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On p-adic Hecke algebras for GL_2 over totally real fields

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0. Introduction

The purpose of this paper is to lay the foundation of the theory of p-adic Hecke algebras for GL_2 over totally real fields. Apart from the flatness of the ordinary part of the universal Hecke algebra over the Iwasawa algebra, almost all results obtained in our previous papers [12] and [14, §1] in the case where $F = \mathbf{Q}$ are generalized to arbitrary totally real fields F. Our result holds without any exception for the prime p; thus, the assumption: $p \geq 5$ which we made in [12] and [14] is now eliminated. One peculiar feature in the treatment of Hilbert modular forms is the existence of multiple weight modular forms; i.e., those forms with the automorphic factor: $\prod_\sigma (c_\sigma z_\sigma + d_\sigma)^{k_\sigma}$ for mutually distinct k_σ's. In order to guarantee the stability of the space of integral cusp forms under Hecke operators, we have to modify a little in Section 3 the definition of Hecke operators $T(\mathfrak{n})$ unless k is parallel (i.e. $k_\sigma = k_\tau$ for all σ, τ). When we consider the congruence subgroups of the adelized GL_2 over F of type

$$\left\{ x \,\middle|\, x \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \bmod p^\alpha \right\},$$

the independence of the p-adic Hecke algebras relative to the weight k no longer holds. We can recover the isomorphism between the ordinary Hecke algebras of different weights k and l only when the difference of k and l is parallel. Thus we have infinitely many distinct Hecke algebras parametrized by the classes of weights modulo parallel ones. The presentation of this phenomenon is one of the motives of this work. Besides the obvious generalizations to totally real fields of the results obtained for $\mathbf{Q}$ in [14] on Galois representations, we hope to discuss in a future occasion how to unify these infinitely many Hecke algebras into the universal one and also to discuss an intimate relation between our results and the theory of cyclotomic $\mathbf{Z}_p$-extensions over $\mathbf{Q}$. In fact, the construction of Galois representations over our big Hecke algebra of parallel weight (i.e. in the case of $v = 0$; see below for details) has already been done by Wiles [40]. His result even covers the totally real fields of even degree.

We shall now give a sketch of our result in the simplest case of *odd* p-power level. We fix throughout the paper a rational prime p and a totally real field F of finite degree over $\mathbf{Q}$. Let I be the set of all embeddings of F into $\mathbf{R}$. The module $\mathbf{Z}[I]$ of weights of F is by definition a free module generated by the elements of I. Each weight $k = \sum_\sigma k_\sigma \cdot \sigma$ can be considered as a quasi character of $F^\times$ assigning $x^k = \prod_\sigma (x^{k_\sigma})^\sigma$ to $x \in F^\times$. We fix an algebraic closure $\overline{\mathbf{Q}}_p$ of the p-adic field $\mathbf{Q}_p$ and take the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$ inside $\mathbf{C}$. We fix once and for all an embedding: $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$, and hence any algebraic number in $\overline{\mathbf{Q}}$ can be considered as a complex number as well as a p-adic number in $\overline{\mathbf{Q}}_p$. Let $\Phi(k)$ be the subfield of $\overline{\mathbf{Q}}$ generated by the values x^k for all $x \in F$, and let $\mathcal{O}(k)$ denote the valuation ring of $\Phi(k)$ corresponding to the embedding: $\Phi(k) \hookrightarrow \overline{\mathbf{Q}}_p$. In $\mathbf{Z}[I]$, there is one specific element $t = \sum_{\sigma \in I} \sigma$ corresponding to the norm map: $F^\times \to \mathbf{Q}^\times$. We write $\xi \sim \eta$ (resp. $\xi \geq \eta$) for two elements $\xi, \eta \in \mathbf{Z}[I]$ if $\xi - \eta \in \mathbf{Z} \cdot t$ (resp. $\xi_\sigma - \eta_\sigma \geq 0$ for all $\sigma \in I$). We shall fix one class of $\mathbf{Z}[I]/\mathbf{Z} \cdot t$ and take the smallest non-negative representative $v \in \mathbf{Z}[I]$ of this class. For each $0 \leq n \in \mathbf{Z}[I]$ with $n \sim -2v$, we put $k = n + 2t$, $w = v + k - t$ and $\hat{w} = t - v = k - w$. Let H be the upper half complex plane. We consider the following automorphic factor:

$$\det(\gamma)^{-\hat{w}} j(\gamma, z)^k = \prod_{\sigma \in I} (a_\sigma d_\sigma - b_\sigma c_\sigma)^{v_\sigma - 1}(c_\sigma z_\sigma + d_\sigma)^{k_\sigma}$$

for $\gamma = \left(\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}\right)_\sigma \in \mathrm{GL}_2(\mathbf{R})^I$ and $z = (z_\sigma)_\sigma \in H^I$. Let $F_\mathbf{A}$ (resp. F_∞, F_f) be the adele ring of F (resp. the infinite part of $F_\mathbf{A}$ and the finite part of $F_\mathbf{A}$). Then $\mathrm{GL}_2(F_\infty)$ can be identified with $\mathrm{GL}_2(\mathbf{R})^I$, and its connected component $\mathrm{GL}_2^+(F_\infty)$ with the identity acts naturally on H^I. Let $C_{\infty+}$ be the stabilizer in $\mathrm{GL}_2^+(F_\infty)$ of $z_0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in H^I$. We denote by $\mathfrak{r}$ the integer ring of F and put $\hat{\mathfrak{r}} = \mathfrak{r} \otimes_\mathbf{Z} \hat{\mathbf{Z}}$, where $\hat{\mathbf{Z}}$ is the product of the l-adic integer ring $\mathbf{Z}_l$ over all rational primes l. Define congruence subgroups of $\mathrm{GL}_2(\hat{\mathfrak{r}})$ by

$$V_1(p^\alpha) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\hat{\mathfrak{r}}) \,\middle|\, c \in p^\alpha\hat{\mathfrak{r}},\ d - 1 \in p^\alpha\hat{\mathfrak{r}}\right\}.$$

Regarding $\hat{\mathfrak{r}}$ as a subring of F_f, we consider $V_1(p^\alpha)$ to be an open-compact subgroup of $\mathrm{GL}_2(F_f)$. We consider modular forms f: $\mathrm{GL}_2(F_\mathbf{A}) \to \mathbf{C}$ satisfying

$$f(axu) = f(x)\det(u_\infty)^{\hat{w}} j(u_\infty, z_0)^{-k} \tag{0.1}$$

for all $a \in \mathrm{GL}_2(F)$ *and all* $u = u_f u_\infty$ *with* $u_f \in V_1(p^\alpha)$ *and* $u_\infty \in C_{\infty+}$.

The space of cusp forms $S_{k,w}^*(p^\alpha; \mathbf{C})$ we consider consists of functions on $\mathrm{GL}_2(F_\mathbf{A})$ satisfying, in addition to (0.1), the holomorphy condition at ∞ and the cuspidal condition (for details, see §2). This space is naturally isomorphic to the space of classical holomorphic Hilbert cusp forms of weight k and of level p^α. Then we can define Hecke operators $T_0(\mathfrak{n})$ for each ideal $\mathfrak{n}$ acting on $S^*_{k,w}(p^\alpha; \mathbf{C})$, and the Hecke algebra $h_{k,w}(p^\alpha; \mathcal{O}(v))$ is by definition the subalgebra of $\mathrm{End}_{\mathbf{C}}(S^*_{k,w}(p^\alpha; \mathbf{C}))$ generated over $\mathcal{O}(v)$ by $T_0(\mathfrak{n})$ for all ideals $\mathfrak{n}$. The precise definition of $T_0(\mathfrak{n})$ will be given in Section 3, which is a slight modification of the classical definition when $v \neq 0$. Let $K/\mathbf{Q}_p$ be a finite extension inside $\overline{\mathbf{Q}}_p$ and suppose that $K \supset \Phi(v)$. Then the p-adic integer ring $\mathcal{O}$ of K contains $\mathcal{O}(v)$. We simply put, for $A = \mathcal{O}$ or K,

$$h_{k,w}(p^\alpha; A) = h_{k,w}(p^\alpha; \mathcal{O}(v)) \otimes_{\mathcal{O}(v)} A.$$

For each pair of integers $\beta > \alpha > 0$, it is well-known that there exists a surjective $\mathcal{O}$-algebra homomorphism ρ^α_β: $h_{k,w}(p^\beta; \mathcal{O}) \to h_{k,w}(p^\alpha; \mathcal{O})$ which takes $T_0(\mathfrak{n})$ to $T_0(\mathfrak{n})$ for all $\mathfrak{n}$. Without having recourse to the theory of p-adic modular forms, we can define the p-adic Hecke algebra by

$$h_{k,w}(p^\infty; \mathcal{O}) = \varprojlim_{\alpha} h_{k,w}(p^\alpha; \mathcal{O}),$$

which, as will be seen in Section 4, acts naturally on the space of p-adic modular forms and is in fact the $\mathcal{O}$-linear dual space of the space of p-adic modular forms (Th. 5.3 in §5). *The ordinary part* $h^{\mathrm{ord}}_{k,w}(p^\alpha; \mathcal{O})$ $(0 < \alpha \leq \infty)$ *is the maximal algebra direct summand of* $h_{k,w}(p^\alpha; \mathcal{O})$ *on which the image of* $T_0(p)$ *is a unit.* Then we have:

THEOREM I. *For any two weights n and n' with $n \sim n' \sim -2v$ and $n \geq n' \geq 0$, there exists an $\mathcal{O}$-algebra isomorphism:*

$$h^{\mathrm{ord}}_{k,w}(p^\infty; \mathcal{O}) \cong h^{\mathrm{ord}}_{k',w'}(p^\infty; \mathcal{O})$$

which takes $T_0(\mathfrak{n})$ to $T_0(\mathfrak{n})$, where $k' = n' + 2t$ and $w' = v + k' - t$.

By this theorem, the ordinary Hecke algebra $h^{\mathrm{ord}}_{k,w}(p^\infty; \mathcal{O})$ depends only on the class of $v \bmod \mathbf{Z} \cdot t$; so, we write $\mathbf{h}^{\mathrm{ord}}_v(1; \mathcal{O})$ instead of $h^{\mathrm{ord}}_{k,w}(p^\infty; \mathcal{O})$. Similar independence of the whole Hecke algebra $h_{k,w}(p^\infty; \mathcal{O})$ with respect to the weights will also be shown in Section 11 in the case where $v = 0$ by some results of Shimura [30] (see also Ohta [26]).

Let Z be the Galois group of the maximal abelian extension $\mathcal{F}_\infty/F$ unramified outside p and ∞. Let $\mathcal{F}_\alpha$ be the maximal ray class field modulo p^α, and put

$$Z_\alpha = \mathrm{Gal}(\mathcal{F}_\infty/\mathcal{F}_\alpha) \subset Z.$$

Let Z_{tor} be the torsion part of Z and decompose $Z = W \times Z_{\mathrm{tor}}$ for a torsion free part W. Then Z_1 and W are p-profinite groups and for sufficiently large α, $Z_\alpha \subset W$. Let $\mathcal{A}$, Λ_α and Λ be the continuous group algebras over $\mathcal{O}$ of Z, Z_α and W, respectively. Then, Λ is isomorphic to the formal power series ring over

$\mathcal{O}$ of several variables (if the Leopoldt conjecture holds for F, then $\Lambda \cong \mathcal{O}[[X]]$ for one indeterminate X). Let $\chi = \chi_t$: $Z \to \mathbf{Z}_p^\times$ be the cyclotomic character, and for each $l \in \mathbf{Z} \cdot t$, with $l = [l]t$ for $[l] \in \mathbf{Z}$, we write χ_l for $\chi^{[l]}$: $Z \to \mathbf{Z}_p^\times$. Since χ_l: $Z_\alpha \to \mathbf{Z}_p^\times$ is a continuous character, we can extend it to an $\mathcal{O}$-algebra homomorphism $\chi_{l,\alpha}$: $\Lambda_\alpha \to \mathcal{O}$. Let $\omega_{l,\alpha} = \mathrm{Ker}(\chi_{l,\alpha})$ which is a prime ideal of Λ_α. Let $\Lambda_{\alpha,l}$ denote the localization of Λ_α at $\omega_{l,\alpha}$. Then we have:

THEOREM II. *There is a natural $\mathscr{A}$-algebra structure on* $\mathbf{h}_v^{\mathrm{ord}}(1; \mathcal{O})$ *such that the natural surjection*: $\mathbf{h}_v^{\mathrm{ord}}(1; \mathcal{O}) \to h_{k,w}^{\mathrm{ord}}(p^\alpha; \mathcal{O})$ *induces an isomorphism*:

$$\mathbf{h}_v^{\mathrm{ord}}(1; \mathcal{O}) \otimes_{\Lambda_\alpha} \Lambda_{\alpha,l}/\omega_{l,\alpha}\Lambda_{\alpha,l} \cong h_{k,w}^{\mathrm{ord}}(p^\alpha; K)$$

for all $n \sim -2v$ $(n \geq 0)$ *and* $\alpha > 0$,

where $l = n + 2v$, $k = n + 2t$ *and* $w = v + k - t$. *Moreover* $\mathbf{h}_v^{\mathrm{ord}}(1; \mathcal{O})$ *is a torsion-free* Λ*-module of finite type and is reduced.*

Let $\mathscr{L}$ be the quotient field of Λ, and fix an algebraic closure $\bar{\mathscr{L}}$ of $\mathscr{L}$. To give an absolutely irreducible component of $\mathrm{Spec}(\mathbf{h}_v^{\mathrm{ord}}(1; \mathcal{O}))$ is equivalent to giving a Λ-algebra homomorphism λ: $\mathbf{h}_v^{\mathrm{ord}}(1; \mathcal{O}) \to \bar{\mathscr{L}}$. We fix such a λ. Let $\mathscr{K}$ be the quotient field of the image of λ and $\mathscr{I}$ be the integral closure of Λ in $\mathscr{K}$. By extending the scalar field if necessary, we may assume that $\overline{\mathbf{Q}}_p \cap \mathscr{I} = \mathcal{O}$. We say that a $\overline{\mathbf{Q}}_p$-valued point P in $\mathrm{Spec}(\mathscr{I})/\mathcal{O}$ is *algebraic* if P is over $\omega_{l,\alpha} \in \mathrm{Spec}(\Lambda_\alpha)$ for some $\alpha > 0$ and $0 \leq l \in \mathbf{Z} \cdot t$. We write this l as $n(P)$, and the minimum of α with the above property as $\alpha(P)$. Regarding algebraic points P as an $\mathcal{O}$-algebra homomorphism: $\mathscr{I} \to \overline{\mathbf{Q}}_p$, we can define an $\mathcal{O}$-algebra homomorphism

$$\lambda_P = P \circ \lambda \colon \mathbf{h}_v^{\mathrm{ord}}(1; \mathcal{O}) \to \overline{\mathbf{Q}}_p.$$

Then we have:

THEOREM III. *For each algebraic point P of* $\mathrm{Spec}(\mathscr{I})$ *with* $n(P) \geq 2v$, $\lambda_P(T_0(\mathfrak{n}))$ *is an algebraic number in* $\overline{\mathbf{Q}}$ *for all ideals* $\mathfrak{n}$, *and there exists a non-trivial complex cusp form* $f_P \in S_{k,w}^*(p^{\alpha(P)}; \mathbf{C})$ *for* $k = n(P) - 2v + 2t$, $w = n(P) - v + t$ *such that* $f_P | \dot{T}_0(\mathfrak{n}) = \lambda_P(T_0(\mathfrak{n}))f_P$ *for all ideals* $\mathfrak{n}$. *This cusp form* f_P *is determined up to constant multiple.*

In fact, we can specify f_P by using the Fourier expansion of f_P. Then, this correspondence: $P \mapsto f_P$ can be extended to an algebraic function on $\mathrm{Spec}(\mathscr{I})(\overline{\mathbf{Q}}_p)$ with values in the space of p-adic modular forms. This parametrization of common eigenforms is universal in the sense that for any given common eigenform $f \in S_{k,w}^*(p^\alpha; \mathbf{C})$ $(k \geq 2t,\ k \sim -2v)$ whose eigenvalue for $T_0(p)$ is a p-adic unit in $\overline{\mathbf{Q}}_p$, we can find λ: $\mathbf{h}_v^{\mathrm{ord}}(1; \mathcal{O}) \to \bar{\mathscr{L}}$ from which f is obtained as in the theorem.

These theorems will be restated for arbitrary level in Section 3 and will be proved in Sections 11 and 12. We shall also prove the finiteness over Λ of the module of congruence and the module of differentials of λ in Section 3, which has at least conjecturally an intimate relation with the special values of L-functions of GL(3) as seen in [14] in the case of $F = \mathbf{Q}$. Our method for proving these theorems relies firstly upon the analysis of the structure, as Λ-module, of cohomology groups of arithmetic subgroups of quaternion algebras over F and secondly upon the theory of p-adic Hilbert modular forms constructed by Deligne, Ribet, Rapoport and Katz. We shall give an exposition of the latter theory in Section 4 and will prove the duality between the Hecke algebras and the space of cusp forms in Section 5. We then analyse the above mentioned cohomology groups in the following Sections 6, 7, 8, 9 and 10 by adopting an idea of Shimura which goes back to 1960's [30]. We shall generalize in these sections the results obtained in [14, §§3, 4 and 5] on cohomology groups coming from $M_2(\mathbf{Q})$ to those coming from quaternion algebras over F which are totally definite, or indefinite but yield Shimura curves. In the hope of having the same type of results for more general quaternion algebras yielding varieties of higher dimension, we have included some results which are irrelevant to our present purpose but are expected to be useful in the higher dimensional case.

