Deforming Galois Representations

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for A. Mazur

Given a continuous homomorphism

$$G_{\mathbf{Q},S} \xrightarrow{\bar{\rho}} GL_2(\mathbf{F}_p)$$

where $G_{\mathbf{Q},S}$ is the Galois group of the maximal algebraic extension of $\mathbf{Q}$ unramified outside the finite set S of primes of $\mathbf{Q}$, the motivating problem of this paper is to study, in a systematic way, the possible liftings of $\bar{\rho}$ to p-adic representations,

$$G_{\mathbf{Q},S} \xrightarrow{\rho_o} GL_2(\mathbf{Z}_p).$$

We use the techniques of deformation theory. There have been numerous studies of the global variation of representations over $\mathbf{C}$ of *finitely generated* groups, cf. the memoir of Lubotzky and Magid **[L-M]** or the recent preprint of Goldman and Millson **[G-M]**. The viewpoint we adopt here is similar, with the exception that in our context (our groups are profinite and our representations are p-adic) it makes sense only to consider formal deformations. We prove that if $\bar{\rho}$ is absolutely irreducible there is a *universal deformation* of $\bar{\rho}$, i.e., a complete noetherian local ring $R = R(\bar{\rho})$ with residue field $\mathbf{F}_p$, and a continuous homomorphism

$$G_{\mathbf{Q},S} \xrightarrow{\boldsymbol{\rho}} \mathrm{GL}_2(R)$$

(well-defined up to conjugation by an element in $\mathrm{GL}_2(R)$ which reduces to the identity matrix modulo the maximal ideal in R) which is *universal* in an evident sense. Under the assumption that $p > 2$ and that S contains the primes p and ∞, we show that the Krull dimension of R/pR is ≥ 1 if $\det(\bar{\rho})$ is even, and it is ≥ 3 if $\det(\bar{\rho})$ is odd, with equality holding if the deformation problem is unobstructed.

I have no examples where there is strict inequality above, or where Spec R is not equidimensional, or where p is nilpotent on an irreducible component of Spec R. It would be of great interest to have some better understanding of the basic geometric features of R, for all, or for a large class of representations $\overline{\rho}$. The viewpoint of the present article is rather to focus on a very special class of representations, the structure of whose universal deformations can be analyzed in some depth.

We give a large number of examples where $\det(\overline{\rho})$ is odd, the image of $\overline{\rho}$ in $\mathrm{GL}_2(\mathbf{F}_p)$ is isomorphic to the symmetric group on three letters, and the deformation problem is unobstructed.

These examples are instances of what we call *special dihedral representations* in Chapter II. If $\overline{\rho}$ is special dihedral, the universal deformation ring R is isomorphic to a power series ring over $\mathbf{Z}_p$ in three variables, and the universal deformation space $X = \mathrm{Hom}(R, \mathbf{Z}_p)$ is a three-dimensional $\mathbf{Q}_p$-analytic manifold, whose points $x \in X$ correspond to representations $\rho_x : G_{\mathbf{Q},S} \longrightarrow \mathrm{GL}_2(\mathbf{Z}_p)$ lifting $\overline{\rho}$ (but taken only up to conjugation by an element in $\mathrm{GL}_2(\mathbf{Z}_p)$ which reduces to the identity matrix $\bmod\ p$). Chapter II is devoted to the 'fine structure' of X. We show that the locus of $x \in X$ such that the image of ρ_x is dihedral, i.e., is contained in the normalizer of a Cartan subgroup in $\mathrm{GL}_2(\mathbf{Q}_p)$, is a *smooth hypersurface* in X. We show that the locus of inertially reducible ρ_x's is a *union of two smooth hypersurfaces in* X. We show that the *ordinary* representations ρ_x (cf. Ch. I, §7) trace out a smooth curve in X; they are approximable by representations attached to modular forms. Are *all* representations ρ_x for $x \in X$ similarly approximable?

We show that the inertially ample locus (i.e., the locus of $x \in X$ for which the image of inertia under ρ_x contains an open subgroup of finite index in $\mathrm{SL}_2(\mathbf{Z}_p)$) is open and dense in X (Ch. II, §7: the "approximation theorem").

Our analysis of the fine structure of X leaves open a number of questions. For example, what precisely is the inertially ample locus in X? How do the three hypersurfaces alluded to in the preceding paragraph intersect? These issues and others will be dealt with for a certain subclass of special dihedral representations in a joint article with Nigel Boston, currently under preparation.

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1. Universal Deformations of Representations.

1.1 Deformations.

Fix a prime number p. A profinite group Π is said to satisfy the *finiteness condition* Φ_p if for every open subgroup of finite index $\Pi_0 \subset \Pi$, the following equivalent conditions hold:

Either

(a) The pro-p-completion of Π_0 is topologically finitely generated, or

(b) The abelianized pro-p-completion of Π_0, given its natural $\mathbf{Z}_p$-module structure, is of finite type over $\mathbf{Z}_p$, or

(c) There are only a finite number of continuous homomorphisms from Π_0 to $\mathbf{F}_p$.

Examples of profinite groups Π satisfying Φ_p for all p, are given by groups arising as algebraic fundamental groups of smooth (geometrically connected) schemes of finite type over $\mathbf{Z}$ ([**K-L**]).

In particular, for K any number field and S any finite set of primes of K, we may take $\Pi = G_{K,S}$ the Galois group of the maximal field extension of K in an algebraic closure, which is unramified outside S. We may also take $\Pi = G_K$, the Galois group of an algebraic closure of any local field K.

In this section, Π will denote a profinite group satisfying condition Φ_p, and k will refer to a finite field of characteristic p. Let $\mathcal{C}$ denote the category of complete noetherian local rings with residue field k. We refer to an object of $\mathcal{C}$ as a "local ring in $\mathcal{C}$." A morphism of the category $\mathcal{C}$ is a homomorphism of complete local rings inducing the identity on residue fields. If A is a local ring in $\mathcal{C}$, then its maximal ideal is denoted m_A.

Let N be a positive integer. If A is a local ring in $\mathcal{C}$, two continuous homomorphisms from Π to $\mathrm{GL}_N(A)$ will be said to be *strictly equivalent* if one can be brought to another by conjugation with an element in the kernel of the reduction map $\mathrm{GL}_N(A) \longrightarrow \mathrm{GL}_N(k)$.

By a *representation* of Π in $\mathrm{GL}_N(A)$ we shall mean a strict equivalence class of continuous homomorphisms from Π to $\mathrm{GL}_N(A)$. Thus, if $A = k$, a representation is nothing more than a continuous homomorphism. By abuse of language, we sometimes write "$\rho_0 : \Pi \longrightarrow \mathrm{GL}_N(A)$" where ρ_0 is a representation.

If $A_1 \longrightarrow A_2$ is a morphism in the category $\mathcal{C}$ and if ρ_1 and ρ_2 are representations of Π in $\mathrm{GL}_N(A_1)$ and in $\mathrm{GL}_N(A_2)$, respectively, we shall say that ρ_1 is a *deformation* of ρ_2 if any homomorphism from Π to $\mathrm{GL}_N(A_1)$ in the strict equivalence class ρ_1, composed with the induced homomorphism $\mathrm{GL}_N(A_1) \longrightarrow \mathrm{GL}_N(A_2)$, yields a homomorphism in the strict equivalence class ρ_2.

By a *residual representation of dimension* N (in a context where Π and k are understood) we shall mean a continuous homomorphism

$$\overline{\rho}:\ \Pi \longrightarrow \mathrm{GL}_N(k),$$

i.e., a representation of Π in $\mathrm{GL}_N(k)$.

Two residual representations are *equivalent* if, as usual, one can be brought into the other by conjugation by an element in $\mathrm{GL}_N(k)$; they are *"twist"-equivalent* if one, after tensoring with a suitable one-dimensional representation, can be made equivalent to the other.

1.2 Existence of universal deformation rings.

Fix Π and k as in §1. The object of this chapter is to establish the existence of a universal deformation of any absolutely irreducible N-dimensional residual representation $\overline{\rho}$. Specifically, there is a complete noetherian local ring

$$R = R(\Pi, k, \overline{\rho}) \in \mathcal{C}$$

with residue field k, together with a deformation

$$\boldsymbol{\rho}:\ \Pi \longrightarrow \mathrm{GL}_N(R)$$

of $\overline{\rho}$ which is universal in the sense that for any $A \in \mathcal{C}$ and deformation ρ_0 of $\overline{\rho}$ to A, there is a unique morphism $R \to A$ in $\mathcal{C}$ such that the induced homomorphism $\mathrm{GL}_N(R) \to \mathrm{GL}_N(A)$ brings $\boldsymbol{\rho}$ to ρ_0. We shall show that the pair $(R, \boldsymbol{\rho})$ is determined up to canonical isomorphism by the twist-equivalence class of $\overline{\rho}$. The ring R also has some other attendant structures. For example, R is endowed in a natural manner with the structure of Λ-algebra where Λ is the local ring in $\mathcal{C}$ described in §4. The ring Λ has a natural co-multiplication law giving it the structure of formal Hopf algebra coming from a commutative formal group over $W(k)$, Φ, as described in

§4. The formal group Φ operates naturally on the local ring R in a manner compatible with its Λ-algebra structure and with the natural action of Φ on Λ. If $\overline{\rho}$ is twist-equivalent to its contragredient, there is an involution (the "duality involution" ι, §5) of R compatible with inversion in the formal group Φ.

The local ring $R = R(\Pi, k, \overline{\rho})$ will be called the *universal deformation ring* of $\overline{\rho}$, and Spec R will be called the *universal deformation space*.

REMARK: If $A \in \mathcal{C}$, then the kernel of $\mathrm{GL}_N(A) \to \mathrm{GL}_N(k)$ is a pro-p-group, and therefore any deformation of $\overline{\rho}$ factors through the maximal quotient $\Pi \twoheadrightarrow \widetilde{\Pi}$ such that the image in $\widetilde{\Pi}$ of the kernel of $\overline{\rho}$ is a pro-p-group. We call $\widetilde{\Pi}$ the *p-completion of* Π *relative to* $\overline{\rho}$. It follows that $R(\Pi, k, \overline{\rho}) = R(\widetilde{\Pi}, k, \overline{\rho})$.

PROPOSITION 1: Existence and Uniqueness. *(a) if $\overline{\rho}$ is absolutely irreducible, a universal deformation ring $R = R(\Pi, k, \overline{\rho})$ and a universal deformation $\boldsymbol{\rho}$ of $\overline{\rho}$ to R exists. The pair $(R, \boldsymbol{\rho})$ is uniquely determined up to canonical isomorphism by the twist-equivalence class of $\overline{\rho}$ in the following sense:*

Given two twist-equivalent residual representations $\overline{\rho}$ and $\overline{\rho}'$, there is a canonical isomorphism

$$r(\overline{\rho}', \overline{\rho}):\ R(\Pi, k, \overline{\rho}) \xrightarrow{\cong} R(\Pi, k, \overline{\rho}')$$

bringing the universal deformation $\boldsymbol{\rho}$ of $\overline{\rho}$ to the universal deformation $\boldsymbol{\rho}'$ of $\overline{\rho}'$. The system of canonical isomorphisms have the homomorphic property:

(i) $r(\overline{\rho}, \overline{\rho})$ *is the identity, for all* $\overline{\rho}$

(ii) $r(\overline{\rho}'', \overline{\rho}') \cdot r(\overline{\rho}', \overline{\rho}) = r(\overline{\rho}'', \overline{\rho})$

(b) if $\overline{\rho}$ is not absolutely irreducible, then a "versal" deformation of $\overline{\rho}$ exists, i.e., there is a hull *in the sense of Schlessinger [***Sch***] which means that we can find an object $R \in \mathcal{C}$ and a deformation $\boldsymbol{\rho}$ of $\overline{\rho}$ to R such that any deformation ρ_0 of $\overline{\rho}$ to any object A in $\mathcal{C}$ is induced by* a not necessarily unique *morphism $R \to A$ of $\mathcal{C}$; however, if A is the "dual numbers" $k[\epsilon]$, the morphism $R \to A$ bringing $\boldsymbol{\rho}$ to ρ_0 is unique.*

The isomorphism-type of the hull R is unique, but R itself is not determined up to canonical *isomorphism.*

In this section we prove the existence of universal and versal deformation rings. The uniqueness assertions will not be established until §3.*b*.2.

Let

$$\begin{array}{ccccc} & & A_3 & & \\ & \swarrow & & \searrow & \\ A_1 & & & & A_2 \\ & \searrow & & \swarrow & \\ & & A_0 & & \end{array}$$

be a cartesian diagram of artinian rings in $\mathcal{C}$, i.e., $A_3 \cong A_1 \times_{A_0} A_2$.

Suppose also that $A_1 \to A_0$ is a "small" extension in the sense of Schlessinger [**Sch**], i.e., it is surjective with kernel a nonzero principal ideal (t) such that $m_{A_1} \cdot (t) = 0$.

Set

$$E_i = \mathrm{Hom}_{\overline{\rho}}(\Pi, \mathrm{GL}_N(A_i)) \text{ for } i = 0, \ldots, 3$$

where the subscript $\overline{\rho}$ means continuous homomorphisms which are liftings of $\overline{\rho}$.

Set

$$G_i = \mathrm{Ker}(\mathrm{GL}_N(A_i) \to \mathrm{GL}_n(k)) \text{ for } i = 0, \ldots, 3.$$

Then G_i acts naturally on E_i by conjugation of the range $\mathrm{GL}_N(A_i)$, and the orbit-space E_i/G_i may be identified with the space of deformations of $\overline{\rho}$ to A_i. We have the natural morphism:

$$b: \ E_3/G_3 \longrightarrow E_2/G_2 \times_{E_0/G_0} E_1/G_1.$$

Since $G_1 \to G_0$ is surjective, one easily checks that b is surjective. A straightforward calculation yields the following criterion for injectivity of b:

Let φ_1 denote an element of E_1 and φ_0 its image in E_0. Set $G_i(\varphi_i) :=$ the subgroup of G_i consisting of all elements commuting with the image of φ_i in $\mathrm{GL}_N(A_i)$, for $i = 0, 1$.

LEMMA 1. *If, for all $\varphi_1 \in E_1$, the natural mapping*

$$G_1(\varphi_1) \to G_0(\varphi_0)$$

is surjective, then b is injective.

A straightforward application of Schur's lemma yields

LEMMA 2. *If $\overline{\rho}$ is absolutely irreducible, then $G_i(\varphi_i)$ consists in the subgroup of scalar matrices in $G_i \subset \mathrm{GL}_N(A_i)$, for $i = 0, 1$.*

REMARK: The automorphism group of our moduli problem is, by lemma 2, the formal completion of the multiplicative group, which is formally smooth. Therefore the hypothesis of Lemma 1 holds and b is injective.

We are now ready to prove our proposition, using Schlessinger's criteria **[Sch]**.

The tangent space of the functor which assigns to $A \in \mathcal{C}$ the set of deformations of $\overline{\rho}$ to A is canonically isomorphic to the k-vector space

$$H^1(\Pi, \mathrm{Ad}(\overline{\rho}))$$

where $\mathrm{Ad}(\overline{\rho})$ is the k-vector space of $N \times N$ matrices with entries in k, viewed as Π-module via an action obtained by composition of $\overline{\rho}$ with the adjoint representation (i.e., conjugation) of $\mathrm{GL}_N(k)$.

Since Π satisfies the condition Φ_p, we easily check that $H^1(\Pi, \mathrm{Ad}(\overline{\rho}))$ is finite-dimensional. In particular, condition (H_3) in Theorem 2.11 of **[Sch]** is always satisfied, as is condition (H_1).

To apply Theorem 2.11, we must check (H_2) in general, and (H_4) when $\overline{\rho}$ is absolutely irreducible. But if $A_0 = k$, $A_1 = k[\epsilon]$, then the morphism in Lemma 1 is clearly surjective, giving (H_2), and if $\overline{\rho}$ is absolutely irreducible, Lemma 2 implies that the morphism in Lemma 1 is surjective for all surjective maps $A_1 \to A_0$ (of the sort we are considering), whence (H_4).

If $\overline{\rho}$ is absolutely irreducible, we refer to $R = R(\pi, k, \overline{\rho})$ as the *universal deformation ring* of $\overline{\rho}$. The universal deformation ring is unique in the sense that it is determined up to canonical isomorphism. In general (i.e., if $\overline{\rho}$ is not necessarily absolutely irreducible) the "versal deformation ring" R is determined up to (noncanonical) isomorphism (which induces the "identity mapping" on Zariski tangent spaces; see **[Sch]**).

