Canonical compactifications of moduli spaces for abelian varieties

For a long time, it was not clear how to define a moduli theoretic compactification of the moduli space of abelian varieties with only 'minor' singularities along the boundary. A major break through in this direction is the work [Alex] of Alexeev. Its review by J. Kollár contains a concise but very readable historical overview, and points out the main new ideas.

A first conceptual step of Alexeev is to consider no longer moduli $\mathcal{A}_{g,d}$ of abelian varieties A of fixed dimension g with a polarization of degree d, but moduli $\mathcal{AP}_{g,d}$ of pairs (A, Θ) where $A \in \mathcal{A}_{g,d}$ and Θ is an ample divisor defining the polarization. In the principally polarized case $d = 1$ the two agree. But in general Alexeev's moduli form a covering of the previously considered moduli with the fibers being the choices of possible polarizations.

The key new insight of Alexeev is how to describe the boundary of his moduli, as a moduli space $\overline{\mathcal{AP}}_{g,d}$. It parametrizes so called semi-abelic 'pairs' which are triples $(G \curvearrowright P, \theta)$ where G is a semiabelian variety of dimension g, P is a projective variety with an ample divisor θ of degree d, G acts on P with finitely many orbits, and P has a few further nice properties. (The 'pair' is (P, θ).)

Alexeev's construction has two drawbacks. The first is that in the non-principal polarized case it might be nice to remove the choice of ample divisor at the end of the day. The second is that the moduli constructed by Alexeev have several components, and only one of them contains the open substack of abelian varieties (with a choice of ample divisor). So a natural compactification is obtained by singling out this one 'main' component.

Solutions to both problems are presented by M. Olsson in [Ols2]. Adding log-structures into Aleexev's moduli, he is able to directly single out the main component in the second problem. He does give a solution to the first problem as well – however it is not clear to the writer of this program to what extend Olsson's approach is based on Alexeev's moduli $\overline{\mathcal{AP}}_{g,d}$. The program will focus on the principally polarized case.

We mainly follow in our seminar §3 of the book [Ols2] of Martin C. Olsson with the same title. The easier accessible §2 which serves as an introduction to §3 has to be consulted at various places to understand the notations.

lecture 1: $\mathcal{A}_g$ the moduli space of abelian varieties

10.04.08, Gebhard Böckle

We review Mumford's construction of moduli given in [Mum1], §1–6.

lecture 2: $\overline{\mathcal{A}_g}$ Mumford's compactification of $\mathcal{A}_g$

17.04.08, Eckart Viehweg

Here the compactification $\mathcal{A}_g$ constructed by Mumford and its properties are explained.

lecture 3: logarithmic algebraic geometry

24.04.08, Franziska Heinloth

The aim of this talk is to present the material of sections 1–3 of Kato's article [Kato]. (We are only interested in fine log structures)

- define (pre) log structures
- construction of the associated log structure of a pre-log structure
- morphisms of (pre) log structures
- f_* and f^*
- Examples 1.5
- log differentials, explain why $(0, 1 \otimes a) = d\log(a)$.
- definition of charts and their existence (étale locally)
- definition of log-smooth/étale, and the basic examples 3.7.1 and 3.7.2
- characterisation of log-smoothness/étaleness (Theorem 3.5)
- connection of log-smoothness and $\omega^1_{X/Y}$.

(The numbers refer to Kato's article.) The main emphasis should be on the examples and definitions, because the time is limited to 90 minutes.

lecture 4: Artin stacks
8.05.08, Georg Hein
We recall the definition of Artin stacks and explain why they naturally appear in the theory of moduli spaces.