Contents

Notation. Throughout this paper, we fix a rational prime p and a totally real algebraic number field F of finite degree. We denote by $\overline{\mathbf{Q}}$ the algebraic closure of $\mathbf{Q}$ inside $\mathbf{C}$. We also fix an algebraic closure $\overline{\mathbf{Q}}_p$ of the p-adic field $\mathbf{Q}_p$

and an embedding: $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. Thus any algebraic number in $\overline{\mathbf{Q}}$ can be considered uniquely as a p-adic number as well as a complex number. The normalized p-adic absolute value of $x \in \overline{\mathbf{Q}}_p$ will be denoted by $|x|_p$. We denote by Ω the completion of $\overline{\mathbf{Q}}_p$ under the norm $|\ |_p$.

By a quaternion algebra over F, we mean a central simple algebra over F of dimension 4; so, we include the matrix algebra $M_2(F)$ in this category. For each quaternion algebra B/F, we denote by $G = G^B$ the linear algebraic group defined over $\mathbf{Q}$ such that $G(\mathbf{Q}) = B^\times$. We denote by $G_{\mathbf{A}}$ the adelization of G, and G_f (resp. $G_\infty, G_{\infty+}$) denotes the finite part of $G_{\mathbf{A}}$ (resp. the infinite part of $G_{\mathbf{A}}$ and the connected component of G_∞ with the identity). Similarly, for each finite extension $K/\mathbf{Q}$, we sometimes consider $K^\times$ (resp. K) as a non-split torus (resp. an additive group) defined over $\mathbf{Q}$ such that $K^\times(\mathbf{Q})$ (resp. $K(\mathbf{Q})$) is isomorphic to the multiplicative group (resp. the additive group) of K, and $K_{\mathbf{A}}$, K_∞, K_f and $K_{\infty+}^\times$ denote the adele ring of K, the infinite part of $K_{\mathbf{A}}$, the finite part of $K_{\mathbf{A}}$ and the connected component of $K_\infty^\times$ of the identity, respectively. We denote by ν: $G \to F^\times$ the reduced norm map which can be viewed as a homomorphism of algebraic groups. For each place σ of F, let F_σ denote the completion of F at σ. For each finite extension $K/\mathbf{Q}$, let $\mathfrak{r}_K$ denote the integer ring of K. We write simply $\mathfrak{r}$ for $\mathfrak{r}_F$. For each integral ideal N of $\mathfrak{r}$, let $F_N = \prod_{\sigma|N} F_\sigma$. We denote by x_N for $x \in G_{\mathbf{A}}$, $F_{\mathbf{A}}$ or $F_{\mathbf{A}}^\times$ the projection of x in $G(F_N)$, F_N or $F_N^\times$. Especially, x_σ (resp. x_∞, x_f) denotes the σ-component (resp. the infinite part, the finite part) of x. We denote by $\mathfrak{r}_N$ (resp. $\mathfrak{r}_\sigma$) the closure of $\mathfrak{r}$ in F_N (resp. F_σ). Then we know that $\mathfrak{r}_N = \prod_{\sigma|N} \mathfrak{r}_\sigma$. We put $\hat{\mathbf{Z}} = \prod_l \mathbf{Z}_l$, where l runs over all rational primes, and we put $\hat{\mathfrak{r}}_K = \mathfrak{r}_K \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$. We can regard $\hat{\mathfrak{r}}_K$ as a subring of K_f. Each fractional ideal $\mathfrak{a}$ of K can be expressed as $x\hat{\mathfrak{r}}_K \cap K$ (in K_f) for some $x \in K_f^\times$. The ideal $x\hat{\mathfrak{r}}_K \cap K$ will be written as $x\mathfrak{r}_K$.

Let I_K be the set of all embeddings of K into $\overline{\mathbf{Q}}$. When $K = F$, we simply write I for I_F. We denote by $A[J]$, for each commutative algebra A and a subset J of I_K, the A-free module generated by the elements of J. The module $A[I_K]$ has a natural right action of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. For $k = \sum_{\sigma \in I} k_\sigma \cdot \sigma \in \mathbf{R}[I]$, we write $k \geq 0$ if $k_\sigma \geq 0$ for all $\sigma \in I$, and $k > 0$ if $k \geq 0$ and $k \neq 0$. This positivity on $\mathbf{R}[I]$ is extended to an order on $\mathbf{R}[I]$ so that $k > k'$ (resp. $k \geq k'$) if $k - k' > 0$ (resp. $k - k' \geq 0$). We define a map k: $\mathbf{C}^I \ni x \mapsto x^k \in \mathbf{C}$ for each $k = \sum_\sigma k_\sigma \cdot \sigma \in \mathbf{Z}[I]$ by $x^k = \prod_{\sigma \in I} x_\sigma^{k_\sigma}$. When $x_\sigma > 0$ for all $\sigma \in I$, we can even define $x^s = \prod_\sigma x^{s_\sigma}$ for $s \in \mathbf{C}[I]$. Since we can consider $F_\infty^\times$ as a subspace of $\mathbf{C}^I$ naturally, the map k induces a quasi-character: $F_\infty^\times \to \mathbf{C}^\times$. We denote by $\Phi(k)$ the subfield of $\overline{\mathbf{Q}}$ generated by x^k for all $x \in F$. Then $\Phi(k)$ is the fixed field of

$$\{\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \,|\, k\sigma = k\}.$$

Let $\mathbf{e} = \mathbf{e}_F \colon F_{\mathbf{A}}/F \to \mathbf{C}$ be the unique additive character such that

$$\mathbf{e}_F(x) = \exp\left(2\pi\sqrt{-1}\sum_{\sigma\in I} x_\sigma\right) \qquad \text{for } x = (x_\sigma) \in F_\infty.$$

By abusing notation, we write for $x = (x_\sigma) \in \mathbf{C}^I$, $\mathbf{e}_F(x) = \exp(2\pi\sqrt{-1}\sum_{\sigma\in I}x_\sigma)$; especially, for $\xi \in F$ and $x \in \mathbf{C}^I$, $\mathbf{e}_F(\xi x) = \exp(2\pi\sqrt{-1}\sum_\sigma \xi^\sigma x_\sigma)$ is well defined. For each $x \in F_{\mathbf{A}}^\times$, we write $|x|_{\mathbf{A}}$ for the module of x as in [38].

For each finite set A, $|A|$ denotes the cardinality of A. For any two sets X and Y, we denote by X^Y the set of all functions of Y into X. There is one exception for this notation: If we are given a group Γ and a Γ-module X, we put

$$X^\Gamma = H^0(\Gamma, X) = \{x \in X \mid \gamma \cdot x = x \text{ for all } \gamma \in \Gamma\}.$$

We trust there will be no confusion about this notation.

1. Modules over quaternion algebras

We take a quaternion algebra B over F and fix a maximal order R of B. Let Σ be the set of all places of F, and put

$$I_B = \{\tau \in I \mid B \otimes_F F_\tau \cong M_2(F_\tau)\},$$
$$r = |I_B|, \qquad \Sigma^B = \{\tau \in \Sigma \mid B \otimes_F F_\tau \not\cong M_2(F_\tau)\}.$$

We consider I as a subset of Σ consisting of infinite places. Take a finite Galois extension $K_0/\mathbf{Q}$ (in $\mathbf{C}$) containing F, and denote by $\mathfrak{r}_0$ its integer ring. Suppose that we have an isomorphism:

(1.1) $B \otimes_{\mathbf{Q}} K_0 \cong M_2(K_0)^I$ *such that* (i) *the projection* $\tau\colon B \to M_2(K_0)$ *at each* $\tau \in I_B$ *takes* B *into* $M_2(K_0 \cap \mathbf{R})$, *and* (ii) $R \otimes_{\mathbf{Z}} \mathfrak{r}_0$ *is sent into* $M_2(\mathfrak{r}_0)^I$.

We can always find an extension K_0 and an isomorphism as in (1.1). For each $\mathfrak{r}_0$-algebra A, we consider the polynomial ring $A[X, Y]$ with $2|I|$ indeterminates: $X = (X_\sigma)_{\sigma\in I}$ and $Y = (Y_\sigma)_{\sigma\in I}$. On each polynomial $P(X, Y)$ with coefficients in A, we get $\gamma = (\gamma_\sigma)_{\sigma\in I} \in M_2(\mathfrak{r}_0)^I$ act via

$$P|\gamma(X, Y) = P\big((X, Y)^t\gamma\big),$$

where $(X, Y)^t\gamma = ((X_\sigma, Y_\sigma)^t\gamma_\sigma)_{\sigma\in I}$. Thus $A[X, Y]$ becomes a right module over the multiplicative semi-group R. Let $L(n; A)$ for $0 \le n \in \mathbf{Z}[I]$ denote the A-submodule of $A[X, Y]$ consisting of all polynomials homogeneous for the variables (X_σ, Y_σ) of degree n_σ at every $\sigma \in I$. Then $L(n; A)$ is stable under the right action of R. For each $v \in \mathbf{Z}[I]$, we shall now twist the action of R on

$L(n; A)$ and define a new action by $P|_v\gamma = \nu(\gamma)^v P|\gamma$. The A-module $L(n; A)$ equipped with this twisted right R-action will be denoted by $L(n, v; A)$. More generally, for each A-module M, we can consider the right R-module $L(n, v; M) = L(n, v; A) \otimes_A M$ whose R-action is induced from $L(n, v; A)$. This action can be naturally extended to a unique action of $G^B(A)$. We can convert the right action of $G^B(A)$ into a left action by

$$\gamma \cdot P = P|_v\gamma^{-1}.$$

This left $G^B(A)$-module will be denoted by ${}^tL(n, v; M)$.

2. Spaces of cusp forms and Hecke operators

Put $I^B = I - I_B = \Sigma^B \cap I$. For each $k \in \mathbf{Z}[I]$, we define $k^B \in \mathbf{Z}[I^B]$ and $k_B \in \mathbf{Z}[I_B]$ by $k^B = \sum_{\sigma \in I^B} k_\sigma \cdot \sigma$ and $k_B = \sum_{\sigma \in I_B} k_\sigma \cdot \sigma$. We denote by $t \in \mathbf{Z}[I]$ the special element $t = \sum_{\sigma \in I}\sigma$. We firstly clarify what kind of conditions we shall impose on the weights of cusp forms. For two elements $k, k' \in \mathbf{Z}[I]$, we write $k \sim k'$ if $k - k' \in \mathbf{Z} \cdot t$. We fix throughout the paper a class in $\mathbf{Z}[I]/\mathbf{Z} \cdot t$ and choose a representative $v \in \mathbf{Z}[I]$ of this class such that $v \geq 0$ and $v_\sigma = 0$ for some $\sigma \in I$. Such an element v is uniquely determined. We take $n \in \mathbf{Z}[I]$ satisfying $n + 2v \sim 0$ and $n \geq 0$ and put

$$k = n + 2t, \qquad w = v + k - t \quad \text{and} \quad \hat{w} = t - v = k - w.$$

Then, we see

$$k \sim 2w, \qquad w \sim \hat{w} \sim -v \quad \text{and} \quad k \sim n. \tag{2.1}$$

Under our choice of n, the unit group

$$E = \left\{ \varepsilon \in \mathfrak{r}^\times \middle| \varepsilon^\sigma > 0 \text{ for all } \sigma \in I \right\} = R^\times \cap F_{\infty+}^\times$$

acts trivially on $L(n, v; A)$ for any commutative algebra A over $\mathfrak{r}_0$.

The isomorphism (1.1) induces an identification: $G_\infty^B = \mathrm{GL}_2(\mathbf{R})^{I_B} \times (\mathbf{H}^\times)^{I^B}$, where $\mathbf{H}$ denotes the Hamilton quaternion algebra over $\mathbf{R}$. Let H be the upper half complex plane, and put $\mathscr{Z} = \mathscr{Z}_B = H^{I_B}$. We can identify H with $\mathrm{GL}_2(\mathbf{R})/O_2(\mathbf{R}) \cdot \mathbf{R}^\times$ by

$$\mathrm{GL}_2(\mathbf{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a\sqrt{-1} + b)/(c\sqrt{-1} + d) \in H$$

$$\text{if } ad - bd > 0 \quad \text{and}$$

$$\mathrm{GL}_2(\mathbf{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (-a\sqrt{-1} + b)/(-c\sqrt{-1} + d) \in H$$

$$\text{if } ad - bc < 0.$$

Thus G_∞ naturally acts on z by $g(z) = (g_\sigma(z_\sigma))_{\sigma \in I_B}$. Especially, $G_{\infty+}$ acts on $\mathscr{Z}$ complex analytically. Put $z_0 = (\sqrt{-1}, \ldots, \sqrt{-1}) \in \mathscr{Z}$ and

$$C_\infty = \{g \in G_\infty | g(z_0) = z_0\} \quad \text{and} \quad C_{\infty+} = C_\infty \cap G_{\infty+}.$$

Then, we can identify

$$C_\infty = \left(\mathbf{R}^\times O_2(\mathbf{R})\right)^{I_B} \times (\mathbf{H}^\times)^{I^B}, \qquad C_{\infty+} = \left(\mathbf{R}^\times SO_2(\mathbf{R})\right)^{I_B} \times (\mathbf{H}^\times)^{I^B}.$$

Especially we know that $C_\infty/C_{\infty+} \cong (\mathbf{Z}/2\mathbf{Z})^{I_B}$ as groups. For each subset J of I_B and $x \in G_\infty$, we define another subset $J^x \subset I_B$ by

$$J^x = \left\{\tau \in I_B \middle| \tau \in J \text{ and } \nu(x_\tau) > 0, \text{ or } \tau \in \bar{J} = I_B - J \text{ and } \nu(x_\tau) < 0\right\}.$$

Then one verifies that this gives an action of G_∞ on subsets of I_B; i.e.,

(2.2a) $$J^{xy} = (J^x)^y \qquad \text{for } x, y \in G_\infty.$$

For each subset J of I_B, we shall define an automorphic factor $j_J(x, z) \in \mathbf{C}^{I_B}$ for $z \in \mathscr{Z}_B$ and $x = \left(\begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}\right)_{\sigma \in I} \in G_\infty$ by

$$j_J(x, z) = \left(c_\sigma z_\sigma^J + d_\sigma\right)_{\sigma \in I_B},$$

where

$$z_\sigma^J = \begin{cases} z_\sigma & \text{if } \sigma \in J, \\ \bar{z}_\sigma & \text{if } \sigma \in \bar{J} = I_B - J. \end{cases}$$

Then, by definition, we have

(2.2b) $$j_J(\gamma, z) = j_{I_B}(\gamma, z^J),$$

and as an element of the semi-simple algebra $\mathbf{C}^{I_B}$, $j_J(\gamma, z)$ satisfies

(2.2c) $$j_J(\gamma\delta, z) = j_{J^\delta}(\gamma, \delta(z)) j_J(\delta, z) \qquad \text{for } \gamma, \delta \in G_\infty.$$

For each function f: $G_\mathbf{A}^B \to L(n^B, v^B; \mathbf{C})$, we define a transform $f|_{k,w,J}u$ of f under $u \in G_\mathbf{A}^B$ by

(2.3a) $$\left(f|_{k,w,J}u\right)(x) = j_{J^u}(u_\infty, z_0)^{-k_B} \nu(u_\infty)^{w_B} f(xu^{-1}) \cdot u_\infty,$$

where u_∞ acts from the right on the value $f(xu^{-1})$ in $L(n^B, v^B; \mathbf{C})$. By (2.2a, b, c), we have the compatibility relation:

(2.3b) $$\left(f|_{k,w,J}x\right)\Big|_{k,w,J}y = f|_{k,w,J}(xy).$$

When it is unlikely to cause misunderstanding, we simply write $f|x$ or $f|_{k,w}x$ for the transform $f|_{k,w,J}x$. Let U be an open-compact subgroup of G_f^B. We now denote by $S_{k,w,J}(U; B; \mathbf{C}) = S_{k,w,J}(U; \mathbf{C})$ the space of functions f: $G_\mathbf{A}^B \to L(n^B, v^B; \mathbf{C})$ satisfying the following conditions (2.4a, b, c, d):

(2.4a) $$f = f|_{k,w,J}u \qquad \textit{for all } u \in UC_{\infty+}, \quad \textit{and}$$
$$f(ax) = f(x) \qquad \textit{for all } a \in G^B(\mathbf{Q}).$$

For each $z \in \mathscr{Z}_B$, we can choose $u_\infty \in G_{\infty+}$ so that $u_\infty(z_0) = z$ and define a function f_x: $\mathscr{Z}_B \to {}^tL(n^B, v^B; \mathbf{C})$ for each $x \in G_f^B$ out of each function f satisfying (2.4a) by

$$f_x(z) = j_J(u_\infty, z_0)^{k_B} \nu(u_\infty)^{-w_B} f(xu_\infty) \cdot u_\infty^{-1}.$$

Then f_x is well defined independently of the choice of u_∞ by (2.4a) as a function on $\mathscr{Z}_B$ with values in ${}^tL(n^B, v^B; \mathbf{C})$. Then we impose

(2.4b)
$$\textit{For all } x \in G_f^B, \quad \frac{\partial f_x}{\partial \bar{z}_\sigma} = 0 \quad \textit{if} \quad \sigma \in J, \quad \textit{and}$$
$$\frac{\partial f_x}{\partial z_\sigma} = 0 \quad \textit{if} \quad \sigma \in \bar{J} = I_B - J.$$

In the extreme cases: $r = 0$ (i.e., B is totally definite) or $B = M_2(F)$, we need to impose some other conditions:

(2.4c)
$$\textit{When } B = M_2(F), \textit{ then } \int_{F_\mathbf{A}/F} f\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} x\right) da = 0 \qquad \textit{for all } x \in G_\mathbf{A}$$
$$\textit{for each additive Haar measure } da \textit{ on } F_\mathbf{A}/F.$$

(When $B = M_2(\mathbf{Q})$, we also add the following condition: $|\mathrm{Im}(z)^{k/2} f_x(z)|$ is uniformly bounded on H for all $x \in \mathrm{GL}_2(\mathbf{Q}_f)$). When B is totally definite and $n = 0$ (then $v = 0$), we firstly consider the space $S(U)$ of all functions on $G_\mathbf{A}^B$ satisfying (2.4a). Let $\mathrm{Inv}(U)$ be the subspace of $S(U)$ consisting of functions of the form $f \circ \nu$ for some function f: $F_\mathbf{A}^\times \to \mathbf{C}$, where ν: $G_\mathbf{A}^B \to F_\mathbf{A}^\times$ is the reduced norm map. Then we put

(2.4d) $\quad S_{2t, t, \phi}(U; B; \mathbf{C}) = S(U)/\mathrm{Inv}(U) \qquad$ *if* $n = 0$ *and* $r = 0$.