Having obtained the universal deformation ring R, it is now an easy matter to construct the universal deformation $\boldsymbol{\rho}$. Specifically, for every power of the maximal ideal m of R, we have a deformation ρ_n of $\overline{\rho}$ to R/m^n which can be realized by a *compatible* family of liftings

$$r_n : \Pi \to \mathrm{GL}_N(R/m^n),$$

using the surjectivity of the homomorphisms $\mathrm{GL}_N(R/m^{n+1}) \to \mathrm{GL}_N(R/m^n)$. The universal deformation $\boldsymbol{\rho}$ of $\overline{\rho}$ is then just the strict equivalence class of the inverse limit $\varprojlim_n r_n$.

1.3 Functoriality.

In this section we shall deal only with absolutely irreducible residual representations. There are analogous statements to be made for general residual representations, but they are somewhat more complicated.

(a) **Change of range:**
Fix Π and k. Let $W(k)$ denote the ring of Witt vectors of k. Let

$$\delta_{/W(k)}: \ \mathrm{GL}_{N_{/W(k)}} \longrightarrow \mathrm{GL}_{M_{/W(k)}}$$

be a homomorphism of group schemes. Let

$$\overline{\rho}: \ \Pi \longrightarrow \mathrm{GL}_N(k)$$

be a residual representation, and let $\overline{\rho}'$ be the composition of $\overline{\rho}$ with $\delta_{/k}$. The composition with δ brings deformations of $\overline{\rho}$ to deformations of $\overline{\rho}'$. If $\overline{\rho}$ and $\overline{\rho}'$ are absolutely irreducible and $R = R(\Pi, k, \overline{\rho}), R' = R(\Pi, k, \overline{\rho}')$, then composition with δ' induces a morphism

$$r(\delta) : R' \to R$$

in the category $\mathcal{C}$. The system of morphisms $\delta \longmapsto r(\delta)$ has the homomorphic property:

$$\text{(i)} \quad r(1) = 1$$
$$\text{and (ii)} \quad r(\delta_1) \cdot r(\delta_2) = r(\delta_1 \delta_2).$$

(a.1) **(conjugation)** In particular, if

$$\delta_g: \ \mathrm{GL}_{N_{/W(k)}} \to \mathrm{GL}_{N_{/W(k)}}$$

is given by conjugation with a fixed element $g \in \mathrm{GL}_N(W(k))$ we obtain an isomorphism in $\mathcal{C}$,

$$r(\delta_g): \ R(\Pi, k, \overline{\rho}) = R(\Pi, k, \overline{\rho}'),$$

where $\overline{\rho}'$ is the residual representation equivalent to $\overline{\rho}$ obtained by conjugation by $\overline{g} \in \mathrm{GL}_N(k)$, the reduction of g. Clearly, the isomorphism $r(\delta_g)$

depends only upon the image of $\overline{g}$ in $\mathrm{PGL}_N(k)$. But since $\overline{\rho}$ is absolutely irreducible, an application of Schur's lemma guarantees that the image of $\overline{g}$ in $\mathrm{PGL}_N(k)$ is completely determined by the pair $(\overline{\rho}',\overline{\rho})$. Consequently, $r(\delta_g)$ depends only upon the pair $(\overline{\rho}',\overline{\rho})$. Put $r(\overline{\rho}',\overline{\rho}) = r(\delta_g)$. We have therefore defined the system of homomorphisms $r(\overline{\rho}',\overline{\rho})$ of the Proposition in §2, for any pair of equivalent (absolutely irreducible) residual representations.

(a.2) **(duality)** Let

$$\tau : \mathrm{GL}_{N_{/W(k)}} \longrightarrow \mathrm{GL}_{N_{/W(k)}}$$

be the outer automorphism "transpose-inverse". Then if

$$\overline{\rho} : \ \Pi \rightarrow \mathrm{GL}_N(k)$$

is an absolutely irreducible residual representation, and $\overline{\rho}^*$ is its composition with τ, i.e., the contragredient representation, and if R, R^* are the universal deformation rings of $\overline{\rho}$ and $\overline{\rho}^*$ respectively, we have morphisms

$$R \xrightarrow{r(\tau)} R^* \xrightarrow{r(\tau)} R$$

which can easily be seen to be two-sided inverses of one another. This establishes canonical identifications of R and R^*.

(a.3) **(determinant)** Let $\overline{\rho}$ be an N-dimensional (absolutely irreducible) residual representation, and let

$$\delta = \det : \ \mathrm{GL}_{N_{/W(k)}} \rightarrow \mathrm{GL}_{1_{/W(k)}}$$

be the determinant homomorphism. We obtain a natural homomorphism

$$R(\Pi, k, \det(\overline{\rho})) \xrightarrow{r(\det)} R(\Pi, k, \overline{\rho})$$

to which we shall return later.

(b) **Tensor product.**
Let $\overline{\rho}_1$: $\Pi \rightarrow \mathrm{GL}_N(k)$ and $\overline{\rho}_2 : \ \Pi \rightarrow \mathrm{GL}_M(k)$ be two residual representations, and let

$$\overline{\rho}_1 \otimes \overline{\rho}_2 : \ \Pi \rightarrow \mathrm{GL}_{N\cdot M}(k)$$

denote their tensor product.

To any pair of deformations, of $\overline{\rho}_1$ to $A_1 \in \mathcal{C}$ and of $\overline{\rho}_2$ to $A_2 \in \mathcal{C}$ we can naturally associate a deformation of $\overline{\rho}_1 \otimes \overline{\rho}_2$ to the completed tensor product $A_1 \widehat{\otimes}_{W(k)} A_2$. Let $\overline{\rho}_1$, $\overline{\rho}_2$, and $\overline{\rho}_1 \otimes \overline{\rho}_2$ be absolutely irreducible. We get a natural homomorphism

$$R(\overline{\rho}_1 \otimes \overline{\rho}_2) \stackrel{h(\overline{\rho}_1, \overline{\rho}_2)}{\longrightarrow} R(\overline{\rho}_1) \widehat{\otimes}_{W(k)} R(\overline{\rho}_2)$$

where $R(\overline{\rho})$ refers to the universal deformation ring $R(\Pi, k, \overline{\rho})$. The system of homomorphisms $(\overline{\rho}_1, \overline{\rho}_2) \longmapsto h(\overline{\rho}_1, \overline{\rho}_2)$ satisfies evident *commutativity* and *associativity* properties whose explicit descriptions we leave to the reader.

(b.1) **(contraction with a lifting of $\overline{\rho}_1$)** Now let $\rho_1 : \Pi \to \mathrm{GL}_N(W(k))$ be a deformation of $\overline{\rho}_1$ to $W(k)$. Thus ρ_1 is induced from the deformation of $\overline{\rho}_1$ via a unique homomorphism $h_{\rho_1} : R(\overline{\rho}) \to W(k)$. Define

$$h(\rho_1, \overline{\rho}_2) \; : \; R(\overline{\rho}_1 \otimes \overline{\rho}_2) \to R(\overline{\rho}_2)$$

to be the composition of $h(\overline{\rho}_1, \overline{\rho}_2)$ with $h_{\rho_1} \otimes 1$.

From the associative property referred to above, one sees that the following diagram

$$\begin{array}{ccc} & R(\overline{\rho}_0 \otimes \overline{\rho}_1 \otimes \overline{\rho}_2) & \\ h(\rho_0, \overline{\rho}_1 \otimes \overline{\rho}_2) \swarrow & & \searrow h(\rho_0 \otimes \rho_1, \overline{\rho}_2) \\ R(\overline{\rho}_1 \otimes \overline{\rho}_2) & \xrightarrow[h(\rho_1, \overline{\rho}_2)]{} & R(\overline{\rho}_2) \end{array}$$

commutes, where the relevant residual representations are absolutely irreducible, and ρ_0, ρ_1 are deformations of $\overline{\rho}_0$, $\overline{\rho}_1$ respectively, to $W(k)$.

(b.2) **(twisting by a character)** In the special case where $\overline{\rho}_1$ is one-dimensional, we refer to $h(\rho_1, \overline{\rho}_2)$ as the *twisting morphism* by ρ_1, and sometimes denote it simply $h(\rho_1)$. From the commutative triangle displayed above, one sees that the twisting morphisms are isomorphisms in the category $\mathcal{C}$ and enjoy the evident homomorphic property in the variable ρ_1.

Now let $\overline{\rho}$ be any absolutely irreducible residual representation, and let $\overline{\rho}'$ be the tensor product of $\overline{\rho}$ with a one-dimensional residual representation $\overline{\rho}_1$. Let ρ_1 denote the Teichmüller lifting to $W(k)^*$ of the character $\overline{\rho}_1$, and let

$$r(\overline{\rho}, \overline{\rho}') \to R(\overline{\rho})$$

be the twisting isomorphism $h(\rho_1, \overline{\rho})$. This, together with the discussion in (a.1), enables us to define canonical isomorphisms $r(\overline{\rho}', \overline{\rho})$ for any pair of twist-equivalent residual representations such that $(\overline{\rho}', \overline{\rho}) \mapsto r(\overline{\rho}', \overline{\rho})$ possesses the homomorphic property stated in the Proposition of §2.

(c) **Change of domain.**
Let

$$\Pi \xrightarrow{\varphi} \Pi'$$

be a continuous homomorphism (between profinite groups satisfying Φ_p). Let

$$\overline{\rho}' : \ \Pi' \to \mathrm{GL}_N(k)$$

be a residual representation. Let $\overline{\rho}$ be the residual representation

$$\overline{\rho} : \ \Pi \to \mathrm{GL}_N(k)$$

obtained by composing $\overline{\rho}'$ with φ. Suppose that both $\overline{\rho}$ and $\overline{\rho}'$ are absolutely irreducible. Set

$$R = R(\Pi, k, \overline{\rho}) \text{ and } R' = R(\Pi', k, \overline{\rho}').$$

Then, composition with φ brings deformation of $\overline{\rho}'$ to deformations of $\overline{\rho}$ and therefore induces a homomorphism

$$r(\varphi) : \ R \to R'$$

in the category $\mathcal{C}$.

The system $\varphi \mapsto R(\varphi)$ is homomorphic in φ. If φ is surjective, then for all A in $\mathcal{C}$, $R(\varphi)$ induces an injection

$$\mathrm{Hom}_{\mathcal{C}}(R', A) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(R, A).$$

(d) **Change of field.**
Let $i: \ k \hookrightarrow k'$ be a morphism of finite fields of characteristic p. Let $\overline{\rho}: \Pi \to \mathrm{GL}_N(k)$ be an absolutely irreducible residual representation and let $\overline{\rho}': \Pi \to \mathrm{GL}_N(k')$ be the representation obtained from $\overline{\rho}$ by extension of scalars via i.

Then tensoring with $W(k')$ over $W(k)$ brings deformations of $\overline{\rho}$ to deformations of $\overline{\rho}'$ and therefore induces a natural morphism

$$R(i) \ : \ R' \to R \otimes_{W(k)} W(k')$$

in the category $\mathcal{C}$, where $R = R(\Pi, k, \overline{\rho})$ and $R' = R(\Pi, k', \overline{\rho}')$.

The morphism $R(i)$ induces an isomorphism on Zariski tangent spaces.

1.4 One-dimensional representations.

Fix Π and k. Since any two one-dimensional residual representations of Π are projectively equivalent, it follows from the proposition of §2 that when $\overline{\rho}$ is one-dimensional, the universal deformation ring $R = R(\Pi, k, \overline{\rho})$ depends (up to canonical isomorphism) only upon Π and k and not upon $\overline{\rho}$. One easily describes this ring:

Put

$\Gamma := \Pi^{ab,p}$ (the abelianized p-completion of Π), and
$\Lambda := W(k)[[\Gamma]]$ (the completed group ring of Γ with coefficients in $W(k)$).

Let $\widetilde{\rho}: \ \Pi \to W(k)^*$ denote the Teichmüller lifting of $\overline{\rho}$; let $\Pi \overset{\gamma}{\twoheadrightarrow} \Gamma$ denote the natural surjection, and $\gamma \mapsto [\gamma]$ the natural injection of Γ into Λ^*. Let $\boldsymbol{\rho}: \ \Pi \to \Lambda^* = \mathrm{GL}_1(\Lambda)$ be the homomorphism given by

$$x \mapsto \widetilde{\rho}(x) \cdot [\gamma(x)] \ \in \ W(k)^* \cdot \Gamma \subset \Lambda^*.$$

Then $\boldsymbol{\rho}$ is a deformation of $\overline{\rho}$ to Λ and is isomorphic to the universal deformation in the sense that the mapping

$$R(\Pi, k, \overline{\rho}) \xrightarrow{\epsilon} \Lambda$$

induced by $\boldsymbol{\rho}$ is an isomorphism in $\mathcal{C}$.

From now on we shall identify Λ with the universal deformation ring $R(\Pi, k, \overline{\rho})$ via ϵ.

If $\overline{\rho}_0$ denotes the principal (i.e., trivial) one-dimensional residual character, then the morphism $h(\overline{\rho}_0, \overline{\rho}_0)$ of §3.b is the "standard" co-multiplication law on Λ:

$$\begin{array}{ccc} R(\overline{\rho}_0) = R(\overline{\rho}_0 \otimes \overline{\rho}_0) & \xrightarrow{h(\overline{\rho}_0, \overline{\rho}_0)} & R(\overline{\rho}_0)\,\widehat{\otimes}_{W(k)} R(\overline{\rho}_0) \\ \epsilon \big\downarrow \cong & & \cong \big\downarrow \epsilon \widehat{\otimes} \epsilon \\ \Lambda & \xrightarrow{\mu} & \Lambda \widehat{\otimes}_{W(k)} \Lambda \end{array}$$

$$[\gamma] \longmapsto [\gamma] \mathbin{\widehat{\otimes}} [\gamma]\,.$$

Denote by Φ the formal group with affine coordinate ring whose formal group law is given by μ. For any absolutely irreducible residual representation, the morphism

$$R(\overline{\rho}) \cong R(\overline{\rho}_o \otimes \overline{\rho}) \xrightarrow{h(\overline{\rho}_0, \overline{\rho})} \Lambda \widehat{\otimes}_{W(k)} R(\overline{\rho})$$

defines an action of the formal group Φ on the formal scheme $\widehat{X}(\overline{\rho})$ ($cf.$ §2), which commutes with the morphism $r(\delta)$ of (a) above.

In particular, for any such $\overline{\rho}$, we may view $R(\overline{\rho})$ as endowed with a natural Λ-algebra structure, given by the composition

$$\Lambda \underset{\cong}{\overset{\epsilon}{\longleftarrow}} R(\Pi, k, \det\ \overline{\rho}) \xrightarrow{r(\det)} R(\Pi, k, \overline{\rho}) = R(\overline{\rho})$$

and this Λ-algebra structure is compatible with the action of Φ on Λ and on $R(\overline{\rho})$.

1.5 The duality involution.

Let $\overline{\rho}$ be absolutely irreducible, and suppose that there is a one-dimensional residual representation χ such that the contragredient $\overline{\rho}^*$ is equivalent to $\overline{\rho} \otimes \chi$. Then we have a canonical involution

$$\iota : R(\overline{\rho}) \underset{\cong}{\longrightarrow} R(\overline{\rho})$$

defined as the composition

$$R(\overline{\rho}) \underset{\cong}{\xrightarrow{r(\tau)}} R(\overline{\rho}^*) \underset{\cong}{\xrightarrow{r(\overline{\rho},\overline{\rho}^*)}} R(\overline{\rho}).$$

The duality involution ι is inversion ($[\gamma] \mapsto [\gamma^{-1}]$) on Λ, and commutes with the natural Λ-algebra structure morphisms $\Lambda \to R(\overline{\rho})$.

1.6 Obstructions.

Let $\overline{\rho} : \Pi \to \mathrm{GL}_N(k)$ be an absolutely irreducible residual representation. Let $R = R(\overline{\rho})$ denote its universal deformation ring. By $\mathrm{Ad}(\overline{\rho})$ we shall mean the Π-representation whose underlying k-vector space is the space of $N \times N$ matrices with entries in k, $M_N(k)$, and whose Π-action is given by application of $\overline{\rho}$ and then conjugation. Denote by d^i the dimension of the k-vector space $H^i(\Pi,\ \mathrm{Ad}(\overline{\rho}))$, and by δ the difference $d^1 - d^2$.

Let $A_1 \to A_0$ be a surjective mapping of artinian local rings in $\mathcal{C}$ with kernel $I \subset A_1$. Suppose that $I \cdot m_{A_1} = 0$, where m_{A_1} is the maximal ideal of A_1. We view I as a k-vector space (necessarily finite dimensional).

