar\rho}^0\}$ and $\mathcal{L}$ the usual $\mathcal{L}$ or $\mathcal{L}^0$, respectively, there is a five term exact sequence

$$0 \longrightarrow H^1_{\mathcal{L}}(G_{F,S}, M) \longrightarrow H^1(G_{F,S}, M) \longrightarrow \bigoplus_{\nu \in S} H^1(G_\nu, M)/L_\nu$$

$$\longrightarrow H^1_{\mathcal{L}^\perp}(G_{F,S}, M^\vee)^* \longrightarrow \text{III}^2_S(M) \longrightarrow 0.$$

By our definition of L_ν^0, we have $H^1(G_\nu, \mathrm{ad}_{\bar\rho}^0)/L_\nu^0 \cong H^1(G_\nu, \mathrm{ad}_{\bar\rho})^\eta/L_\nu^\eta$. From the exact sequence (8) and the above 5-term sequence we thus obtain the 5-term sequence

$$0 \longrightarrow H^1_{\mathcal{L}}(G_{F,S}, \mathrm{ad}_{\bar\rho})^{(\eta)} \longrightarrow H^1(G_{F,S}, \mathrm{ad}_{\bar\rho})^{(\eta)} \longrightarrow \bigoplus_{\nu \in S} H^1(G_\nu, \mathrm{ad}_{\bar\rho})^{(\eta)}/L_\nu^{(\eta)}$$

(9)

$$\longrightarrow H^1_{\mathcal{L}^{(\eta),\perp}}(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee)^* \longrightarrow \text{III}^2_S(\mathrm{ad}_{\bar\rho}^{(0)}) \longrightarrow 0.$$

As in Section 3, one can compare local and global presentations of deformation rings also for the more restricted deformation problems.

$$0 \longrightarrow \langle \widetilde{J}_\nu^{(\eta)} : \nu \in S \rangle \longrightarrow \mathcal{O}[[T_{\nu,1}, \ldots, T_{\nu,\widetilde{h}_\nu^{(\eta)}} \mid \nu \in S]] \longrightarrow \hat{\otimes}_{\nu \in S} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

$$0 \longrightarrow \widetilde{J}^{(\eta)} \longrightarrow \mathcal{O}[[T_1, \ldots, T_{\widetilde{h}^{(\eta)}}]] \longrightarrow \widetilde{R}_{S,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

Theorem 5.2 *As an ideal, $\widetilde{J}^{(\eta)}$ is generated by the images of the ideals $\widetilde{J}_\nu^{(\eta)}$, $\nu \in S$ together with at most $\dim_{\mathbb{F}} H^1_{\mathcal{L}^{(\eta)}, \perp}(G_{F,S}, \mathrm{ad}_{\bar{\rho}}^{(0)})$ other elements. In particular*

$$\mathrm{gen}(\widetilde{J}^{(\eta)}) \leq \sum_{\nu \in S} \mathrm{gen}(\widetilde{J}_\nu^{(\eta)}) + \dim_{\mathbb{F}} H^1_{\mathcal{L}^{(\eta)}, \perp}(G_{F,S}, \mathrm{ad}_{\bar{\rho}}^{(0)}).$$

PROOF: Let us first consider the local situation. The following diagram compares the local presentations (3) and (6) for the functors $\mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)}$ and $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{(\eta)}$, respectively:

$$0 \longrightarrow J_\nu^{(\eta)} \longrightarrow \mathcal{O}[[T_{\nu,1}, \ldots, T_{\nu,h_\nu^{(\eta)}}]] \longrightarrow R_{\nu,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

$$0 \longrightarrow \widetilde{\mathcal{J}}_\nu^{(\eta)} \longrightarrow \mathcal{O}[[T_{\nu,1}, \ldots, T_{\nu,h_\nu^{(\eta)}}]] \longrightarrow \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

$$0 \longrightarrow \widetilde{J}_\nu^{(\eta)} \longrightarrow \mathcal{O}[[T_{\nu,1}, \ldots, T_{\nu,\widetilde{h}_\nu^{(\eta)}}]] \longrightarrow \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)} \longrightarrow 0.$$

The ideal $\widetilde{\mathcal{J}}_\nu^{(\eta)}$ is the kernel of the composite $\mathcal{O}[[T_{\nu,1}, \ldots, T_{\nu,h_\nu^{(\eta)}}]] \to R_{\nu,\mathcal{O}}^{(\eta)} \to \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}$. The epimorphism π_ν is chosen so that the lower right square commutes. We may rearrange the variables in such a way that π_ν is concretely given by mapping $T_{\nu,i}$ to $T_{\nu,i}$, for $i \leq \widetilde{h}_\nu^{(\eta)}$, and by mapping $T_{\nu,i}$ to zero for $i > \widetilde{h}_\nu^{(\eta)}$. Let moreover denote by $f_{\nu,1}, \ldots, f_{\nu,r_\nu}$ a minimal set of generators of $\widetilde{J}_\nu^{(\eta)}$. Then a set of generators of $\widetilde{\mathcal{J}}_\nu^{(\eta)}$ is formed by the elements

$$f_{\nu,1}, \ldots, f_{\nu,r_\nu}, T_{\nu,\widetilde{h}_\nu^{(\eta)}+1}, \ldots, T_{\nu,h_\nu^{(\eta)}}.$$

Now we turn to the global situation. By Theorem 3.1 the relation ideal in the presentation (4) of $R_{S,\mathcal{O}}^{(\eta)}$ is generated by local relations together with at most $r := \dim_{\mathbb{F}} \mathrm{III}_S^2(M)$ further elements $f_1, \ldots, f_r$. Let the α_ν and $\alpha = \prod_\nu \alpha_\nu$ be homomorphisms as in diagram (5). For the ring $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ we have the following two presentations. First, since $\widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{(\eta)} \subset \mathrm{Def}_{S,\mathcal{O}}^{(\eta)}$ is defined by imposing local conditions, we may take the presentation of $R_{S,\mathcal{O}}^{(\eta)}$ and consider its quotient by further local relations. Second, we have the presentation (7). We obtain

$$0 \longrightarrow \langle \{\alpha_\nu(\widetilde{\mathcal{J}}_\nu^{(\eta)}); \nu \in S\} \cup \{f_1, \ldots, f_r\} \rangle \longrightarrow \mathcal{O}[[T_1, \ldots, T_{h^{(\eta)}}]] \longrightarrow \widetilde{R}_{S,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

$$0 \longrightarrow \widetilde{J}^{(\eta)} \longrightarrow \mathcal{O}[[T_1, \ldots, T_{\widetilde{h}^{(\eta)}}]] \longrightarrow \widetilde{R}_{S,\mathcal{O}}^{(\eta)} \longrightarrow 0.$$

Since $\widetilde{h}^{(\eta)} = \dim \mathfrak{t}_{\widetilde{R}_{S,\mathcal{O}}^{(\eta)}}$, the homomorphism π is surjective. By properly choosing the coordinate functions T_i, we may thus assume that π is given as $T_i \mapsto T_i$ for $i = 1\ldots, \widetilde{h}^{(\eta)}$ and $T_i \mapsto 0$ for $i > \widetilde{h}^{(\eta)}$.

To further understand π, we interprete the $\mathbb{F}$-dual of sequence (9) as an assertion on the variables of our local and global presentations. Defining Δ via

$$0 \longrightarrow \text{III}_S^2(\text{ad}_{\bar\rho}^{(0)})^* \longrightarrow H^1_{\mathcal{L}^{(\eta),\perp}}(G_{F,S}, (\text{ad}_{\bar\rho}^{(0)})^\vee) \longrightarrow \Delta \longrightarrow 0,$$

we have

$$0 \longrightarrow \Delta \longrightarrow \bigoplus_{\nu\in S}(H^1(G_\nu, M)^{(\eta)}/L_\nu^{(\eta)})^* \longrightarrow (H^1(G_{F,S}, M)^{(\eta)})^* \longrightarrow (H^1_{\mathcal{L}}(G_{F,S}, M)^{(\eta)})^* \longrightarrow 0.$$

For $R \in \mathcal{C}_\mathcal{O}$ we have $\mathfrak{t}_R^* = \mathfrak{m}_R/(\mathfrak{m}_\mathcal{O} + \mathfrak{m}_R^2)$. This gives an interpretation for the $H^1(\ldots)^*$-terms:

- The (images of the) elements $T_1, \ldots, T_{\widetilde{h}^{(\eta)}}$ form an $\mathbb{F}$-basis of $(H^1_{\mathcal{L}}(G_{F,S}, M)^{(\eta)})^*$.

- The (images of the) elements $T_1, \ldots, T_{h^{(\eta)}}$ form an $\mathbb{F}$-basis of $(H^1(G_{F,S}, M)^{(\eta)})^*$.

- The (images of the) elements $T_{\nu, l_\nu^{(\eta)}+1}, \ldots, T_{\nu, h_\nu^{(\eta)}}$ form an $\mathbb{F}$-basis of $(H^1(G_\nu, M)^{(\eta)}/L_\nu^{(\eta)})^*$.

Thus in the set $V := \bigcup_{\nu\in S}\alpha_\nu(\{T_{\nu, \widetilde{h}_\nu^{(\eta)}+1}, \ldots, T_{\nu, h_\nu^{(\eta)}}\})$ we may choose $h^{(\eta)} - \widetilde{h}^{(\eta)}$ many elements which form a basis of the $\mathbb{F}$-span of $\{T_{\widetilde{h}^{(\eta)}+1}, \ldots, T_{h^{(\eta)}}\}$. Using the freedom we have in choosing the variables $T_{\widetilde{h}^{(\eta)}+1}, \ldots, T_{h^{(\eta)}}$, we may assume that these are precisely the chosen ones from V. Hence under π, these chosen variables all map to zero.