We have now finished defining the space of cusp forms for each B/F. In our definition, we have (implicitly) assumed that $k \geq 2t$ since $n \geq 0$. It is well-known that the space $S_{k, w, J}(U; \mathbf{C})$ is of finite dimension.

By the approximation theorem, we can find $t_i \in G_\mathbf{A}$ $(i = 1, \ldots, h)$ with $(t_i)_\infty = (t_i)_N = 1$ for any given ideal N of $\mathfrak{r}$ such that $G_\mathbf{A}^B = \coprod_{i=1}^h G_\mathbf{Q} t_i U G_{\infty+}$. When B is indefinite, the number h is equal to $|F^\times \backslash F_\mathbf{A}^\times / \nu(U) F_{\infty+}^\times|$ and is independent of B by the strong approximation theorem. When B is (totally) definite, h depends on U and B. We put, by fixing such a decomposition,

$$\Gamma^i(U) = G_\mathbf{Q}^B \cap t_i U G_{\infty+}^B t_i^{-1}, \qquad \bar{\Gamma}^i(U) = \Gamma^i(U)/\Gamma^i(U) \cap F^\times.$$

Then $\Gamma^i(U)$ is a discrete arithmetic subgroup of $G_{\infty+}$. We then consider the space of cusp forms with respect to $\Gamma^i(U)$, which is written as $S_{k, w, J}(\Gamma^i(U); \mathbf{C})$

and consists of functions $f\colon \mathscr{X}_B \to {}^tL(n^B, v^B; \mathbf{C})$ satisfying the following conditions:

(2.5a) $f(\gamma(z)) = \nu(\gamma)^{-w_B} j_J(\gamma, z)^{k_B}(\gamma \cdot f(z))$ *for all* $\gamma \in \Gamma^i(U)$,

(2.5b) $\dfrac{\partial f}{\partial \bar{z}_\sigma} = 0$ *if* $\sigma \in J$, *and* $\dfrac{\partial f}{\partial z_\sigma} = 0$ *if* $\sigma \in \bar{J}$.

When $B = M_2(F)$, we suppose the cuspidal condition:

(2.5c) *f vanishes at all cusps of* $\Gamma^i(U)$.

(This condition means that for all $\alpha \in \mathrm{SL}_2(F)$, $f|\alpha(z) = \nu(\alpha)^w j_J(\alpha, z)^{\ k} f(\alpha(z))$ has Fourier expansion of the form $\sum_{\xi \in L_+} a(\xi)\mathbf{e}_F(\xi z)$ ($z \in \mathscr{X} = H^I$) for a lattice L depending on α, where L_+ is the subset of all totally positive elements of L.) Then the correspondence: $f \mapsto (f_{t_i})_i$ gives an isomorphism:

(2.6a) $$S_{k,w,J}(U; \mathbf{C}) \cong \bigoplus_{i=1}^{h} S_{k,w,J}(\Gamma^i(U); \mathbf{C})$$

if $k > 2t$ $(n > 0)$ or B is indefinite,

and

$$S(U) \cong \bigoplus_{i=1}^{h} S_{2t,t,\phi}(\Gamma^i(U); \mathbf{C}) \qquad \textit{if } k = 2t \quad \textit{and} \quad B \textit{ is totally definite}.$$

Note that if B is totally definite, we have the trivial identity

(2.6b) $H^0(\Gamma^i(U), {}^tL(n, v; \mathbf{C})) = S_{k,w,\phi}(\Gamma^i(U); \mathbf{C})$ *if* $k > 2t$.

The assertion (2.6a) follows from the following formula (2.6c): We define for $f\colon \mathscr{X}_B \to {}^tL(n^B, v^B; \mathbf{C})$ another function $f|_{k,w,J}\gamma\colon \mathscr{X}_B \to {}^tL(n^B, v^B; \mathbf{C})$ for $\gamma \in G_{\mathbf{Q}}^B$ by

$$(f|_{k,w,J}\gamma)(z) = \nu(\gamma)^{w_B} j_{J\gamma}(\gamma, z)^{-k_B}(\gamma^{-1} \cdot f(\gamma(z))).$$

Put, for each $x \in G_f C_\infty$, $U^x = x^{-1}Ux$ and decompose

$$G_{\mathbf{A}}^B = \coprod_i G_{\mathbf{Q}}^B t_i' U^x G_{\infty+}^B$$

(for example, the choice $t_i' = t_i x$ works well). Then, if $t_j x \in \gamma t_i' U^x G_{\infty+}$ for $\gamma \in G_{\mathbf{Q}}^B$, then

(2.6c) $$(f|_{k,w,J}x)_{t_i'} = (f_{t_j})|_{k,w,J}\gamma.$$

For the proof of (2.6c), see for example [29, §3] and [8, §1].

We shall now define the Hecke operators on $S_{k,w,J}(U; \mathbf{C})$. Let U and U' be two open compact subgroups of G_f. For each $x \in G_f C_\infty$, we shall define a linear operator $[UxU']$: $S_{k,w,J}(U;\mathbf{C}) \to S_{k,w,J^x}(U';\mathbf{C})$ as follows: Decompose $(UC_{\infty+})x(UC_{\infty+}) = \coprod_i (UC_{\infty+})x_i$ as a disjoint union of finitely many left cosets. Then we define

$$f|[UxU'] = \sum_i f|_{k,w,J}x_i.$$

By (2.4a), this operator is independent of the choice of the representative set $\{x_i\}$ of the left cosets. By (2.2a, c) and (2.6c), the operator $[UxU']$ takes $S_{k,w,J}(U;\mathbf{C})$ into $S_{k,w,J^x}(U';\mathbf{C})$. Note that $Ux_fU' = \coprod_i U(x_i)_f$ if and only if $(UC_{\infty+})x(UC_{\infty+}) = \coprod_i U(x_i)_f x_\infty$. Because of this fact, we have used the symbol $[UxU']$ to denote this operator instead of $[UC_{\infty+}xUC_{\infty+}]$. Through the operator $[Ux_\infty U]$ for $x_\infty \in C_\infty$, the finite group $C_\infty/C_{\infty+}$ acts on the sum $\oplus_{J\subset I_B} S_{k,w,J}(U;\mathbf{C})$. In the extreme case of $r = 0$ and $n = 0$, the transformation: $f \mapsto f|x$ preserves by definition the subspace $\mathrm{Inv}(U)$. Thus the operator $[UxU']$ induces a linear operator:

$$S_{2t,t,\phi}(U;\mathbf{C}) \to S_{2t,t,\phi}(U';\mathbf{C}),$$

which is again denoted by $[UxU']$.

If $U' \subset U$, $S_{k,w,J}(U;\mathbf{C})$ is naturally contained in $S_{k,w,J}(U';\mathbf{C})$. Thus we can take the injective limit:

$$S_{k,w,J}(B;\mathbf{C}) = \varinjlim_U S_{k,w,J}(U;B;\mathbf{C}) \tag{2.7}$$

over the partially ordered set of all open compact subgroups of G_f. By (2.3b), for each $f \in S_{k,w,J}(B;\mathbf{C})$, the transform $f|_{k,w,J}x$ is again contained in $S_{k,w,J}(B;\mathbf{C})$ for $x \in G_f$. Thus G_f naturally acts on $S_{k,w,J}(B;\mathbf{C})$. When B is unramified at every finite place (then $r \equiv [F:\mathbf{Q}] \bmod 2$), we shall fix an isomorphism of $\hat{R} = R \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$ with $M_2(\hat{\imath})$. Then every open compact subgroup U of $\mathrm{GL}_2(F_f)$ can be regarded as an open compact subgroup of G_F^B. Then, by virtue of a result of Jacquet, Langlands and Shimizu, we have:

THEOREM 2.1. *Let B and B' be quaternion algebras over F unramified at all finite places. Suppose that $n \geq 0$ (i.e. $k \geq 2t$) and $k + 2v \sim 0$. Then for open compact subgroups U of $\mathrm{GL}_2(F_f)$, there is a system of isomorphisms*

$$i_U\colon S_{k,w,I_B}(U;B;\mathbf{C}) \cong S_{k,w,I_{B'}}(U;B';\mathbf{C})$$

such that (i) *for* $U \supset U'$, *there is a commutative diagram*:

$$\begin{array}{ccc} S_{k,w,I_B}(U;B;\mathbf{C}) & \overset{i_U}{\cong} & S_{k,w,I_{B'}}(U;B';\mathbf{C}) \\ \downarrow & & \downarrow \\ S_{k,w,I_B}(U';B;\mathbf{C}) & \overset{i_{U'}}{\cong} & S_{k,w,I_{B'}}(U';B';\mathbf{C}), \end{array}$$

and (ii) *if* $i = \varinjlim_U i_U$: $S_{k,w,I_B}(B;\mathbf{C}) \cong S_{k,w,I_{B'}}(B';\mathbf{C})$, *then* i *is an isomorphism of* $\mathrm{GL}_2(F_f)$*-module*; *in particular*,

$$i_{U'} \circ [UxU'] = [UxU'] \circ i_U$$

for every pair of open compact subgroup (U, U') *of* $\mathrm{GL}_2(F_f)$ *and every* $x \in \mathrm{GL}_2(F_f)$.

This follows from a result in [18, §16]. An exposition can be found in [8, §2].

As we have already remarked, for each subset J of I_B, we can find $c = c(J) \in C_\infty$ such that $I_B = J^{c(J)}$. Since $c_f = 1$, the operator $[UcU]$: $S_{k,w,J}(U;\mathbf{C}) \to S_{k,w,I_B}(U;\mathbf{C})$ commutes with $[UxU]$ for all $x \in G_f^B$. Note that $[UcU]^2 = \mathrm{id}$, and thus, $[UcU]$ is actually an isomorphism. Namely, we have:

THEOREM 2.2. *For each subset* J *of* I_B, *the map*

$$[Uc(J)U]: S_{k,w,J}(U;\mathbf{C}) \to S_{k,w,I_B}(U;\mathbf{C})$$

is a surjective isomorphism satisfying

$$[Uc(J)U] \circ [UxU] = [UxU] \circ [Uc(J)U] \qquad \textit{for all } x \in G_f^B.$$

Let ι: $G_{\mathbf{A}} \to G_{\mathbf{A}}$ denote the involution defined by $xx^\iota = \nu(x)$. Note that we have defined $\hat{w} = t - v = k - w$. Thus $k \sim 2w$ and we may consider the space $S_{k,\hat{w},J}(U;\mathbf{C})$. Put $U^\iota = \{x^\iota | x \in U\}$. For each $f \in S_{k,w,J}(U;\mathbf{C})$, we define

$$f^*: G_{\mathbf{A}} \to {}^tL(n^B, -n^B - v^B;\mathbf{C}) \qquad \text{by } f^*(x) = f(x^{-\iota}),$$

where we have written $x^{-\iota}$ for $(x^{-1})^\iota = (x^\iota)^{-1}$.

PROPOSITION 2.3. *The correspondence*: $f \mapsto f^*$ *induces an isomorphism*:

$$S_{k,w,J}(U;\mathbf{C}) \cong S_{k,\hat{w},J}(U^\iota;\mathbf{C})$$

which satisfies $(f|[UxU])^* = f^*|[U^\iota x^{-\iota} U^\iota]$ *for all* $x \in UC_\infty$.

Proof. The U^ι-invariance of f^* as in (2.4a) follows from a direct calculation. The cuspidal condition for f^* in (2.4c) is obvious since $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}^\iota = \begin{pmatrix} 1 & -a \\ 0 & 1 \end{pmatrix}$. We shall show the analyticity of f^* at infinite places. A straightforward computation shows that $(f^*)_x(z) = f_{x^{-\iota}}(z)$. This shows the expected analyticity. The last assertion follows simply from the definition of the operator $[UxU]$.

For each integral ideal N of F disjoint from Σ^B, we may suppose that the isomorphism (1.1) induces another isomorphism

$$R_N = R \otimes_{\mathfrak{r}} \mathfrak{r}_N \cong M_2(\mathfrak{r}_N).$$

Then we shall define standard open compact subgroups of $\hat{R}^\times$ (for $\hat{R} = R \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$) by

$$U_0(N) = \left\{ x \in \hat{R}^\times \text{ with } x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| c \in N\mathfrak{r}_N \right\},$$

$$U_1(N) = \left\{ x \in U_0(N) \text{ with } x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a - 1 \in N\mathfrak{r}_N \right\},$$

$$V_1(N) = \left\{ x \in U_0(N) \text{ with } x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| d - 1 \in N\mathfrak{r}_N \right\},$$

$$U_E(N) = E \cdot U_1(N) \quad \text{and} \quad V_E(N) = E \cdot V_1(N),$$

where $E = \{\varepsilon \in \mathfrak{r}^\times | \varepsilon^\sigma > 0 \text{ for all } \sigma \in I\}$ and the product of E with $U_1(N)$ and $V_1(N)$ is taken in G_f^B. Suppose that $f \in S_{k,w,J}(U_1(N); \mathbf{C})$ or $f \in S_{k,w,J}(V_1(N); \mathbf{C})$. Then we see from (2.4a) that

$$f(xc_\infty) = c_\infty^{2w-k} f(x) \qquad \text{for } c \in F_\infty^\times.$$

We know from the fact: $2w - k \in \mathbf{Z} \cdot t$ that $\varepsilon^{2w-k} = 1$ for all $\varepsilon \in E$. Therefore $f(x) = f(\varepsilon x) = f(x\varepsilon_f \varepsilon_\infty) = f(x\varepsilon_f)$ for $\varepsilon \in E$. This shows that

$$\begin{aligned} S_{k,w,J}(U_1(N); \mathbf{C}) &= S_{k,w,J}(U_E(N); \mathbf{C}), \\ S_{k,w,J}(V_1(N); \mathbf{C}) &= S_{k,w,J}(V_E(N); \mathbf{C}). \end{aligned} \tag{2.8}$$

We simply write $S^*_{k,w,J}(N; B; \mathbf{C})$ for $S_{k,\hat{w},J}(V_1(N); B; \mathbf{C})$ and $S_{k,w,J}(N; B; \mathbf{C})$ for $S_{k,w,J}(U_1(N); B; \mathbf{C})$. Now we shall define Hecke operators $T(\mathfrak{n})$ and $T(\mathfrak{n}, \mathfrak{n})$ on these spaces. Put, for $\hat{R} = R \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$,

$$\Delta_1(N) = \left\{ x \in \hat{R} \cap G_f^B \text{ with } x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| a - 1 \in N\mathfrak{r}_N,\ c \in N\mathfrak{r}_N \right\} \quad \text{and}$$

$$\Delta_E(N) = E \cdot \Delta_1(N).$$

Then these are semi-groups containing $U_1(N)$ or $U_E(N)$, and we can form the abstract Hecke ring $R(U_1(N), \Delta_1(N))$ and $R(U_E(N), \Delta_E(N))$ as in [36, III] (see also [7, §2]). Then the following facts can be verified in exactly the same manner as in [36] and [7]:

(2.9a) $$R(U_1(N), \Delta_1(N)) \cong R(U_E(N), \Delta_E(N)) \cong R\left(V_E(N), \Delta_E(N)^{-\iota}\right)$$
$$\cong R\left(V_1(N), \Delta_1(N)^{-\iota}\right)$$
$$U_1(N)xU_1(N) \mapsto U_E(N)xU_E(N) \mapsto V_E(N)x^{-\iota}V_E(N) \mapsto V_E(N)x^{-\iota}V_E(N),$$

(2.9b) *For each ideal $M \subsetneqq \mathfrak{r}$ disjoint from Σ^B, we have an isomorphism of rings*:
$$R(U_1(NM^\alpha), \Delta_1(NM^\alpha)) \cong R\left(U_1(NM^\beta), \Delta_1(NM^\beta)\right) \qquad \textit{for all } \alpha \geq \beta > 0$$
$$U_1(NM^\alpha)xU_1(NM^\alpha) \mapsto U_1(NM^\beta)xU_1(NM^\beta).$$

For each ideal $\mathfrak{m} \subset \mathfrak{r}$, put $\mathcal{T}(\mathfrak{m}) = \{x \in \Delta_1(N) \mid \nu(x)\mathfrak{r} = \mathfrak{m}\}$. We decompose $\mathcal{T}(\mathfrak{m})$ into a disjoint union $\coprod_j U_1(N)x_jU_1(N)$ of finitely many double cosets, which is always possible, and we use the same symbol $\mathcal{T}(\mathfrak{m})$ for the element $\sum_j U_1(N)x_jU_1(N)$ in $R(U_1(N), \Delta_1(N))$. By choosing an element $m \in \hat{\mathfrak{r}} \cap F_f^\times$ such that $m_{\mathfrak{r}} = \mathfrak{m}$ and $m - 1 \in N\hat{\mathfrak{r}}$ (then $m \in \Delta_1(N)$) if $\mathfrak{m}$ is prime to N and disjoint from Σ^B, we put

$$\mathcal{T}(\mathfrak{m}, \mathfrak{m}) = \begin{cases} U_1(N)mU_1(N) & \textit{if } \mathfrak{m} \textit{ is prime to } N \textit{ and is disjoint from } \Sigma^B, \\ 0 & \textit{otherwise}. \end{cases}$$

Then $\mathcal{T}(\mathfrak{m}, \mathfrak{m})$ is determined independently of the choice of m, and we have

(2.9c) $R(U_1(N), \Delta_1(N))$ *is isomorphic to the polynomial ring over* $\mathbf{Z}$ *of variables* $\mathcal{T}(\ell, \ell)$ $(\ell \nmid N,\ \ell \notin \Sigma^B)$ *and* $\mathcal{T}(\ell)$ *for all prime ideals* ℓ,

(2.9d) $\mathcal{T}(\ell)^2 - \mathcal{T}(\ell^2) = \mathcal{N}_{F/\mathbf{Q}}(\ell)\mathcal{T}(\ell, \ell)$ *for each prime ideal* ℓ *outside* N *and* Σ^B.