Given any deformation $\rho_0 : \Pi \to \mathrm{GL}_N(A_0)$ of $\overline{\rho}$ to A_0 there is a canonical obstruction class $\mathcal{O}(\rho_0) \in H^2(\Pi,\ \mathrm{Ad}(\overline{\rho})) \otimes I$ which depends only upon the deformation ρ_0, and which vanishes if and only if there is a deformation ρ_1 of $\overline{\rho}$ to A_1, which when projected to A_0 yields the deformation ρ_0. The construction of $\mathcal{O}(\rho_0)$ is standard: Fix a *set-theoretic* mapping $\gamma_1 : \Pi \to \mathrm{GL}_N(A_1)$ which when projected to $\mathrm{GL}_N(A_0)$ yields a homomorphism in the strict equivalence class of ρ_0. Then form the obstruction cocycle:

$$c(g_1, g_2) = \gamma_1(g_1 g_2)\ \gamma_1(g_2)^{-1}\ \in\ 1 + I \otimes M_N(k) \cong I \otimes \mathrm{Ad}(\overline{\rho}).$$

The cohomology class of c in $H^2(\Pi,\ I \otimes \mathrm{Ad}(\overline{\rho})) = H^2(\Pi,\ \mathrm{Ad}(\overline{\rho})) \otimes I$ can be seen to depend only upon the deformation ρ_0 (and not on the chosen mapping γ_1) and is denoted $\mathcal{O}(\rho_0)$.

If ρ_0 does lift to a deformation ρ_1 to A_1, then the set of all such liftings is, in a natural way, a principal homogeneous set under the action of $H^1(\Pi, \mathrm{Ad}(\overline{\rho})) \otimes I$. It follows that the set of deformations of $\overline{\rho}$ to the ring of "dual numbers" $k[\epsilon]$ is canonically isomorphic to the k-vector space $H^1(\Pi, \mathrm{Ad}(\overline{\rho}))$. The set of deformations of $\overline{\rho}$ to $k[\epsilon]$ is naturally endowed (cf. **[Sch]**) with the structure of k-vector space (the "reduced Zariski tangent space" of R, or equivalently: the Zariski tangent space of R/pR) and as such, is the k-dual of $m_R/(m_R^2, p)$. We therefore have a perfect k-duality between $H^1(\Pi, \mathrm{Ad}(\overline{\rho}))$ and $m_R/(m_R^2, p)$.

PROPOSITION 2. *We have the inequality:*

$$\text{Krull}\dim(R/pR) \geq \delta = d^1 - d^2.$$

If $d^2 = 0$ (i.e., the lifting problem for $\overline{\rho}$ is unobstructed) then we have equality above, and moreover R is a formal power series ring in d^1 parameters over $W(k)$.

PROOF: The ring R/pR is the universal deformation ring for characteristic p deformations of $\overline{\rho}$. Let F be a power series ring in d^1 variables over k, and let $F \to R/pR$ be a continuous homomorphism which induces an isomorphism on Zariski tangent spaces. Then $F \to R/pR$ is surjective. Let $J \subset F$ denote its kernel; let $\mathfrak{m} = m_F$, the maximal ideal.

Consider the exact sequence:

$$0 \to J/\mathfrak{m} \cdot J \longrightarrow F/\mathfrak{m} \cdot J \longrightarrow R/pR \longrightarrow 0.$$

Let ρ_0 denote the deformation of $\overline{\rho}$ to R/pR induced from the universal deformation ρ of $\overline{\rho}$ to R. One can define, as above, an obstruction class $\mathcal{O}(\rho_0) \in H^2(\Pi, \mathrm{Ad}(\overline{\rho})) \otimes J/\mathfrak{m} \cdot J$. If V denotes the dual k-vector space to $J/\mathfrak{m} \cdot J$, then $f \mapsto (1 \otimes f)\mathcal{O}(\rho_0))$ defines a homomorphism from V to $H^2(\Pi, \mathrm{Ad}(\overline{\rho}))$. The inequality stated in the proposition follows from the fact that this homomorphism is *injective*, a fact which we shall now prove. If $f \in V$ goes to zero in $H^2(\Pi, \mathrm{Ad}(\overline{\rho}))$, let R' denote the quotient of $F/\mathfrak{m} \cdot J$ by the kernel of f. We suppose that $f \neq 0$. Then we have an exact sequence,

$$0 \longrightarrow I \longrightarrow R' \longrightarrow R/pR \longrightarrow 0$$

where I is isomorphic to k, and for which the obstruction to lifting ρ_0 to a deformation to R' vanishes. But R' is of characteristic p, and by universality of R, it follows that the above exact sequence splits which contradicts the fact that the mapping $R' \to R/pR$ is an isomorphism on Zariski tangent spaces and $I \neq 0$.

REMARK: The above argument is a standard one in deformation theory, as is the statement of the proposition in the unobstructed case.

The Lie Algebra structure on $\mathrm{Ad}(\overline{\rho})$ induces, via cup-product, a graded Lie algebra structure on $H^*(\Pi,\ \mathrm{Ad}(\overline{\rho}))$, and, in particular, a symmetric bilinear pairing,

$$H^1(\Pi,\ \mathrm{Ad}(\overline{\rho})) \times H^1(\Pi,\ \mathrm{Ad}(\overline{\rho})) \to H^2(\Pi,\ \mathrm{Ad}(\overline{\rho}))$$

which gives the "quadratic relations" (up to higher terms) satisfied by a minimal set of formal parameters of R/pR, if $p \neq 2$.

1.7 Ordinary representations.

Fix Π and k. Fix $I \subset \Pi$ a closed subgroup. A 2-dimensional representation

$$\rho_0 : \Pi \to \mathrm{GL}_2(A) \qquad\qquad (A \in \mathcal{C})$$

is said to be ***ordinary at I*** if for $M = A \times A$ given a Π-module structure via a homomorphism in the strict equivalence class of ρ_0, the sub-A-module of I-invariant elements M^I is a direct summand in M and free of rank 1 over A.

If the subgroup $I \subset \Pi$ is understood, we shall simply say that ρ is *ordinary*.

PROPOSITION 3. *Let $\overline{\rho}$ be an ordinary, absolutely irreducible, 2-dimensional residual representation. Then a universal* ordinary *deformation of $\overline{\rho}$ exists. That is, there is a local ring*

$$R^\circ = R^\circ(\Pi, k, \overline{\rho}) \in \mathcal{C}$$

and an ordinary deformation ρ° of $\overline{\rho}$ to R° such that any ordinary deformation of $\overline{\rho}$ to A is induced from ρ° by a unique morphism $R^\circ \to A$. The pair (R°, ρ°) depends, up to canonical isomorphism, only upon the equivalence class of $\overline{\rho}$.

PROOF: Similar to the Proposition of §2. Note that the notion of ordinariness is, in general, destroyed by twisting by a one-dimensional character.

The Zariski tangent space of R° may be identified with the subspace of $H^1(\Pi,, \mathrm{End}(M))$ consisting of cohomology classes representable by 1-cocycles $c : \Pi \to \mathrm{End}(M)$ for which all $g \in I$ have as value endomorphisms $c(g) : M \to M$ in $\mathrm{End}(M)$ with the property that M^I is contained in the kernel of $c(g)$. We refer to this subspace as the A-module of *ordinary 1-cohomology* in $H^1(\Pi, \mathrm{End}(M))$.

There is a natural morphism

$$R \longrightarrow R^\circ$$

(where $R = R(\Pi, k, \overline{\rho})$ is the universal deformation ring of $\overline{\rho}$).

1.8 Schur-type results.

Let $M_N(k)^0$ denote the k-vector space of $N \times N$ matrices with entries in k, of trace zero. For $H \subset \mathrm{GL}_N(k)$, let Ad^0_H denote $M_N(k)^0$ endowed with the adjoint action of H.

PROPOSITION 4. *Let*

$$\overline{\rho} : \Pi \to \mathrm{GL}_N(k)$$

be a residual representation (absolutely irreducible). Let $H \subset \mathrm{GL}_N(k)$ be the image of Π under $\overline{\rho}$. Suppose that

$$H^1(H, \mathrm{Ad}^0_H) = 0.$$

Let $R = R(\overline{\rho})$ be the universal deformation ring of $\overline{\rho}$ and let $R_{tr} \subset R$ denote the smallest closed $W(k)$-subalgebra containing the traces of all $\rho(g)$ for $g \in \Pi$, where ρ is the universal deformation.

Then:

$$R_{tr} = R.$$

PROOF: It suffices to show surjectivity of reduced tangent spaces. That is, it suffices to show that there are no non-constant deformations of $\overline{\rho}$ to $k[\epsilon]$ ($\epsilon^2 = 0$) with traces lying in $k \subset k[\epsilon]$.

Let $\widetilde{H} \subset \mathrm{GL}_N(k[\epsilon])$ be the image of Π under a lifting of $\overline{\rho}$ to $k[\epsilon]$ with traces lying in $k \subset k[\epsilon]$.

LEMMA 3. *The natural mapping $\widetilde{H} \to H$ is an isomorphism.*

First note that if the lemma is true, we are done, for then (by virtue of our assumption of the vanishing of $H^1(H, \mathrm{Ad}^0_H)$) any lifting of $\overline{\rho}$ with traces in $k \subset k[\epsilon]$ is conjugate to the standard lifting (i.e., the constant deformation induced by the inclusion $k \subset k[\epsilon]$).

To prove the lemma, let U denote the k-linear subspace of $M_N(k)$ such that

$$1 \to 1 + \epsilon U \to \widetilde{H} \to H \to 1$$

is exact. We view $M_N(k)$ as $H \times H$-module with the natural "left-right" action. We claim that if $V \subset M_N(k)$ is the smallest $H \times H$ submodule containing U, then V is contained in the hyperplane $M_N(k)^0 \subset M_N(k)$. But since $H \to \mathrm{GL}_N(k)$ is an absolutely irreducible representation, it follows that $M_N(k)$ is an absolutely irreducible $H \times H$-module (Exer. 2 of §27.28 in [**C-R**]). Therefore V is zero, and so is U. To prove the claim, let $v \in V$. Then v can be written as a summation

$$v = \sum_i h_i u_i g_i$$

for $h_i, g_i \in H$ and $u_i \in U$.

We shall show that the trace of $h\,u\,g$ is zero for any $h, g \in H$ and $u \in U$. To see this, let $\widetilde{h}, \widetilde{g} \in \widetilde{H}$ be liftings in $\widetilde{H}$ of h, g, respectively, and note that:

$$\widetilde{h}(1 + \epsilon u)\widetilde{g} - \widetilde{h}\widetilde{g} = \epsilon h\,u\,g (\text{in } M_N(k[\epsilon])).$$

But the two terms on the lefthand side are elements of $\widetilde{H}$ and therefore have traces in $k \in k[\epsilon]$. It follows that the righthand side has trace zero.

REMARK: The requirement that $H^1(H, \mathrm{Ad}^0_H)$ vanish is not very restrictive. It holds, for example, if H has order prime to p, or if k has cardinality ≥ 7 and $H = \mathrm{GL}_N(k)$ or $H = \mathrm{SL}_N(k)$ ([**C-P-S**] Thm. 4.2).

COROLLARY 1. *Let $A' \subset A$ be an inclusion of complete local noetherian $W(k)$-algebras with residue field equal to k. Let $\overline{\rho} : \Pi \to \mathrm{GL}_N(k)$ be an absolutely irreducible residual representation admitting a deformation $\rho_0 : \Pi \to \mathrm{GL}_N(A)$, to A. Let H be the image of $\overline{\rho}$ and suppose that $H^1(H, \mathrm{Ad}^0_H)$ vanishes. Suppose that the traces of $\rho_0(g)$ for $g \in \Pi$ lie in A'.*

Then there is a deformation $\rho_0' : \Pi \to \mathrm{GL}_N(A')$ of $\overline{\rho}$ to A' which induces the deformation ρ_0 when composed with the homomorphism coming from the inclusion $A' \subset A$.

COROLLARY 2. *Let $A' \subset A$ be an inclusion of complete noetherian semilocal $W(k)$-algebras, with A' local. Let $A = \prod_i A_i$ be the factorization of*

A into a product of local rings. We suppose that A' and the A_i's all have k as residue field. Let $r : \Pi \to \mathrm{GL}_N(A)$ be a continuous homomorphism such that the residual representations

$$\overline{\rho}_i : \Pi \xrightarrow{r} \mathrm{GL}_N(A) \xrightarrow{\mathrm{proj}_i} \mathrm{GL}_N(A_i) \longrightarrow \mathrm{GL}_N(k)$$

are all equivalent, and are absolutely irreducible. Suppose that the trace of $r(g)$ lies in $A' \subset A$ for all $g \in \Pi$. Suppose, finally, that if H is the image of one of the $\overline{\rho}_i$, $H^1(H, \mathrm{Ad}^0_H)$ vanishes.

Then there is a continuous homomorphism $r' : \Pi \to \mathrm{GL}_N(A')$ such that the induced representation in $\mathrm{GL}_N(A)$ is A-equivalent to r.

PROOF: It suffices to prove Cor. 2. Let us first apply an A-equivalence to replace r by a representation such that all the residual representations $\overline{\rho}_i$ are equal to a fixed one, $\overline{\rho}$. Then consider $R = R(\overline{\rho})$ and note that by the universality property, the deformations given by the strict equivalence classes of the r_i determine mappings $R \to A_i$ and taking the product over i, we get a mapping $R \to A$. Corollary 2 will follow if we show that R maps into A'. But under the hypotheses of Corollary 2, $R_{tr} = R$ and R_{tr} maps into A'. □

1.9 A few simple examples.

Let Π be a subgroup of $\mathrm{GL}_N(k)$, where k is a finite field of characteristic p, and let $\overline{\rho}$ denote the inclusion homomorphism. Let $R = R(\Pi, k, \overline{\rho})$.

(a) If the order of Π is prime to p, then $\overline{\rho}$ admits a unique deformation to $W(k)$, and this deformation is the universal deformation of $\overline{\rho}$. In particular, $R = W(k)$.

(b) Let $\Pi = \mathrm{SL}_N(k)$, and $\overline{\rho} : \mathrm{SL}_N(k) \hookrightarrow \mathrm{GL}_N(k)$ the natural inclusion.

If the cardinality of k is ≥ 7, then $H^1(\Pi,\ \mathrm{Ad}(\overline{\rho})) = 0$ by [**C-P-S**, Thm. 4.2] *and consequently $R/pR = k$. An argument when $k = \mathbf{F}_p$ $(p \geq 7)$ derived directly from* [**Serre 1**, IV 3.4, Lemma 3, and Ex. 1a on IV-27] *then gives that $R = k$, i.e., $\overline{\rho}$ is rigid. When $k = \mathbf{F}_5$, (and $\Pi = \mathrm{SL}_2(\mathbf{F}_5)$), however, one can show that $R = \mathbf{Z}_5[\sqrt{5}]$.*

1.10 Global Galois Representations.

Let K be a finite field extension of $\mathbf{Q}$, of degree n. Let $\overline{K}$ be an algebraic closure of K, and S a finite set of places of K containing S_∞, the set of all archimedean places of K, and also containing the places dividing the rational prime number p. Let $G_{K,S}$ denote the Galois group of the maximal intermediate extension of $\overline{K}$ which is unramified outside S. Then, as previously remarked, $\Pi = G_{K,S}$ satisfies the condition Φ_p of §1.

Let

$$\overline{\rho} : G_{K,S} \to \mathrm{GL}_N(k)$$

be an absolutely irreducible residual representation, where k is a finite field of characteristic p.

Let $R = R(\overline{\rho})$ be the universal deformation ring of $\overline{\rho}$.

Recall the notation $d^i = \dim_k H^i(G_{K,S}, \mathrm{Ad}(\overline{\rho}))$ and $\delta = d^1 - d^2$. For each $v \in S$ fix $G_v \subset G_{K,S}$ a "decomposition group" at v, so that G_v is of order 2 if v is real, and is trivial if v is complex.

PROPOSITION 5.

$$\mathrm{Krull}\dim(R/pR) \geq \delta = n \cdot N^2 + 1 - \sum_{v \in S_\infty} H^0(G_v,\ Ad(\overline{\rho})).$$

REMARKS: We have no examples where the Krull dimension of R/pR is strictly greater than δ. We have no examples of Global Galois representation $\overline{\rho}$ such that Spec R/pR possesses an irreducible component of Krull dimension different from δ. We also have no example of a Global Galois representation $\overline{\rho}$ such that Spec R possesses an irreducible component on which p is nilpotent, i.e., which "doesn't lift to characteristic zero".

PROOF: Recall Tate's Global Euler Characeristic Formula (cf. Theorem 2 of 3.1 in **[H]**, but note that we have corrected a typographical error that occurs in formula (1) in the statement of the theorem): Let $G_{K,S}$ be as above, with S containing S_∞, and let M be a finite $G_{K,S}$-module, of order which is an S-unit in K. Then:

$$\frac{[H^0(G_{K,S}, M)] \cdot [H^2(G_{K,S}, M)]}{[H^1(G_{K,S}, M)]} = \frac{\prod_{v \in S_\infty} [H^0(G_v, M)]}{[M]^n}$$

where $n = [K : \mathbf{Q}]$ and [ ] denotes cardinality.