We may therefore conclude the following: The ideal $\widetilde{J}^{(\eta)}$ is spanned by the images of the relations $f_{\nu,j}$, $\nu \in S$, $j = 1, \ldots, l_\nu^{(\eta)}$, i.e., the local relations in a minimal presentation of $\widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}$, together with the images of the elements f_j, $j = 1, \ldots, r$, and together with the

$$d := \sum_{\nu\in S}(h_\nu^{(\eta)} - \widetilde{h}_\nu^{(\eta)}) - (h^{(\eta)} - \widetilde{h}^{(\eta)})$$

further elements in V which may or may not map to zero under π. Since $d = \dim_{\mathbb{F}} \Delta$, and $d + r = \dim_{\mathbb{F}} H^1_{\mathcal{L}^{(\eta),\perp}}(G_{F,S}, \text{ad}_{\bar\rho}^{(0)})$, the assertion of the theorem is shown. ∎

Corollary 5.3 *For the presentation*

$$0 \longrightarrow \widetilde{J}^{(\eta)} \longrightarrow \mathcal{O}[[T_1, \ldots, T_{\widetilde{h}^{(\eta)}}]] \longrightarrow \widetilde{R}_{S,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

one has

$$\widetilde{h}^{(\eta)} - \text{gen}(\widetilde{J}^{(\eta)}) \geq h^0(G_{F,S}, \text{ad}_{\bar\rho}^{(0)}) - h^0(G_{F,S}, (\text{ad}_{\bar\rho}^{(0)})^\vee) - \delta(G_{F,S}, \text{ad}_{\bar\rho})^{(\eta)}$$

$$(10) \qquad\qquad + \sum_{\nu\in S}\left(\widetilde{h}_\nu^{(\eta)} + \delta(G_\nu, \text{ad}_{\bar\rho})^{(\eta)} - h^0(G_\nu, \text{ad}_{\bar\rho}^{(0)}) - \text{gen}(\widetilde{J}_\nu^{(\eta)})\right).$$

PROOF: Following Wiles, cf. [18] (8.6.20), and using (8) we have

$$
\begin{aligned}
\widetilde{h}^{(\eta)} &+ \delta(G_{F,S}, \mathrm{ad}_{\bar{\rho}})^{(\eta)} - \dim_{\mathbb{F}} H^1_{\mathcal{L}^{(\eta)}, \perp}(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^{(0)})^{\vee}) \\
&= h^0(G_{F,S}, \mathrm{ad}_{\bar{\rho}}^{(0)}) - h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^{(0)})^{\vee}) + \sum_{\nu \in S} (\widetilde{h}_{\nu}^{(0)} - h^0(G_{\nu}, \mathrm{ad}_{\bar{\rho}}^{(0)})).
\end{aligned}
$$

By our definition of $\widetilde{h}_{\nu}^{(0)}$ we have $\widetilde{h}_{\nu}^{(0)} = \widetilde{h}_{\nu}^{(\eta)} + \delta(G_{\nu}, \mathrm{ad}_{\bar{\rho}})^{(\eta)}$. Subtracting the bound for $\mathrm{gen}(J^{(\eta)})$ from Theorem 5.2 from the quantity $\widetilde{h}^{(\eta)}$ yields the desired estimate.   ∎

Remark 5.4 Because of Remark 3.2, we expect the above estimate to be optimal in the case that $h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^{(0)})^{\vee}) = 0$. If F contains ℓ-th roots of unity, the same remark shows that for $\mathrm{ad}_{\bar{\rho}}$ the above estimate will not be optimal. (For $\mathcal{G} = \mathrm{GL}_1$, i.e, for class field theory, the reader may easily verify this.) If $\mathrm{ad}_{\bar{\rho}} = \mathrm{ad}_{\bar{\rho}}^0 \oplus \mathbb{F}$ this problem can be remedied since then the universal ring $\widetilde{R}_{S,\mathcal{O}}$ is the completed tensor product of $\widetilde{R}_{S,\mathcal{O}}^{\eta}$ with the deformation ring for one-dimensional representations. By class field theory (and Leopoldt's conjecture) the latter is well-understood.

6   General remarks and the case $\mathcal{G} = \mathrm{GL}_2$

The aim of this section is to analyze the terms occurring in estimate 10 given in Corollary 5.3 for the number of variables minus the number of relations in a presentation of $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$. After some initial general remarks we shall soon focus on the case $\mathcal{G} = \mathrm{GL}_2$. The main result is Theorem 6.8.

For many naturally defined subfunctors $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{(\eta)} \subset \mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)}$ (for $\nu \in S$) (for instance for the examples presented below) one has the following:

(i) If $\nu \nmid \ell$, then $\widetilde{h}_{\nu}^{(0)} - h^0(G_{\nu}, \mathrm{ad}_{\bar{\rho}}^{(0)}) - \mathrm{gen}(\widetilde{J}_{\nu}^{(\eta)}) \geq 0$.

(ii) If one imposes a suitable semistability condition on deformations at places $\nu | \ell$, and a suitable parity condition at places above ∞, then

$$
\sum_{\nu | \ell \, \mathrm{or} \, \nu | \infty} \left(\widetilde{h}_{\nu}^{(0)} - h^0(G_{\nu}, \mathrm{ad}_{\bar{\rho}}^{(0)}) - \mathrm{gen}(\widetilde{J}_{\nu}^{(\eta)}) \right) \geq 0.
$$

The estimate in (i) is typically easy to achieve, and without any requirements on the restriction of $\bar{\rho}$ to G_{ν}. This is presently not so for (ii) at places $\nu | \ell$: If $\bar{\rho}$ satisfies some ordinariness condition at ν, then the ring parameterizing deformations satisfying a similar ordinariness conditions is relatively well understood. If on the other hand $\bar{\rho}$ is flat at ν, then suitable deformation rings are only well understood and well-behaved if the order of ramification of $\bar{\rho}$ at ν is relatively small.

We now turn to some examples, first for the local situation:

Example 6.1 $\nu \nmid \ell, \infty$:

(a) At such places one has

$$h^1(G_\nu, M) - h^0(G_\nu, M) - h^2(G_\nu, M) = 0$$

for the local Euler-Poincaré characteristic for any finite $\mathbb{F}[G_\nu]$-module M. Thus for $\mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)}$ on obtains $\widetilde{h}_\nu^{(0)} - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) - \mathrm{gen}(J_\nu^{(\eta)}) \geq 0$.

(b) For the local deformation problems defined by Ramakrishna in [19], Prop. 1, p. 122, the ring $\widetilde{R}_{\nu,\mathcal{O}}^{\eta}$ is smooth over $\mathcal{O}$ of relative dimension $h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) = h^1(G_\nu, \mathrm{ad}_{\bar\rho}^0) - h^2(G_\nu, \mathrm{ad}_{\bar\rho}^0)$ over $\mathcal{O}$; cf. the remark in [19], p. 124. Here $\widetilde{h}_\nu^0 - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) - \mathrm{gen}(J_\nu^{\eta}) = 0$.

(c) The local deformation problem defined in [10], Prop. 2.2, is smooth of relative dimension 1 over $\mathcal{O}$ and again one has $\widetilde{h}_\nu^0 - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) - \mathrm{gen}(J_\nu^{\eta}) = 0$.

(d) The local deformation problem in [6], p. 141, in the definition of $R^\flat$ at places $\nu \in P$, i.e., at prime number p with $p \equiv -1 \pmod{\widetilde{h}}$ again defines a local deformation problem with versal representing ring smooth of relative dimension 1 over $\mathcal{O}$. As in the previous cases one has $\widetilde{h}_\nu^0 - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) - \mathrm{gen}(J_\nu^{\eta}) = 0$.

Remark 6.2 Let $\bar\rho \colon G_{F_\nu} \to \mathrm{GL}_2(\mathbb{F})$ be arbitrary. Building on previous work by Boston, Mazur and Taylor-Wiles it is shown for $\nu \nmid \ell$ and in [2] that the Krull dimension of $R_{\nu,\mathcal{O}}^{\eta}$ is for any choice of $\bar\rho$ equal to $h^1(G_\nu, \mathrm{ad}_{\bar\rho}^0) - h^2(G_\nu, \mathrm{ad}_{\bar\rho}^0)$. In [2] the same is shown for $\nu | \ell$ for all possible $\bar\rho$. Thus by Remark 2.3 in these cases the rings $R_{\nu,\mathcal{O}}^{\eta}$ are known to be complete intersections of the expected dimensions. So the estimate in part (a) above is optimal for $d = \det \colon \mathcal{G} = \mathrm{GL}_2 \to \mathcal{T} = \mathrm{GL}_1$. For parts (b), (c) and (d) the same can be shown (e.g. by explicit calculation).

Example 6.3 $\nu | \infty$, $\ell > 2$: If ν is real, then G_ν is generated by a complex conjugation c_ν (of order 2). For $\ell > 2$ and $R \in \mathcal{C}_\mathcal{O}$, the group ring $R[\mathbb{Z}/(2)]$ has idempotents for the two R-projective irreducible representations of c_ν. Hence for any deformation $[\rho]$ of $\bar\rho$ to R, the lift of $\bar\rho_{G_\nu}$ is unique up to isomorphism. Therefore $\widetilde{h}_\nu^{(\eta)} = 0 = \mathrm{gen}(J_\nu^{(\eta)})$, and $h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)})$ depends on the action of c_ν on $\mathrm{ad}_{\bar\rho}^{(0)}$, more precisely on the conjugacy class of $\bar\rho(c_\nu)$. For $\ell = 2$ the problem is more subtle, cf. Example 6.4.

If ν is complex, then G_ν acts trivially, and again $\widetilde{h}_\nu^{(\eta)} = 0 = \mathrm{gen}(J_\nu^{(\eta)})$ (this also holds for $\ell = 2$). Clearly one has $h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) = \dim_\mathbb{F} \mathrm{ad}_{\bar\rho}^{(0)}$

For cases with $\nu | \ell$ we refer to Examples 6.5 and 7.1. For a case with $\nu | \infty$ and $\ell = 2$, we refer to Example 6.4.

For the remainder of this section, we assume that $d = \det \colon \mathcal{G} = \mathrm{GL}_2 \to \mathcal{T} = \mathrm{GL}_1$.

One calls a residual representation $\bar\rho$ **odd**, if for any real place ν of F one has $\det \bar\rho(c_\nu) = -1$. Note that for $\ell = 2$, the condition $\det \bar\rho(c_\nu) = -1$ is vacuous.