lecture 5: 3.1 (part one) from [Ols2]
15.05.08, Alex Küronya
Give the definitions and results from the first half of 3.1 in [Ols2] (3.1.1 –3.1.11). Here the focus should be in understanding the definitions and constructions for the examples $X = \mathbb{Z}$, for the paving obtained by the quadratic function $a(n) = n^2$, and for $X = \mathbb{Z}^2$ for the pavings obtained by the quadratic functions $a_t(n,m) = n^2 + tmn + m^2$ for $t \in \{-1, 0, 1\}$. To show the finite generatedness and sharpness of H_S in these examples (or at least one of them) will probably help more than to present the proofs of these statements.
Try to explain the construction 3.1.10 for the example $X = \mathbb{Z}$ and give some hints (speculations) about the geometry in the other examples. End with Lemma 3.1.11.
Note: In 3.1.1 there appears twice a Q which should be replaced by $X_{\mathbb{R}}$.

lecture 6: 3.1 (part two) and 3.6 from [Ols2]
29.05.08, Manuel Blickle
As in the previous talk we think, that it would help in understanding the presentation in [Ols2], if we understood this construction for the easiest examples. That is, even to explain the most basic example $B = \mathrm{Spec}(K)$, and $X = \mathbb{Z}$ would be a valuable contribution.
Maybe you start with the definition of the moduli problem from 3.6 in [Ols2] and show how the standard families constructed in 3.1.22 fit into the given moduli problem.

under this line there is work to be done for the organizers

lecture 5: Semiabelic pairs and linearization of torus actions
15.05.08, Alex Küronya
This talk is based on [Alex]. Start by giving the precise definition of *seminormal variety* and of *semiabelic pair* (also in the relative situation) from [Alex], §1.1. Explain briefly why in the case that the toric part is trivial, a semiabelic pair of degree 1 is simply an abelian variety. (cf. [Alex], Cor 3.0.9.)
The main part should explain Ch.4 of Alexeev's work. The key result is Thm. 4.3.1. It says that if one has torus action on a pair (P, L) with P a 'nice' proper scheme and L ample on P, there is a cover $(\tilde{P}, \tilde{L})$ with an action by the character group X of T and quotient (P, L) **and** such that now T acts on $\tilde{P}$ **and** on $\tilde{L}$. (this goes back to some ad hoc constructions of Mumford from [Mum2]).
Along the way there appears a version of the theorem of the square for semiabelian varieties (4.1.6, 4.1.7m 4.1.18) which probably needs to be stated without proof, and its consequence 4.1.22 which is needed in proving the desired correspondence.

lecture 6:
29.05.08, Manuel Blickle
The main aim of the present lecture is to reveal the usefulness of the content of the previous one. Again the source is [Alex]. Abelic pairs $(G \curvearrowright P, L)$ with a linear torus action possess a useful combinatorial description – in spirit similar to that of toric varieties. In this talk, the semiabelian variety G and a polarization of its abelian quotient A are fixed. Moreover one should stick to principal polarized A.
The talk starts with §5.2 of Alexeev. It seems very useful to describe the entire section in some detail. Many things are based on it. The second half of the talk should cover §5.3 where the non-linearized case is described using the results from the previous talk. It culminates in Thm. 5.3.8. Unfortunately the precise combinatorial description is kind of a mess. It requires definitions from 1.1.16 to 1.1.29, §2.1 and §2.2 and §5.1. For simplicity, it should be assumed that $\rho \colon |\widetilde{\Delta}| \to X_{\mathbb{R}}$ is injective. – Alexeev's description of the (co-)homology of cell complexes with values in local systems seems 'heavy-handed' – for injective ρ, hopefully, the situation should simply considerably. It is enough to give the audience an idea of the combinatorics without laying it all out in front of us.
At the very end, the extensions Thm 5.4.1 and 5.4.3 to the case of semiabelic pairs, $(G \curvearrowright P, \theta)$, could be mentioned.

lectures 7 and 8: Degenerating abelian varieties over complete rings
5.06.08 and 12.06.08, N.N.
These two talks should explain the content of [Mum2]. At the end it would be good to come back to Alexeev's work and integrate (in some way) [Alex], §5.6.