Now set $M = \mathrm{Ad}(\overline{\rho})$. The above formula is applicable since, for our proposition, we have assumed that all primes above p lie in S. Since $\overline{\rho}$

is absolutely irreducible, we have the $H^0(G_{K,S}, M)$ is a vector space of dimension one over k. One immediately computes the equality asserted in the proposition from the above formula and remark. The inequality comes directly from the Proposition of §7.

Consequences:

(1) $N = 1$: In this case $\delta = r_2 + 1$, where r_2 is the number of complex places of K.

LEMMA 4. *Leopoldt's conjecture for the field K and the prime p is equivalent to the statement that the Krull dimension of R/pR is equal to δ, for one, and hence all, residual representations with $N = 1$.*

PROOF: Note that R/pR is isomorphic to $k[[G_{K,S}^{ab,p}]]$ where the superscripts *ab* and p refer to "maximal abelian quotient" and "p-completion" respectively. The Krull dimension of R/pR is therefore equal to the dimension of the $\mathbf{Q}_p$-vector space $\mathbf{Q}_p \otimes_{\mathbf{Z}_p} G_{K,S}^{ab,p}$ where we view the abelian pro-p-group $G_{K,S}^{ab,p}$ as $\mathbf{Z}_p$-module in the natural way. A standard calculation using class field theory gives that the dimension of this vector space is $r_2 + 1$ if and only if Leopoldt's conjecture for (K, p) is valid.

(2) $N = 2, K = \mathbf{Q}$, and $p > 2$. Our residual representation is then a homomorphism

$$G_{\mathbf{Q},S} \xrightarrow{\overline{\rho}} \mathrm{GL}_2(k).$$

We say that $\overline{\rho}$ is *even* or *odd* depending upon whether the image of one (and hence all) complex conjugation involutions under $\overline{\rho}$ is a *scalar* or a *nonscalar* matrix, i.e., in the even case its image would be one of the matrices $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and in the odd case it would be equivalent to the diagonal matrix $\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$. Equivalently, $\overline{\rho}$ is *even* if and only if the splitting field of $\det(\overline{\rho})$ is totally real.

An immediate consequence of our proposition is that if $\overline{\rho}$ is as above, and $R = R(\overline{\rho})$, then:

COROLLARY 3.

$$\mathrm{Krull\,dim}(R/pR) \geq \begin{cases} 3 & \text{if } \overline{\rho} \text{ is odd} \\ 1 & \text{if } \overline{\rho} \text{ is even.} \end{cases}$$

We shall give examples below of odd Galois representations, such that the Krull dimension of R/pR is equal to 3.

1.11 Remarks on Galois Representations to $\mathrm{SL}_2(\mathbf{F}_p)$.

These are relatively hard to come by. A consequence of Remark (2) at the end of the preceding paragraph is that in the "unobstructed case", i.e., when $d^2 = 0$, a Galois representation of even determinant is "rigid" in the sense that its only deformations in characteristic p come from twisting by wild, one-dimensional characters. The following example is meant to illustrate this, and deserves further study.

Let

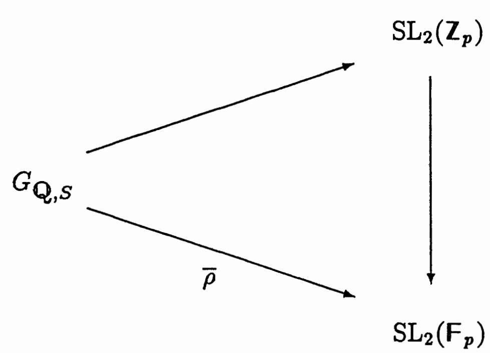
$$\overline{\rho} : G_{\mathbf{Q},S} \to \mathrm{SL}_2(\mathbf{F}_p)$$

be a surjective Galois representation. Thus its splitting field extension $L/\mathbf{Q}$ has Galois group isomorphic to $\mathrm{SL}_2(\mathbf{F}_p)$ and is unramified outside S.

PROPOSITION 6. *Let $\overline{\rho}$ be as above, and suppose that $p \geq 7$. Then one of the two conditions below holds:*

(A) *There is a Galois extension $M/\mathbf{Q}$ unramified outside S, containing $L/\mathbf{Q}$, with Galois group isomorphic to $\mathrm{SL}_2(\mathbf{F}_p[\epsilon])$.*

(B) *There is a* **unique** *lifting (up to strict equivalence) of $\overline{\rho}$ to $\mathbf{Z}_p$:*



PROOF: In remark (2) at the end of §10 we see that, viewing $\overline{\rho}$ as a representation into $\mathrm{GL}_2(\mathbf{F}_p)$, we have $\delta \geq 1$ with equality if $d^2 = 0$. Thus either $d^2 = 0$ and $d^1 = 1$ which gives (B) or $d^1 > 1$ which yields a nontrivial deformation of $\overline{\rho}$ to $\mathrm{SL}_2(\mathbf{F}_p[\epsilon]) \subset \mathrm{GL}_2(\mathbf{F}_p[\epsilon])$. But since $p \geq 7$, there are no proper subgroups of $\mathrm{SL}_2(\mathbf{F}_p[\epsilon])$ which project surjectively to $\mathrm{SL}_2(\mathbf{F}_p)$; therefore we are in case (A).

REMARKS: There are very few surjective Galois representations from $G_{\mathbf{Q},S}$ to $\mathrm{SL}_2(\mathsf{F}_p)$ known, for $p \geq 7$. I am grateful to Walter Feit who informed me of some examples for $p = 7$ due to Zeh-Marschke, [**ZM**] whose construction depends upon a family of $\mathrm{PSL}_2(\mathsf{F}_p)$-extensions of $\mathbf{Q}$ found by La Macchia [**LM**]. Does case (A) or case (B) apply for the examples of Zeh-Marschke?

See also the paper of Feit [**F**] in which an infinity of $\mathrm{SL}_2(\mathsf{F}_5)$- extensions are constructed.

1.12 Neat Residual Representations.

Let, as above, S denote a finite set of places of K containing the archimedean places and all places above p. Let $\overline{\rho} : G_{K,S} \to \mathrm{GL}_N(k)$ be an absolutely irreducible representation whose image has order prime to p. Let L/K be its splitting field over K, and let $G = \mathrm{Gal}(L/K)$. Say that an $\mathsf{F}_p[G]$-module is "relatively prime to $\mathrm{Ad}(\overline{\rho})$" if

$$\mathsf{F}_p[G] \otimes_{\mathsf{F}_p} \mathrm{Ad}(\overline{\rho})$$

does not contain the identity representation of G, when viewed as $k[G]$-module.

We say that $\overline{\rho}$ is *neat* if the order of its image is prime to p and if the following three $\mathsf{F}_p[G]$-modules are relatively prime to $\mathrm{Ad}(\overline{\rho})$:

(1) The cokernel of $\mu_p(L) \to \oplus_w \mu_p(L_w)$, where the summation is taken over all nonarchimedean places w of L lying over places in S. Here μ_p means the group of p-th roots of 1.
(2) The kernel of $\mathcal{O}_L^*/\mathcal{O}_L^{*p} \to \oplus_w \mathcal{O}_{L_w}^*/\mathcal{O}_{L_w}^{*p}$, where the summation over w is as in (1) above.
(3) $\mathrm{Pic}(\mathcal{O}_L)[p]$, i.e., the subgroup of elements killed by p in the ideal class group of L.

Let $d^i = \dim_k H^i(G_{K,S}, \mathrm{Ad}(\overline{\rho}))$ as before, and $\delta = d^1 - d^2$.

PROPOSITION 7. *If $\overline{\rho}$ is a neat residual representation whose image has order prime to p, then $d^2 = 0$. The universal deformation ring R of $\overline{\rho}$ is isomorphic to a power series ring over $W(k)$ in $\delta = d^1$ parameters.*

PROOF: Let S' denote the set of places of L lying above the places S of K. Let $Y_L = \mathrm{Spec}(\mathcal{O}_L)$ where $\mathcal{O}_L$ is the ring of integers in L and let

$Y_{L,S'} \subset Y_L$ denote the open subscheme which is the complement of the closed subscheme $S' - S'_\infty$. From the standard long exact sequence for étale cohomology for the pair $(Y_L, Y_{L,S'})$ with coefficients in F_p, one computes, using (1),(2), (3), that $H^2(Y_{L,S'}, \mathsf{F}_p)$ is, as $\mathsf{F}_p[G]$-module, relatively prime to $\mathrm{Ad}(\overline{\rho})$. But by [**H**, Appendix 2, 3.3.1] we have the isomorphism $H^2(Y_{L,S'}, \mathsf{F}_p) \cong H^2(G_{L,S'}, \mathsf{F}_p)$.

Also, since G has order prime to p, we have the isomorphism

$$H^2(G_{K,S}, \mathrm{Ad}(\overline{\rho})) \cong H^0(G, H^2(G_{L,S'}, \mathsf{F}_p) \otimes \mathrm{Ad}(\overline{\rho}))$$

from which our proposition follows.

1.13 Neat S_3-extensions of Q.

Let $K_1/\mathbf{Q}$ be a noncyclic cubic extension of discriminant $\pm p \equiv 1 \mod 4$ for p a prime number ≥ 5. Let $L/\mathbf{Q}$ be the Galois closure of $K_1/\mathbf{Q}$, so that L contains $K_i \quad i = (1,2,3)$ the fields conjugate to K_1, and L contains the quadratic field $\mathbf{Q}(\sqrt{\pm p})$. Let $G = \mathrm{Gal}(L/\mathbf{Q}) \quad (\cong S_3$, the symmetric group on three letters). Let $K = \mathbf{Q}$, $S = \{p, \infty\}$ and let

$$\overline{\rho} : G_{\mathbf{Q},\{p,\infty\}} \longrightarrow \mathrm{GL}_2(k)$$

be a residual representation obtained by choosing a suitable finite field k of characteristic p, and imbedding of G in $\mathrm{GL}_2(k)$, and then composing with the natural projection $G_{\mathbf{Q},\{p,\infty\}} \twoheadrightarrow G$.

If $1, \epsilon, \rho_2$ denote the three inequivalent irreducible k-representations of G (i.e., 1 is the trivial representation, ϵ is the nontrivial one-dimensional sign representation, and ρ_2 is the irreducible two-dimensional representation given by the above imbedding), then $\mathrm{Ad}(\overline{\rho}) = 1 \oplus \epsilon \oplus \rho_2$.

An $\mathsf{F}_p[G]$-module is relatively prime to $\mathrm{Ad}(\overline{\rho})$, in the sense of §12, if and only if it vanishes. Let us consider the three $\mathsf{F}_p[G]$- modules (1), (2), (3) of §12 and the conditions under which they vanish.

(1) If w is a place of L lying over p, $\mu_p(L_w) = 0$, so the $\mathsf{F}_p[G]$-module (1) of §12 vanishes.

(2) **(Special cubic fields).** We shall exhibit a family of S_3-extensions of $\mathbf{Q}$ of discriminant $-p$ which have the property that the G-representation (2) of §12 vanish. Note first that since the discriminant is negative, we have that L is a totally complex sextic field, and consequently the representation

of G on the vector $\mathbf{Q}$-vector space $\mathbf{Q} \otimes \mathcal{O}_L^*$ is the irreducible 2-dimensional representation. If the natural imbedding of $\mathcal{O}_L^* \mapsto \prod_{\wp | p} \mathcal{O}_{L,\wp}^*$ does not have the property that the image is contained in the subgroup of p-th powers, then condition (2) of §12 holds.

Consider cubic polynomials $f(X) = X^3 + aX + 1$ for integers a such that $27 + 4a^3$ is a prime number. Set $p = 27 + 4a^3$.

DEFINITION. *A special cubic field* (**of discriminant** $-p$) *is a field* $K_1 = \mathbf{Q}(x)$ *where* x *is a root of* $f(x)$.

It is, indeed, the case that K_1 has discriminant $-p$. Moreover (see [**Art**, pp. 169-171] the full group of units of K_1 is generated by $\pm x$. Here is the list of $p < 1,000,000$ such that $-p$ is the discriminant of a special cubic: 23, 31, 59, 283, 1399, 4027, 5351, 11003, 16411, 32027, 97583, 119191, 157243, 202639, 275711, 415319, 562459, 665527.

Let K_1 be a special cubic field of discriminant $-p$. Let L be the Galois closure of K_1 over $\mathbf{Q}$, i.e., a splitting field of f over $\mathbf{Q}$, containing x. Since the constant term of the polynomial f is 1, the element x is a unit in the ring, $\mathcal{O}_{L_1}$, of integers in K_1.

The polynomial f factors (mod p) as follows:

$$f(X) \equiv (X + 3/2a)^2 \cdot (X - 3/a) \qquad (\text{mod } p).$$

One calculates the norm $N_{K_1/\mathbf{Q}}(x + 3/2a)$ to be $p/8a^3$. It follows that if we put $\pi = x + 3/2a$ and $\eta = x - 3/a$ we may write the prime decomposition of p in $\mathcal{O}_{K_1}$ as follows: $(p) = \wp_1 \cdot {\wp_2}^2$ where $\wp_1 = (p, \eta)$ is the unramified prime lying above p, and $\wp_2 = (p, \pi)$ is the ramified prime lying above p. If $\mathcal{O}_{K_1,\wp_2}$ is the completion at $\wp_2$, then $\mathcal{O}_{K_1,\wp_2} = \mathbf{Z}_p[\Pi]$ with Π equal to the image of π. If we let x_2 denote the image of x in $\mathcal{O}_{K_1,\wp_2}^*$ (i.e., $x_2 = -3/2a + \Pi$) then x_2 is not a p-th power in $\mathcal{O}_{K_1,\wp_2}^*$ since no p-th power in $\mathcal{O}_{K_1,\wp_2}^*$ can be expressed as an integer plus Π times a unit.

By considering the mapping

$$\mathcal{O}_{K_1,\wp_1}^* \times \mathcal{O}_{K_1,\wp_2}^* \longmapsto \prod_{\wp | p} \mathcal{O}_{L,\wp}^*$$

one sees then that the image of x in $\prod_{\wp | p} \mathcal{O}_{L,\wp}^*$ is not a p-th power, and consequently the G-representation space given by (2) of §12 vanishes.

(3) Let K_1 be any cubic field of discriminant $-p$ for p a prime number. We shall show that the class number of L is prime to p and hence the $\mathbf{F}_p[G]$-module (3) of §12 vanishes. For this it suffices to prove that the class number of $\mathbf{Q}(\sqrt{-p})$ (for $-p \equiv 1 \mod 4$) is $< p$ (a fact which follows from the standard inequalities coming from the Minkowski bounds) and that the class number h_1 of K_1 is also $< p$. To see that $h_1 < p$, we use an explicit bound for $h_1 R_1$ (the product of class number and regulator of K_1) given in a more general situation by Lavrik (cf. [**Na**] and the discussion on page 401 in §3 of Chap. VIII there for detailed statements and references) but which can be computed in our situation to yield

$$h_1 R_1 \leq (0.01765155) \cdot (1 + 1/\log p)^4 \cdot \log^2 p \cdot \sqrt{p}.$$

Throwing out the case $p = 23$ (where $h_1 = 1$) we have the following lower bound for the regulator (cf. [**Art**, p. 170]):

$$R_1 > \sqrt[3]{\log(\frac{p-24}{4})}$$

which yields the desired inequality (with a wide margin) for h_1.

DEFINITION. *A representation* $\overline{\rho} : G_{\mathbf{Q},\{p,\infty\}} \to \mathrm{GL}_2(\mathbf{F}_p)$ *is called* a special S_3-representation *if it is a residual representation constructed as in the beginning of this section starting with* $K_1/\mathbf{Q}$ a special *cubic field, where* $K_1 = \mathbf{Q}(x)$ *and* $x^3 + ax + 1 = 0$ *with* $p = 27 + 4a^3$.

To summarize:

PROPOSITION 8. *For each prime number p of the form* $27 + 4a^3$ *with* $a \in \mathbf{Z}$ *there is a unique special* S_3*-representation (up to equivalence),*

$$\overline{\rho} : G_{\mathbf{Q},\{p,\infty\}} \to \mathrm{GL}_2(\mathbf{F}_p).$$

The special S_3*-representations are neat (in the sense of §12 above). If* $R = R(\overline{\rho})$ *denotes the universal deformation ring of a special* S_3*-representation, then* R *is a power series ring on 3 parameters over* $\mathbf{Z}_p$.

2. The Internal Structure of Universal Deformation Spaces.

2.1 General Glossary.

Let $\overline{\rho} : G_{K,S} \to \mathrm{GL}_N(k)$ be an absolutely irreducible residual representation, and let $R = R(\overline{\rho})$ be its universal deformation ring. We shall be interested in studying $X = \mathrm{Spec}(R)$, which we refer to as the *universal deformation space* of $\overline{\rho}$.