Example 6.4 Suppose ν is real and $\bar\rho$ is odd. If $\ell \neq 2$, then $h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) = 1$, and so from the remarks in Example 6.3 it is clear that

$$\widetilde{h}_\nu^0 - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) - \mathrm{gen}(J_\nu^{\eta}) = -1.$$

If $\ell = 2$, the main interest lies in deformation which are odd, i.e., for which the image of c_ν is non-trivial. The following two cases are the important ones for $\mathcal{G} = \mathrm{GL}_2$, and say with η fixed:

Case I: $\bar{\rho}(c_\nu)$ is conjugate to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. The versal hull for this problem is then given by

$$\rho_{\nu,\mathcal{O}}^{\eta} : \mathbb{Z}/(2) = \langle c_\nu \rangle \longrightarrow \mathrm{GL}_2(\mathcal{O}) : c_\nu \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right).$$

We have $\widetilde{h}_\nu^\eta = 0$, $h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) = 2$ and $\mathrm{gen}(J_\nu^\eta) = 0$, so that

$$\delta(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta + \widetilde{h}_\nu^\eta - h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) - \mathrm{gen}(J_\nu^\eta) = 1 + 0 - 2 - 0 = -1.$$

Case II: $\bar{\rho}(c_\nu)$ is conjugate to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then the versal hull of a good deformation problem at ν (so that the deformations are odd whenever this is reasonable) is given by

$$\rho_{\nu,\mathcal{O}}^{\eta} : \mathbb{Z}/(2) = \langle c_\nu \rangle \longrightarrow \mathrm{GL}_2(\mathcal{O}[[a,b,c]]/(a^2 + 2a + bc)) : c_\nu \mapsto \left(\begin{smallmatrix} 1+a & b \\ c & -1-a \end{smallmatrix}\right).$$

We have $\widetilde{h}_\nu^\eta = 3$, $h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) = 3$, $\mathrm{gen}(J_\nu^\eta) = 1$ and (!) $\delta(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta = 0$, so that

$$\delta(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta + \widetilde{h}_\nu^\eta - h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) - \mathrm{gen}(J_\nu^\eta) = 0 + 3 - 3 - 1 = -1.$$

Example 6.5 We now turn to the case $\nu | \ell$.
Case I: $F_\nu = \mathbb{Q}_\ell$, $h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) = 0$, and $\bar{\rho}_{G_K}$ is flat at ν for some finite extension K of $\mathbb{Q}_\ell$ of ramification degree at most $\ell - 1$ so that the corresponding group scheme and its Cartier dual are both connected. Then by [5] and [20], one has $\widetilde{R}_{\nu,\mathcal{O}}^{\eta} \cong \mathcal{O}[[T]]$ for a suitable flat deformation functor $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{\eta}$. Because $\ell \neq 2$ by Remark 2.5 we have $\delta(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta = 0$. Hence

$$\delta(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta + \widetilde{h}_\nu^\eta - h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) - \mathrm{gen}(J_\nu^\eta) = 1 = [F_\nu : \mathbb{Q}_\ell].$$

Case II: $\bar{\rho}$ is ordinary at ν. We recall the computation of the obstruction theoretic invariants for $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{\eta} \subset \mathrm{Def}_{\nu,\mathcal{O}}^{\eta}$ (much of these computations is contained in work of Mazur and Wiles): Suppose we are given a residual representation

$$(11) \qquad \bar{\rho} : G_\nu \to \mathrm{GL}_2(\mathbb{F}) : \sigma \mapsto \left(\begin{smallmatrix} \bar{\chi} & \bar{b} \\ 0 & \bar{\chi}^{-1}\bar{\eta}_\nu \end{smallmatrix}\right),$$

where $\bar{\chi}$ is unramified, and where $\bar{\eta}_\nu$ denotes the mod $\mathfrak{m}_\mathcal{O}$ reduction of $\eta_\nu = \eta_{|G_\nu}$.
We make the following (standard) hypotheses: The image $\mathrm{Im}(\bar{\rho})$ is not contained in the set of scalar matrices, and, if $\bar{\chi} = \bar{\chi}^{-1}\bar{\eta}_\nu$, then (after possibly twisting by a character) we assume that $\bar{\chi} = \bar{\chi}^{-1}\bar{\eta}_\nu$ is the trivial character. In particular this means that if $\bar{b} = 0$, then $\bar{\chi}^2 \neq \bar{\eta}_\nu$.
By an ordinary lift of fixed determinant we mean a lift of the form

$$\rho : G_\nu \to \mathrm{GL}_2(R) : \sigma \mapsto \left(\begin{smallmatrix} \chi & b \\ 0 & \chi^{-1}\bar{\eta}_\nu \end{smallmatrix}\right),$$

where χ is unramified. Since $\mathrm{Im}(\bar{\rho})$ is not contained in the set of scalar matrices, passing to strict equivalence classes of such lifts defines a relatively representable subfunctor $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{\eta} \subset \mathrm{Def}_{\nu,\mathcal{O}}^{\eta}$ in the sense of Definition 4.1.
To compute the mod $\mathfrak{m}_\mathcal{O}$ tangent space of the corresponding ring $\widetilde{R}_{\nu,\mathcal{O}}^{\eta}$, and a bound on the number of relations in a minimal presentation we distinguish several subcases: (i) $\ell = 2$ (ii) $\ell \neq 2$ and $\bar{\chi}^2 \neq \bar{\eta}_\nu$, (iii) $\ell \neq 2$ and $\bar{\chi}^2 = \bar{\eta}_\nu$ (and so by our assumptions on $\bar{\rho}$, we have $\bar{\chi} = \bar{\eta}_\nu = 1$.).

Subcase (i), $\ell = 2$: Let us denote by ρ a lift to $\mathbb{F}[\varepsilon]/(\varepsilon^2)$, and use $\bar{\rho}$ also to denote the trivial lift. Then

$$(12) \qquad \sigma \mapsto \rho(\sigma)\bar{\rho}(\sigma)^{-1} = \begin{pmatrix} \chi(\sigma)\bar{\chi}^{-1}(\sigma) & \bar{\eta}_\nu^{-1}(\sigma)(b(\sigma)\bar{\chi}(\sigma) - \bar{\chi}(\sigma)b(\sigma)) \\ 0 & \chi^{-1}(\sigma)\bar{\chi}(\sigma) \end{pmatrix} =: \begin{pmatrix} 1 + \varepsilon c_1(\sigma) & \varepsilon c_2(\sigma) \\ 0 & 1 - \varepsilon c_1(\sigma) \end{pmatrix}$$

defines a 1-cocycle into the upper triangular matrices in $\mathrm{ad}_{\bar{\rho}}^0$. Because we assume $\ell = 2$, one may in fact verify that the matrix entries c_1 and c_2 are also 1-cocycles for a suitable module. In fact they yield classes $[c_1] \in H^1(G_\nu/I_\nu, \mathbb{F})$ and $[c_2] \in H^1(G_\nu, \bar{\chi}^2\bar{\eta}_\nu^{-1})$. Since $G_\nu/I_\nu \cong \hat{\mathbb{Z}}$, one has $H^1(G_\nu/I_\nu, \mathbb{F}) \cong \mathbb{F}$.

If ρ and ρ' are lifts to $\mathbb{F}[\varepsilon]/(\varepsilon^2)$ of the required form, such that ρ' and ρ are conjugate by $1 + \varepsilon a$ for some $a \in \mathrm{ad}_{\bar{\rho}}^0$ which is upper triangular, then 1-cocycles for ρ and ρ' give rise to the same cohomology classes. Conversely, if to a given pair of classes, one chooses different 1-cocycles, the resulting lifts ρ', ρ differ by conjugation by a $1 + \varepsilon a$ for some $a \in \mathrm{ad}_{\bar{\rho}}^0$ which is upper triangular.

Regarding strict equivalence one has the following easy if tedious result: For arbitrary $a \in \mathrm{ad}_{\bar{\rho}}^0$ the conjugate of any lift ρ to $\mathbb{F}[\varepsilon]/(\varepsilon^2)$ under $1 + \varepsilon a$ is again a lift of the required form if and only if one of the following happens:

(a) $a \in \mathrm{ad}_{\bar{\rho}}^0$ is arbitrary if $\bar{\chi}^2 = \bar{\eta}_\nu$ and $\bar{b}(I_\nu) = \{0\}$

(b) $a \in \mathrm{ad}_{\bar{\rho}}^0$ is upper triangular otherwise.

Case (a) means that the image of $\bar{\rho}$ is an ℓ-group and that $\bar{\rho}$ is unramified. We define $\delta_\nu^{\ell,\mathrm{unr}}$ to be 1 in case (a) and 0 in case (b). Then we have

$$\widetilde{h}_\nu^0 = \dim_{\mathbb{F}} \mathfrak{t}_{\widetilde{R}_{\nu,\mathcal{O}}^\eta} = 1 + h^1(G_\nu, \bar{\chi}^2\bar{\eta}_\nu^{-1}) - \delta_\nu^{\ell,\mathrm{unr}}.$$

Similarly, one can compute the obstruction to further lift a representation

$$\rho \colon G_\nu \to \mathrm{GL}_2(R) \colon \sigma \mapsto \begin{pmatrix} \chi & b \\ 0 & \chi^{-1}\eta_\nu \end{pmatrix}$$

to a representation ρ' given by $\begin{pmatrix} \chi' & b \\ 0 & (\chi')^{-1}\eta_\nu \end{pmatrix}$ for a small surjection $R' \to R$. Letting χ' be an unramified character which lifts χ (and always exists since $G_\nu/I_\nu \cong \hat{\mathbb{Z}}$ is of cohomological dimension one) and b' a set-theoretic continuous lift, as is standard, one shows that

$$(s,t) \mapsto \rho'(st)\rho'(t)^{-1}\rho'(s)^{-1} =: \begin{pmatrix} 1 & c_{s,t} \\ 0 & 1 \end{pmatrix}$$

defines a 2-cocycle of G_ν with values in $\bar{\chi}^2\bar{\eta}_\nu^{-1}$, and so we obtain a class in $H^2(G_\nu, \bar{\chi}^2\bar{\eta}_\nu^{-1})$. This gives the bound $\mathrm{gen}(\widetilde{J}_{\nu,\mathcal{O}}^\eta) \le h^2(G_\nu, \bar{\chi}^2\bar{\eta}_\nu^{-1})$.

As a last ingredient, we compute $h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0)$. This leads to the identity

$$\bar{\rho}\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\bar{\rho}^{-1} = \begin{pmatrix} \alpha + \gamma\bar{b}\bar{\chi}^{-1} & -2\alpha\bar{\chi}\bar{b}\bar{\eta}_\nu^{-1} + \beta\bar{\chi}^2\bar{\eta}_\nu^{-1} - \gamma\bar{b}^2\bar{\eta}_\nu^{-1} \\ \gamma\bar{\chi}^{-2}\bar{\eta}_\nu & -\alpha - \gamma\bar{b}\bar{\chi}^{-1} \end{pmatrix} \overset{!}{=} \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix},$$

and so gives the conditions

$$\gamma\bar{b} = 0, \gamma(1 - \bar{\eta}_\nu\bar{\chi}^{-2}) = 0, \beta(1 - \bar{\eta}_\nu\bar{\chi}^{-2}) = 2\alpha\bar{\chi}^{-1}\bar{b} = 0.$$

Since under our hypotheses we cannot have $\bar{b} = 0$ and $\bar{\eta}_\nu = \bar{\chi}^2$ simultaneously, we obtain $\gamma = 0$. From the last condition we see that the vanishing of β depends on $\bar{\eta}_\nu = \bar{\chi}^2$ or not. So we find $h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) = 1 + h^0(G_\nu, \bar{\chi}^2\bar{\eta}_\nu^{-1})$.