Now we identify all the Hecke rings by the isomorphism (2.9a) and write it simply as $\mathcal{R} = \mathcal{R}(N)$. Then

(2.9e) $\mathcal{R}(N)$ *acts on* $S_{k,w,J}(N; B; \mathbf{C})$ *via* $U_1(N)xU_1(N) \mapsto [U_1(N)xU_1(N)]$ *and on* $S^*_{k,w,J}(N; B; \mathbf{C})$ *via* $U_1(N)xU_1(N) \mapsto [V_1(N)x^{-\iota}V_1(N)]$.

The operator corresponding to $\mathcal{T}(\mathfrak{n})$, $\mathcal{T}(\mathfrak{n}, \mathfrak{n})$ on these spaces will be denoted by $\mathcal{T}(\mathfrak{n})$ and $\mathcal{T}(\mathfrak{n}, \mathfrak{n})$. Then by Proposition 2.3, we have a commutative diagram

$$
(2.10)\qquad \begin{array}{ccc} S_{k,w,J}(N;\mathbf{C}) & \cong & S^*_{k,w,J}(N;\mathbf{C}) \\ \Big\downarrow {\scriptstyle T(\mathfrak{n}),\, T(\mathfrak{n},\mathfrak{n})} & & \Big\downarrow {\scriptstyle T(\mathfrak{n}),\, T(\mathfrak{n},\mathfrak{n})} \\ S_{k,w,J}(N;\mathbf{C}) & \cong & S^*_{k,w,J}(N;\mathbf{C}). \end{array}
$$

By our definition of the transformation: $f \mapsto f|x$ in (2.3a), the action of $T(\mathfrak{n})$ and $T(\mathfrak{n}, \mathfrak{n})$ on $S^*_{k,w,J}(N;\mathbf{C})$ given here coincides with the Hecke operators on the space of holomorphic cusp forms defined by Shimura [29], [34] and Weil [39] for a suitable choice of w for each k. For example, in [34, §2], $k/2$ is taken as w. This choice is not appropriate, when $k \not\equiv 0 \bmod 2\mathbf{Z}[I]$, for the analysis of integrality of Hecke operators which will be done in the coming sections. The use of integral w was initiated by Shimura in [29] and [30].

3. Results on Hecke algebras

In order to define the Hecke algebra for the space of cusp forms, we shall modify a little the Hecke operators $T(\mathfrak{n})$ and $T(\mathfrak{n}, \mathfrak{n})$ when $v \neq 0$. The case: $v = 0$ corresponds to the classical parallel weight cusp forms and in this case, no modification is necessary. For each $\xi \in \mathbf{Z}[I]$, let

$$\mathscr{G}(\xi) = \left\{\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) | \xi\sigma = \xi\right\}.$$

Then it is easy to see that $\mathscr{G}(\xi) = \mathscr{G}(\eta)$ if $\xi \sim \pm\eta$, since $\mathscr{G}(t)$ is the whole group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. Let $\Phi(\xi) = \{x \in \overline{\mathbf{Q}} | x^\sigma = x \text{ for all } \sigma \in \mathscr{G}(\xi)\}$. We then consider a quasi-character ξ: $F^\times \to \overline{\mathbf{Q}}^\times$ given by $x \to x^\xi = \prod_{\sigma \in I} x^{\xi_\sigma \sigma}$. Then $\Phi(\xi)$ is the subfield of $\overline{\mathbf{Q}}$ generated by x^ξ for all $x \in F^\times$, and thus the quasi-character ξ has values in the finite extension $\Phi(\xi)$. Therefore the character ξ extends by continuity to characters: $F_{\mathbf{A}}^\times \to \Phi(\xi)_{\mathbf{A}}^\times$, $F_p^\times \to \overline{\mathbf{Q}}_p^\times$ and $F_\infty^\times \to \mathbf{C}^\times$. We write simply $\mathfrak{r}(v)$ for $\mathfrak{r}_{\Phi(v)}$. Let A be an $\mathfrak{r}(v)$-algebra inside $\mathbf{C}$, and suppose

(3.1) *For every $x \in F_f^\times$, the A-ideal $x^v A = (x^v \mathfrak{r}(v))A$ is generated by a single element in A.*

We can find a finite extension $K_0/\Phi(v)$ such that the integer ring $\mathfrak{r}_0$ of K_0 satisfies (3.1). In fact, by choosing elements $\{a_i\}_{i=1,\dots,h}$ of $F_f^\times$ such that $a_i\mathfrak{r}$ gives a complete representative set for all ideal classes of F, we can take $\alpha_i \in \mathfrak{r}$ so that $\alpha_i\mathfrak{r} = (a_i\mathfrak{r})^h$. Then, as an example of such an extension, we can take the field generated over $\Phi(v)$ by $\sqrt[h]{\alpha_i^v}$ for all $i = 1, \dots, h$. As another example of A satisfying (3.1), we may of course take the valuation ring of $\Phi(v)$ at each finite place. When $v = 0$ (i.e. $w \sim 0$), the rational integer ring $\mathbf{Z}$ satisfies the condition (3.1), and thus the condition (3.1) imposes no restriction to the ring A.

For each prime ideal ℓ of $\mathfrak{r}$, we take $x \in F_f^\times$ such that $\ell = x\mathfrak{r}$ and choose, once and for all, a generator $\{x^v\} = \{\ell^v\}$ of x^vA. For a general ideal $\mathfrak{a}$ of F, we decompose $\mathfrak{a} = \prod_\ell \ell^{e(\ell)}$ as a product of prime ideals and put

$$\{\mathfrak{a}^v\} = \prod_\ell \{\ell^v\}^{e(\ell)},$$

which gives a generator of the ideal x^vA for $x \in F_f^\times$ with $x\mathfrak{r} = \mathfrak{a}$. We also write $\{x^v\}$ for $\{\mathfrak{a}^v\}$ if $\mathfrak{a} = x\mathfrak{r}$ for $x \in F_f^\times$. The correspondence: $\mathfrak{a} \mapsto \{\mathfrak{a}^v\}$ gives a multiplicative map of the ideal group of F into the quotient field of A but is not necessarily a Hecke character. We shall define operators

$$(U_1(N)xU_1(N)) \colon S_{k,w,J}(N;B;\mathbf{C}) \to S_{k,w,J}(N;B;\mathbf{C}) \quad \text{by}$$

$$f|(U_1(N)xU_1(N)) = \{\nu(x)^v\}^{-1} f|[U_1(N)xU_1(N)] \qquad \text{for } x \in \Delta_1(N).$$

Similarly, we define $(U_1(N)xU_1(N)) \in \operatorname{End}_{\mathbf{C}}(S^*_{k,w,J}(N;B;\mathbf{C}))$ by

$$f|(U_1(N)xU_1(N)) = \{\nu(x)^v\}^{-1} f|[V_1(N)x^{-\iota}V_1(N)] \qquad \text{for } x \in \Delta_1(N).$$

By decomposing $\mathcal{T}(\mathfrak{n}) = \coprod_i U_1(N)x_iU_1(N)$ (then, $\nu(x_i)\mathfrak{r} = \mathfrak{n}$ for all i), we put

$$T_0(\mathfrak{n}) = \sum_i (U_1(N)x_iU_1(N)) = \{\mathfrak{n}^v\}^{-1}T(\mathfrak{n}), \qquad T_0(\mathfrak{n},\mathfrak{n}) = \{\mathfrak{n}^{2v}\}^{-1}T(\mathfrak{n},\mathfrak{n})$$

as operators on $S_{k,w,J}(N;\mathbf{C})$ and $S^*_{k,w,J}(N;\mathbf{C})$. Hereafter in this section, we suppose

(3.2) $\qquad$ *B is unramified at all finite places* (i.e. $\Sigma^B \subset I$),

and we shall identify $\hat{R} = R \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$ with $M_2(\hat{\mathfrak{r}})$. Then the Hecke algebra $h_{k,w}(N;A)$ is by definition the A-subalgebra of $\operatorname{End}_{\mathbf{C}}(S_{k,w,J}(N;B;\mathbf{C}))$ generated over A by the operators $T_0(\mathfrak{n})$ for all integral ideals $\mathfrak{n}$. By Theorems 2.1 and 2.2, $h_{k,w}(N;A)$ is determined independently of the choice of the quaternion algebra B under (3.2) and the subset J of I_B. It is also plain that the A-algebra $h_{k,w}(N;A)$ is independent of the choice of the map: $\mathfrak{a} \mapsto \{\mathfrak{a}^v\}$. By (2.9d), we have the relation:

$$T_0(\ell)^2 - T_0(\ell^2) = \mathcal{N}_{F/\mathbf{Q}}(\ell)T_0(\ell,\ell) \qquad \textit{for prime ideals } \ell \nmid N.$$

Thus if $\mathcal{N}_{F/\mathbf{Q}}(\ell)$ is invertible in A, $T_0(\ell,\ell)$ is contained in $h_{k,w}(N;A)$ (this statement is actually true without the assumption that $\mathcal{N}_{F/\mathbf{Q}}(\ell)^{-1} \in A$ as will be seen later in this section). The following fact may be well-known, but we will give a proof in Section 7:

THEOREM 3.1. *In addition to* (3.1), *suppose one of the following conditions for* A:

(i) A *is the integer ring of a finite extension of* $\Phi(v)$,
(ii) A *is a discrete valuation ring of* $\Phi(v)$,
(iii) A *is a field extension of* $\Phi(v)$.

Then

(3.3a) *For any* A*-algebra* D *in* $\mathbf{C}$, $h_{k,w}(N; D) \cong h_{k,w}(N; A) \otimes_A D$,

(3.3b) $h_{k,w}(N; A)$ *is a flat* A*-module of finite type*.

Let $\mathcal{O}(v)$ be the valuation ring of $\Phi(v)$ corresponding to the fixed embedding: $\Phi(v) \hookrightarrow \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. For any $\mathcal{O}(v)$-algebra A not necessarily inside $\mathbf{C}$, we put

$$h_{k,w}(N; A) = h_{k,w}(N; \mathcal{O}(v)) \otimes_{\mathcal{O}(v)} A. \tag{3.4}$$

By (3.3a), this definition is compatible with the base change of the ring A. Let $\hat{\mathcal{O}}(v)$ be the p-adic closure of $\mathcal{O}(v)$ in $\overline{\mathbf{Q}}_p$. Fix a valuation ring $\mathcal{O}$ containing $\hat{\mathcal{O}}(v)$ which is finite flat over $\mathbf{Z}_p$. We now fix an integral ideal N prime to p. Then for $\alpha \geq \beta > 0$, we have a commutative diagram (cf. [36, III], [7, §2]):

$$\begin{tikzcd} S_{k,w,J}(Np^{\beta}; \mathbf{C}) \arrow[r] \arrow[d, "T_0(n)"] & S_{k,w,J}(Np^{\alpha}; \mathbf{C}) \arrow[d, "T_0(n)"] \\ S_{k,w,J}(Np^{\beta}; \mathbf{C}) \arrow[r] & S_{k,w,J}(Np^{\alpha}; \mathbf{C}) \end{tikzcd} \tag{3.5}$$

for all ideals n. Thus the restriction of operators in $h_{k,w}(Np^{\alpha}; A)$ to the subspace $S_{k,w,J}(Np^{\beta}; \mathbf{C})$ induces a surjective A-algebra homomorphism:

$$h_{k,w}(Np^{\alpha}; A) \to h_{k,w}(Np^{\beta}; A) \qquad \text{for } \alpha \geq \beta > 0, \tag{3.6a}$$

which takes $T_0(n)$ to $T_0(n)$. By (3.3b), $h_{k,w}(Np^{\alpha}; \mathcal{O})$ is a p-adically complete semi-local ring and hence is a product of finitely many local rings. Let $h^{\mathrm{ord}}_{k,w}(Np^{\alpha}; \mathcal{O})$ be the product of all local factors of $h_{k,w}(Np^{\alpha}; \mathcal{O})$ on which the projected image of $T_0(p)$ is a unit. The change of the map: $a \mapsto \{a^v\}$ affects $T_0(p)$ by multiplication of an $\mathcal{O}$-unit; hence, $h^{\mathrm{ord}}_{k,w}(Np^{\alpha}; \mathcal{O})$ is well-defined independently of the choice of the map: $a \mapsto \{a^v\}$. Let $e = e_\alpha$ be the idempotent of $h^{\mathrm{ord}}_{k,w}(Np^{\alpha}; \mathcal{O})$ in the Hecke algebra $h_{k,w}(Np^{\alpha}; \mathcal{O})$. If one chooses a suitable integer $\gamma > 0$ and put $m = p^{\gamma} - 1$, e_α can be explicitly given by a p-adic limit:

$$e_\alpha = \lim_{n \to \infty} T_0(p)^{p^n m} \qquad \text{in } h_{k,w}(Np^{\alpha}; \mathcal{O}).$$

By the commutativity of (3.5), if $\alpha \geq \beta > 0$, e_α is sent to e_β under the projection map (3.6a). Thus (3.6a) induces a surjective $\mathcal{O}$-algebra homomorphism

$$(3.6\mathrm{b}) \qquad h^{\mathrm{ord}}_{k,w}(Np^\alpha; \mathcal{O}) \to h^{\mathrm{ord}}_{k,w}(Np^\beta; \mathcal{O}).$$

We now take the projective limit of the morphisms (3.6a, b)

$$(3.7) \qquad \begin{aligned} h_{k,w}(Np^\infty; \mathcal{O}) &= \varprojlim_\alpha h_{k,w}(Np^\alpha; \mathcal{O}), \\ h^{\mathrm{ord}}_{k,w}(Np^\infty; \mathcal{O}) &= \varprojlim_\alpha h^{\mathrm{ord}}_{k,w}(Np^\alpha; \mathcal{O}). \end{aligned}$$

Put $U_F(Np^\alpha) = U_1(Np^\alpha) \cap F_f^\times = V_1(Np^\alpha) \cap F_f^\times$ and

$$\mathrm{Cl}_F(Np^\alpha) = F_{\mathbf{A}}^\times / F^\times U_F(Np^\alpha) F_{\infty+}^\times,\ \overline{\mathrm{Cl}}_F(Np^\alpha) = F_{\mathbf{A}}^\times / F^\times U_F(Np^\alpha) F_\infty^\times.$$