Although at times we are interested in the A-valued points of the scheme X for A *any* object of the category $\mathcal{C}$, if we refer simply to a *point* x of X, we shall mean a D-valued point of the scheme X, where D is some complete discrete valuation ring, finite as $W(k)$-algebra which is fixed in the discussion. Thus to a point x is associated a local- ring homomorphism $\phi_x : R \to D$, and consequently we also get an induced D-valued representation,

$$\rho_x : G_{K,S} \to \mathrm{GL}_N(D).$$

The "points" of X are then also points of the analytic space over the field of fractions of D, determined by the scheme X. We refer to this analytic space as X^{an}. If the representation ρ_x satisfies any *property* P, we shall say that x *has property* P. A property P is called a *quotient group property* if there is a commutative diagram

$$\begin{array}{ccc} G_{K,S} & \xrightarrow{\quad\pi\quad} & \Omega \\ & \overline{\rho} \searrow \quad \swarrow \overline{\rho}' & \\ & \mathrm{GL}_N(k) & \end{array} \qquad (0.1)$$

where the horizontal map is a surjection of profinite groups and such that

x has property P if and only if ρ_x factors through Ω.

Given such a diagram (0.1) one can then define the property P for any A-valued point of X by the same requirement, and an application of the proposition of §2 of Chap. 1 gives us that there is a closed affine subscheme (call it X_P) in X with the characterization that an A-valued point of X has property P if and only if it lies in X_P.

Specifically, take X_P to be the universal deformation space $X(\overline{\rho}')$ and the mapping π induces a closed immersion, $X_P \hookrightarrow X$.

Here is a "glossary" of properties which we shall be concerned with in this chapter. *We assume* $N = 2$.

I. **Inertial Properties.** Fix D, a finite discrete valuation ring extension of $\mathbf{Z}_p$.

If x is a point of X and v a place in S, let I_v denote an inertia group at v contained in $G_{K,S'}$ and let $\rho_{x,v}$ be the restriction of ρ_x to I_v.

A point x is said to be *inertially finite at* v if the image of I_v under $\rho_{x,v}$ is finite. A point x is *inertially dihedral at* v if $\rho_{x,v}$ is a generalized dihedral representation of I_v in the sense that its image in $\mathrm{GL}_2(F)$, where F is the field of fractions of D, is contained in the normalizer of a Cartan subgroup of $\mathrm{GL}_2(F)$.

It is *inertially reducible at* v if $\rho_{x,v}$ is reducible in the sense that if $M = D \times D$ is the D-module upon which the group I_v is made to act via $\rho_{x,v}$ composed with the standard representation of $\mathrm{GL}_2(D)$, then there is a free D-module of rank 1 contained in M which is left stable under the action of I_v. It is *ordinary at* v if ρ_x is ordinary at I_v, in the sense of §7 of Chap. 1. Ordinary points are inertially reducible. If x is a D-valued point, it is said to be *inertially D-ample at* v if the image of $\rho_{x,v}$ contains an open subgroup of finite index in $SL_2(D) \subset \mathrm{GL}_2(D)$.

Recall that a profinite group is said to be metabelian if it contains a closed abelian normal subgroup whose associated quotient group is also abelian. A point x is said to be *inertially metabelian at* v if the image of I_v under ρ_x is metabelian; it is said to be *inertially abelian at* v if that image is abelian. Both properties (inertially metabelian and inertially abelian at v) are quotient group properties.

II. **Global properties.** We shall call x *globally dihedral* if ρ_x is a dihedral representation of $G_{K,S}$.

We shall say that x is *automorphic* if there is an automorphic representation π of $\mathrm{GL}_{2/K}$, unramified outside S, whose Hecke eigenvalues are contained in the ring of integers $\mathcal{O}_F$ of a number field $F \subset \mathbf{C}$, and there is a homomorphism $i : \mathcal{O}_F \to D$ such that if v is a place of K not in S, and λ_v is the eigenvalue of the Hecke operator T_v on π, then $i(\lambda_v) = \text{trace } \rho_x(\phi_v)$ where ϕ_v is a (geometric) Frobenius element at v in $G_{K,S}$.

More specially, if $K = \mathbf{Q}$ and the automorphic representation π can be generated from a holomorphic modular form ω of weight $w \geq 1$ on $\Gamma_1(Np^n)$ for some integer $N \geq 1$ (prime to p) and for $n \geq 1$, where ω is an eigenform for the Hecke operators T_ℓ (for $\ell \nmid Np$), for the operators U_q (for $q|Np$) and for the "diamond operators" $<r>$, $r \in (\mathbf{Z}/Np^n\mathbf{Z})^*$, we say that x is *modular* (*of weight* w *and level* Np^n). In the case of weight $w = 2$ the correspondence to modular eigenforms of such representations ρ_x is due to Eichler-Shimura-Igusa (cf. [**Sh**]); the generalization to the case of weight $w \geq 2$ is due to Deligne [**D**]; for weight $w = 1$ it is due to Deligne-Serre [**D-S**].

The reader should note that the form ω is *not* necessarily uniquely determined up to multiplication by scalars by the point x to which it gives rise.

Say that x is *pro-automorphic* in $X(D)$ if there is a sequence of automorphic D-valued points of X converging to x (in the natural topology of $X(D)$). Define *pro-modular* similarly. What is the locus of pro-automorphic points in $X(D)$? Of pro-modular points?

III. **Twists.** The universal deformation ring R is naturally a Λ-algebra as described in Chap. 1. The algebra Λ has a natural Hopf algebra structure, and if Φ is the formal group scheme whose associated affine Hopf algebra is Λ, then $X = \text{Spec}(R)$ is endowed with a natural action of Φ. If x is any point of X, and y is in the orbit of x under the action of Φ we shall call y a *twist* of x, insofar as the representation ρ_y is the twist of ρ_x by a one-dimensional (wild) character.

In certain cases "a twist by a tame character" may also induce an automorphism of X. Specifically, let

$$\overline{\rho} : G_{K,S} \to \text{GL}_2(k)$$

be a dihedral, absolutely irreducible, residual representation. Let L/K be the quadratic field extension of K which cuts out the normal subgroup of index two in $G_{K,S}$ determined by the dihedral representation $\overline{\rho}$. Let $\epsilon_{L/K} : G_{K,S} \to (\pm 1)$ be the quadratic character determining the field extension L/K. If $\overline{\rho}' = \epsilon_{L/K} \cdot \overline{\rho}$ one immediately sees that $\overline{\rho}$ and $\overline{\rho}'$ have the same trace and determinant, and consequently they are equivalent. We get natural isomorphisms

$$R(\overline{\rho}) \underset{\alpha}{\cong} R(\overline{\rho}) \underset{\beta}{\cong} R(\overline{\rho})$$

where α is obtained as in Chap. 1, §2 because $\overline{\rho}$ and $\overline{\rho}'$ are equivalent. Let γ denote the composition $\beta \circ \alpha$ which one easily sees to be a (nontrivial) involution of R and consequently, of X. If $\overline{\rho}$ is dihedral, we refer to γ as the *inner twist* of X. If $\overline{\rho}$ is dihedral, the inner twist of X preserves the locus of points of X satisfying each of the properties we have listed above, except for the property of being *ordinary*.

2.2 Special dihedral representations.

We will analyze the universal deformation spaces of certain absolutely irreducible residual representations in some detail.

Specifically, let $p \geq 5$, and let

$$\overline{\rho} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{F}_p)$$

be an absolutely irreducible residual representation which is odd and has as image a dihedral group of order $2h$, where h is supposed prime to $2p$. Let $L/\mathbf{Q}$ be the dihedral extension of $\mathbf{Q}$ which is the splitting field of $\overline{\rho}$. We shall say that such a $\overline{\rho}$ is a *special dihedral representation* if $S = \{p, \infty\}$, the class number of L is prime to p, $\overline{\rho}$ is neat, cf. Chap. I, §12, and the restriction of ρ to $G_{\mathbf{Q}(\sqrt{-p}),S'}$ is unramified (where S' consists in the primes of $\mathbf{Q}(\sqrt{-p})$ lying above S).

There exist special dihedral representations: for example, we may take the special S_3-representations studied in Chapter I.

The action of an inertia group $I_p \subset G_{\mathbf{Q},S}$ via a special dihedral representation $\overline{\rho}$ is through its unique quotient group of order two, the nontrivial element in that quotient group having as image under $\overline{\rho}$ a matrix equivalent to $\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbf{F}_p)$. It follows that special dihedral representations are ordinary.

From the theory developed in Chapter I, it follows that the universal deformation ring of such a $\overline{\rho}$ is a power series ring in three variables over $\mathbf{Z}_p$.

From now on, we *fix such a special dihedral representation* $\overline{\rho}$. Let R be its universal deformation ring, and $X = \mathrm{Spec}(R)$, its universal deformation space. Let $D_{2h} \subset \mathrm{GL}_2(\mathbf{F}_p)$ denote the image of $\overline{\rho}$.

2.3 The origin.

Up to strict equivalence, there is a unique lifting

$$\rho_0 : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{Z}_p)$$

of $\overline{\rho}$, such that its image in $\mathrm{GL}_2(\mathbf{Z}_p)$ is isomorphic to D_{2h}, the image of $\overline{\rho}$ in $\mathrm{GL}_2(\mathbf{F}_p)$.

Let $x_0 \in X$ denote the $\mathbf{Z}_p$-valued point of X corresponding to ρ_0, which we take as our *origin.*

Clearly, x_0 is ordinary, and hence also inertially reducible; x_0 is also dihedral.

PROPOSITION 9. *The point x_0 is the unique inertially finite $\mathbf{Z}_p$-valued point of X.*

PROOF: Let $x \in S$ be a $\mathbf{Z}_p$-valued inertially finite point. Since the image of I_p under ρ_x is finite, it follows that its image is isomorphic to its image under $\overline{\rho}$, the isomorphism being given by the natural projection $\mathrm{GL}_2(\mathbf{Z}_p) \twoheadrightarrow \mathrm{GL}_2(\mathbf{F}_p)$. It follows that ρ_x restricted to $G_{\mathbf{Q}(\sqrt{-p})}$ is unramified. Consequently, restricted to G_L it is also unramified. But $\rho_x\,|_{G_L}$ has image in the kernel of $\mathrm{GL}_2(\mathbf{Z}_p) \to \mathrm{GL}_2(\mathbf{F}_p)$ which is a pro-p group. Since the class number of L is prime to p, it then follows that ρ_x is trivial on G_L, i.e., the image of ρ_x is isomorphic to D_{2h}. Since, up to strict equivalence, there is a unique lifting of D_{2h} from $\mathrm{GL}_2(\mathbf{F}_p)$ to $\mathrm{GL}_2(\mathbf{Z}_p)$, it then follows that $x = x_0$.

REMARK: The same argument gives the fact that an infinitesimal deformation of $\overline{\rho}$ is constant if and only if the restriction to I_p is constant. To be precise,

LEMMA 5. *Let $\tilde{\rho} : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{F}_p[\epsilon])$ be a deformation of $\overline{\rho}$ to the ring of dual numbers. Then $\tilde{\rho}$ is constant (i.e., obtained from $\overline{\rho}$ by pullback via the imbedding $\mathbf{F}_p \hookrightarrow \mathbf{F}_p[\epsilon]$) if and only if $\widetilde{\rho}\,|_{I_p}$ is constant.*

PROOF: If $\widetilde{\rho}\,|_{I_p}$ is constant, it follows as in the proof of the previous proposition that $\widetilde{\rho}$ restricted to G_L, the Galois group of the splitting field of $\overline{\rho}$, is an everywhere unramified mapping of G_L to a p-group, which must be trivial since L has class number prime to p.

PROPOSITION 10. *The "origin" $x_0 \in X$ is modular. The representation ρ_0 associated to x_0 comes from a modular form ω of weight one on $\Gamma_0(p)$*

which is an eigenvector for the Hecke operators T_ℓ ($\ell \neq p$) and for U_p, and which is p-ordinary, in the sense that the U_p-eigenvalue is $+1$, and hence a p-unit.

PROOF: The basic statement, that ρ_0 is attached to a modular form θ of the above description is due, essentially, to Hecke. Here is a description of the modular form θ. Let

$$\chi_0 : \mathrm{Gal}(L/\mathbf{Q}(\sqrt{-p})) \to D^*$$

be the one-dimensional character whose induction to $\mathrm{Gal}(L/\mathbf{Q})$ gives ρ_0, where D is either $\mathbf{Z}_p$ (in the split dihedral case) or an extension of degree two over $\mathbf{Z}_p$ (in the nonsplit dihedral case). By Class Field Theory, χ_0 corresponds to an ideal class character χ (indeed, an unramified ideal class character, by our assumptions on L). Then

$$\theta = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) q^{N\mathfrak{a}}$$

where $\mathfrak{a}$ runs through all ideals in the rings of integers of $\mathbf{Q}(\sqrt{-p})$. Note that θ is actually a power series in q with coefficients in $\mathbf{Z}_p$. Note also that, by ([**Serre 2**, §8]; cf. also [**Hecke**]), θ is of weight one with Legendre character on $\Gamma_0(p)$. It is an eigenvector for the Hecke operators T_ℓ, $\ell \neq p$ and U_p, and, moreover, its U_p-eigenvalue is 1. Consequently θ is a p-ordinary eigenform.

2.4 The globally dihedral locus.

Let Π_D (the subscript D is for "dihedral") denote the maximal profinite quotient group of $G_{\mathbf{Q},S}$ in which the image of $G_{\mathbf{Q}(\sqrt{-p}),S'}$ is abelian. Let $\overline{\rho}_D : \Pi_D \to \mathrm{GL}_2(\mathbf{F}_p)$ denote the residual representation such that composition with the natural projection $G_{\mathbf{Q},S} \to \Pi$ yields $\overline{\rho}$. Let R_D be the universal deformation ring of $\overline{\rho}_D$ and $X_D = \mathrm{Spec}(R_D)$ its universal representation space. We have a natural closed immersion,

$$X_D \hookrightarrow X.$$

LEMMA 6. *A $\mathbf{Z}_p$-valued point $x \in X$ is globally dihedral if and only if it is in X_D.*

REMARK: In particular, the property of being globally dihedral is a "quotient group property" in the sense of §1.

PROOF: If $x \in X_D$, then ρ_x is an odd representation whose image is a dihedral subgroup in $\mathrm{GL}_2(\mathbf{Z}_p)$. Also, since $\overline{\rho}$ is absolutely irreducible, so is ρ_x. It follows that ρ_x is a dihedral representation.

PROPOSITION 11. *The universal deformation ring R_D is a power series ring in two variables over $\mathbf{Z}_p$. The subscheme X_D is a smooth hypersurface in X.*

PROOF: Let $s : D_{2h} \twoheadrightarrow (\pm 1)$ denote the natural surjective homomorphism given by the dihedral structure of D_{2h}. Let Γ_+, Γ_- be two free pro-p-groups on one generator (i.e., $\cong \mathbf{Z}_p$) endowed with continuous D_{2h}-actions as follows:

(a) D_{2h} acts trivially on Γ_+.
(b) An element $g \in D_{2h}$ acts on Γ_- by multiplication by $s(g)$.

Let Δ denote the semi-direct product $\Gamma_+ \times \Gamma_- \rtimes D_{2h}$ where the action of D_{2h} on Γ_+ and Γ_- is given as in (a),(b). An elementary exercise in class field theory shows that there exists a (noncanonical) surjective homomorphism $\Pi_D \to \Delta$ such that any lifting $\rho : \Pi_D \to \mathrm{GL}_2(A)$ of $\overline{\rho}_D$ for any $A \in \mathcal{C}$ factors through Δ. Thus, if we fix such a surjective homomorphism $a : \Pi_D \twoheadrightarrow \Delta$, and let

$$\overline{\rho}_\Delta : \Delta \to \mathrm{GL}_2(\mathbf{F}_p)$$

be the residual representation such that composition with a yields $\overline{\rho}_D$ then R_D is isomorphic to the universal deformation ring of $\overline{\rho}_\Delta$. But the latter universal deformation ring is immediately seen to be isomorphic to $Z_p[[\Gamma_+ \times \Gamma_-]]$ since $\overline{\rho}_\Delta \mid_{D_{2h}}$ admits a unique deformation to A for any $A \in \mathcal{C}$. □

2.5 The ordinary locus.

Let R° be the universal deformation ring of *ordinary* deformations of $\overline{\rho}$ (Chap. I, §7) and let $X^\circ = \mathrm{Spec}(R^\circ)$.

We have a natural mapping

$$R \to R^\circ$$

which brings ρ to the deformation

$$\rho^\circ : G_{\mathbf{Q},\{p,\infty\}} \to \mathrm{GL}_2(R^\circ).$$

i.e., to the universal ordinary deformation of $\overline{\rho}$. If $M^\circ = R^\circ \times R^\circ$ is the free R°-module of rank 2 endowed with a $G_{\mathbf{Q},\{p,\infty\}}$ action via composition of ρ° with the natural action of $\mathrm{GL}_2(R^\circ)$, then the submodule $(M^\circ)^{I_p}$ of inertial invariants is a free R°-module of rank 1, which is a direct factor in M°.