Using the formula for the local Euler-Poincaré characteristic at a place $\nu | \ell$ one obtains $\sum_{i=0}^{2} h^i(G_\nu, \bar\chi^2 \bar\eta_\nu^{-1}) = -[F_\nu : \mathbb{Q}_\ell]$. Hence

$$
\begin{aligned}
\widetilde{h}_\nu^0 - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) - \mathrm{gen}(J_\nu^\eta) \;&\geq\; h^0(G_\nu, \bar\chi^2 \bar\eta_\nu^{-1}) - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) + [F_\nu : \mathbb{Q}_\ell] + 1 - \delta_\nu^{\ell,\mathrm{unr}} \\
&=\; [F_\nu : \mathbb{Q}_\ell] - \delta_\nu^{\ell,\mathrm{unr}}
\end{aligned}
$$

for the corresponding ring $\widetilde{R}_{\nu,\mathcal{O}}^\eta$.

From now on, we assume $\ell \neq 2$. In this case $\delta(G_\nu, \mathrm{ad}_{\bar\rho})^\eta = 0$ by Remark 2.5, and so $\widetilde{h}_\nu^0 = \widetilde{h}_\nu^\eta$. Now for $\ell \neq 2$, the 1-cocycle defined in (12) cannot be decomposed in two independent 1-cocycles, and so one proceeds differently: Let $(\mathfrak{n} \subset) \mathfrak{b} \subset \mathrm{ad}_{\bar\rho}^0$ denote the subrepresentations on (strictly) upper triangular matrices of $\mathrm{ad}_{\bar\rho}^0$. Following Wiles, we see that the cocycle defines a cohomology class in

$$
H_{\mathrm{str}}^1 := \mathrm{Ker}(H^1(G_\nu, \mathfrak{b}) \longrightarrow H^1(I_\nu, \mathfrak{b}/\mathfrak{n})).
$$

One observes that $H^1(G_\nu, \mathfrak{b}) \longrightarrow H^1(I_\nu, \mathfrak{b}/\mathfrak{n})$ factors via $H^1(G_\nu, \mathfrak{b}/\mathfrak{n})$, and that the action of G_ν on $\mathfrak{b}/\mathfrak{n} \cong \mathbb{F}$ is trivial. Using the left exact inflation-restriction sequence one finds that H_{str}^1 is the pullback of the diagram

$$
\begin{array}{c}
H^1(G_\nu/I_\nu, \mathfrak{b}/\mathfrak{n}) \\
\downarrow \\
H^1(G_\nu, \mathfrak{b}) \longrightarrow H^1(G_\nu, \mathfrak{b}/\mathfrak{n}).
\end{array}
$$

Case (ii), $\ell \neq 2$ and $\bar\chi^2 \neq \bar\eta_\nu$. We claim that $H^1(G_\nu, \mathfrak{b}) \to H^1(G_\nu, \mathfrak{b}/\mathfrak{n})$ is surjective: Using the long exact sequence of cohomology it suffices to show that

$$
H^2(G_\nu, \mathfrak{n}) \longrightarrow H^2(G_\nu, \mathfrak{b}) \longrightarrow H^2(G_\nu, \mathfrak{b}/\mathfrak{n}) \longrightarrow 0
$$

is also exact on the left. Using Tate local duality, one has $h^2(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) = h^0(G_\nu, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee)$. The formulas

$$
h^2(G_\nu, \mathfrak{n}) = \begin{cases} 1 & \text{if } \bar\chi^2 = \bar\eta_\nu \\ 0 & \text{else} \end{cases} \qquad
h^2(G_\nu, \mathfrak{b}/\mathfrak{n}) = \begin{cases} 1 & \text{if } \bar\chi_{\mathrm{cyc}} \text{ is trivial} \\ 0 & \text{else} \end{cases}
$$

follow readily. Representing matrices $\begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}$ in $\mathfrak{b}$ by column vectors $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, the representation of $\sigma \in G_\nu$ on $\mathfrak{b}$ is given by

$$
\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ -2\bar\chi(\sigma)\bar{b}(\sigma) & \bar\eta_\nu^{-1}(\sigma)\bar\chi^2(\sigma) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.
$$

The invariants of the $\vee$-dual of $\mathfrak{b}$ are the solutions (α, β) in $\mathbb{F}^2$ to the equations

$$
\begin{pmatrix} 1 & 2\bar\chi^{-1}(\sigma)\bar{b}(\sigma)\bar\eta_\nu(\sigma) \\ 0 & \bar\eta_\nu(\sigma)\bar\chi^{-2}(\sigma) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \bar\chi_{\mathrm{cyc}}^{-1}(\sigma) \begin{pmatrix} \alpha \\ \beta \end{pmatrix},
$$

where σ ranges over all elements of G_ν. For fixed σ, the dimension of the solution space is 2 minus the rank of the matrix

$$
\begin{pmatrix} \bar\chi_{\mathrm{cyc}}(\sigma)-1 & 2\bar\chi^{-1}(\sigma)\bar\chi_{\mathrm{cyc}}(\sigma)\bar{b}(\sigma)\bar\eta_\nu(\sigma) \\ 0 & \bar\eta_\nu(\sigma)\bar\chi^{-2}(\sigma)\bar\chi_{\mathrm{cyc}}(\sigma)-1 \end{pmatrix}.
$$

If $\bar{\chi}_{\mathrm{cyc}}(\sigma)$ is non-trivial and different from $\bar{\eta}_\nu^{-1}(\sigma)\bar{\chi}^2(\sigma)$, it follows that $H^0(G_\nu, (\mathrm{ad}_{\bar\rho}^0)^\vee) = 0$. For varying σ, the maximal rank has to be non-zero, since otherwise we would have $1 = \bar{\chi}_{\mathrm{cyc}} = \bar{\eta}_\nu^{-1}\bar{\chi}^2$, contradicting our hypotheses. One concludes that if one of the identities $1 = \bar{\chi}_{\mathrm{cyc}}$ or $1 = \bar{\eta}_\nu^{-1}\bar{\chi}^2$ holds, then the rank of $H^0(G_\nu, (\mathrm{ad}_{\bar\rho}^0)^\vee)$ is 1, and otherwise, it is 2. It follows that

$$h^2(G_\nu, \mathfrak{b}) - h^2(G_\nu, \mathfrak{n}) - h^2(G_\nu, \mathfrak{b}/\mathfrak{n}) = 0,$$

and so the claim is shown.

By the claim the horizontal homomorphism in the above pullback diagram is surjective. By the inflation restriction sequence the vertical homomorphism $H^1(G_\nu/I_\nu, \mathfrak{b}/\mathfrak{n}) \to H^1(G_\nu, \mathfrak{b}/\mathfrak{n})$ is injective. Because $H^1(G_\nu/I_\nu, \mathfrak{b}/\mathfrak{n}) \cong H^1(\hat{\mathbb{Z}}, \mathbb{F}) \cong \mathbb{F}$, we therefore deduce

$$\dim_{\mathbb{F}} H_{\mathrm{str}}^1 = h^1(G_\nu, \mathfrak{b}) - h^1(G_\nu, \mathfrak{b}/\mathfrak{n}) + 1.$$

Using the local Euler-Poincaré formula and the above results, we find

$$\begin{aligned}
\dim_{\mathbb{F}} H_{\mathrm{str}}^1 &= [F_\nu : \mathbb{Q}_\ell] + 1 + h^2(G_\nu, \mathfrak{b}) - h^2(G_\nu, \mathbb{F}) + h^0(G_\nu, \mathfrak{b}) - h^0(G_\nu, \mathbb{F}) \\
&= [F_\nu : \mathbb{Q}_\ell] + h^2(G_\nu, \mathfrak{n}) + h^0(G_\nu, \mathfrak{b}).
\end{aligned}$$

One easily shows that $h^0(G_\nu, \mathfrak{b}) = h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0)$.

To compute $\widetilde{h}_\nu^\eta$ there is as in case (i) the question about strict equivalence. The definition of H_{str}^1 only take conjugation by upper triangular elements into account. However, from $\chi^2 \neq \bar{\eta}_\nu$, one may easily deduce (as in case (i)) that conjugating by a matrix of the form $1 + \varepsilon a$, $a \in \mathrm{ad}_{\bar\rho}^0$, preserves the upper diagonal form of a lift to $\mathbb{F}[\varepsilon]/(\varepsilon^2)$ only if a lies in $\mathfrak{b}$. Hence in fact H_{str}^1 does describe the mod $\mathfrak{m}_\mathcal{O}$ tangent space of the versal hull of $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^\eta \subset \mathrm{Def}_{\nu,\mathcal{O}}^\eta$. Thus $\widetilde{h}_\nu^\eta = \dim_{\mathbb{F}} H_{\mathrm{str}}^1$.

The computation of possible obstructions proceeds as in case (i). The analysis given there does not depend on $\ell = 2$. Again one finds $\mathrm{gen}(\widetilde{J}_{\nu,\mathcal{O}}^\eta) \leq h^2(G_\nu, \mathfrak{n})$. Combining the above results, we find

$$\widetilde{h}_\nu^\eta - \mathrm{gen}(\widetilde{J}_{\nu,\mathcal{O}}^\eta) - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) \geq [F_\nu : \mathbb{Q}_\ell].$$

Case (iii), $\ell \neq 2$ and $\bar{\chi}^2 = \bar{\eta}_\nu$. In this case the image of $\bar\rho$ is an elementary abelian ℓ-group. Therefore all the lifts factor via the pro-ℓ quotient $\widehat{G}_\nu^\ell$ of G_ν. This group is known to a sufficient degree, as to yield a precise estimate for $\widetilde{h}_\nu^\eta - \mathrm{gen}(\widetilde{J}_{\nu,\mathcal{O}}^\eta) - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0)$ by direct computation. This could be deduced from [2]. But for completeness we chose to give a simple direct argument.