Via the correspondence: $x \mapsto x^{-1}\mathfrak{r}$ for $x \in F_f^\times$ with $x_{Np} = 1$, we can identify these groups with the narrow or usual ray class group of F modulo Np^α. We shall define p-profinite groups $\mathrm{Z} = \mathrm{Z}(N) = \varprojlim_\alpha \mathrm{Cl}_F(Np^\alpha)$, $\overline{\mathrm{Z}} = \overline{\mathrm{Z}}(N) = \varprojlim_\alpha \overline{\mathrm{Cl}}_F(Np^\alpha)$, and we denote by Z_α (resp. $\overline{\mathrm{Z}}_\alpha$) the kernel of the projection map $\mathrm{Z} \to \mathrm{Cl}_F(Np^\alpha)$ (resp. $\overline{\mathrm{Z}} \to \overline{\mathrm{Cl}}_F(Np^\alpha)$). We shall define a continuous character: $\overline{\mathrm{Z}} \to h_{k,w}(Np^\alpha; \mathcal{O})$ for $\alpha = 1, 2, \ldots, \infty$ in order to regard Hecke algebras as algebras over the continuous group algebra of $\overline{\mathrm{Z}}$. To do this, let $\mathrm{Il}(M)$ be the set of all integral ideals prime to M for each ideal M of $\mathfrak{r}$. Then we have a natural map: $\mathrm{Il}(Np) \to \mathrm{Z}$ with dense image. (There might be a non-trivial kernel of the map: $\mathrm{Il}(Np) \to \mathrm{Z}$; so, we shall correct the statement in [13], page 140, line 9 from the bottom as follows: "the set $\mathcal{I}(Np)$ is a dense subset" should read "the image of the set $\mathcal{I}(Np)$ is a dense subset", where we have written $\mathcal{I}(Np)$ instead of $\mathrm{Il}(Np)$ here. This error in [13] does not affect the result obtained there.) We denote by $[\mathfrak{a}]$ the image of $\mathfrak{a} \in \mathrm{Il}(Np)$ in Z. If we write $U_F(Np^\infty) = \{x \in U_F(N) \mid x_p = 1\}$, then we have a natural isomorphism

$$(3.8) \qquad \mathrm{Z}(N) = F_{\mathbf{A}}^\times / \overline{F^\times U_F(Np^\infty) F_{\infty+}^\times},$$

where "$\overline{}$" indicates the closure in $F_{\mathbf{A}}^\times$. If we write $\mathfrak{a} = x\mathfrak{r}$ for each $\mathfrak{a} \in \mathrm{Il}(Np)$ with $x \in F_f^\times$ and $x_{Np} = 1$, then under the isomorphism (3.8), $[\mathfrak{a}]$ corresponds to the class of x^{-1} on the right-hand side. More generally, if one identifies $F_{\mathbf{A}}^\times / \overline{F^\times F_{\infty+}^\times}$ with the Galois group over F of the maximal abelian extension F_{ab} of F by class field theory, $[\mathfrak{a}]$ for $\mathfrak{a} \in \mathrm{Il}(Np)$ coincides with the Artin symbol of $\mathfrak{a}$ on the subfield of F_{ab} fixed by $\overline{F^\times U_F(Np^\infty) F_{\infty+}^\times} / \overline{F^\times F_{\infty+}^\times}$. Let μ_{p^α} be the group of all p^α-th roots of unity and put $\mu_{p^\infty} = \varinjlim_\alpha \mu_{p^\alpha}$. The action of $\mathrm{Z}(N)$ as the Galois group over F on μ_{p^∞} gives a continuous character

$$\chi\colon \mathrm{Z}(N) \to \mathrm{Aut}(\mu_{p^\infty}) \cong \mathbf{Z}_p^\times$$

such that $\chi([\mathfrak{a}]) = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})$. For each $\xi \in \mathbf{Z} \cdot t$, we write $\xi = [\xi]t$ for $[\xi] \in \mathbf{Z}$, and put $\chi_\xi = \chi^{[\xi]}: Z(N) \to \mathbf{Z}_p^\times$. The groups Z and $\overline{Z}$ can be decomposed as $Z = W \times Z_{\mathrm{tor}}$ and $\overline{Z} = \overline{W} \times \overline{Z}_{\mathrm{tor}}$ for $\mathbf{Z}_p$-free groups W and $\overline{W}$ and finite groups Z_{tor} and $\overline{Z}_{\mathrm{tor}}$. Put $W_\alpha = W \cap Z_\alpha$ and $\overline{W}_\alpha = \overline{W} \cap \overline{Z}_\alpha$. We may then identify W and $\overline{W}$ under the natural map: $Z \to \overline{Z}$. Then for sufficiently large α, we have $Z_\alpha = W_\alpha = \overline{W}_\alpha = \overline{Z}_\alpha$. This assertion holds if $\alpha \geq 1$ when p is an odd prime. We denote by Λ, $\mathscr{A}$ and $\overline{\mathscr{A}}$ the continuous group algebras over $\mathscr{O}$ of W, Z and $\overline{Z}$, respectively. Namely,

$$\Lambda = \varprojlim_\alpha \mathscr{O}[W/W_\alpha],$$

$$\mathscr{A} = \varprojlim_\alpha \mathscr{O}[\mathrm{Cl}_F(Np^\alpha)] \quad \text{and} \quad \overline{\mathscr{A}} = \varprojlim_\alpha \mathscr{O}[\overline{\mathrm{Cl}}_F(Np^\alpha)].$$

In order to give an $\overline{\mathscr{A}}$-algebra structure on the Hecke algebras $h_{k,w}(Np^\alpha; \mathscr{O})$ $(\alpha = 1, 2, \ldots, \infty)$, we let $F_{\mathbf{A}}^\times$ act on $S_{k,w,J}(Np^\alpha; B; \mathbf{C})$ by $f \to f|_{k,w}a$ for $a \in F_{\mathbf{A}}^\times$. This action coincides with the operator $[U_1(N)aU_1(N)]$ and thus gives an action of $F_{\mathbf{A}}^\times / U_F(Np^\alpha)F_\infty^\times$. Since $f(ax) = f(x)$ and $f|_{k,w}a_\infty = f$ by (2.4a) for $a \in F^\times$, we have that

$$(*) \qquad f|_{k,w}a_f = f|_{k,w}a = a^{2w-k}f = a^{n+2v}f.$$

For $\mathfrak{a} \in \mathrm{Il}(Np)$, by taking $a \in F_f^\times$ with $a\mathfrak{r} = \mathfrak{a}$ and $a_{Np} = 1$, we define $\langle \mathfrak{a} \rangle_n \in \mathrm{End}_{\mathbf{C}}(S_{k,w,J}(Np^\alpha; B; \mathbf{C}))$ by

$$(3.9) \qquad f|\langle \mathfrak{a} \rangle_n = \chi_{-n-2v}(\mathfrak{a}) f|_{k,w}a.$$

Then $(*)$ shows that if $\mathfrak{a}$ is trivial in $\mathrm{Cl}_F(Np^\alpha)$, then $f|\langle \mathfrak{a} \rangle_n = f$; namely, the finite group $\mathrm{Cl}_F(Np^\alpha)$ acts on $S_{k,w,J}(Np^\alpha; \mathbf{C})$ via $f \mapsto f|\langle \mathfrak{a} \rangle_n$. As an operator on $S_{k,w,J}(Np^\alpha; \mathbf{C})$, $T(\mathfrak{a}, \mathfrak{a}) = \chi_{n+2v}(\mathfrak{a})\langle \mathfrak{a} \rangle_n$.

Now we shall show that $T(\mathfrak{a}, \mathfrak{a})$ and $T_0(\mathfrak{a}, \mathfrak{a})$ is contained in $h_{k,w}(Np^\alpha; A)$ for arbitrary A (if $n \geq 0$). Since $\langle \mathfrak{a} \rangle_n$ depends only on the class of $\mathfrak{a}$ in $\mathrm{Cl}_F(Np^\alpha)$, we can choose two prime ideals ℓ and $\mathfrak{q}$ of F such that $\mathscr{N}_{F/\mathbf{Q}}(\ell)$ is prime to $\mathscr{N}_{F/\mathbf{Q}}(\mathfrak{q})$ but $\langle \ell \rangle_n = \langle \mathfrak{q} \rangle_n$. Then we can find integers x, y such that

$$x\chi_{n+2v+t}(\ell) + y\chi_{n+2v+t}(\mathfrak{q}) = 1.$$

Then by (2.9d), we have that

$$\begin{aligned} \langle \ell \rangle_n &= x\chi(\ell)T(\ell, \ell) + y\chi(\mathfrak{q})T(\mathfrak{q}, \mathfrak{q}) \\ &= x\big(T(\ell)^2 - T(\ell^2)\big) + y\big(T(\mathfrak{q})^2 - T(\mathfrak{q}^2)\big) \in h_{k,w}(Np^\alpha; A). \end{aligned}$$

Thus for any $\mathfrak{a} \in \mathrm{Il}(Np)$, $\langle \mathfrak{a} \rangle_n \in h_{k,w}(Np^\alpha; A)$ and thus

$$T(\mathfrak{a}, \mathfrak{a}) = \chi_{n+2v}(\mathfrak{a})\langle \mathfrak{a} \rangle_n \in h_{k,w}(Np^\alpha; A),$$

$$T_0(\mathfrak{a}, \mathfrak{a}) = \{\mathfrak{a}^{-2v}\}\chi_{n+2v}(\mathfrak{a})\langle \mathfrak{a} \rangle_n \in h_{k,w}(Np^\alpha; A)$$

because $\{\mathfrak{a}^{-2v}\}\chi_{n+2v}(\mathfrak{a})$ is always an element in A.

We now know from (*) and (3.9) that the correspondence: $\mathrm{Il}(Np) \ni a \mapsto T(a, a) \in h_{k,w}(Np^\alpha; \mathcal{O})$ factors through the image of $\mathrm{Il}(Np)$ in $\overline{Z}(N)$ and is continuous under the topology induced by $\overline{Z}(N)$. Then, by continuity, this homomorphism of the semi-group $\mathrm{Il}(N)$ extends to a continuous character: $\overline{Z}(N) \to h_{k,w}(Np^\alpha; \mathcal{O})$. The image of $z \in Z(N)$ or $a \in \mathrm{Il}(Np)$ under this character will be written as $\langle z \rangle$ or $\langle a \rangle \in h_{k,w}(Np^\alpha; \mathcal{O})$ $(\alpha = 1, 2, \dots)$. This character is naturally compatible with the projection map: $h_{k,w}(Np^\alpha; \mathcal{O}) \to h_{k,w}(Np^\beta; \mathcal{O})$ for $\alpha \geq \beta > 0$. By taking the projective limit, we have a continuous character: $\overline{Z}(N) \to h_{k,w}(Np^\infty; \mathcal{O})$. By the universality of the continuous group algebra, $h_{k,w}(Np^\alpha; \mathcal{O})$ becomes an algebra over $\mathcal{A}$ and Λ via this character for $\alpha = 1, 2, \dots, \infty$.

Theorem 3.2. *For each $k \in \mathbf{Z} \cdot t$ $(t = \sum_\sigma \sigma)$ with $k \geq 2t$* (i.e. $v = 0$ *and* $n \geq 0$), *there exists an $\mathcal{A}$-algebra isomorphism*:

$$h_{2t,t}(Np^\infty; \mathcal{O}) \cong h_{k,k-t}(Np^\infty; \mathcal{O}),$$

which takes $T(n)$ to $T(n)$ for all ideals n of $\mathfrak{r}$.

The implication of this theorem in terms of p-adic modular forms will be explained in Section 5. The same type of assertion is also expected to be true for $h_{k,w}(Np^\infty; \mathcal{O})$ even for general $k \in \mathbf{Z}[I]$ (or $v \in \mathbf{Z}[I]$). The author hopes to come back to this problem in the future. As for the ordinary part $h^{\mathrm{ord}}_{k,w}(Np^\infty; \mathcal{O})$, we have the following general result:

Theorem 3.3. *If k and k' in $\mathbf{Z}[I]$ satisfy $k \geq k' \geq 2t$ and $k \sim k' \sim -2v$, then there exists an $\mathcal{A}$-algebra isomorphism*:

$$h^{\mathrm{ord}}_{k,w}(Np^\infty; \mathcal{O}) \cong h^{\mathrm{ord}}_{k',w'}(Np^\infty; \mathcal{O})$$

which takes $T_0(n)$ to $T_0(n)$ for all ideals n, where $w = v + k - t$ and $w' = v + k' - t$. Moreover, $h^{\mathrm{ord}}_{k,w}(Np^\infty; \mathcal{O})$ is a torsion-free Λ-module of finite type.

Theorems 3.2 and 3.3 will be proved in Section 11 after the analysis of the structure of cohomology groups in Sections 8, 9 and 10, where we shall employ Shimura's method in [30] as a key technique. Since $h^{\mathrm{ord}}_{k,w}(Np^\infty; \mathcal{O})$ only depends on $v \bmod \mathbf{Z} \cdot t$ $(w = v + k - t)$, we hereafter write $\mathbf{h}^{\mathrm{ord}}_v(N; \mathcal{O})$ for $h^{\mathrm{ord}}_{k,w}(Np^\infty; \mathcal{O})$.

Let λ: $W \to \overline{\mathbf{Q}}_p^\times$ be a continuous character. Then we can extend λ to an algebra homomorphism λ: $\Lambda \to \overline{\mathbf{Q}}_p$. Let P_λ denote the point of $\mathrm{Spec}(\Lambda)_{/\mathcal{O}}(\overline{\mathbf{Q}}_p)$ corresponding to this algebra homomorphism. For each finite order character ε: $W \to \overline{\mathbf{Q}}_p^\times$ and $m \in \mathbf{Z} \cdot t$, we write $P_{m,\varepsilon}$ for $P_{\chi_m \varepsilon}$. We also define, if ε factors through W/W_α and $W_\alpha = Z_\alpha$,

$$S^*_{k,w,J}(Np^\alpha, \varepsilon; \mathbf{C}) = \left\{ f \in S^*_{k,w,J}(Np^\alpha; \mathbf{C}) \middle| f \middle| \langle w \rangle_n = \varepsilon(w) f \text{ for all } w \in W \right\}.$$

Here, note that the action: $W \ni w \mapsto \langle w \rangle_n$ factors through the finite group

W/W_α. For any $\mathcal{O}(v)$-algebra A in $\mathbf{C}$ containing the values of ε, we define $h_{k,w}(Np^\alpha, \varepsilon; A)$ to be the A-subalgebra of $\mathrm{End}_{\mathbf{C}}(S^*_{k,w,J}(Np^\alpha, \varepsilon; \mathbf{C}))$ generated over A by $T_0(n)$ for all ideals n. Let $\mathcal{O}(v, \varepsilon)$ be the valuation ring of the field generated over $\Phi(v)$ by the values of ε corresponding to the embedding: $\overline{\mathbf{Q}} \to \overline{\mathbf{Q}}_p$. If ε has values in $\mathcal{O}$, $\mathcal{O}$ contains $\mathcal{O}(v, \varepsilon)$. We then put

$$h_{k,w}(Np^\alpha, \varepsilon; \mathcal{O}) = h_{k,w}(Np^\alpha, \varepsilon; \mathcal{O}(v, \varepsilon)) \otimes_{\mathcal{O}(v,\varepsilon)} \mathcal{O}.$$

There is natural surjection: $h_{k,w}(Np^\alpha; \mathcal{O}) \to h_{k,w}(Np^\alpha, \varepsilon; \mathcal{O})$ which sends $T_0(n)$ to $T_0(n)$. Let K be the quotient field of $\mathcal{O}$. Then, we can define

$$h^{\mathrm{ord}}_{k,w}(Np^\alpha, \varepsilon; \mathcal{O}) = e_\alpha h_{k,w}(Np^\alpha, \varepsilon; \mathcal{O}),$$

$$h^{\mathrm{ord}}_{k,w}(Np^\alpha, \varepsilon; K) = h^{\mathrm{ord}}_{k,w}(Np^\alpha, \varepsilon; \mathcal{O}) \otimes_{\mathcal{O}} K.$$

By the following theorem, $h^{\mathrm{ord}}_{k,w}(Np^\alpha, \varepsilon; K)$ and hence $h^{\mathrm{ord}}_{k,w}(Np^\alpha, \varepsilon; \mathcal{O})$ is independent of the choice of α such that ε factors through W/W_α and $W_\alpha = Z_\alpha$:

THEOREM 3.4. *Let n be an element of $\mathbf{Z}[I]$ with $n \sim -2v$ and $n \geq 0$, and let $\varepsilon\colon W/W_\alpha \to \mathcal{O}^\times$ be a finite order character. Suppose that $W_\alpha = Z_\alpha$. Write P for $P_{n+2v,\varepsilon}$, and let Λ_P denote the localization of Λ at P. Then there is a canonical isomorphism*:

$$\mathbf{h}^{\mathrm{ord}}_v(N; \mathcal{O}) \otimes_\Lambda \Lambda_P/P\Lambda_P \cong h^{\mathrm{ord}}_{k,w}(Np^\alpha, \varepsilon; K)$$

$$\text{for } k = n + 2t \quad \text{and} \quad w = v + k - t,$$

which takes $T_0(n)$ to $T_0(n)$ for all n. Especially, $\mathbf{h}^{\mathrm{ord}}_v(N; \mathcal{O}) \otimes_\Lambda \Lambda_P$ is free of finite rank over Λ_P, and the dimension of $h^{\mathrm{ord}}_{k,w}(Np^\alpha, \varepsilon; K)$ over K is independent of ε and n (or k) and is equal to the dimension of $\mathbf{h}^{\mathrm{ord}}_v(N; \mathcal{O}) \otimes_\Lambda \mathcal{L}$ over $\mathcal{L}$ for the quotient field $\mathcal{L}$ of Λ.