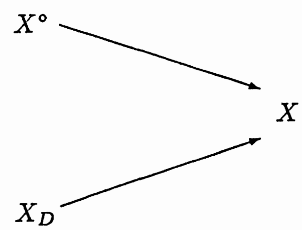
PROPOSITION 12. *The natural mapping $R \to R^\circ$ is surjective.*

PROOF: Let $R' \subset R^\circ$ denote the image of R. The universal deformation ρ of ρ' to R induces a deformation ρ' of $\overline{\rho}$ to R'. Let $M = R' \times R'$ be the module upon which $G_{\mathbf{Q},S}$ acts by composition of ρ' with the standard action of $\mathrm{GL}_2(R')$. The induced action of $G_{\mathbf{Q},S}$ on $M^\circ = M \otimes_{R'} R^\circ$ is ordinary: in fact, it is the universal ordinary deformation ρ° of $\overline{\rho}$.

Let Π denote the p-completion of $G_{\mathbf{Q},S}$ relative to $\overline{\rho}$ (Chap. I, §2). Then Π is an extension of the dihedral group D_{2h} by a pro-p-group. By Schur-Zassenhaus (compare [**B**, Prop. 2.6], it is a semi-direct product. Let $\mathcal{I}$ be the image of an inertia group at p in Π. Then $\mathcal{I}$ is a semi-direct product of a cyclic group of order two by a pro-p-group. Let $\sigma \in \mathcal{I}$ be an element of order two. Since $p \neq 2$, the R'-module M decomposes as a direct sum $M = M^+ \oplus M^-$ where the R'-modules $M^\pm$ are the $\pm$-eigenspaces of the involution σ. Since σ is not a scalar matrix, it follows that $M^\pm$ are both free R'-modules of rank 1. Consequently, $M^\pm \otimes_{R'} R^\circ$ are also free R°-modules of rank 1. Since σ acts like -1 on M^- it follows that the invariants under $\mathcal{I}$ in $M^\circ = M \otimes_{R'} R^\circ$ are contained in $M^+ \otimes_{R'} R^\circ$. But since ρ° is ordinary, the module of $\mathcal{I}$-invariants in M° is free of rank 1 over R° and a direct summand in M°. It follows that $M^+ \otimes_{R'} R^\circ$ *is* the module of $\mathcal{I}$-invariants in M°, and consequently, M^+ is the module of $\mathcal{I}$-invariants in M. But then we have shown that the module of $\mathcal{I}$-invariants in M is free of rank 1 and a direct summand in M. It follows that ρ' is ordinary. Since R° is the universal ring of ordinary deformations of $\overline{\rho}$, we then have a canonical homomorphism $R^\circ \to R'$ with respect to which, $\overline{\rho}$ is induced from ρ°. One easily sees that the composition $R^\circ \to R' \subset R^\circ$ is the identity, giving equality, $R' = R^\circ$, and our proposition.

We have the following diagram of subschemes (after §4, and prop. 12 above):

(0.2)



and we now show that the ordinary locus X° and the globally dihedral locus X_D intersect *transversally* in X. Specifically, abbreviating $k = \mathsf{F}_p$, we show:

LEMMA 7. *Any deformation of $\overline{\rho}$ to the ring of dual numbers $k[\epsilon]$ which is both globally dihedral and ordinary is constant.*

PROOF: Let $\tilde{\rho}$ be such a deformation, and let $M = k[\epsilon] \times k[\epsilon]$ be the $k[\epsilon]$-module on which $G_{\mathbf{Q},S}$ operates via $\tilde{\rho}$. Let H be the (finite) quotient group of $G_{\mathbf{Q},S}$ through which $\tilde{\rho}$ operates faithfully. We may write

$$H = A \rtimes D_{2h} \qquad \text{(semi-direct product)}$$

where D_{2h} acts on A via an automorphism factoring through the projection of D_{2h} to (± 1), and A is a finite abelian p-group. If $C_h \triangleleft D_{2h}$ is the kernel of the projection to (± 1), we have, evidently,

$$H = A \times C_h \rtimes (\pm 1).$$

For this description of things, we have used that $\tilde{\rho}$ is globally dihedral.

We may coordinatize H as above so that the image of the inertia group at p, I in H is given by $A \times (0) \rtimes (\pm 1)$. Here we use that L has class number prime to p.

Now we make use of the assumption that $\tilde{\rho}$ is also ordinary. Thus M^I is a free $k[\epsilon]$-submodule of M, of rank 1. Since A is contained in the image of I, $M^I \subseteq M^A$. Let us first note that M^I cannot be equal to M^A. For if we had equality, since $D_{2h} \subset H$ normalizes A, it stabilizes M^A, hence also

M^I, which would give us that M^I is stabilized by all of $\tilde{\rho}$, which is clearly not the case ($\tilde{\rho}$ is absolutely irreducible). For the rest of this argument, we make use only of the k-vector space structure of M (and not its $k[\epsilon]$-module structure). M is a k-vector space of dimension 4, and M^I is of dimension 2. Since M^I is properly contained in M^A, we have that $\dim_k M^A$ is either 3 or 4. But M, viewed as k-representation space for $D_{2h} \subset H$ is a sum of two copies of an absolutely irreducible representation of dimension 2. In particular, there are no D_{2h}-stable subspaces of the k-vector space M of dimension 3. Since M^A is stabilized by D_{2h}, it then follows that M^A is all of M; but A acts faithfully on M, giving $A = 0$, and our lemma.

Recall that the reduced Zariski tangent space of a local ring A with residue field k is the k-vector space which is k-dual to $m_A/(m_A^2, p)$ where m_A is the maximal ideal in A. An easy consequence of the above lemma is:

PROPOSITION 13. *The reduced Zariski tangent space of X° has dimension ≤ 1 over $k = \mathbf{F}_p$.*

PROOF: Consider the diagram of subschemes (0.2) above. Since the reduced Zariski tangent space of X is of dimension 3, and the reduced Zariski tangent space of X_D is of dimension 2 (Prop. of §4) the transversality lemma above implies our proposition.

A consequence of Proposition 2 and Nakayama's lemma is that if $\mathbf{Z}_p[[t]]$ is a power series ring in the variable t over $\mathbf{Z}_p$, there is a surjection of rings

$$\mathbf{Z}_p[[t]] \twoheadrightarrow R^\circ.$$

Fix such a surjection and let $J \subseteq \mathbf{Z}_p[[t]]$ denote its kernel.

It follows that

LEMMA 8.

$$(\ddagger) \qquad \text{Krull dimension } (R^\circ) \leq 2.$$

We have equality in (‡) if and only if $J = 0$, i.e., if and only if R° is isomorphic to a power series ring in one variable over $\mathbf{Z}_p$.

To proceed further in our argument, we must bring in Hida's theory [**Hida**]. By proposition 2 of §4, the origin $x_0 \in X$ is modular, and moreover there is a p-ordinary modular form ω of weight one whose associated representation is ρ_{x_0}.

We view ω as having Fourier coefficients in $\mathbf{Z}_p$, and now invoke the theory developed in [**Hida**] and [**M-W**]. (See also the Ph.D. Thesis of Gouvêa [**G**, § 2.4]) Let $\mathbf{T}$ be the Hecke algebra associated to the prime number p, as defined in §8 of [**M-W**]. Then there is a homomorphism $f : \mathbf{T} \to \mathbf{Z}_p$ such that $T_\ell \in \mathbf{T}$ goes to the eigenvalue of the action of T_ℓ on ω for all $\ell \neq p$, and $U_p \in \mathbf{T}$ goes to the eigenvalue of U_p on ω. Let $\overline{f} : \mathbf{T} \to \mathbf{Z}/p\mathbf{Z}$ denote the reduction mod p of f, and let $\mathfrak{m} \subseteq \mathbf{T}$ denote the kernel of $\overline{f}$. If $\mathbf{T}_\mathfrak{m}$ denotes the completion of $\mathbf{T}$ at $\mathfrak{m}$, then by Prop. 2 of §8 of [**M-W**] we have a two-dimensional ordinary representation

$$\rho_\mathfrak{m} : G_{\mathbf{Q},S} \longrightarrow \mathrm{GL}_2(\mathbf{T}_\mathfrak{m})$$

which is a lifting of $\overline{\rho}$. The ordinary-ness of $\rho_\mathfrak{m}$ (i.e., the fact that the module of inertial *invariants* in its representation space is a free $\mathbf{T}_\mathfrak{m}$- module of rank one and a direct factor) is linked to our requirement—cf. [**M-W**]—that T_ℓ be the trace of a *geometric* Frobenius element in the representation space of $\rho_\mathfrak{m}$. If we had chosen, instead, to work with the *arithmetic* Frobenius, the inertial *coinvariants* would enjoy the property described in the parenthesis above.

LEMMA 9. *There is a surjective homomorphism*

$$R^\circ \twoheadrightarrow \mathbf{T}_\mathfrak{m}$$

of Λ-algebras such that $\rho_\mathfrak{m}$ is induced via this homomorphism from ρ°, the universal ordinary deformation of $\overline{\rho}$.

PROOF: Existence (and uniqueness) of the asserted homomorphism comes from the universality property of ρ° and the fact that $\rho_\mathfrak{m}$ is ordinary. For prime numbers $\ell \neq p$, set $T_\ell^\circ =$ the trace (over R°) of (geometric) Frobenius at ℓ acting on the free R°-module $M^\circ = R^\circ \times R^\circ$ via ρ°, and let $U_p^\circ \in R^\circ$ denote the eigenvalue of a choice of Frobenius at p, acting on the free R°-module of rank 1 consisting in the inertial invariants in M°. It follows easily that the natural homomorphism $R^\circ \to \mathbf{T}_\mathfrak{m}$ is a homomorphism of Λ-algebras which sends T_ℓ° to T_ℓ for all $\ell \neq p$ and U_p° to U_p. Since the Λ-algebra $\mathbf{T}_\mathfrak{m}$ is generated by the Hecke operators T_ℓ for $\ell \neq p$ and by U_p, the surjectivity assertion of lemma 9 follows.

Since $\mathbf{T}_\mathfrak{m}$ is finite and flat over Λ, [**Hida**, Theorem 3.1] and nonzero, we have that

Lemma 10. *Krull Dimension* $(\mathbf{T}_{\mathfrak{m}})$ = *Krull dimension* $(\Lambda) = 2$.

Proposition 14. *The natural mapping*

$$R^o \to \mathbf{T}_{\mathfrak{m}}$$

is an isomorphism of rings, which are noncanonically isomorphic to power series rings in one variable over $\mathbf{Z}_p$.

Proof: Putting lemmas 8, 9, 10 together, we get, firstly, that the Krull dimension of R^o is equal to 2. Then, by lemma 8, R^o is isomorphic to $\mathbf{Z}_p[[t]]$. But then, by lemma 9, there is a surjection

$$(R^o \cong)\mathbf{Z}_p[[t]] \twoheadrightarrow \mathbf{T}_{\mathfrak{m}}$$

which, by Lemma 10, is an isomorphism. We have thus proved our proposition.

Corollary 4. *Let* D *be any finite discrete valuation ring extension of* $\mathbf{Z}_p$. *Then any ordinary deformation of* $\overline{\rho}$ *to* D *is pro-modular.*

Corollary 5. *The* Λ*-algebra* R^o *is finite and flat.*

Both corollaries are immediate consequences of Proposition 14. Indeed, since $R^o \cong \mathbf{T}_{\mathfrak{m}}$ any ordinary deformation of $\overline{\rho}$ to D "comes from" a p-adic p-ordinary cuspidal eigenform. Corollary 5 is also immediate, and it is recorded here because we see no way of proving it without appeal to Hida's theory.

Problem: Let $\tau(\overline{\rho})$ denote the rank of the finite flat Λ-algebra $R^o \cong \mathbf{T}_{\mathfrak{m}}$. What can be said about $\tau(\overline{\rho})$? Is it ever greater than one?

For some information about $\tau(\overline{\rho})$ see forthcoming joint publications with Nigel Boston.

2.6 The inertially reducible locus.

The purpose of this section is to show that the locus of inertially reducible points in X consists in the union of two regular hypersurfaces (Prop. 18 below). The intersection of these hypersurfaces consists in the locus of inertially abelian points (Prop. 16 below).

Let $X^o \subset X$ be the ordinary locus as in §5. Let $\widehat{X}^o \subset \widehat{X}$ denote the associated formal affine schemes. Recall the action of the formal group Φ

on $\widehat{X}$ ("twists by wild one-dimensional characters"; see Chap. I, §4). Let $\widehat{Z}^o \subset \widehat{X}$ be the saturation of $\widehat{X}^o$ under the action of Φ, i.e., $\widehat{Z}$ is the smallest closed formal subscheme of $\widehat{X}$ stable under the action of Φ, and containing $\widehat{X}^o$. In particular, we have a "dominant morphism" $\Phi \times \widehat{X}^o \xrightarrow{\mu} \widehat{Z}^o$. If $\mathcal{A}(-)$ denotes the affine ring of a formal affine scheme, then, using the morphism μ, we may view $\mathcal{A}(\widehat{Z}^o)$ as a subring of $\mathcal{A}(\Phi \times \widehat{X}^o) = \Lambda \widehat{\otimes} R^o$ which (using the propositions of §5) is a power series ring on two parameters over $\mathbf{Z}_p$.

PROPOSITION 15. *The morphism*

$$\mu : \Phi \times \widehat{X}^o \to \widehat{Z}^o$$

is an isomorphism of formal affine schemes. The affine ring $\mathcal{A}(\widehat{Z}^o)$ is a power series ring on two parameters over $\mathbf{Z}_p$.

PROOF: The idea is simply that $\widehat{X}^o$ is transversal to the orbits of Φ in the sense that the natural morphism induced by μ on $k[\epsilon]$-points yields an injection

$$\Phi(k[\epsilon]) \times \widehat{X}^o(k[\epsilon]) \to \widehat{X}(k[\epsilon]).$$

This can be checked using the fact that if $\tilde{\rho}$ is any ordinary deformation of $\overline{\rho}$ to $k[\epsilon]$, and χ is any (wild) *nontrivial* character with values in $1+\epsilon k \subseteq k[\epsilon]^*$, then $\chi \otimes \tilde{\rho}$ is no longer ordinary.

It follows that

$$\mu : \Phi(k[\epsilon]) \times \widehat{X}^o(k[\epsilon]) \to \widehat{Z}^o(k[\epsilon])$$

is injective, and consequently, the injective homomorphism

$$\mu : \mathcal{A}(\widehat{Z}^o) \to \mathcal{A}(\Phi \times \widehat{X}^o)$$

induces a surjection on Zariski cotangent spaces. It follows that μ is an isomorphism.

Now let $\gamma : \widehat{X} \to \widehat{X}$ denote the *inner twist involution* (cf. §1) acting on the formal scheme $\widehat{X}$.

Put

$$\widehat{Z}^{oo} := \gamma \cdot \widehat{Z}^o.$$

Since γ is an automorphism of formal schemes, we have that the affine ring $\mathcal{A}(\widehat{Z}^{oo})$, as well, is isomorphic to a power series ring on two parameters over $\mathbf{Z}_p$.

Let $X_{i.a.} \subset X$ be the closed subscheme representing deformations of $\overline{\rho}$ which are inertially abelian, i.e., deformations with respect to which I_p acts through an abelian quotient, and let $\widehat{X}_{i.a.}$ be the formal completion of $X_{i.a.}$.

Let $\widehat{z}$ denote the saturation under the action of Φ of the reduced formal subscheme in $\widehat{X}$ consisting of the origin x_o.

Thus, since Φ acts principally on $\widehat{X}$, we have that $\mathcal{A}(\widehat{z})$ is a power series ring on one parameter over $\mathbf{Z}_p$.

Clearly, $\widehat{z} \subset \widehat{X}_{i.a.}$. Is this inclusion an equality?

PROPOSITION 16. *We have an equality of formal subschemes of* $\widehat{X}$*:*

$$\widehat{Z}^o \cap \widehat{Z}^{oo} = \widehat{X}_{i.a.}.$$

PROOF: Let A be an artinian object of the category $\mathcal{C}$ and let

$$\rho_A : G_{\mathbf{Q},S} \to \mathrm{GL}_2(A)$$

be a deformation of $\overline{\rho}$ to A. Let $M = A \times A$ be given a $G_{\mathbf{Q},S}$-module structure via ρ_A.