There are two cases. Suppose first that F_ν does not contain a primitive ℓ-th root of unity. Then by [14], Thm. 10.5, the group $\widehat{G}_\nu^\ell$ is a free pro-ℓ group on (topological) generators $s, t_1, \ldots, t_n$, $n = [F_\nu : \mathbb{Q}_\ell]$, such the normal closure of the t_i form the inertia subgroup of $\widehat{G}_\nu^\ell$. By our hypothesis on $\bar\rho$, for any lift to some ring R in $\mathcal{C}_\mathcal{O}$ one has

$$s \mapsto \begin{pmatrix} 1+\alpha & \beta \\ 0 & (1+\alpha)^{-1}\eta_\nu(s) \end{pmatrix}, t_1 \mapsto \begin{pmatrix} 1 & \tau_1 \\ 0 & \eta_\nu(t_1) \end{pmatrix}, \ldots, t_n \mapsto \begin{pmatrix} 1 & \tau_n \\ 0 & \eta_\nu(t_n) \end{pmatrix}.$$

Obviously there are no obstructions to lifting. The element α lies in $\mathfrak{m}_R$. The elements τ_i lie in $\mathfrak{m}_R$ precisely is $\bar\rho(t_i)$ is trivial. Not having taken strict equivalence into account we have therefore $n + 2$ independent variables.

The analysis of the effect of strict equivalence proceeds as in case (i) and leads to the same cases (a) and (b) as described there. Hence one finds

$$\widetilde{h}_\nu^\eta = n + 2 - (1 + \delta_\nu^{\ell,unr}) = [F_\nu : \mathbb{Q}_\ell] + 1 - \delta_\nu^{\ell,unr}.$$

Moreover $h^0(\mathrm{ad}_{\bar{\rho}}^0) = 1$, and since there are no obstructions to lifting, we find

$$\widetilde{h}_{\nu}^{\eta} - \mathrm{gen}(\widetilde{J}_{\nu,\mathcal{O}}^{\eta}) - h^0(G_{\nu}, \mathrm{ad}_{\bar{\rho}}^0) = [F_{\nu} : \mathbb{Q}_{\ell}] - \delta_{\nu}^{\ell,unr}.$$

Let us now assume that F_{ν} does contain a primitive ℓ-th root of unity, and let q be the largest ℓ-power so that F_{ν} contains a primitive q-th root of unity. Let F be the free pro-ℓ group on (topological) generators $s, t_0, \ldots, t_n$, $n = [F_{\nu} : \mathbb{Q}_{\ell}]$. For closed subgroups H, K of F, let $[H, K]$ denote the closed subgroup of F generated by commutators $[h, k] = h^{-1}k^{-1}hk$, $h \in H$, $k \in K$, and denote by N the closed normal subgroup $[F, F] \cap F^{q^2}[F^q, F][F^q[F, F], F]$. Then by [14], Thm. 10.9, the group $\widehat{G}_{\nu}^{\ell}$ is the quotient of F by the closed normal subgroup generated by the element

$$r := t_0^{q_0} t_1^{q_1} [t_0, s][t_1, t_2] \ldots [t_{n-1}, t_n] r'$$

for some ℓ-powers q_0, q_1 which are divisible by q, and for some element $r' \in N$. The isomorphism may be chosen, so that the closed normal subgroup generated by the t_i maps to the inertia subgroup of $\widehat{G}_{\nu}^{\ell}$. By our hypothesis on $\bar{\rho}$, for any lift to some ring R in $\mathcal{C}_{\mathcal{O}}$ one has

$$s \mapsto \begin{pmatrix} 1+\alpha & \beta \\ 0 & (1+\alpha)^{-1}\eta_{\nu}(s) \end{pmatrix}, t_0 \mapsto \begin{pmatrix} 1 & \tau_1 \\ 0 & \eta_{\nu}(t_0) \end{pmatrix}, \ldots, t_n \mapsto \begin{pmatrix} 1 & \tau_n \\ 0 & \eta_{\nu}(t_n) \end{pmatrix}.$$

If the variables α, β and τ_i are chosen arbitrarily, the image of r in $\mathrm{GL}_2(R)$ is of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ for some $x \in \mathfrak{m}_R$. The reason is as follows: The image of such a ρ is upper triangular. Passing to the quotient modulo the unipotent upper triangular normal subgroup gives a representation into $R^* \times R^*$, which by our hypothesis factors via G_{ν}/I_{ν}. Now r lies in the inertia subgroup of $\widehat{G}_{\nu}^{\ell}$, and so its image in $R^* \times R^*$ is zero.

In fact the expression x is computable in terms of α, β and the τ_i up to some error coming from r'. It is useful to write each variable ξ as $\xi_0 + \tilde{\xi}$, where ξ_0 is the Teichmüller lift of the reduction mod $\mathfrak{m}_R$ of ξ, and $\tilde{\xi} \in \mathfrak{m}_R$ for $R = \mathcal{O}[[\tilde{\alpha}, \tilde{\beta}, \tilde{\tau}_0, \ldots, \tilde{\tau}_n]]$. Working modulo $\mathfrak{m}_R^3$, one can evaluate x and show that it does not vanish. It may happen that x lies in $\mathfrak{m}_R \smallsetminus (\mathfrak{m}_R^2, \mathfrak{m}_{\mathcal{O}})$. Therefore in a minimal presentation (6) of $\widetilde{R}_{\nu,\mathcal{O}}^{\eta}$ one has $\mathrm{gen}(\widetilde{R}_{\nu,\mathcal{O}}^{\eta}) \leq 1$, and moreover $\widetilde{h}_{\nu}^{\eta} - \mathrm{gen}(\widetilde{R}_{\nu,\mathcal{O}}^{\eta})$ is $n + 3 - 1 = n - 2$ minus the number of variables that disappear when passing from lifts to deformations, i.e., to strict equivalence classes of lifts.

The analysis of the effect of strict equivalence is as in the case where no primitive ℓ-th root of unity lies in F_{ν}, and so we need to subtract $(1 + \delta_{\nu}^{\ell,\mathrm{unr}})$ from $n - 2$. This yields

$$\widetilde{h}_{\nu}^{\eta} - \mathrm{gen}(\widetilde{R}_{\nu,\mathcal{O}}^{\eta}) = n - 1 - \delta_{\nu}^{\ell,\mathrm{unr}}.$$

Since $h^0(G_{\nu}, \mathrm{ad}_{\bar{\rho}}^0) = 1$ in case (iii), we found, independently of a primitive ℓ-th root of unity being in F_{ν} or not, that

$$\widetilde{h}_{\nu}^{\eta} - \mathrm{gen}(\widetilde{J}_{\nu,\mathcal{O}}^{\eta}) - h^0(G_{\nu}, \mathrm{ad}_{\bar{\rho}}^0) = n - \delta_{\nu}^{\ell,\mathrm{unr}} = [F_{\nu} : \mathbb{Q}_{\ell}] - \delta_{\nu}^{\ell,\mathrm{unr}}.$$

Let us summarize our results:

Proposition 6.6 *Suppose* $\bar{\rho}$ *is ordinary at* ν, *i.e., of the form (11). Let* $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{\eta} \subset \mathrm{Def}_{\nu,\mathcal{O}}^{\eta}$ *denote the subfunctor of ordinary deformations with fixed determinant. Define* $\delta_{\nu}^{\ell,\mathrm{unr}}$ *to be 1 if at the same time* $\bar{\rho}$ *is unramified and* $\mathrm{Im}(\bar{\rho})$ *is an* ℓ-*group, and to be zero otherwise. Then*

$$\widetilde{h}_{\nu}^{\eta} - \mathrm{gen}(\widetilde{J}_{\nu,\mathcal{O}}^{\eta}) - h^0(G_{\nu}, \mathrm{ad}_{\bar{\rho}}^0) \geq [F_{\nu} : \mathbb{Q}_{\ell}] - \delta_{\nu}^{\ell,\mathrm{unr}}.$$

Example 6.7 Lastly, we need to discuss the global terms in the estimate (10). Let $\overline{G}$ denote the quotient of $\mathrm{Im}(\bar{\rho})$ modulo its intersection with the center of $\mathrm{GL}_2(\mathbb{F})$, and assume that $\overline{G}$ is non-trivial.

We first give the results for $\ell = 2$. There, independently of F, by explicit computation one finds:

$$h^0(G_{F,S}, \mathrm{ad}_{\bar{\rho}}) = h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}})^\vee) = \begin{cases} 2 & \text{if } \overline{G} \text{ is abelian,} \\ 1 & \text{otherwise.} \end{cases}$$

$$h^0(G_{F,S}, \mathrm{ad}_{\bar{\rho}}^0) = \begin{cases} 2 & \text{if } \overline{G} \text{ is a 2-group (and hence abelian),} \\ 1 & \text{otherwise.} \end{cases}$$

$$\delta(G_{F,S}, \mathrm{ad}_{\bar{\rho}})^\eta = \begin{cases} 0 & \text{if } \overline{G} \text{ is of order prime to 2,} \\ 1 & \text{otherwise.} \end{cases}$$

$$h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^0)^\vee) = \begin{cases} 2 & \text{if } \overline{G} \text{ is a 2-group,} \\ 1 & \text{if } \overline{G} \text{ is dihedral or of Borel type and not a 2-group,} \\ 0 & \text{otherwise.} \end{cases}$$

For $\ell \neq 2$, the result depends on F, because $\bar{\chi}_{\mathrm{cyc}}$ will in general be non-trivial. If $\bar{\rho}$ when considered over $\mathbb{F}^{\mathrm{alg}}$ is reducible, let $\bar{\chi}_2$ denote the character of $G_{F,S}$ on a one-dimensional quotient and $\bar{\chi}_1$ on the corresponding 1-dimensional subspace, and define $\bar{\chi} = \bar{\chi}_1 \bar{\chi}_2^{-1}$. One finds:

$$h^0(G_{F,S}, \mathrm{ad}_{\bar{\rho}}) - 1 = h^0(G_{F,S}, \mathrm{ad}_{\bar{\rho}}^0) = \begin{cases} 1 & \text{if } \overline{G} \text{ is abelian,} \\ 0 & \text{otherwise.} \end{cases}$$