This theorem will be proved in Section 12. Let us now indicate an important implication of the theorem and the duality theorem (Th. 5.3 below). Let $\mathcal{L}$ be the quotient field of Λ. We fix an algebraic closure $\overline{\mathcal{L}}$ of $\mathcal{L}$ and consider $\overline{\mathbf{Q}}_p$ as a subfield of $\overline{\mathcal{L}}$. We shall fix a Λ-algebra homomorphism $\lambda\colon \mathbf{h}^{\mathrm{ord}}_v(N; \mathcal{O}) \to \overline{\mathcal{L}}$. Let $\mathcal{K}$ be the quotient field of the image of λ. Then, by Theorem 3.3, $\mathcal{K}$ is a finite extension of $\mathcal{L}$. Let $\mathcal{I}$ be the integral closure of Λ in $\mathcal{K}$. Then again by Theorem 3.3, λ in fact has values in $\mathcal{I}$. Put $\mathcal{O}_{\mathcal{I}} = \overline{\mathbf{Q}}_p \cap \mathcal{I}$. Then $\mathcal{O}_{\mathcal{I}}$ is a valuation ring, finite flat over $\mathcal{O}$. Let $\mathcal{X} = \mathcal{X}(\mathcal{I})$ be the space of all $\overline{\mathbf{Q}}_p$-valued points of $\mathrm{Spec}(\mathcal{I})_{/\mathcal{O}}$. Thus

$$\mathcal{X} = \mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathcal{I}, \overline{\mathbf{Q}}_p).$$

Let $\mathcal{X}_{\mathrm{alg}}(\Lambda) = \{P_{n,\varepsilon} \in \mathcal{X}(\Lambda) \mid n \in \mathbf{Z}\cdot t,\ n \geq 0$ and $\varepsilon\colon W \to \overline{\mathbf{Q}}^\times$ be a finite order character$\}$. We have a natural morphism $\pi\colon \mathrm{Spec}(\mathcal{I})_{/\mathcal{O}} \to \mathrm{Spec}(\Lambda)_{/\mathcal{O}}$ of schemes over $\mathcal{O}$. Put $\mathcal{X}_{\mathrm{alg}}(\mathcal{I}) = \pi^{-1}(\mathcal{X}_{\mathrm{alg}}(\Lambda))$. If one considers $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ as an $\mathcal{O}$-algebra homomorphism P: $\mathcal{I} \to \overline{\mathbf{Q}}_p$, then $P|_\Lambda = P_{n,\varepsilon}$ for some $n \in \mathbf{Z} \cdot t$ and a finite order character ε: $W \to \overline{\mathbf{Q}}^\times$. We write this n as $n(P)$ and this ε as ε_P. The minimum of α such that $\mathrm{Ker}(\varepsilon_P) \supset Z_\alpha$ will be denoted by $\alpha(P)$. For each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$, regarding P as an $\mathcal{O}$-algebra homomorphism P: $\mathcal{I} \to \overline{\mathbf{Q}}_p$, we can consider an $\mathcal{O}$-algebra homomorphism λ_P: $\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \to \overline{\mathbf{Q}}_p$ given by $\lambda_P = P \circ \lambda$. Then λ_P factors through

$$h_{k,w}^{\mathrm{ord}}\left(Np^{\alpha(P)}, \varepsilon_P; K\right) \cong \mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \otimes_\Lambda \Lambda_{\bar{P}}/\bar{P}\Lambda_{\bar{P}} \qquad \text{for } \bar{P} = P|_\Lambda$$

by Theorem 3.4 and can be considered as a K-algebra homomorphism

$$\lambda_P\colon h_{k,w}^{\mathrm{ord}}\left(Np^{\alpha(P)}, \varepsilon_P; K\right) \to K$$
$$\text{for } k = n(P) - 2v + 2t \quad \text{and} \quad w = -v + n(P) + t.$$

Then we have:

COROLLARY 3.5. *For each $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ with $n(P) \geq 2v$, the value $\lambda_P(T(\mathfrak{n}))$ is contained in $\overline{\mathbf{Q}}$ for all $\mathfrak{n}$, and since $\lambda_P(T(\mathfrak{n}))$ is a complex number by the fixed embedding of $\overline{\mathbf{Q}}$ into $\mathbf{C}$, there is a non-zero cusp form $f_P \in S_{k,w,I_B}^*(Np^{\alpha(P)}, \varepsilon_P; \mathbf{C})$ for $k = n(P) - 2v + 2t$ and $w = -v + n(P) + t$ such that $f_P|T(\mathfrak{n}) = \lambda_P(T(\mathfrak{n}))f_P$ for all ideals $\mathfrak{n}$. The cusp form f_P is uniquely determined by the above conditions up to constant factors. Conversely, suppose that there is a common eigenform f of all Hecke operators $T(\mathfrak{n})$ in $S_{k,w,I_B}^*(Np^\alpha; \mathbf{C})$ whose eigenvalue for $T_0(p)$ is a p-adic unit in $\overline{\mathbf{Q}}_p$ and whose weight $k \geq 2t$ satisfies $k \sim -2v$. Then there exists a Λ-algebra homomorphism λ: $\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \to \bar{\mathcal{L}}$ and $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ such that*

$$f|T(\mathfrak{n}) = \lambda_P(T(\mathfrak{n}))f \qquad \textit{for all } \mathfrak{n} \textit{ prime to } p.$$

We shall prove in Section 5 a slightly stronger result than the statement of Corollary 3.5 after proving duality theorems between Hecke algebras and the space of cusp forms.

Let λ: $\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \to \bar{\mathcal{L}}$ be a Λ-algebra homomorphism. We consider the set:

$$\left\{\lambda'\colon \mathbf{h}_v^{\mathrm{ord}}(N'; \mathcal{O}) \to \bar{\mathcal{L}} \,\middle|\, \lambda'(T_0(\ell)) = \lambda(T_0(\ell)) \right.$$
$$\left. \text{except for finitely many prime ideals } \ell \right\},$$

where the level N' varies. Obviously in this set, there exists a Λ-algebra homomorphism λ_0: $\mathbf{h}_v^{\mathrm{ord}}(C; \mathcal{O}) \to \bar{\mathcal{L}}$ with the smallest level C. This λ_0 is called *primitive* or a *primitive* homomorphism associated with λ. The level C is called the *conductor* of λ. We can make an analogous definition in the finite level case: Let ϕ: $h_{k,w}(M; \mathcal{O}) \to \overline{\mathbf{Q}}_p$ be an $\mathcal{O}$-algebra homomorphism for an $\mathfrak{r}$-ideal M. We consider the set

$$\left\{\phi' \colon h_{k,w}(M';\mathcal{O}) \to \overline{\mathbf{Q}}_p \middle| \phi(T(\ell)) = \phi'(T(\ell)) \text{ for almost all prime ideals } \ell\right\}.$$

Then in this set, we can find a *unique* ϕ_0: $h_{k,w}(C;\mathcal{O}) \to \overline{\mathbf{Q}}_p$ with the smallest level C. This ϕ_0 is called *primitive* or the *primitive* homomorphism associated with ϕ. The level C is called the conductor of ϕ, which is a *divisor* of M. This fact can be deduced from a result of Miyake [23] in view of the equivalence between the following two conditions:

(3.10a) *ϕ is primitive of conductor C,*

(3.10b) *the cusp form f in $S^*_{k,w}(C;\mathbf{C})$ satisfying $f|T(n) = \phi_0(T(n))f$ for all n is a new form of exact level C in the sense of* [23].

Note that the existence of f as in (3.10b) will be guaranteed by Theorem 5.3 in Section 5.

We have decomposed $\overline{Z}(N) = W \times \overline{Z}_{\mathrm{tor}}$. Let ψ: $\overline{Z}_{\mathrm{tor}} \to \overline{\mathbf{Q}}$ be the combination: $\overline{Z}_{\mathrm{tor}} \hookrightarrow \overline{Z}(N) \to \mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \xrightarrow{\lambda} \overline{\mathscr{L}}$. Then ψ can be considered to be a finite order character of $F_{\mathbf{A}}^\times / F^\times F_{\infty+}^\times$. This character ψ will be called the character of λ. Similarly, for each homomorphism ϕ: $h_{k,w}(M,\mathcal{O}) \to \overline{\mathbf{Q}}_p$, the correspondence: $\mathrm{Il}(M) \ni a \to \phi(\langle a \rangle_n) \in \overline{\mathbf{Q}}$ gives a character of $\mathrm{Cl}_F(M)$. This character is called the character of ϕ. By Corollary 3.5 and (3.9), we have:

(3.11) *The character of λ_P $\left(P \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})\right)$ is given by $\varepsilon_P \psi \cdot (\omega \circ \chi_{-n(P)})$,*

where ω: $\mathbf{Z}_p^\times \to \overline{\mathbf{Q}}^\times$ is the Teichmüller character. Then we have:

THEOREM 3.6. *Let λ: $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \to \overline{\mathscr{L}}$ be a Λ-algebra homomorphism. Then the primitive homomorphism λ_0 associated with λ is unique, and its conductor C is a divisor of N. If λ itself is primitive, then for all $P \in \mathscr{X}_{\mathrm{alg}}(\mathscr{I})$, the conductor of $\lambda_P = P \circ \lambda$ is divisible by N. If the conductor of λ_P is moreover divisible by every prime factor of p, then λ_P itself is primitive.*

COROLLARY 3.7. *Let λ: $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \to \overline{\mathscr{L}}$ be a primitive homomorphism, and let $\mathscr{K}$ be the quotient field of the image of λ. Then, we can decompose*

$$\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \otimes_\Lambda \mathscr{K} = \mathscr{K} \oplus \mathscr{B}$$

as an algebra direct sum so that the projection to the first factor $\mathscr{K}$ coincides with λ on $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O})$.

Theorem 3.6 together with Corollary 3.7 will be proved in Section 12. Now we touch briefly on the module of congruence and the module of differentials attached to λ, which have an intimate relation with a certain L-function of cusp forms at least in the case: $F = \mathbf{Q}$ as disclosed in [12], [14] and [15]. Let us define $\hat{\lambda}$: $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \otimes_\Lambda \mathscr{I} \to \mathscr{I}$ by the combination of $\lambda \otimes \mathrm{id}$: $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \otimes_\Lambda \mathscr{I} \to \mathscr{I} \otimes_\Lambda \mathscr{I}$ with the multiplication map: $\mathscr{I} \otimes_\Lambda \mathscr{I} \to \mathscr{I}$. Suppose λ to be primitive, and let $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \otimes_\Lambda \mathscr{K} = \mathscr{K} \oplus \mathscr{B}$ be the decomposition as in Corollary 3.7. Let

$\mathrm{pr}_{\mathscr{B}}$: $\mathbf{h}_v^{\mathrm{ord}}(N; \mathscr{O}) \otimes_\Lambda \mathscr{I} \to \mathscr{B}$ be the projection map, and define

$$\delta\colon \mathbf{h}_v^{\mathrm{ord}}(N; \mathscr{O}) \otimes_\Lambda \mathscr{I} \to \mathscr{I} \oplus \mathrm{pr}_{\mathscr{B}}\big(\mathbf{h}_v^{\mathrm{ord}}(N; \mathscr{O}) \otimes_\Lambda \mathscr{I}\big)$$

by the diagonal map of λ and $\mathrm{pr}_{\mathscr{B}}$. The module of congruence $\mathscr{C}_0(\lambda)$ is defined by

$$\mathscr{C}_0(\lambda) = \mathrm{Coker}(\delta).$$

The module of differentials $\mathscr{C}_1(\lambda)$ is defined by

$$\mathscr{C}_1(\lambda) = \Omega^1_{\mathbf{h}/\mathscr{I}} \otimes_{\mathbf{h}} \mathscr{I},$$

where we have written simply $\mathbf{h}$ for $\mathbf{h}_v^{\mathrm{ord}}(N; \mathscr{O}) \otimes_\Lambda \mathscr{I}$ and $\mathscr{I}$ is regarded as an $\mathbf{h}$-module via $\hat{\lambda}\colon \mathbf{h} \to \mathscr{I}$.

COROLLARY 3.8. *The modules $\mathscr{C}_0(\lambda)$ and $\mathscr{C}_1(\lambda)$ are torsion $\mathscr{I}$-modules of finite type. Moreover, we have the identity of support of these modules*:

$$\mathrm{Supp}_{\mathscr{I}}(\mathscr{C}_0(\lambda)) = \mathrm{Supp}_{\mathscr{I}}(\mathscr{C}_1(\lambda)).$$

The author hopes to clarify the relation of these modules with certain p-adic L-functions of cusp forms on a future occasion.

Proof. Since $\mathbf{h} = \mathbf{h}_v^{\mathrm{ord}}(N; \mathscr{O}) \otimes_\Lambda \mathscr{I}$ is of finite type over $\mathscr{I}$, the first assertion is obvious. Write $\mathscr{S}$ for $\mathrm{pr}_{\mathscr{B}}(\mathbf{h})$, and let $\mathfrak{a}$ be the kernel of $\hat{\lambda}$. Then, from [13, Lemma 1.1], we know that $\mathscr{C}_1(\lambda) \cong \mathfrak{a}/\mathfrak{a}^2$. We take the intersection $\mathfrak{a}_0 = \mathrm{Im}(\delta) \cap \mathscr{S}$ inside $\mathscr{I} \oplus \mathscr{S}$. Then δ induces a surjection: $\mathfrak{a} \to \mathfrak{a}_0$ and hence we have a surjection: $\mathscr{C}_1(\lambda) \to \mathfrak{a}_0/\mathfrak{a}_0^2$. Note that

$$C_0(\lambda) = (\mathscr{I} \oplus \mathscr{S})/\mathrm{Im}(\delta) = \mathscr{S} + \mathrm{Im}(\delta)/\mathrm{Im}(\delta) \cong \mathscr{S}/\mathscr{S} \cap \mathrm{Im}(\delta) = \mathscr{S}/\mathfrak{a}_0.$$

This shows that $\mathrm{Supp}(\mathscr{C}_1(\lambda)) \supset \mathrm{Supp}(\mathscr{C}_0(\lambda))$. If we have a vanishing

$$\mathscr{C}_0(\lambda)_P = \mathscr{C}_0(\lambda) \otimes_{\mathscr{I}} \mathscr{I}_P = 0$$

for the localization $\mathscr{I}_P$ of $\mathscr{I}$ at a prime ideal P, then we have a decomposition:

$$\mathbf{h}_P = \mathbf{h} \otimes_{\mathscr{I}} \mathscr{I}_P = \mathscr{I}_P \oplus \big(\mathscr{S} \otimes_{\mathscr{I}} \mathscr{I}_P\big)$$

Thus $\mathscr{I}_P$ is $\mathbf{h}_P$-projective and $\mathscr{C}_1(\lambda) \otimes_{\mathscr{I}} \mathscr{I}_P = \mathrm{Tor}_1^{\mathbf{h}_P}(\mathscr{I}_P, \mathscr{I}_P) = 0$ by [13, Lemma 1.1]. This shows that $\mathrm{Supp}(\mathscr{C}_0(\lambda)) \supset \mathrm{Supp}(\mathscr{C}_1(\lambda))$, which finishes the proof.

Let us give here a few words about the filtrations $\overline{\mathrm{Z}}_\alpha \subset \overline{\mathrm{Z}}$ and $\mathrm{Z}_\alpha \subset \mathrm{Z}$. There is a commutative diagram of natural maps:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Z}_\alpha \arrow[r] \arrow[d, "a"] & \mathrm{Z} \arrow[r] \arrow[d, "b"] & \mathrm{Cl}_F(Np^\alpha) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \overline{\mathrm{Z}}_\alpha \arrow[r] & \overline{\mathrm{Z}} \arrow[r] & \overline{\mathrm{Cl}}_F(Np^\alpha) \arrow[r] & 0.
\end{tikzcd}$$

Since the morphisms b and c are surjective and their kernels are killed by a power of 2, a is an isomorphism if $\alpha \geq 1$ and $p > 2$. Since the kernels of b and c are generated by the image of $F_\infty^\times$, the morphism a is always surjective. The finiteness of the kernel of b implies that a gives a surjective isomorphism for sufficiently large α even if $p = 2$. Especially, a induces a surjective isomorphism between the $\mathbf{Z}_p$-torsion-free parts of Z and $\overline{Z}$.

4. Stability of integral cusp forms under Hecke operators

Throughout this section and the next, unless otherwise mentioned, we suppose that $B = M_2(F)$. We shall give an exposition of two methods for proving stability under Hecke operators $T_0(\mathfrak{n})$ of the space of rational or even integral cusp forms. One is due to Shimura's theory of canonical models [31] and [34], and the other is derived from the moduli theoretic interpretation of integral modular forms and the q-expansion principle studied by Deligne and Ribet [3], Rapoport [27] and Katz [19]. We shall do this because the former method may be generalized to a vast class of algebraic groups for which canonical models exist and the latter gives a stronger result concerning integral cusp forms.

Firstly, we shall define rational subspaces of $S^*_{k,w,I}(N; \mathbf{C})$ for each ideal N of $\mathfrak{r}$. Let A be a subalgebra of $\mathbf{C}$ satisfying (3.1) for the fixed $v \in \mathbf{Z}[I]$. For each ideal $\mathfrak{a}$, we choose a generator $\{\mathfrak{a}^v\}$ of $\mathfrak{a}^v A$ as in Section 3. Then the symbol $\{\mathfrak{a}^v\}$ satisfies the multiplicative relation: $\{\mathfrak{a}^v\}\{\mathfrak{b}^v\} = \{(\mathfrak{a}\mathfrak{b})^v\}$. For $\xi \in F^\times$, we write $\{\xi^v\}$ for $\{(\xi\mathfrak{r})^v\}$. We take $n, k, w \in \mathbf{Z}[I]$ satisfying $0 \leq n \sim -2v$, $k = n + 2t \geq 2t$, $w = v + k - t$ and $\hat{w} = k - w = t - v$. For each weight $\eta \in \mathbf{Q} \cdot t$, we write $\eta = [\eta]t$ for $[\eta] \in \mathbf{Q}$. For the symbols which are not defined here, see the introduction.

PROPOSITION 4.1. *Let $\mathfrak{d}$ be the different of $F/\mathbf{Q}$, and put*

$$F_+^\times = \{\xi \in F^\times \mid \xi^\sigma > 0 \textit{ for all } \sigma \in I\}.$$

Then each element $f \in S^*_{k,w,I}(N; \mathbf{C})$ has the Fourier expansion of the following form:

$$f\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = |y_f|_{\mathbf{A}} y_\infty^{\hat{w}} \sum_{\xi \in F_+^\times} a(\xi y\mathfrak{d}, f)\{(\xi y\mathfrak{d})^v\}\xi^{-v}\mathbf{e}_F(\sqrt{-1}\,\xi y_\infty)\mathbf{e}_F(\xi x),$$

where the function: $\mathfrak{a} \mapsto a(\mathfrak{a}, f) \in \mathbf{C}$ is a function on the group of fractional ideals of F which vanishes unless $\mathfrak{a}$ is integral.