If ρ_A is in $\widehat{X}_{i.a.}(A)$, then the action of I_p on M is through $I_p^{ab} \cong \Gamma \times \mathbf{F}_p^*$ (the isomorphism being given by local class field theory). The A-module M is completely reducible for the $\mathbf{F}_p^*$-action, and splits into the direct sum of two free A-modules of rank 1, $M = M^+ \oplus M^-$ where $\mathbf{F}_p^*$ acts trivially on M^+ and through the character of order two on M^-. Since the action of Γ commutes with the $\mathbf{F}_p^*$-action, M^+ and M^- are Γ-stable, hence I_p-stable. Let $\chi^{\pm}$ be the inverses of the wild character with values in A^* giving the action of Γ on $M^{\pm}$. Twisting ρ_A by χ^+ yields an ordinary representation, hence ρ_A is in $\widehat{Z}^o(A)$. Twisting ρ_A by χ^- and then applying the "inner twist" γ also gives an ordinary representation, hence ρ_A is in $\widehat{Z}^{oo}(A)$ as well.

If ρ_A is in $\widehat{Z}^o(A)$ and in $\widehat{Z}^{oo}(A)$, we may find A-submodules $M^{\pm} \subset M$ which are direct factors of rank 1, and on M^+, I_p acts via a wild character with values in A^* while on M^-, I_p acts via ϵ times such a wild character, where ϵ is the basic quadratic character. It follows that the natural homomorphism $M^+ \oplus M^- \to M$ is an isomorphism and consequently I_p acts through an abelian quotient group.

PROPOSITION 17. *Let x be a $\mathbf{Z}_p$-valued point of X. Then x is inertially reducible if and only if x is a $\mathbf{Z}_p$-valued point of $\widehat{Z}^o \cup \widehat{Z}^{oo}$.*

PROOF: It is evident that any ordinary $\mathbf{Z}_p$-valued point is inertially reducible, and that inertially reducible points remain so after tensoring with one-dimensional characters. It follows that any $\mathbb{Z}_p$-valued point of $\widehat{Z}^o \cup \widehat{Z}^{oo}$ is inertially reducible. Now let x be an inertially reducible $\mathbb{Z}_p$- valued point of X. We can choose a homomorphism $\rho_x : G_{\mathbf{Q},\{p,\infty\}} \to \mathrm{GL}_2(\mathbb{Z}_p)$ in the strict equivalence class of deformations of $\overline{\rho}$ corresponding to x so that its restriction to I_p is a "triangular representation":

$$\rho_x : I_p \longrightarrow \mathrm{GL}_2(\mathbb{Z}_p)$$
$$g \longmapsto \begin{pmatrix} \eta_p(g) & * \\ 0 & * \end{pmatrix}$$

where $\eta_p : I_p \to \mathbb{Z}_p^*$ is some continuous character. Let $\overline{\eta}_p : I_p \to \mathbb{F}_p^*$ be the reduction of η_p mod p. Since ρ_x is a deformation of $\overline{\rho}$ we have that $\overline{\eta}_p$ is either the trivial character, or else it is the quadratic character ϵ attached to the field $\mathbf{Q}(\sqrt{-p})$, restricted to the inertia group I_p.

Replacing x by $y = \gamma(x)$ if necessary, we may suppose that $\overline{\eta}_p$ is trivial, i.e., that η_p is wild. Let

$$\eta : G_{\mathbf{Q},\{p,\infty\}} \longrightarrow 1 + p\mathbb{Z}_p \subset \mathbb{Z}_p^*$$

be the unique global character of the above form, whose restriction to I_p is η_p.

Tensoring ρ_X with η^{-1} gives us an ordinary representation. It follows that y is a $\mathbb{Z}_p$-valued point of $\widehat{Z}^o$ and our original x lies in either $Z^o \subset X$ or $Z^{oo} \subset X$. Let $X_{i.r.} \subset X$ denote the union of these subschemes. To summarize:

PROPOSITION 18. *$X_{i.r.}$ is the union of two regular hypersurfaces. A $\mathbb{Z}_p$-valued point of X is inertially reducible if and only if x is a point of $X_{i.r.}$.*

2.7 The inertially metabelian and the inertially dihedral locus.

The property of being *inertially metabelian at p* is a quotient group property.

Specifically, let

$$G_{\mathbf{Q},S} \to \Pi_{i.m.}$$

denote the maximal quotient group of $G_{\mathbf{Q},S}$ in which the image of I_p is metabelian ("i.m." stands for *inertially metabelian*). Note that $\overline{\rho}$ factors through a residual representation,

$$\overline{\rho}_{i.m.} : \Pi_{i.m.} \to \mathrm{GL}_2(\mathbb{Z}_p).$$

Form the universal representation ring $R_{i.m.} := R(\overline{\rho}_{i.m.})$ and universal representation space $X_{i.m.} = \mathrm{Spec}(R_{i.m.})$ attached to $\overline{\rho}_{i.m.}$.

We have that $X_{i.m.} \subset X$ is a closed subscheme, and by construction any A-valued point $x \in X$ lies in $X_{i.m.}$ if and only if it is *inertially metabelian* in the sense that the image of I_p under ρ_x is metabelian. Since any closed metabelian subgroup of $\mathrm{GL}_2(\mathbf{Z}_p)$ is either contained in a group conjugate to the group of upper triangular matrices

$$\left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Z}_p)\right\} \qquad \text{(the "Borel subgroup")}$$

or in the normalizer of a Cartan subgroup of $\mathrm{GL}_2(\mathbf{Q}_p)$ we have the following

PROPOSITION 19. *Any $\mathbf{Z}_p$-valued point of $X_{i.m.}$ is either inertially reducible or inertially dihedral.*

Since the group of upper triangular matrices in $\mathrm{GL}_2(A)$ is metabelian, we have that any inertially reducible A-valued points of X is contained in $X_{i.m.}$.

COROLLARY 6. *The closed subscheme $X_{i.r.}$ is contained in $X_{i.m.}$.*

PROOF: It suffices to show that Z^o and Z^{oo} are separately contained in $X_{i.m.}$. But each are smooth hypersurfaces in which the $\mathbf{Z}_p$-valued points are Zariski-dense. Since the $\mathbf{Z}_p$-valued points of Z^o and of Z^{oo} are contained in $X_{i.m.}$, the Corollary follows.

Since any closed subgroup G in $\mathrm{GL}_2(\mathbf{Z}_p)$ which projects to the two-element subgroup $\begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$ of $\mathrm{GL}_2(\mathbf{F}_p)$ is either metabelian, or else it contained an open subgroup of $\mathrm{SL}_2(\mathbf{Z}_p)$, we have the following

PROPOSITION 20. *A $\mathbf{Z}_p$-valued point x of X is inertially ample if and only if it does not lie in $X_{i.m.}$.*

We shall conclude this section with a proof of

PROPOSITION 21. *$X_{i.m.}$ is not equal to X,*

yielding the following Corollary, in the light of Prop. 20:

APPROXIMATION THEOREM. *The inertially ample points in $X(\mathbf{Z}_p)$ are open and dense. (Equivalently:) Given any deformation*

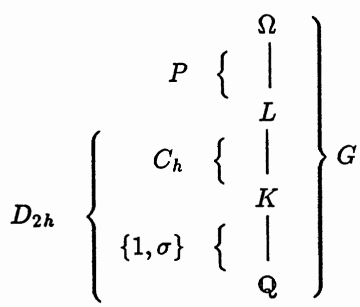
$$\rho' : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{Z}/p^N\mathbf{Z})$$

of $\overline{\rho}$ to $\mathbf{Z}/p^N\mathbf{Z}$ (any $N \geq 1$), there is an inertially ample deformation

$$\rho_x : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{Z}_p)$$

of ρ' to $\mathbf{Z}_p$.

We now prepare for the proof of Proposition 21. Let G denote the p-completion of $G_{\mathbf{Q},\{p,\infty\}}$ relative to $\overline{\rho}$ as defined in the **Remark** of Chap. I, §2 and also in [B], and [B-M]. By Schur-Zassenhaus (e.g., Prop. 2.6 of [B]) G can be expressed as a semi-direct product $D_{2h} \rtimes P$, where P is an open (and closed) normal pro-p subgroup in G (P is *the* pro-p-Sylow subgroup in G). Moreover, we have that the deformation theory of $\overline{\rho}$ factors through G. Write the dihedral group D_{2h} explicitly as a semi-direct product, $D_{2h} = \{1,\sigma\} \rtimes C_h$ where C_h is a cyclic subgroup of order h. We have the following diagram of fields:



Here $K = \mathbf{Q}(\sqrt{-p})$ and Ω is the maximal p-extension of L unramified outside the set of primes of L lying above p. Note that Ω contains $\mathbf{Q}_\infty$, the unique $\mathbf{Z}_p$-extension of $\mathbf{Q}$, and consequently we have a canonical surjection $G \twoheadrightarrow Gal(\mathbf{Q}_\infty/\mathbf{Q})$.

Since L/K is an everywhere unramified abelian extension, and the prime ideal $(\sqrt{-p})$ in $\mathcal{O}_K$ is principal, it follows that it splits completely in L. Let $q_1, q_2, \ldots, q_h$ denote the primes lying above it in $\mathcal{O}_L$ where we have chosen $q = q_1$ to be the prime ideal stabilized by σ.

LEMMA 11. *There are two elements u, v contained in the image of an inertia group at q in P satisfying the following properties:*

(a) *The closed subgroup of P generated by u and v is equal to the image* *of that inertia group at* q.

(b) *The closed subgroup in* P *generated by the elements* $\tau(u), \tau(v)$ *for* τ *ranging through the elements of* C_h *is all of* P.

(c) $\sigma(u) = u$ *and* $\sigma(v) = v^{-1}$.

(d) *The image of* u *under the natural projection* $G \twoheadrightarrow Gal(\mathbf{Q}_\infty/\mathbf{Q})$ *is of infinite order, or equivalently: is* $\neq 0$.

PROOF: We use the techniques of [**B**]. Let Q_i be a choice of prime of Ω lying above q_i, for each $i = 1, 2, \ldots, h$. Let G_i denote the inertia group at Q_i in G. Let $P_i \subset G_i$ denote the (pro-) p Sylow subgroup (in fact it is the unique closed normal subgroup of index 2) in G. We have that P_i is contained in P, and we have an induced mapping $\overline{P}_i \to \overline{P}$ on Frattini (p-) quotients.

Recall that the Frattini p-quotient of a group is the maximal abelian quotient of exponent p (cf. [**B**] for a detailed relevant discussion).

The action of D_{2h} on P induces an action on $\overline{P}$ and the subgroup C_h acts transitively on the set of h subgroups {image $\overline{P}_i$}. Let L'/L denote the intermediate field extension in Ω/L such that the quotient group $\overline{P}$ is identified with $Gal(L'/L)$. Then L' is an abelian extension of exponent p, unramified outside the primes q_i, $i = 1, 2, \ldots, h$. The set of subgroups image $\overline{P}_i \subset \overline{P}$ generate all of $\overline{P}$, since L has class number prime to p. By local class field theory, we have a surjection $\overline{U} \twoheadrightarrow \overline{P}_1$, where $\overline{U}$ is the Frattini p-quotient of U, with U the group of units in $\mathbf{Z}_p[\sqrt{-p}]$.

One sees that $\overline{U}$ is an $\mathbf{F}_p$-vector space of dimension 2, and the involution σ acts semi-simply on it, in a nonscalar manner. It follows that one can find two elements $\overline{u}, \overline{v} \in \overline{P}_1$ which generate $\overline{P}_1$, and such that $\sigma(\overline{u}) = \overline{u}$, $\sigma(\overline{v}) = \overline{v}^{-1}$. The element $\overline{u}$ is nontrivial as can be seen from the fact that it maps to a generator of the Frattini p-quotient of $Gal(\mathbf{Q}_\infty/\mathbf{Q})$, via the natural projection signalled above.

By Theorem 2.8 of [**B**] (applied to "G" $= P_1$ and $A = \{1, \sigma\}$) we may find two elements $u, v \in P_1$ which generate P_1 as a pro-p group, i.e, which satisfy property (a), which also satisfy property (c) of our lemma, and which lift $\overline{u}, \overline{v} \in \overline{P}$. By the paragraph above, property (d) holds, as well. By Burnside's lemma [**B**, Prop. 2.2] and the fact that the subgroups $\{\tau(\text{image } \overline{P}_i); \tau \in C_h\}$ generate $\overline{P}$, property (b) also holds, establishing our lemma.

We now introduce two other groups,

$$G' = D_{2h} \rtimes P' \qquad \text{and} \qquad G'' = D_{2h} \rtimes P''$$

to compare with G.

Definition of G': Let P' denote the free pro-p group on $2h$ generators which are labelled $\tau(U)$, $\tau(V)$ where τ ranges through the elements of C_h. We make the abbreviations $U = 1(U)$ and $V = 1(V)$ where 1 is the identity element in C_h. Let C_h act on the free pro-p group P' in the expected manner, i.e., $\lambda \cdot \tau(U) = (\lambda\tau)(U)$ for all $\lambda, \tau \in C_h$, and similarly for V. Extend the action of C_h on P' to an action of D_{2h} on P' by requiring σ to fix U and to send V to its inverse. There is a unique surjection $P' \twoheadrightarrow P$ which sends U to u and V to v and which respects D_{2h}-actions. This surjection extends to a surjection $G' \twoheadrightarrow G$ which is the identity on the subgroups D_{2h}.

Definition of G'': Let P'' denote the quotient of P' obtained by imposing the commutation relations $[\tau(U), \tau(V)] = 1$ for each $\tau \in C_h$. The action of D_{2h} on P' stabilizes the normal subgroup generated by these relations, and therefore induces an action on P''. Let $G'' = D_{2h} \rtimes P''$ be the semi-direct product formed via this action.

Now consider the three residual representations

$$\overline{\rho} : G \longrightarrow \mathrm{GL}_2(\mathbf{F}_p), \quad \overline{\rho}' : G' \longrightarrow \mathrm{GL}_2(\mathbf{F}_p), \quad \overline{\rho}'' : G'' \longrightarrow \mathrm{GL}_2(\mathbf{F}_p)$$

obtained by projecting the three groups G, G', G'' to D_{2h} and then imbedding D_{2h} in $\mathrm{GL}_2(\mathbf{F}_p)$ by the identity. There is no loss of generality in assuming that the involution in D_{2h} maps to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}_2(\mathbf{F}_p)$, and we assume this. Moreover, we fix a lifting of this imbedding, $D_{2h} \to \mathrm{GL}_2(\mathbf{Z}_p)$ where the involution maps to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbf{Z}_p)$. Such a lifting is unique up to strict equivalence (indeed, up to strict equivalences fixing $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$).

It is now an elementary matter to obtain an explicit description of R', the universal deformation ring of $\overline{\rho}'$, and a description of $\rho' : G' \to \mathrm{GL}_2(R')$, the universal deformation of $\overline{\rho}'$. Specifically, we may coordinatize R' as $R' := \mathbf{Z}_p[[T_1, T_2, T_3, T_4]]$, and take ρ' to be the homomorphism which

(a) when restricted to $D_{2h} \subset G'$ is the chosen lifting of D_{2h} to $\mathrm{GL}_2(\mathbf{Z}_p)$ composed with the natural imbedding $\mathrm{GL}_2(\mathbf{Z}_p) \subset \mathrm{GL}_2(R')$.

(b) $\rho'(U) = \begin{pmatrix} 1+T_1 & T_2 \\ T_2 & 1+T_1 \end{pmatrix}$

(c) $\rho'(V) = \begin{pmatrix} 1+T_3 & T_4 \\ -T_4 & 1+S \end{pmatrix}$ where S is chosen to make the determinant of the matrix equal to 1, i.e., $S = -(T_4^2 + T_3)/(1 + T_3)$.

What about R'', the universal deformation ring of the residual representation $\overline{\rho}''$? This is simply the quotient ring of R' given by the two relations $T_2T_4 = 0$ and $T_2(T_3 - S) = 0$ [these relations expressing the fact that the images of U and V commute].

We now consider again R, the universal deformation ring of $\overline{\rho}$. The surjection $G' \twoheadrightarrow G$ makes R a quotient ring of R'. We already know that R is smooth of Krull dimension 4. Let X, X', X'' denote the Spectra of the rings R, R', R'', respectively.

Thus, we have a diagram of closed immersions,

$$X \hookrightarrow X' \hookleftarrow X''$$

where X' is smooth of Krull dimension 5, and as can immediately be seen from the relations defining R'', X'' is the union of two smooth irreducible components of Krull dimensions 3 and 4.

From the defining property of X'', one sees that the D-valued inertially metabelian points of X all are precisely the D-valued points of the subscheme $X \cap X''$ in X'. To show that the inertially metabelian locus is of codimension ≥ 1 in X, it then suffices to show that X is not contained in X'', as subscheme of X'. Suppose, then, that X is contained in X''. It would then follow that X is contained in (indeed is equal to) the irreducible component of X'' of Krull dimension 4. Consequently, the relation $T_2 = 0$ holds for all $\mathbf{Z}_p$-valued points x of X, i.e., $\rho_x(u)$ is a scalar matrix for any such $\mathbf{Z}_p$-valued point x. If x is ordinary, it follows that $\rho_x(u) = 1$. But there exist ordinary points $x \in X(\mathbf{Z}_p)$ such that ρ_x has determinant $\chi^k \cdot \eta$ where η is a character of finite order and χ is the cyclotomic character, and k is a nonzero rational integer. Since $\chi(u)$ is of infinite order (as follows from property (d) of the Lemma above) we have our contradiction.