$h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}})^\vee) = h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^0)^\vee) + h^0(G_{F,S}, \mathbb{F}(1))$, and

$$h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^0)^\vee) = \begin{cases} 2 & \text{if } \overline{G} \cong \mathbb{Z}/(2) \text{ and } \bar{\chi}_{\mathrm{cyc}} = \bar{\chi}, \\ 1 & \text{if } \overline{G} \cong \mathbb{Z}/(2) \text{ and } \bar{\chi}_{\mathrm{cyc}} = 1, \\ 1 & \text{if } \overline{G} \not\cong \mathbb{Z}/(2) \text{ is abelian and } \bar{\chi}_{\mathrm{cyc}} \in \{\bar{\chi}, \bar{\chi}^{-1}, 1\}. \\ 1 & \text{if } \overline{G} \text{ is non-abelian of Borel type and } \bar{\chi}_{\mathrm{cyc}} = \bar{\chi}. \\ 1 & \text{if } \overline{G} \text{ is dihedral, } [F(\zeta_\ell) : F] = 2 \text{ and } \bar{\rho}_{|G_{F(\zeta_\ell)}} \text{ is reducible,} \\ 0 & \text{otherwise.} \end{cases}$$

Combining the above results, we obtain the following general theorem in the case $\mathcal{G} = \mathrm{GL}_2$:

Theorem 6.8 *Suppose F is totally real and $\bar{\rho}$ is odd. Suppose further that*

(a) *At $\nu \nmid \ell, \infty$ the local deformation problem satisfies $\widetilde{h}_\nu^\eta - h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) - \mathrm{gen}(J_\nu^\eta) \geq 0$.*

(b) *At $\nu | \infty$ we choose either of the versal hulls in Example 6.4 depending on whether $\bar{\rho}(c_\nu)$ is trivial or not.*

(c) *At $\nu | \ell$, either (i) $F_\nu = \mathbb{Q}_\ell$, and $\bar{\rho}$ satisfies the requirements in 6.5 case I, and $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^\eta$ is the functor of "flat deformations", or (ii) $\bar{\rho}$ is ordinary and $\delta_\nu^{\ell,\mathrm{unr}} = 0$, and $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^\eta$ is the functor of ordinary deformations with fixed determinant.*

(d) *$h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^0)^\vee) = 0$. (cf. Example 6.7 for explicit conditions.)*

Then $\widetilde{R}_{S,\mathcal{O}}^\eta$ has a presentation $\mathcal{O}[[T_1, \ldots, T_n]]/(f_1, \ldots, f_n)$ for suitable $f_i \in \mathcal{O}[[T_1, \ldots, T_n]]$.

PROOF: Example 6.7 and (d) imply that

$$h^0(G_{F,S}, \mathrm{ad}_{\bar\rho}^0) - \delta(G_{F,S}, \mathrm{ad}_{\bar\rho}^0)^\eta = 0.$$

Hence, if view of (a) it suffices to show that the joint contribution in (10) from the places above ℓ and ∞ under the stated hypotheses is zero. Using Proposition 6.6 and Example 6.4 yields

$$\sum_{\nu | \ell \text{ or } \nu | \infty} \left(\widetilde{h}_\nu^0 - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^0) - \mathrm{gen}(J_\nu^\eta) \right) \;=\; \sum_{\nu | \ell}[F_\nu : \mathbb{Q}_\ell] - \sum_{\nu | \infty} 1 = [F : \mathbb{Q}] - [F : \mathbb{Q}] = 0. \quad \blacksquare$$

7   Comparison to the results by Tilouine and Mauger

In this section we will apply the estimate from Corollary 5.3 to obtain another approach to the results by Tilouine and Mauger in [23, 15] on presentations for universal deformations. While their main interest was in representations into symplectic groups, their results are rather general. If $h^0(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee) = 0$, we completely recover their results with fewer hypothesis. If not, a comparison is less clear. It seems however, that in most cases where their results are applicable, the term $h^0(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee)$ will be zero. Our main result is Theorem 7.6.

Example 7.1 Let $d \colon \mathcal{G} \to \mathcal{T}$ be arbitrary and let $S_{\mathrm{ord}} \subset S$ be a set of places of F which contains all places above ℓ and none above ∞. For each $\nu \in S_{\mathrm{ord}}$, we fix a smooth closed $\mathcal{O}$-subgroup scheme $\mathcal{P}_\nu \subset \mathcal{G}$.
For each place ν in S_{ord}, we consider the subfunctor $\mathrm{Def}_{\nu,\mathcal{O}}^{\nu\text{-n.o.}} \subset \mathrm{Def}_{\nu,\mathcal{O}}$ of deformations $[\rho_\nu \colon G_\nu \to \mathcal{G}(R)]$ such that there exists some $g_\nu \in \mathcal{G}(R)$, whose reduction mod $\mathfrak{m}_R$ is the identity, such that $g_\nu \rho_\nu g_\nu^{-1}(G_\nu) \subset \mathcal{P}_\nu(R)$. For this subfunctor to make sense, one obviously requires that $\bar\rho(G_\nu) \subset \mathcal{P}_\nu(\mathbb{F})$.
Following [23], a deformation $[\rho]$ is called $\mathcal{P}$-**nearly ordinary** (at S_{ord}) (where $\mathcal{P}$ stands for the family $(\mathcal{P}_\nu)_{\nu \in S_{\mathrm{ord}}}$) if for each $\nu \in S_{\mathrm{ord}}$ the restriction $[\rho_{|G_\nu}]$ satisfies the above condition. By $\widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{S_0\text{-n.o.}} \subset \mathrm{Def}_{S,\mathcal{O}}$ we denote the global global deformation functor of deformations which are $\mathcal{P}$-nearly ordinary at $S_{\mathrm{ord}} \subset S$, and are described by some other relatively representable functors $\widetilde{\mathrm{Def}}_{\nu,\mathcal{O}}^{\eta} \subset \mathrm{Def}_{\nu,\mathcal{O}}^{\eta}$ at places $\nu \in S \setminus S_{\mathrm{ord}}$. If one furthermore fixes a lift

$$\eta \colon G_{F,S} \longrightarrow \mathcal{T}(\mathcal{O})$$

of $d \circ \bar\rho \colon G_{F,S} \to \mathcal{G}(\mathbb{F}) \to \mathcal{T}(\mathbb{F})$, the corresponding subfunctor is

$$\widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{S_0\text{-n.o.},\eta} := \widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{S_0\text{-n.o.}} \cap \mathrm{Def}_{S,\mathcal{O}}^{\eta}.$$

For each $\nu \in S_{\mathrm{ord}}$, let $\mathfrak{p}_\nu \subset \mathfrak{g}$ denote the Lie-subalgebra of $\mathfrak{g}$ which corresponds to $\mathcal{P}_\nu \subset \mathcal{G}$. It carries a natural $\mathcal{P}_\nu$-action, so that $\mathfrak{g}/\mathfrak{p}_\nu(\mathbb{F})$ is a finite $\mathcal{P}_\nu$-module. Again following [23], we define the condition

$$(\mathbf{Reg}) : \text{ For all } \nu \in S_{\mathrm{ord}} : h^0(G_\nu, \mathfrak{q}/\mathfrak{p}_\nu(\mathbb{F})) = 0.$$

One has the following simple result whose proof we omit:

Lemma 7.2 *If the condition* (**Reg**) *holds, for all* $\nu \in S_{\mathrm{ord}}$, *the subfunctor* $\mathrm{Def}_{\nu,\mathcal{O}}^{\nu\text{-n.o.}} \subset \mathrm{Def}_{\nu,\mathcal{O}}$ *is relatively representable. Hence in this case* $\widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{S_0\text{-n.o.},(\eta)}$ *has a versal hull*

$$\rho_{S,\mathcal{O}}^{S_0\text{-n.o.},(\eta)} : G_{F,S} \to \mathcal{G}(R_{S,\mathcal{O}}^{S_0\text{-n.o.},(\eta)}).$$

Locally at $\nu \in S_{\mathrm{ord}}$ denote by $\mathrm{Def}_{\mathcal{P}_\nu,\mathcal{O}}^{(\eta)}$ the functor of deformations for representations of G_ν into $\mathcal{P}_\nu$ (possibly with the additional condition the deformations are compatible with the chosen η.) Let

$$\rho_{\mathcal{P}_\nu,\mathcal{O}}^{(\eta)} : G_{F,S} \to \mathcal{P}_\nu(R_{\mathcal{P}_\nu,\mathcal{O}}^{(\eta)})$$

denote a corresponding versal hull and define $\mathfrak{p}_\nu^0$ as the Lie-Algebra of the kernel of the composite $\mathcal{P}_\nu \hookrightarrow \mathcal{G} \xrightarrow{d} \mathcal{T}$. By Theorems 2.2 and 2.4 we find:

Proposition 7.3 *The mod* $\mathfrak{m}_{\mathcal{O}}$ *tangent space of* $R_{\mathcal{P}_\nu,\mathcal{O}}^{(\eta)}$ *is isomorphic to*

$$H^1(G_\nu, \mathfrak{p}_\nu)^{(\eta)} := \mathrm{Im}(H^1(G_\nu, \mathfrak{p}_\nu^{(0)}) \to H^1(G_\nu, \mathfrak{p}_\nu)).$$

Let $h_{\mathcal{P}_\nu}^{(\eta)} := \dim_{\mathbb{F}} H^1(G_\nu, \mathfrak{p}_\nu)^{(\eta)}$. *Then there exists a presentation*

$$0 \longrightarrow J_{\mathcal{P}_\nu}^{(\eta)} \longrightarrow \mathcal{O}[[T_1, \dots, T_{h_{\mathcal{P}_\nu}^{(\eta)}}]] \longrightarrow R_{\mathcal{P}_\nu,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

for some ideal $J_{\mathcal{P}_\nu}^{(\eta)} \subset \mathcal{O}[[T_1, \dots, T_{h_{\mathcal{P}_\nu}^{(\eta)}}]]$ *with* $\mathrm{gen}(J_{\mathcal{P}_\nu}^{(\eta)}) \leq \dim_{\mathbb{F}} H^2(G_\nu, \mathfrak{p}_\nu^{(0)})$.

The two functors $\mathrm{Def}_{\nu,\mathcal{O}}^{\nu\text{-n.o.},(\eta)}$ and $\mathrm{Def}_{\mathcal{P}_\nu,\mathcal{O}}^{(\eta)}$ essentially describe the same deformation problem, except that a priori they work with a different notion of strict equivalence.