Proof. For each f as in the proposition, we define another function f_0: $G_{\mathbf{A}} \to \mathbf{C}$ by

$$f_0(g) = |\det(g)|_{\mathbf{A}}^{[w-(k/2)]} f(g).$$

Then we can easily check that f_0 is contained in $S^*_{k,k/2,I}(N;\mathbf{C})$, and by [34, (2.18)], f_0 has the (unique) Fourier expansion of the form:

$$f_0\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \sum_{\xi \in F_+^\times} a_0(\xi y \mathscr{d}, f)\xi^{k/2} y_\infty^{k/2} \mathbf{e}_F(i\xi y_\infty)\mathbf{e}_F(\xi x),$$

where "$\mathscr{a} \mapsto a_0(\mathscr{a}, f) \in \mathbf{C}$" is a function on fractional ideals of F vanishing outside integral ideals. If we put

$$a(\mathscr{a}, f) = a_0(\mathscr{a}, f)\mathscr{N}_{F/\mathbf{Q}}(\mathscr{a}\mathscr{d}^{-1})^{1-[(k/2)-w]}\{\mathscr{a}^{-v}\} \tag{4.1}$$

$$\left(\{\mathscr{a}^{-v}\} = \{\mathscr{a}^{v}\}^{-1} = \{(\mathscr{a}^{-1})^{v}\}\right),$$

we obtain the result.

The following corollary can be deduced from [34, (2.23)] by (4.1) or from an easy calculation by use of an explicit decomposition of $\mathscr{T}(\mathscr{n})$ into the disjoint union of left cosets of $U_1(N)$:

COROLLARY 4.2. *For each $f \in S^*_{k,w,I}(N;\mathbf{C})$,*

$$\begin{aligned} a(\mathscr{m}, f|T_0(n)) &= \sum_{\substack{\ell|\mathscr{m},\ \ell|\mathscr{n} \\ \ell+N=\mathscr{r}}} \mathscr{N}_{F/\mathbf{Q}}(\ell)a(\mathscr{m}\mathscr{n}/\ell^2, f|T_0(\ell,\ell)) \\ &= \sum_{\substack{\ell|\mathscr{m},\ \ell|\mathscr{n} \\ \ell+N=\mathscr{r}}} \mathscr{N}_{F/\mathbf{Q}}(\ell)\{\ell^{2v}\}^{-1}a(\mathscr{m}\mathscr{n}/\ell, f|\langle\ell\rangle). \end{aligned} \tag{4.2}$$

For a while, we shall deal with a general quaternion algebra B unramified at every finite place not necessarily equal to $M_2(F)$. We fix a complete representative set $\{a_i\}_{i=1,\dots,h}$ for $\mathrm{Cl}_F(1) = F_{\mathbf{A}}^\times/F^\times\hat{\mathscr{r}}^\times F_{\infty+}^\times$ such that $a_i \in F_f^\times \cap \hat{\mathscr{r}}$ and $(a_i)_N = 1$. We define $t_i \in \mathrm{GL}_2(F_f)$ by $t_i = \begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix}$. Then, if B is indefinite, we have a disjoint decomposition:

$$G_{\mathbf{A}}^B = \coprod_{i=1}^h G_{\mathbf{Q}}^B t_i^{-\iota} U_1(N) G_{\infty+}^B = \coprod_{i=1}^h G_{\mathbf{Q}}^B t_i V_1(N) G_{\infty+}^B.$$

When B is totally definite, we just choose $\{t_i\}_{i=1,\dots,h}$ in G_f^B which satisfies the above type of decomposition; so, in this case, h may be different from the narrow ray class number of F. Then, we know if B is indefinite,

$$G_{\mathbf{A}}^B = \coprod_{i=1} G_{\mathbf{Q}}^B t_i x V_1(N) x^{-1} G_{\infty+}^B \qquad \text{for all } x \in G_{\mathbf{A}}^B.$$

Put

$$\Gamma_E^i(N) = t_i^{-\iota} U_E(N) G_{\infty+}^B t_i^{\iota} \cap G_{\mathbf{Q}}^B = t_i V_E(N) G_{\infty+}^B t_i^{-1} \cap G_{\mathbf{Q}}^B. \tag{4.3}$$

Now we return to the analysis of $B = M_2(F)$. Then by (2.6a), we have a canonical isomorphism:

$$S^*_{k,w,I}(N;\mathbf{C}) \cong \bigoplus_{i=1}^{h} S_{k,\hat{w},I}\big(\Gamma^i_E(N);\mathbf{C}\big).$$

For each $f \in S^*_{k,w,I}(N;\mathbf{C})$, we denote f_{t_i} by f_i as in (2.4b), which is an element in $S_{k,\hat{w},I}(\Gamma^i_E(N);\mathbf{C})$. Then, by definition of the above isomorphism, we have

Corollary 4.3. *Let* $\mathfrak{a}_i = \mathfrak{a}_i\mathfrak{r}$, *and put* $\mathfrak{a}^*_i = \mathfrak{a}_i^{-1}\mathfrak{d}^{-1}$ *and*

$$\mathfrak{a}^*_{i+} = \{\xi \in \mathfrak{a}^*_i \,|\, \xi^\sigma > 0 \text{ for all } \sigma \in I\}.$$

Then, for each $f \in S^*_{k,w,I}(N;\mathbf{C})$, $f_i(z) \in S_{k,\hat{w},I}(\Gamma^i_E(N);\mathbf{C})$ *has the following Fourier expansion*:

$$f_i(z) = c_{v,i} \sum_{\xi \in \mathfrak{a}^*_{i+}} a(\xi\mathfrak{a}_i\mathfrak{d}, f)\{\xi^v\}\xi^{-v}e_F(\xi z),$$

where $c_{v,i} = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a}_i)^{-1}\{(\mathfrak{a}_i\mathfrak{d})^v\}$.

Here note that $\{\xi^v\}\xi^{-v}$ is a unit in A.

Now we shall define the space of A-integral cusp forms. To do this, let us prepare some notation. For each ideal $\mathfrak{a}$, we put

$$\mathfrak{a}_+ = \{\xi \in \mathfrak{a} \,|\, \xi^\sigma > 0 \text{ for all } \sigma \in I\}.$$

Then we consider the formal power series ring

$$A[[q]]_{\mathfrak{a}} = \left\{ \sum_{\xi \in \mathfrak{a}_+ \cup \{0\}} a(\xi)q^\xi \,\middle|\, a(\xi) \in A \right\}.$$

Especially, we write $A[[q]]_i$ for $A[[q]]_{\mathfrak{a}^*_i}$. Then, replacing $\mathbf{e}_F(\xi z)$ by q^ξ, we have an embedding for each congruence subgroup Γ of $GL^+_2(F) = GL_2(F) \cap G_{\infty+}$:

$$S_{k,w;I}(\Gamma;\mathbf{C}) \hookrightarrow \mathbf{C}[[q]]_{\mathfrak{a}}$$

for a suitable choice of $\mathfrak{a}$. By this embedding, we shall regard $S_{k,w,I}(\Gamma;\mathbf{C})$ as a subspace of $\mathbf{C}[[q]]_{\mathfrak{a}}$. We then put

$$S_{k,w,I}(\Gamma;A) = A[[q]]_{\mathfrak{a}} \cap S_{k,w,I}(\Gamma;\mathbf{C}), \qquad S_{k,w,I}(A) = \varinjlim_{\Gamma} S_{k,w,I}(\Gamma;A).$$

Then, for any $\alpha \in GL^+_2(F)$, $f \mapsto f|_{k,w}\alpha$ gives an endomorphism of $S_{k,w,I}(\mathbf{C})$, where

$$(4.4) \qquad (f|_{k,w}\alpha)(z) = \det(\alpha)^w j_I(\alpha,z)^{-k} f(\alpha(z)).$$

As for the spaces of functions on $G_{\mathbf{A}}$, we define

$$\text{(4.5a)}\qquad S^*_{k,w,I}(N;A) = S^*_{k,w,I}(N;M_2(F);A)$$
$$= \left\{f \in S^*_{k,w,I}(N;\mathbf{C}) \middle| a(\mathfrak{a}, f) \in A \text{ for all } \mathfrak{a} \in \mathrm{Il}(1)\right\}.$$

Then the isomorphism (2.6a) induces

$$\text{(4.5b)}\qquad S^*_{k,w,I}(N;A) \cong \bigoplus_{i=1}^{h} c_{v,i} S_{k,\hat{w},I}\left(\Gamma^i_E(N);A\right).$$

Now we shall give an exposition of Shimura's method to prove the stability of $S^*_{k,s,I}(N;K)$ under Hecke operators for each finite extension $K/\Phi(v)$. Put

$$\text{(4.6a)}\quad \mathcal{G}_+ = \left\{x \in \mathrm{GL}_2(F_{\mathbf{A}}) \middle| \det(x_\infty) \in F^\times_{\infty+} \text{ and } \det(x) \in \mathbf{Q}^\times_{\mathbf{A}} F^\times F^\times_{\infty+}\right\}.$$

Since $\mathbf{Q}^\times_{\mathbf{A}} F^\times F^\times_{\infty+}/F^\times F^\times_{\infty+} \cong \mathbf{Q}^\times_{\mathbf{A}}/\mathbf{Q}^\times\mathbf{Q}^\times_{\infty+} \cong \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ canonically for the maximal abelian extension $\mathbf{Q}_{ab}/\mathbf{Q}$, we can let $x \in \mathcal{G}_+$ act on $\mathbf{Q}_{ab}$ via $\det(x)$. The corresponding element of $\mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ to $x \in \mathcal{G}_+$ will be written as $\rho(x)$. Define

$$\text{(4.6b)}\qquad \mathcal{G}_+(K) = \left\{x \in \mathcal{G}_+ \middle| \rho(x) \text{ acts trivially on } K \cap \mathbf{Q}_{ab}\right\}.$$

One may consider $\rho(x)$ for $x \in \mathcal{G}_+(K)$ to be an element of $\mathrm{Gal}(K\mathbf{Q}_{ab}/K)$. It has been shown by [34, Prop. 1.7]:

(4.7a) *For each extension $K/\Phi(v)$ inside* $\mathbf{C}$, *we have a natural isomorphism*:

$$S_{k,\hat{w},I}(K) \cong S_{k,\hat{w},I}(\Phi(v)) \otimes_{\Phi(v)} K.$$

This shows especially that if we define an action of $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\Phi(v))$ on $f = \sum_{\xi \in \mathfrak{a}_+} a(\xi) q^\xi \in \overline{\mathbf{Q}}[[q]]_{\mathfrak{a}}$ by $f^\sigma = \sum_{\xi \in \mathfrak{a}_+} a(\xi)^\sigma q^\xi$, then we have

(4.7b) $\mathrm{Gal}(\Phi(v)\mathbf{Q}_{ab}/\Phi(v))$ *acts on* $S_{k,\hat{w},I}(\Phi(v)\mathbf{Q}_{ab})$ *via* $f \mapsto f^\sigma$.

For each positive integer N, we put

$$U'_N = G_{\infty+} \times \left\{x \in \mathrm{GL}_2(\hat{\imath}) \middle| x \equiv \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \bmod NM_2(\hat{\imath})\right\},$$

$$U_N = \left\{x \in U'_N \middle| \det(x) \in \mathbf{Q}^\times_{\mathbf{A}}\right\} \cdot G_{\infty+}, \qquad \Gamma_N = \mathrm{GL}_2(F) \cap U_N.$$

Note that for each $\gamma \in \Gamma_N$, $\det(\gamma) \in \mathbf{Q}^\times_{\mathbf{A}}$ by definition of U_N. This implies that Γ_N is a subgroup of $\mathrm{SL}_2(F)$, and hence, Γ_N is the principal congruence subgroup modulo N of $\mathrm{SL}_2(\imath)$. By the strong approximation theorem (cf. [31, Prop. 3.4]), we know that $\mathcal{G}_+ = U_N \cdot \mathrm{GL}^+_2(F)$. Then, for each $x \in \mathcal{G}_+$, we decompose $x = u\alpha$ for $\alpha \in \mathrm{GL}^+_2(F)$ and $u \in U_N$ and consider the correspondence: $x \mapsto \det(\alpha)^{\hat{w}}$. If $x = u\alpha = u'\alpha'$ for another choice of $u' \in U_N$ and $\alpha' \in$

$\mathrm{GL}_2(F)$, then $u'^{-1}u = \alpha'\alpha^{-1} \in \Gamma_N$. Especially, $\det(\alpha) = \det(\alpha')$. Thus the function: $x \mapsto \det(\alpha)^{\hat{w}} \in \Phi(v)$ gives a continuous quasi-character of $\mathscr{G}_+$. Now we shall present a slight modification of a result of Shimura:

THEOREM 4.4. *For each finite extension K of $\Phi(v)$, there is a continuous right action of $\mathscr{G}_+(K)$ on $S_{k,\hat{w},I}(K\mathbf{Q}_{ab})$ which is characterized by the following properties*:

(4.8a) $$(f+g)^x = f^x + g^x, \qquad (f^x)^y = f^{xy},$$

(4.8b) $$f^\alpha = f|_{k,\hat{w}}\alpha \quad \textit{as in } (4.4) \qquad \textit{if } \alpha \in \mathrm{GL}_2^+(F),$$

(4.8c) $$f^x = f^{\rho(x)} \qquad \textit{if } x = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \quad \textit{with } t \in \hat{\mathbf{Z}}^\times.$$

Here the continuity of the action means that for each $f \in S_{k,\hat{w},I}(K\mathbf{Q}_{ab})$, we can find a positive integer N so that

$$f^u = f \text{ for all } u \in \left(U_N \cap \mathrm{SL}_2(F_{\mathbf{A}})\right) \cdot G_{\infty+}.$$

It might be necessary to explain how to deduce this theorem from a result of Shimura in [34, Th. 1.5] where a similar but different action: $f \mapsto f^{[x]}$ is given. The definition of this action is as follows: For each $f \in S_{k,\hat{w},I}(K\mathbf{Q}_{ab})$, the set $\{f^\sigma | \sigma \in \mathrm{Gal}(K\mathbf{Q}_{ab}/K)\}$ has only finitely many elements ([34, Prop. 1.3] or else (4.7a) in the text); hence, we can find a positive integer N so that $f^\sigma \in S_{k,\hat{w},I}(\Gamma_N; K\mathbf{Q}_{ab})$ for all $\sigma \in \mathrm{Gal}(\mathbf{Q}_{ab}K/K)$. This is possible since Γ_N is the principal congruence subgroup of $\mathrm{SL}_2(\mathfrak{r})$ modulo N as already remarked. By the strong approximation theorem (cf. [31, Prop. 3.4]), for each $x \in \mathscr{G}_+(K)$, we can decompose $x = u\alpha$ for $u \in U_N$ and $\alpha \in \mathrm{GL}_2^+(F)$. Then we define

$$f^{[x]} = f^{\rho(x)}|_{k,0}\alpha.$$

Our action is defined by

$$f^x = f^{\rho(x)}|_{k,\hat{w}}\alpha = \det(\alpha)^{\hat{w}} f^{[x]}.$$

Since as already remarked, $x \mapsto \det(\alpha)^{\hat{w}}$ gives a continuous quasi-character of $\mathscr{G}_+$ into $\Phi(v)$, the action $f \mapsto f^x$ is well defined. The verification of the properties (4.8b, c) is left to the reader.

For each open compact subgroup U of $\mathrm{GL}_2(F_f)$, we put

$$D_U = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \in U \,\middle|\, t \in \hat{\mathbf{Z}}^\times \right\}, \qquad U(K) = UG_{\infty+} \cap \mathscr{G}_+(K),$$
$$D_{U(K)} = D_U \cap U(K), \qquad \Gamma_U = UG_{\infty+} \cap \mathrm{GL}_2(F).$$

COROLLARY 4.5. *Let U be an open compact subgroup of $\mathrm{GL}_2(F_f)$, and suppose that*

(i) $\rho\colon D_U G_{\infty+} \to \mathrm{Gal}(\mathbf{Q}_{ab}/\mathbf{Q})$ *is surjective*,

(ii) $D_U \Gamma_U G_{\infty+}$ *is dense in* $UG_{\infty+} \cap \mathscr{G}_+$.

Then we have that $S_{k,\hat{w},I}(\Gamma_U; K) = H^0(U(K), S_{k,\hat{w},I}(K\mathbf{Q}_{ab}))$ *and thus*,

$S_{k,\hat{w},I}(\Gamma_U; K) \cong S_{k,\hat{w},I}(\Gamma_U; \Phi(v)) \otimes_{\Phi(v)} K$. *Especially, for* t_i *as in* (4.3), $t_i V_E(N) t_i^{-1}$ *and* $t_i V_1(N) t_i^{-1}$ *satisfy conditions* (i) *and* (ii).

Proof. The last assertion is obvious from the definitions of t_i, $V_1(N)$ and $V_E(N)$, and the second assertion follows from the first and (4.7a); so, we prove the first assertion. Let $\hat{\Gamma}_U$ be the closure of $\Gamma_U G_{\infty+}$ in $\mathcal{G}_+$. Then, by the continuity of the action of Theorem 4.4 and (4.8b), we have

$$S_{k,\hat{w},I}(\Gamma_U; K\mathbf{Q}_{ab}) = H^0\big(\hat{\Gamma}_U, S_{k,\hat{w},I}(K\mathbf{Q}_{ab})\big).$$

Since $U(K) = \hat{\Gamma}_U D_{U(K)}$ by (i), (4.8c) shows the result.