lecture 9: $\overline{\mathcal{AP}}_{g,d}$ is a moduli stack
19.06.08, N.N.
The main assertion is Thm. 5.10.1. We suggest to give a detailed proof of properness, which is Theorem 5.7.1. This should be the main part of the talk. In the remaining time. the speaker could go over (without going into many details) the further results are needed. A brief review of Artin's axioms might help. Whatever time is left could be spend on saying somthing about §5.9 in which nice families covering the stack are constructed. The construction 'is as in' [Mum2], but perhaps not much beyond that idea can be explained.

lecture 10: Olsson's standard family
26.06.08, N.N.
Cover 3.1 from [Ols2]. As in talk 6, the combinatorics are not light to digest. Perhaps this is intrinsic to the rather difficult subject. The new feature is the appearance of a log structure.
Olsson's works with different objects than Alexeev. Olsson's central combinatorial object is the monoid H_S. To give some insight into H_S, some examples seem more useful than proving properties. We recommend $X = \mathbb{Z}$, for the paving obtained by the quadratic function $a(n) = n^2$, and $X = \mathbb{Z}^2$ for the pavings obtained by the quadratic functions $a_t(n,m) = n^2 + tmn + m^2$ for $t \in \{-1, 0, 1\}$. It is not recommended to give proofs for 3.1.4–3.1.9 but the point them out int the example.
Once H_S is 'understood', the family in 3.1.10 is easy to describe. Perhaps it is useful to prove 3.1.11 and 3.1.13. The content of 3.1.14-3.1.21 was essentially covered in talks 5 and 6. So one should be able to move rapidly from 3.1.13 to 3.1.22, i.e., to stating the standard family. It would be nice if the talk ends by stating Olsson's moduli problem described in §3.6 and explaining how the the standard families constructed in 3.1.22 fit. (they describe the strata of certain degeneration types).
Note: In 3.1.1 there appears twice a Q which should be replaced by $X_{\mathbb{R}}$.

lecture 11: Deformations and automorphisms
3.07.08, N.N. It would be good if the talk was given by someone with a good conceptual understanding of deformation theory.
This is an outrageous task. Namely to cover 3.2–3.4 from [Ols2]. The recommendation is to simply give precise statement of the results on automorphisms as given in §3.2 and §3.4, i.e. of Prop. 3.2.2 and Prop 3.4.2, and then the spend the majority of the time explaining the deformation theory of §3.3. The key result is Prop. 3.3.3. It would be good to briefly recall something about deformation theory and a theory of obstructions and then to explain parts of the proof.

lecture 12: Versal families and Alexeev's main component
10.07.08, N.N.
Cover 3.5 in [Ols2]. Describe Olsson's versal family over $\mathcal{W}_3$ and the action of the group $\mathcal{G}$. Olsson stops short of redoing §5.9 from [Alex] which in turn is based on [Mum2] who algebraizes a formal construction. After explaining the objects, the main emphasis should be on relating W_3 (at least its infinitesimal versions) to the normalization $\widetilde{\mathcal{Q}}$ of Alexeev's main component $\mathcal{Q}$: 3.5.15-3.5.20. Since the families constructed here cover Olsson's space $\mathcal{K}_g$ this indicates that one might expect a morphism $\mathcal{K}_g \to \widetilde{\mathcal{Q}}$.

lecture 13: A morphism $\widetilde{\mathcal{Q}} \to \mathcal{K}_g$ and its isomorphy
17.07.08, N.N.
Again, the source is [Ols2]. A morphism $\widetilde{\mathcal{Q}} \to \mathcal{K}_g$ is defined and discussed in 3.7.4 and 3.7.5. After explaining this, it is up to the speaker to either explain §3.8 on approximation, or to explain the proof of the main theorem of Ch.3 by Olsson, namely of Thm. 3.6.2 assume the results from §3.8.

under this line you find the organizers

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The material can be found at: `http://www.uni-essen.de/~mat903/sem/ss08/`

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