Lemma 7.4 *The obvious surjection* $\mathrm{Def}_{\mathcal{P}_\nu,\mathcal{O}}^{(\eta)}(\mathbb{F}[\varepsilon]/(\varepsilon^2)) \longrightarrow \mathrm{Def}_{\nu,\mathcal{O}}^{\nu\text{-n.o.},(\eta)}(\mathbb{F}[\varepsilon]/(\varepsilon^2))$ *is a bijection provided that* (**Reg**) *holds.*

PROOF: Clearly every lift ρ of $\bar{\rho}$ to $\mathbb{F}[\varepsilon]/(\varepsilon^2)$ whose class lies in $\mathrm{Def}_{\nu,\mathcal{O}}^{\nu\text{-n.o.},(\eta)}(\mathbb{F}[\varepsilon]/(\varepsilon^2))$ can by definition be conjugated to take its image inside $\mathcal{P}_\nu(\mathbb{F}[\varepsilon]/(\varepsilon^2))$. Moreover the notion of strict equivalence for $\mathrm{Def}_{\mathcal{P}_\nu,\mathcal{O}}^{(\eta)}$ is an a priori weaker one than for $\mathrm{Def}_{\nu,\mathcal{O}}^{\nu\text{-n.o.},(\eta)}$, so that the orbits under the second notion of strict equivalence may be larger. This shows that the map in the lemma is well-defined and surjective. Let us now show injectivity, i.e., that the orbits under both notions of strict equivalence agree.
Let $\rho = (1 + \varepsilon a)\bar{\rho}$ be a lift of $\bar{\rho}$ to $\mathbb{F}[\varepsilon]/(\varepsilon^2)$ with image inside $\mathcal{P}_\nu(\mathbb{F}[\varepsilon]/(\varepsilon^2))$, so that $a : G_\nu \to \mathfrak{p}_\nu$ is a 1-cocycle. Let $g = 1 + \varepsilon b$ be arbitrary with $b \in \mathfrak{g}$. We need to show that the set of those b for which $g\rho g^{-1}$ lies in $\mathcal{P}_\nu(\mathbb{F}[\varepsilon]/(\varepsilon^2))$ (for all a as above) is exactly the set $\mathfrak{p}_\nu$: One computes explicitly

$$g\rho g^{-1} = (1 + \varepsilon(a + gbg^{-1} - b))\bar{\rho}.$$

So independently of a, the element $gbg^{-1} - g$ must lie in $\mathfrak{p}_\nu$ for all $g \in \bar{\rho}(G_\nu)$. Equivalently, the image of b under the surjection $\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{p}_\nu$ must lie in $H^0(G_\nu, \mathfrak{g}/\mathfrak{p}_\nu)$. By (**Reg**) the latter set is zero, and so b lies indeed in $\mathfrak{p}_\nu = \mathrm{Ker}(\mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{p}_\nu)$. ∎

Recall that $0 \leq \delta(G_\nu, \mathfrak{p}_\nu)^\eta = h^1(G_\nu, \mathfrak{p}_\nu^0) - \dim H^1(G_\nu, \mathfrak{p})^\eta$. The formula for the local Euler-Poincaré characteristic yields:

Proposition 7.5 *For $\nu \in S_0$ and the functor $\mathrm{Def}_{\nu,\mathcal{O}}^{(\eta)} = \mathrm{Def}_{\nu,\mathcal{O}}^{\nu\text{-n.o.},(\eta)}$ one has*

$$\widetilde{h}_\nu^{(\eta)} + \delta(G_\nu, \mathfrak{p}_\nu)^{(\eta)} - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) - \mathrm{gen}(J_\nu^{(\eta)}) = \begin{cases} 0, & \textit{if } \nu \nmid \ell, \\ [F_\nu : \mathbb{Q}_\ell] \dim_{\mathbb{F}} \mathfrak{p}_\nu, & \textit{if } \nu \mid \ell. \end{cases}$$

In [23, 15] there is never chosen a lift of $d \circ \bar\rho$. Hence the term $\delta(G_\nu, \mathfrak{p}_\nu)^{(\eta)}$ is not present in their formulas.

Combining the above with Corollary 5.3 shows:

Theorem 7.6 *Fix $\mathcal{P} = (\mathcal{P}_\nu)_{\nu \in S_0}$ as above, and assume that:*

(a) $\bar\rho \in \widetilde{\mathrm{Def}}_{S,\mathcal{O}}^{S_0\text{-n.o.},(\eta)}(\mathbb{F})$.

(b) *At $\nu \in S \setminus (S_{\mathrm{ord}} \cup \{\nu : \nu|\infty\})$ we have $\widetilde{h}_\nu^{(0)} - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) - \mathrm{gen}(J_\nu^{(\eta)}) \geq 0$.*

(c) *The condition (**Reg**) is satisfied.*

Then for the presentation

$$0 \longrightarrow \widetilde{J}^{(\eta)} \longrightarrow \mathcal{O}[[T_1, \ldots, T_{\widetilde{h}^{(\eta)}}]] \longrightarrow \widetilde{R}_{S,\mathcal{O}}^{S_0\text{-n.o.},(\eta)} \longrightarrow 0$$

one has

$$\widetilde{h}^{(\eta)} - \mathrm{gen}(J^{(\eta)}) \geq h^0(G_{F,S}, \mathrm{ad}_{\bar\rho}^{(0)}) - h^0(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee) - \delta(G_{F,S}, \mathrm{ad}_{\bar\rho})^{(\eta)}$$

$$(13) \qquad + \sum_{\nu|\ell} [F_\nu : \mathbb{Q}_\ell] \dim_{\mathbb{F}} \mathfrak{p}_\nu^{(0)} + \sum_{\nu|\infty} \left(\widetilde{h}_\nu^{(\eta)} + \delta(G_\nu, \mathrm{ad}_{\bar\rho})^{(\eta)} - h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) - \mathrm{gen}(\widetilde{J}_\nu^{(\eta)}) \right).$$

If $\ell \neq 2$, or if no constraints are imposed for the deformation at the infinite places, then their contribution in the above formula simplifies to $-\sum_{\nu|\infty} h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)})$.

Remark 7.7 Since in [23] or [15] no homomorphism η is fixed, and there are no conditions at ∞, the above is (philosophically) the same formula as that in [23], Prop. 7.3 or [15], Prop. 3.9, except for the term $-h^0(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee)$. As noted in Remark 5.4, we expect that usually this term is not present in the formula – but that technically we are not able to remove it.

By 'philosophically' we mean that their formula was used primarily to bound the Krull dimension of some deformation ring. Our formula can obviously serve the same purpose.

Our hypotheses and those in [23, 15] are however different. If $h^0(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee) = 0$ our result holds under much weaker hypotheses, namely without the hypothesis (**Reg′**) in [15], Prop. 3.9. The latter seems to be rather hard to verify in practice.

If $h^0(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee)$ is non-zero the comparison is less clear. The non-vanishing either means that we are in the case $\mathrm{ad}_{\bar\rho}$ and F contains a primitive ℓ-th root of unity, or that $\mathrm{ad}_{\bar\rho}^0$ surjects onto a one-dimensional quotient representation on which $G_{F,S}$ acts by the inverse of the mod ℓ-cyclotomic character. In the former case we'd expect that the $\mathfrak{p}_\nu$ typically also contain a trivial subrepresentation, and then the terms $h^0(G_\nu, (\mathfrak{p}_\nu^{(0)})^\vee) = h^2(G_\nu, \mathfrak{p}_\nu^{(0)})$ would be non-zero, so that the hypothesis (**Reg′**) in [15], Prop. 3.9, would not be satisfied. In the latter case it is not clear to us whether this one-dimensional quotient will typically also occur as a quotient of one of the $\mathfrak{p}_\nu^{(0)}$. In any case, if $\bar\rho$ is 'highly irreducible' which is the generic case, the second case is unlikely to occur.

8   Relative presentations

In this last section we deduce some results on presentations of global deformation rings as quotients of power series rings over the completed tensor product of the corresponding local versal deformation rings from the results in Section 5. This is inspired by M. Kisin's theory of framed deformations. The results below are due to Kisin [13], who has given a different more direct approach.

This section makes no reference to Sections 6 and 7. *We let the notation be as in Section 5.*

Lemma 8.1 *There exists a presentation*

$$0 \longrightarrow \widetilde{J} \longrightarrow \left(\widehat{\bigotimes}_{\nu \in S} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}\right)[[T_1, \ldots, T_s]] \longrightarrow \widetilde{R}_{S,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

with $\mathrm{gen}(\widetilde{J})$ *being bounded by*

$$s + \delta(G_{F,S}, \mathrm{ad}_{\bar{\rho}})^{(\eta)} - h^0(G_{F,S}, \mathrm{ad}_{\bar{\rho}}^{(0)}) + h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^{(0)})^\vee) + \sum_{\nu \in S} h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^{(0)}) - \delta(G_\nu, \mathrm{ad}_{\bar{\rho}})^{(\eta)}.$$

PROOF: The proof of Theorem 5.2 yields the following commutative diagram

$$0 \rightarrowtail \widetilde{J}_{\mathrm{loc}} := \langle \{\widetilde{J}_\nu^{(\eta)} : \nu \in S\}\rangle \rightarrowtail \mathcal{O}[[T_{\nu,j} : \nu \in S, j = 1, \ldots, \widetilde{h}_\nu^{(\eta)}]] \longrightarrow \mathcal{R}_{\mathrm{loc}} := \left(\widehat{\bigotimes}_{\nu \in S} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}\right) \twoheadrightarrow 0$$

$$\downarrow \qquad\qquad \downarrow{\scriptstyle \pi} \qquad\qquad \downarrow$$

$$0 \rightarrowtail \langle \{\pi(\widetilde{J}_{\mathrm{loc}}) \cup \{g_1, \ldots, g_r\}\rangle \longrightarrow \mathcal{O}[[T_1, \ldots, T_{\widetilde{h}^{(\eta)}}]] \longrightarrow \widetilde{R}_{S,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

for $r = \dim_{\mathbb{F}} H^1_{\mathcal{L}^{(\eta),\perp}}(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^{(0)})^\vee)$ and suitable functions $g_j \in \mathcal{O}[[T_1, \ldots, T_{\widetilde{h}^{(\eta)}}]]$. The failure of the surjectivity of $\mathcal{R}_{\mathrm{loc}} \to \widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ can be measured by considering the induced homomorphism on mod $\mathfrak{m}_{\mathcal{O}}$ tangent spaces. Let s denote the dimension of the cokernel of $\mathfrak{t}_{\mathcal{R}_{\mathrm{loc}}} \to \mathfrak{t}_{\widetilde{R}_{S,\mathcal{O}}^{(\eta)}}$. (It is not difficult to show that $s = \dim \mathrm{III}^1(\mathrm{ad}_{\bar{\rho}}^{(0)})$), but we do not need this.) Then there is a surjective homomorphism $\mathcal{R}_{\mathrm{loc}}[[U_1, \ldots, U_s]] \to \widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ for variables U_i. Abbreviating $\mathcal{S}_{\mathrm{loc}} := \mathcal{O}[[T_{\nu,j} : \nu \in S, j = 1, \ldots, \widetilde{h}_\nu^{(\eta)}]]$, there is a commutative diagram