2.8 Loci of constant p-adic Hodge Type.

Up to this point we have considered "algebraic structures" in X (e.g., the closed subschemes $X_D, X_{i.m.}, X_{i.a.}, X_{i.r.}, X^o$, etc.) and "formal algebraic structures" related to the formal completion $\hat{X}$ (e.g., the action of the formal group Φ). We shall now study the associated analytic space

$X^{an} = X(\mathbf{Z}_p)$ and introduce a locally analytic mapping provided to us by Sen [**Sen 2**] from X^{an} to the two- dimensional $\mathbf{Q}_p$-analytic manifold of monic polynomials of degree two with coefficients in $\mathbf{Q}_p$.

Let us briefly recall the part of Sen's theory that is relevant to our situation.

If V is a vector space over $\mathbf{Q}_p$ of dimension two, endowed with a continuous $G_{\mathbf{Q}_p}$-action, let $V \otimes \mathbf{C}_p$ denote the tensor product over $\mathbf{Q}_p$ with $\mathbf{C}_p$, the completion of the algebraic closure of $\mathbf{Q}_p$. The action of $G_{\mathbf{Q}_p}$ on $V \otimes \mathbf{C}_p$ is taken to be the diagonal action, with $G_{\mathbf{Q}_p}$ acting on $\mathbf{C}_p$ in the unique manner which continuously extends its natural action on $\overline{\mathbf{Q}}_p$. We refer to this action of $G_{\mathbf{Q}_p}$ as the "semi-linear action". It is a consequence of the theory of Sen Sen 1 that (taking as our choice of χ the canonical cyclotomic character, cf. [**Sen 2**] §1) we may associate to the $G_{\mathbf{Q}_p}$-module V an operator $\phi \in M_2(K)$, where K is some finite extension of $\mathbf{Q}_p$. The characteristic polynomial

$$f_V(t) = \det_K(t \cdot I - \phi)$$

has coefficients in $\mathbf{Q}_p$ and is dependent only upon the isomorphism class of $V \otimes \mathbf{C}_p$ viewed as semi-linear G_K-module for K any finite extension of $\mathbf{Q}_p$ in $\overline{\mathbf{Q}}_p$.

Moreover, we have the following (sufficient) criterion for $f_V(t)$ to be reducible as a polynomial over $\mathbf{Q}_p$. If $K \subset \overline{\mathbf{Q}}_p$ is a finite field extension containing the $2p$-th roots of 1, and if s is a p-adic integer, then $\mathbf{C}_p(s)$ denotes the semi-linear G_K-module obtained by twisting $\mathbf{C}_p$, given the natural G_K-action, by the character $\chi^s : G_K \to 1 + 2p\,\mathbf{Z}_p \subset \mathbf{Z}_p^*$, where χ is the natural cyclotomic character restricted to G_K. The s-th power χ^s makes sense, since χ takes its values in $1 + 2p\,\mathbf{Z}_p$.

LEMMA 12. *Let K be a field extension of $\mathbf{Q}_p$ as above, and let $V \otimes \mathbf{Q}_p$ denote the semi-linear G_K-module obtained by restriction from $G_{\mathbf{Q}_p}$. Let s_1, s_2 be p-adic integers and suppose that there exists an exact sequence of semi-linear G_K-modules*

$$(\S) \qquad 0 \to \mathbf{C}_p(s_1) \to V \otimes \mathbf{C}_p \to \mathbf{C}_p(s_2) \to 0$$

Then $f_V(t)$ is reducible over $\mathbf{Q}_p$, with roots s_1, s_2.

PROOF: This follows directly from the construction of [**Sen 1,Sen 2**].

Recall further that the sequence of G_K-modules (§) splits if s_1 differs from s_2. If the exact sequence (§) splits, we say that V has *semi-simple (generalized) p-adic Hodge structure*. If $s_1 = s_2 = 0$, then V has semi-simple (generalized) p-adic Hodge structure if and only if the inertia group at p acts through a finite quotient group on V [**Sen 1**].

Let us now return to the analytic space $X^{an} = X(\mathbf{Z}_p)$. For $x \in X^{an}$, consider the representation $\rho_x : G_{\mathbf{Q},S} \to \mathrm{GL}_2(\mathbf{Z}_p)$ associated to X, and let V_x denote the two-dimensional $\mathbf{Q}_p$-vector space $\mathbf{Q}_p \times \mathbf{Q}_p$ with $G_{\mathbf{Q}_p}$- action obtained by allowing $G_{\mathbf{Q},S}$ to act on $\mathbf{Q}_p \times \mathbf{Q}_p$ via composition of ρ_x with the standard action of $\mathrm{GL}_2(\mathbf{Z}_p)$ and then restricting to the image of $G_{\mathbf{Q}_p}$ in $G_{\mathbf{Q},S}$. Let $f_{V_x}(t)$ denote the characteristic polynomial constructed by Sen, as described above, and let $b(x)$, $c(x)$ denote its coefficients:

$$f_{V_x}(t) = t^2 + b(x) \cdot t + c(x).$$

By the *Sen mapping* $\mathcal{S} : X^{an} \to \mathbf{Q}_p \times \mathbf{Q}_p$ we mean the mapping which sends $x \in X^{an}$ to the pair $(b(x), c(x))$.

PROPOSITION 22 (Sen). *The mapping $\mathcal{S} : X^{an} \to \mathbf{Q}_p \times \mathbf{Q}_p$ is locally analytic.*

PROOF: This is a particular case of the main result in [**Sen 2**].

For a pair of p-adic numbers $(b, c) \in \mathbf{Q}_p \times \mathbf{Q}_p$, let $X(b, c)$ denote the inverse image of (b, c) under $\mathcal{S}$ in $X^{an} = X(\mathbf{Z}_p)$. Since $\mathcal{S}$ is locally analytic with domain a p-adic analytic manifold of dimension three and range of dimension two it immediately follows that

PROPOSITION 23. *Let $(b, c) \in \mathbf{Q}_p \times \mathbf{Q}_p$. The set $X(b, c)$ inherits the structure of p-adic analytic variety from X^{an}. The analytic variety $X(b, c)$ may be empty, but for each smooth point $x \in X(b, c)$, there is a neighborhood of x in $X(b, c)$ which is a p-adic analytic manifold of dimension 1.*

If $X(b, c)$ contains a smooth point, it contains an uncountable number of points.

PROOF: By "smooth point" I mean: point at which the Sen mapping has a jacobian of maximal rank (i.e., of rank 2). The second sentence of the proposition is then evident by the implicit function theorem, as is the third sentence.

Proposition 24. *The restriction of $\mathcal{S}$ to $Z^o(\mathbf{Z}_p)$ gives us a finite-to-one mapping*

$$\mathcal{S} : Z^o(\mathbf{Z}_p) \to \mathbf{Z}_p \times \mathbf{Z}_p \subset \mathbf{Q}_p \times \mathbf{Q}_p$$

[to be described "explicitly" below].

Proof: By Prop. 15 of §6, we have:

$$Z^o(\mathbf{Z}_p) \underset{\mu}{\cong} \mathrm{Hom}(\Lambda \widehat{\otimes} R^o; \mathbf{Z}_p).$$

Let $\Lambda \xrightarrow{\iota} R^o$ be the structural homomorphism and consider the composition, $\mathcal{S}'$:

$$\begin{array}{ccc}
Z^o(\mathbf{Z}_p) \cong \mathrm{Hom}(\Lambda \widehat{\otimes} R^o, \mathbf{Z}_p) & \xrightarrow{(1 \otimes \iota)^*} & \mathrm{Hom}(\Lambda \widehat{\otimes} \Lambda; \mathbf{Z}_p) \\
\Big\downarrow \mathcal{S}' & & \Big\downarrow \cong \\
 & & \mathrm{Hom}(\Gamma, \mathbf{Z}_p^*) \times \mathrm{Hom}(\Gamma, \mathbf{Z}_p^*) \\
 & & \Big\downarrow \cong \\
\mathbf{Z}_p \times \mathbf{Z}_p & \xleftarrow{\quad q \quad} & \mathbf{Z}_p \times \mathbf{Z}_p
\end{array}$$

where to obtain the two right-hand vertical isomorphisms we use the natural isomorphisms,

$$\mathrm{Hom}(\Lambda, \mathbf{Z}_p) \cong \mathrm{Hom}(\Gamma, \mathbf{Z}_p^*)$$

(where $\Gamma \subset \mathbf{Z}_p^*$ is the subgroup of 1-units and "Hom" means "continuous homomorphisms") and

$$\mathbf{Z}_p \underset{\cong}{\xrightarrow{\psi}} \mathrm{Hom}(\Gamma, \mathbf{Z}_p^*)$$

(defined in the usual manner: for $s \in \mathbf{Z}_p, \psi(s)(\gamma) = \gamma^s$).

The mapping $q : \mathbf{Z}_p \times \mathbf{Z}_p \to \mathbf{Z}_p \times \mathbf{Z}_p$ sends (s_1, s_2) to the coefficients (b, c) of the polynomial $t^2 + bt + c = (t - s_1)(t - s_1 - s_2)$, i.e., $b = -2s_1 - s_2, c = s_1(s_1 + s_2)$.

By the corollary to prop. 14 of §5, R^0 is a finite flat Λ-algebra. Consequently, $(1 \otimes \iota)^*$ is finite-to-one.

Since q is a quadratic mapping, and since all the mappings other than $(1 \otimes \iota)^*$ and q (contributing to the composition $\mathcal{S}'$) are isomorphisms, it follows that $\mathcal{S}'$ is finite-to-one.

From the lemma of this section it follows that $\mathcal{S} = \mathcal{S}'$.

Remark. Note that in the important case where $\Lambda \xrightarrow{\iota} R^o$ is an isomorphism, we have that the mapping of proposition 24 is surjective.

COROLLARY 7. *The restriction of $\mathcal{S}$ to $X_{i.r.}(\mathbf{Z}_p)$ is finite-to-one.*

PROOF: $\mathcal{S}(\gamma \cdot x) = \mathcal{S}(x)$ and

$$X_{i.r.}(\mathbf{Z}_p) = Z^o(\mathbf{Z}_p) \cup \gamma Z^o(\mathbf{Z}_p).$$

By the *"weight-one locus"*, we mean the locally analytic subvariety $X(0,0)$ in $X^{an}(\mathbf{Z}_p)$. Since the origin x_0 is in $X(0,0)$, $X(0,0)$ is nonempty, but x_0 is not a smooth point for the mapping $\mathcal{S}$. Are there other $\mathbf{Z}_p$-valued points of p-adic Hodge type $(0,0)$? In any event, we have

PROPOSITION 25. *All points x in $X(0,0) - \{x_0\}$ have non-semi-simple p-adic Hodge type. None of these points are inertially dihedral. All but a finite number of points in $X(0,0)$ are inertially ample.*

PROOF: Let $x \in X(0,0)$ and let V_x denote the associated $G_{\mathbf{Q}_p}$-representation as in the discussion at the beginning of this section. By the theorem of Sen already quoted, V_x has semi-simple p-adic Hodge type if and only if the action of I_p on V_x factors through a finite quotient group, i.e., if and only if x is inertially finite. But by Proposition 9, the origin x_0 is the only inertially finite $\mathbf{Z}_p$-valued point of X. It follows that all points in $X(0,0) - \{x_0\}$ have non-semi-simple p-adic Hodge type.

Now the property of having non-semi-simple p-adic Hodge type is insensitive to finite base change. If x were inertially dihedral, then ρ_x restricted to the decomposition group $G_{\mathbf{Q}_p(\sqrt{-p})}$ of the prime above p in the quadratic extension field $\mathbf{Q}(\sqrt{-p})$ would be, after a possible change of scalars, a direct sum of two one-dimensional representations and hence would have semi-simple p-adic Hodge type.

It follows that no point in $X(0,0) - \{x_0\}$ is inertially dihedral. Note also, that since the image of $X(0,0)$ under $\mathcal{S}$ is a single point, and since

(Proposition 24) $\mathcal{S}$ is finite-to-one on $X_{i.r.}(\mathbf{Z}_p)$ we have that $X(0,0)$ has at most a finite number of points in common with $X_{i.r.}(\mathbf{Z}_p)$. By Prop. 19 it then follows that $X(0,0)$ has only a finite number of points in common with $X_{i.m.}(\mathbf{Z}_p)$. By Proposition 20, we then have that all but a finite number of points in $X(0,0)$ are inertially ample.

Further Questions.

Is it generally true that $X(a,b)$ is (either empty or) a 1-dimensional p-adic analytic variety?

Are there *irreducible polynomials* (over $\mathbf{Q}_p$) t^2+bt+c for which $X(b,c)$ is nonempty?

Is it the case that whenever $X(b,c)$ is nonempty, it contains an open dense subspace of inertially ample points?

For integers $k \geq 2$, how many modular points (necessarily of weight k) does $X(1-k,0)$ contain?

REFERENCES

[Art] E. Artin, "Theory of Algebraic Numbers," notes by G. Wurges from lectures at the Mathematisches Institut, Göttingen, translated and distributed by G. Striker, Göttingen, 1959.

[B] N. Boston, N., "Deformation Theory of Galois Representations," Harvard Ph.D. Thesis, 1987.

[B-M] N. Boston, and B. Mazur, *Explicit universal deformations of Galois representations*, (to appear).

[C-P-S] E. Cline, B. Parshall and L. Scott, *Cohomology of finite groups of Lie type I*, Publ. Math. IHES **45** (1975), 169-191.

[C-R] C. Curtis and I. Reiner, "Representation Theory of Finite Groups and Associated Algebras," Interscience, New York-London, 1962.

[D] P. Deligne, *Formes modulaires et représentations de* GL(2), Lecture Notes in Math. **349** (1973), 55-105, Springer-Verlag.

[D-S] P. Deligne and J.-P. Serre, *Formes modulaires de poids 1*, Annales scientifiques de l'E.N.S. **7** (1974), 507-530.

[F] W. Feit, $\widetilde{A}_5$ *and* $\widetilde{A}_7$ *are Galois groups over number fields*, J. Alg. **104** (1986), 231-260.

[G-M] W. Goldman and J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds.*

[G] F. Gouvêa, "Arithmetic of p-adic Modular Forms," Harvard Ph.D. Thesis, 1987; Lecture Notes in Mathematics **1304** (1988), Springer.

[H] K. Haberland, "Galois Cohomology of Algebraic Number Fields," VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[Hecke] E. Hecke, *Zur Theorie der elliptischen Modulfunktionen*, in "Mathematische Werke," (no. 23), Vandenhoeck & Ruprecht, Göttingen, 1970, pp. 428-453.

[Hida] H. Hida, *Iwasawa modules attached to congruences of cusp forms*, Ann. Scient. Ed. Norm. Sup. **19** (1986), 231-273.

[K-L] N. Katz and S. Lang, *Finiteness theorems in geometric classfield theory*, L'Enseignement mathématique **XXVII** (1981), 285-319.

[LM] S. LaMacchia, *Polynomials with Galois group* PSL(2, 7), Comm. Alg. **8** (1980), 983-992.

[L-M] A. Lubotzky and A. Magid, *Varieties of representations of finitely generated groups*, Memoirs of the A.M.S. **58** (1985), 336.

[M-W] B. Mazur and A. Wiles, *On p-adic analytic families of Galois representations*, Comp. Math. **59** (1986), 231-264.

[Na] W. Narkiewicz, "Elementary and Analytic Theory of Algebraic Numbers," PWN - Polish Scientific Publishers, Warsaw, 1974.

[Sch] M. Schlessinger, *Functors on Artin rings*, Trans. A.M.S. **130** (1968), 208-222.

[Sen 1] S. Sen, *Continuous cohomology and p-adic Galois representations*, Inv. Math. **62** (1981), 89-116.

[Sen 2] S. Sen, *The analytic variation of p-adic Hodge structure*, to appear in Annals of Math..

[Serre 1] J.-P. Serre, *Abelian ℓ-adic Representations and Elliptic Curves*, Benjamin, New York.

[Serre 2] J.-P. Serre, *Modular forms of weight one and Galois representations*, in "Algebraic Number Fields," edited by A. Frohlich, Acad. Press, 1977, pp. 193-268.

[Sh] G. Shimura, "Introduction to the Arithmetic Theory of Automorphic Functions," Princeton Univ. Press, 1971.

[ZM] A. Zeh-Marschke, $SL_2(7)$ *als Galoisgruppe über* $\mathbf{Q}$, to appear.

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