$$0 \longrightarrow \langle \widetilde{J}_{\mathrm{loc}}\rangle \longrightarrow \mathcal{S}_{\mathrm{loc}}[[U_1, \ldots, U_s]] \longrightarrow \mathcal{R}_{\mathrm{loc}}[[U_1, \ldots, U_s]] \twoheadrightarrow 0$$

$$\downarrow \qquad\qquad \downarrow{\scriptstyle \widetilde{\pi}} \qquad\qquad \downarrow$$

$$0 \rightarrowtail \langle \{\widetilde{\pi}(\widetilde{J}_{\mathrm{loc}}) \cup \{g_1, \ldots, g_r\}\rangle \rightarrowtail \mathcal{O}[[T_1, \ldots, T_{\widetilde{h}^{(\eta)}}]] \longrightarrow \widetilde{R}_{S,\mathcal{O}}^{(\eta)} \longrightarrow 0$$

with surjective middle and right vertical homomorphisms. Since $\mathcal{S}_{\mathrm{loc}}$ is a power series ring over $\mathcal{O}$, the kernel of $\widetilde{\pi}$ is generated by

$$u := s + \sum_{\nu \in S} \widetilde{h}_\nu^{(\eta)} - \widetilde{h}^{(\eta)}$$

elements $H_1, \ldots, H_u$. Because $\widetilde{\pi}$ is smooth, we may choose elements $G_1, \ldots, G_r$ in the ring $\mathcal{S}_{\mathrm{loc}}[[U_1, \ldots, U_s]]$ whose images in $\mathcal{O}[[T_1, \ldots, T_{\widetilde{h}^{(\eta)}}]]$ agree with $g_1, \ldots, g_r$. Thus $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$

is the quotient of $\mathcal{R}_{\mathrm{loc}}[[U_1, \ldots, U_s]]$ by the ideal $\widetilde{J}$ generated by the images of the elements $G_1, \ldots, G_r, H_1, \ldots, H_u$. Using the first formula in the proof of Corollary 5.3, we have

$$\mathrm{gen}(\widetilde{J}) - s = \sum_{\nu \in S} \widetilde{h}_\nu^{(\eta)} - \widetilde{h}^{(\eta)} + \dim_{\mathbb{F}} H^1_{\mathcal{L}(\eta), \perp}(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee)$$

$$= \delta(G_{F,S}, \mathrm{ad}_{\bar\rho})^{(\eta)} - h^0(G_{F,S}, \mathrm{ad}_{\bar\rho}^{(0)}) + h^0(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee) + \sum_{\nu \in S} h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) - \delta(G_\nu, \mathrm{ad}_{\bar\rho})^{(\eta)}. \blacksquare$$

If R is flat over $\mathcal{O}$, its relative Krull dimension over $\mathcal{O}$ is denoted $\dim_{\mathrm{Krull}/\mathcal{O}} R$.

Corollary 8.2 *Suppose that*

(a) $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}/(\ell)$ *is finite.*

(b) *The rings* $\widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}$, $\nu \in S$ *are flat over* $\mathcal{O}$.

(c) $\dim_{\mathrm{Krull}/\mathcal{O}} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)} \geq h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) - \delta(G_\nu, \mathrm{ad}_{\bar\rho})^{(\eta)}$ *for* $\nu \nmid \ell, \infty$.

(d) *One has* $\sum_{\nu | \ell \, \mathrm{or} \, \nu | \infty} \dim_{\mathrm{Krull}/\mathcal{O}} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)} \geq \sum_{\nu | \ell \, \mathrm{or} \, \nu | \infty} (h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) - \delta(G_\nu, \mathrm{ad}_{\bar\rho})^{(\eta)})$.

(e) $\delta(G_{F,S}, \mathrm{ad}_{\bar\rho})^{(\eta)} - h^0(G_{F,S}, \mathrm{ad}_{\bar\rho}^{(0)}) + h^0(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee) = 0$.

Then the ℓ-torsion of $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ is finite, and $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ modulo its ℓ-torsion is non-zero over $\mathcal{O}$. Hence this quotient is non-zero and finite flat over $\mathcal{O}$.

PROOF: Since $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ is noetherian the ℓ-torsion submodule of $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ is finitely generated. Therefore there exists some $m \geq 0$ such that the ℓ-torsion submodule injects into $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}/(\ell^m)$. By condition (a) (and noetherianess of $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$) the latter is finite. To complete the proof of the corollary, it suffices to show that the Krull dimension of $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ is at least one.

We first compute the relative Krull dimension of the middle term in the presentation of Lemma 8.1:

$$\dim_{\mathrm{Krull}/\mathcal{O}} \left(\widehat{\bigotimes}_{\nu \in S} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}\right) \overset{(b)}{=} \sum_{\nu \in S} \dim_{\mathrm{Krull}/\mathcal{O}} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}$$

$$\overset{(c),(d)}{\geq} \sum_{\nu \in S} (h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) - \delta(G_\nu, \mathrm{ad}_{\bar\rho})^{(\eta)}).$$

Using (e), the relative Krull dimension of $\widehat{\bigotimes}_{\nu \in S} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}[[U_1, \ldots, U_s]]$ over $\mathcal{O}$ is therefore at least

$$s + \delta(G_{F,S}, \mathrm{ad}_{\bar\rho})^{(\eta)} - h^0(G_{F,S}, \mathrm{ad}_{\bar\rho}^{(0)}) + h^0(G_{F,S}, (\mathrm{ad}_{\bar\rho}^{(0)})^\vee) + \sum_{\nu \in S} h^0(G_\nu, \mathrm{ad}_{\bar\rho}^{(0)}) - \delta(G_\nu, \mathrm{ad}_{\bar\rho})^{(\eta)}.$$

This is also the bound on $\mathrm{gen}(\widetilde{J})$ in the presentation of Lemma 8.1. Now the quotient of a local ring by a number of relations decreases the Krull dimension of the ring by at most this number (unless the quotient is zero). Since the Krull dimension is one more than the relative Krull dimension over $\mathcal{O}$, it follows that the Krull dimension of

$$\widetilde{R}_{S,\mathcal{O}}^{(\eta)} \cong \left(\widehat{\bigotimes}_{\nu \in S} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}\right)[[U_1, \ldots, U_s]]/\widetilde{J}$$

is at least one, as was to be shown. $\blacksquare$

Remark 8.3 The following is a ring theoretic example which shows that without any further hypotheses, one cannot rule out the possibility of ℓ-torsion in the ring $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$:

Suppose $F = \mathbb{Q}$, $S = \{\ell, \infty\}$, $\widetilde{R}_{\ell,\mathcal{O}}^{(\eta)} \cong \mathbb{Z}_\ell[[S,T]]/((\ell + S)T, T^2)$, that (d) and (e) of the corollary are satisfied, and that $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ is the quotient of $\widetilde{R}_{\ell,\mathcal{O}}^{(\eta)}$ by the ideal (S). Then $\widetilde{R}_{\ell,\mathcal{O}}^{(\eta)} \cong \mathbb{Z}_\ell[[T]]/(\ell T, T^2)$ has ℓ-torsion, although the remaining assertions (a) and (b) of the corollary hold ((c) holds trivially).

However, if in addtion to (a)–(e) one imposes the further condition that $\widehat{\bigotimes}_{\nu \in S} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)}$ is Cohen-Macaulay, then from standard results in commutative algebra one may indeed deduce that $\widetilde{R}_{S,\mathcal{O}}^{(\eta)}$ is flat over $\mathcal{O}$. This was pointed out by M. Kisin.

We now apply the previous corollary to the situation of Theorem 6.8, where however we relax the condition at the places above ℓ:

Theorem 8.4 *Suppose* $d = \det : \mathcal{G} = \mathrm{GL}_2 \to \mathcal{T} = \mathrm{GL}_1$, *$F$ is totally real and $\bar{\rho}$ is odd. Suppose further that*

(a) $\widetilde{R}_{S,\mathcal{O}}^{\eta}/(\ell)$ *is finite.*

(b) *The rings $\widetilde{R}_{\nu,\mathcal{O}}^{\eta}$, $\nu \in S$ are flat over $\mathcal{O}$.*

(c) *At $\nu \nmid \ell, \infty$ the local deformation problem satisfies $\widetilde{h}_\nu^0 - h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) - \mathrm{gen}(J_\nu^\eta) \geq 0$.*

(d) *At $\nu | \infty$ we choose either of the versal hulls in Example 6.4 depending on whether $\bar{\rho}(c_\nu)$ is trivial or not.*

(e) *At $\nu | \ell$, one has $\dim_{\mathrm{Krull}/\mathcal{O}} \widetilde{R}_{\nu,\mathcal{O}}^{\eta} = [F_\nu : \mathbb{Q}_\ell] + h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) - \delta(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta$.*

(f) *$h^0(G_{F,S}, (\mathrm{ad}_{\bar{\rho}}^0)^\vee) = 0$. (cf. Example 6.7 for explicit conditions.)*

Then $\widetilde{R}_{S,\mathcal{O}}^{\eta}$ has finite ℓ-torsion, and its quotient modulo ℓ-torsion is non-zero and finite flat over $\mathcal{O}$.

PROOF: It suffices to verify the hypothesis of Corollary 8.2. Conditions (a), (b), (c) and (f) imply conditions (a), (e), (b) and (d) of Corollary 8.2, respectively. At the infinite places ν, condition (d) implies $\dim_{\mathrm{Krull}/\mathcal{O}} \widetilde{R}_{\nu,\mathcal{O}}^{(\eta)} = h^0(G_\nu, \mathrm{ad}_{\bar{\rho}}^0) - \delta(G_\nu, \mathrm{ad}_{\bar{\rho}})^\eta - 1$. Because of the identity $\sum_{\nu | \ell} [F_\nu : \mathbb{Q}_\ell] = [F : \mathbb{Q}] = \sum_{\nu | \infty} 1$, the latter observation combined with condition (e) implies condition (c) of Corollary 8.2. ∎

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