For simplicity, we shall write for a moment U, V, V^i and G^i for $U_E(N)$, $V_E(N)$, $t_i V_E(N) G_{\infty+} t_i^{-1}$ and $V^i \cap \mathrm{GL}_2(F)$, respectively. We shall decompose for $x \in \Delta_1(N)$, $UxU = \coprod_l U x_l$; i.e., $V x^{-\iota} V = \coprod_l V x_l^{-\iota}$. We then find an index i, γ_l and γ in $\mathrm{GL}_2^+(F)$ such that $t_j x_l^{\iota} \in \gamma_l t_i V G_{\infty+}$ and $t_j x^{\iota} \in \gamma t_i V G_{\infty+}$. The index i is determined independently of l and the correspondence: $i \leftrightarrow j$ is bijective.

LEMMA 4.6. *Under the above notation, put* $V^i(K) = V^i \cap \mathcal{G}_+(K)$. *Then we have a disjoint decomposition*:

$$V^i(K)\gamma^{-1}V^j(K) = \coprod_l V^i(K)\gamma_l^{-1} \quad \textit{if } x \in \Delta_1(N).$$

Proof. We have that $V x^{-\iota} V = V t_i^{-1} \gamma^{-1} t_j V$, and thus $V^i \gamma^{-1} V^j = V^i t_i x^{-\iota} t_j^{-1} V^j$ and $V x_l^{-\iota} t_j^{-1} = V t_i^{-1} \gamma_l^{-1}$; that is, $V^i t_i x^{-\iota} t_j^{-1} = V^i \gamma_l^{-1}$. This shows that $V^i \gamma^{-1} V^j = \coprod_l V^i \gamma_l^{-1}$. Thus γ_l^{-1} can be written as $v\gamma^{-1}v'$ with $v \in V^i$ and $v' \in V^j$. By the strong approximation theorem, we may suppose that $v \in \mathrm{SL}_2(F_f)$ and therefore $v' \in V^j(K)$. In fact, since $\det(V^i \cap \gamma^{-1} V^j \gamma) = \det(V^i)$, we can find $w \in V^i \cap \gamma^{-1} V^j \gamma$ so that $\det(w) = \det(v)$. Then $v\gamma^{-1}v' = vw^{-1}\gamma^{-1}\gamma w \gamma^{-1} v'$ and $\gamma w \gamma^{-1} v' \in V^j$, $vw^{-1} \in V^i$. By replacing v by vw^{-1}, we may thus assume that $v \in \mathrm{SL}_2(F_f)$. This shows that

$$V^i(K)\gamma^{-1}V^j(K) \supset \coprod_l V^i(K)\gamma_l^{-1}.$$

We shall show the other inclusion:

$$V^i(K)\gamma^{-1}V^j(K) \subset \coprod_l V^i(K)\gamma_l^{-1}.$$

For each $v\gamma^{-1}v' \in V^i(K)\gamma^{-1}V^j(K)$, we can find $v'' \in V^i$ such that $v\gamma^{-1}v' = v''\gamma_l^{-1}$ for some l. By taking the determinants, we learn that $\det(v'') \in \det(v'v'')F^\times F_{\infty+}^\times$ and $\rho(v'')$ fixes $K \cap \mathbf{Q}_{ab}$; hence $v'' \in V^i(K)$. This shows the assertion.

LEMMA 4.7. *If* $x \in \Delta_1(N)$, *then* $\det(V^j(K) \cap \gamma V^i(K) \gamma^{-1}) = \det(V^j(K))$. *Moreover*,

$$V^i(K)\gamma^{-1}V^j(K) = V^i(K)\gamma^{-1}\Gamma^j \quad \textit{and} \quad \Gamma^i\gamma^{-1}\Gamma^j = \coprod_l \Gamma^i \gamma_l^{-1}.$$

Proof. The last assertion follows from the first by [31, (2.19.3)]; so, we shall prove the first assertion. From the definition of $\Delta_1(N)$, we see easily that $\det(V \cap x^{\iota}Vx^{-\iota}) = \det(V)$. Then we have that

$$V^j \cap \gamma V^i \gamma^{-1} = t_j V t_j^{-1} \cap t_j x^{\iota} V x^{-\iota} t_j^{-1} = t_j (V \cap x^{\iota} V x^{-\iota}) t_j^{-1}.$$

This shows that $\det(V^j \cap \gamma V^i \gamma^{-1}) = \det(V^j)$. We shall show that

$$\left(V^j(K) \cap \gamma V^i(K) \gamma^{-1}\right) \supset \left(V^j \cap \gamma V^i \gamma^{-1} \cap \mathscr{G}_+(K)\right).$$

In fact, if $u = \gamma u' \gamma^{-1}$ for $u \in V^j(K)$ and $u' \in V^i$, then $u' = \gamma^{-1} u \gamma$ is contained in $V^i(K)$. This shows the above inclusion. The other inclusion is obvious, and hence, we have $V^j(K) \cap \gamma V^i(K) \gamma^{-1} = V^j \cap \gamma V^i \gamma^{-1} \cap \mathscr{G}_+(K)$. Then, we see that

$$\begin{aligned}\det\left(V^j(K) \cap \gamma V^i(K) \gamma^{-1}\right) &= \det\left(V^j \cap \gamma V^i \gamma^{-1} \cap \mathscr{G}_+(K)\right)\\ &= \det\left(V^j \cap \gamma V^i \gamma^{-1}\right) \cap \det\left(\mathscr{G}_+(K)\right)\\ &= \det\left(V^j\right) \cap \det\left(\mathscr{G}_+(K)\right) = \det\left(V^j(K)\right).\end{aligned}$$

For any field extension $K/\Phi(v)$ (not necessarily inside $\mathbf{C}$), let

$$S_{k,\hat{w},I}\left(\Gamma_E^i(N); K\right) = S_{k,\hat{w},I}\left(\Gamma_E^i(N); \Phi(v)\right) \otimes_{\Phi(v)} K \subset K[[q]]_i.$$

For any $\mathfrak{r}(v)$-subalgebra A of K,

$$S_{k,\hat{w},I}\left(\Gamma_E^i(N); A\right) = S_{k,\hat{w},I}\left(\Gamma_E^i(N); K\right) \cap A[[q]]_i.$$

For an $\mathfrak{r}(v)$-subalgebra A of K satisfying (3.1) for the given v in $\mathbf{Z}[I]$, we choose the map: $\mathfrak{a} \mapsto \{\mathfrak{a}^v\} \in A$, and put

$$S^*_{k,w,I}(N; A) = \bigoplus_{i=1}^{h} c_{v,i} S_{k,\hat{w},I}\left(\Gamma_E^i(N); A\right),$$

where $c_{v,i}$ is the constant defined in Corollary 4.3. We can naturally identify $S^*_{k,w,I}(N; K)$ with $S^*_{k,w,I}(N; \Phi(v)) \otimes_{\Phi(v)} K$ by Corollary 4.5 and $S^*_{k,w,I}(N; A)$ with an A-submodule of $S^*_{k,w,I}(N; K)$.

THEOREM 4.8. *If $K/\Phi(v)$ is a field extension inside* $\mathbf{C}$, *then $S^*_{k,w,I}(N; K)$ is stable under the Hecke operator $[V_1(N) x^{-\iota} V_1(N)]$ as in Section 2 for all $x \in \Delta_1(N)$. Especially, $S^*_{k,w,I}(N; K)$ is stable under $T(\mathfrak{n})$ and $T_0(\mathfrak{n})$ for all $\mathfrak{n}$.*

Proof. With the notation of Lemma 4.6, we have by (2.6c) that

$$\begin{aligned}\left(f \middle| \left[V_1(N) x^{-\iota} V_1(N)\right]\right)_j &= \sum_l f_i|_{k,\hat{w}} \gamma_l^{-1}\\ &= \sum_l (f_i)^{\gamma_l^{-1}}.\end{aligned}$$

Since $V^i(K) \gamma^{-1} V^j(K) = \coprod V^i(K) \gamma_l^{-1}$ by Lemma 4.6, we know from Theorem

4.4 and Corollary 4.5 that if $f_i \in H^0(V^i(K), S_{k,\hat{w},I}(K\mathbf{Q}_{ab})) = S_{k,\hat{w},I}(\Gamma_E^i(N); K)$, then

$$\sum_l (f_i)^{\gamma_l^{-1}} \quad \text{is contained in} \quad H^0\big(V^j(K), S_{k,\hat{w},I}(K\mathbf{Q}_{ab})\big)$$

which coincides with $S_{k,\hat{w},I}(\Gamma_E^j(N); K)$ by Corollary 4.5. This shows the assertion.

By this theorem, we can naturally extend the action of the Hecke operators $T(\mathfrak{n})$, $T_0(\mathfrak{n})$ and $T(\mathfrak{n}, \mathfrak{n})$ to $S^*_{k,w,I}(N; K)$ for any field extension $K/\Phi(v)$ (not necessarily in $\mathbf{C}$). The extended operators $T_0(\mathfrak{n})$ and $T(\mathfrak{n}, \mathfrak{n})$ satisfy the same formula as (4.2).

Now we shall generalize Theorem 4.8 to A-integral cusp forms. We suppose that A is a flat $\mathfrak{r}(v)$-algebra satisfying (3.1). Firstly we recall the definition of the Hilbert-Blumenthal abelian varieties over F, which will be abbreviated as HBAV. The details of what follows can be found in [27, §1]. An HBAV X over a base scheme S is an abelian scheme (in the sense of [25]) over S with an algebra homomorphism $m\colon \mathfrak{r} \to \mathrm{End}(X/S)$ making $\mathrm{Lie}(X/S)$ into a locally free sheaf of rank 1 over $\mathfrak{r} \otimes_{\mathbf{Z}} \mathcal{O}_S$, where $\mathcal{O}_S$ denotes the structure sheaf of S. Then m is injective, and the relative dimension of X over S is equal to the absolute degree of F. Let $X^* = \mathrm{Pic}^0(X/S)$ and define $m^*(a) \in \mathrm{End}(X^*/S)$ by the adjoint of $m(a)$ for $a \in \mathfrak{r}$. Then, (X^*, m^*) becomes naturally an HBAV. The polarization module of X is by definition

$$\mathcal{P}(X) = \{\lambda \in \mathrm{Hom}_S(X, X^*) \,|\, \lambda = \lambda^*,\ m^*(a)\circ\lambda = \lambda\circ m(a) \text{ for all } a \in \mathfrak{r}\},$$

where "*" indicates the adjoint. Then "$T \mapsto \mathcal{P}(X \times_S T/T)$" is known to be a locally constant sheaf on the étale site over S, which has values in the category of projective $\mathfrak{r}$-modules of rank 1 ([27, Prop. 1.17]). Let $\mu\colon X \times_S X \to X$ be the multiplication morphism on X and $p_1, p_2\colon X \times_S X \to X$ be the two projections. Consider the following sheaf on $X \times_S X$ made out of each invertible sheaf L on X:

$$\psi(L) = \mu^*(L) \otimes p_1^*(L)^{-1} \otimes p_2^*(L)^{-1}.$$

Note that the value at X of the relative Picard functor for X/S is given by

$$\mathcal{P}ic_{X/S}(X) = \frac{\{\text{group of invertible sheaves on } X \times_S X\}}{\{\text{subgroup of sheaves of the form } p_2^*(M) \text{ for an invertible sheaf } M/X\}}.$$

Then $\psi(L)$ gives an X-valued point of $\mathcal{P}ic_{X/S}(X)$. Now suppose that L is relatively ample. Then, it is known that $\psi(L)$ is contained in the connected component $\mathcal{P}ic^{\tau}_{X/S}(X)$ ([25, VI.2]). Since $\mathcal{P}ic^{\tau}_{X/S}(X) \cong \mathrm{Hom}_S(X, X^*)$

([25, VI]), we have a homomorphism corresponding to $\psi(L)$:

$$\Lambda(L)\colon X \to X^* \quad \text{over } S.$$

By definition, a polarization of X is an S-homomorphism $\lambda\colon X \to X^*$ such that, for every geometric point s of S, the induced $\bar{\lambda}\colon \bar{X} \to \bar{X}^*$ for the fibers $\bar{X}$ and $\bar{X}^*$ at $s \in S$ is of the form $\Lambda(\bar{L})$ for some *ample* invertible sheaf $\bar{L}$ on $\bar{X}$ ([25, I.2.6.3]). Put

$$\mathscr{P}_+(X) = \{\lambda \in \mathscr{P}(X) \mid \lambda \text{ is a polarization for } X\}.$$

Then, it is well-known that there is an $\mathfrak{r}$-linear embedding $i\colon \mathscr{P}(X) \to F$ such that i induces an isomorphism: $\mathscr{P}_+(X) \cong i(\mathscr{P}(X)) \cap F_+^\times$. Thus $\mathscr{P}_+(X)$ gives a notion of positivity on $\mathscr{P}(X)$ ([27, 1.15] [3, §4]).

In order to give the definition of the spaces of modular forms in this context, we fix a positive integer N_0 and a fractional ideal $\mathfrak{c}$. Let $\mathfrak{d}$ be the absolute different of F. We shall consider the moduli problem of quadruples (X, λ, ω, i), where X is an HBAV over a ring A, $\lambda\colon \mathscr{P}(X) \cong \mathfrak{c}$ which induces an isomorphism: $\mathscr{P}_+(X) \cong \mathfrak{c}_+$, ω is a base of $\mathfrak{r} \otimes_{\mathbf{Z}} A$-module $H^0(X, \Omega^1_{X/A})$ and $i\colon N_0^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} \otimes_{\mathbf{Z}} \mu_{N_0} \hookrightarrow X$ is a $\Gamma_{00}(N_0)$-structure over A (cf. [3, §5]). Here μ_{N_0} is a finite flat group scheme over $\mathbf{Z}$ which is the kernel of the multiplication by N_0 on $\mathbf{G}_m$. To give ω is equivalent to giving an $\mathfrak{r}$-linear isomorphism: $\mathrm{Lie}(X_{/A}) \cong \mathfrak{d}^{-1} \otimes_{\mathbf{Z}} A$. Let $k, v \in \mathbf{Z}[I]$ be as in Section 3, and suppose that A is an $\mathfrak{r}(v)$-algebra. Then the modular forms for $\Gamma_{00}(N_0)$ of weight k over A are functions f of isomorphism classes of quadruples $(X, \lambda, \omega, i)_{/A'}$ for all A-algebras A' such that

(4.9a) $$f((X, \lambda, \omega, i)_{/A'}) \in A',$$

(4.9b) $$f((X, \lambda, a\omega, i)_{/A'}) = a^{-k} f((X, \lambda, \omega, i)_{/A'}) \quad \text{for } a \in (\mathfrak{r} \otimes_{\mathbf{Z}} A')^\times,$$

(4.9c) *If* $\rho\colon A' \to A''$ *is a homomorphism of* A*-algebras*,

$$f((X, \lambda, \omega, i)_{/A'} \times A'') = \rho(f((X, \lambda, \omega, i)_{/A'})).$$

When $F = \mathbf{Q}$, an additional condition on the holomorphy at each cusp is necessary. However, the case of $F = \mathbf{Q}$ has already been treated in [12, §1]; so, we hereafter assume that $F \neq \mathbf{Q}$. The space of modular forms over A will be denoted by $\mathscr{M}_k(\Gamma_{00}(N_0), \mathfrak{c}; A)$.

An (unramified) cusp on $\Gamma_{00}(N_0)$ over A is given by the following data ([3, §5], [19, 1.1]):

(4.10a) *two ideals* $\mathfrak{a}, \mathfrak{b}$ *with* $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$,

(4.10b) *an* $\mathfrak{r}$*-linear isomorphism* $\varepsilon\colon N_0^{-1}\mathfrak{r}/\mathfrak{r} \cong N_0^{-1}\mathfrak{a}^{-1}/\mathfrak{a}^{-1}$,

(4.10c) *an* $\mathfrak{r} \otimes_{\mathbf{Z}} A$*-linear isomorphism* $j\colon \mathfrak{a}^{-1} \otimes_{\mathbf{Z}} A \cong \mathfrak{r} \otimes_{\mathbf{Z}} A$.

The evaluation of modular forms at the generalized Tate curve: $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, \omega_{\mathrm{can}}(j), i_{\mathrm{can}}(\varepsilon))$ as in [19, (1.1.13–17)] (or [27, §4]) gives the following q-expansion of $f \in \mathcal{M}_k(\Gamma_{00}(N_0), \mathfrak{c}; A)$:

$$f(\mathfrak{a}, \mathfrak{b}, \varepsilon, j) = \sum_{\xi \in \mathfrak{a}\mathfrak{b}_+ \cup \{0\}} a(\xi; \mathfrak{a}, \mathfrak{b}, \varepsilon, j; f) q^{\xi} \in A[[q]]_{\mathfrak{a}\mathfrak{b}}.$$

If $\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})$ is a unit in A, then we have a canonical isomorphism

$$j_{\mathrm{can}}\colon \mathfrak{a}^{-1} \otimes_{\mathbf{Z}} A \cong \mathfrak{r} \otimes_{\mathbf{Z}} A.$$

For each $a \in F_{\mathbf{A}}^{\times}$ with $\mathfrak{a} = a\mathfrak{r}$, we can choose an isomorphism

$$\text{(4.11)}\qquad \begin{array}{ccccccc} \varepsilon_a\colon N_0^{-1}\mathfrak{r}/\mathfrak{r} = & N_0^{-1}\hat{\mathfrak{r}}/\hat{\mathfrak{r}} & \cong & N_0^{-1}\mathfrak{a}^{-1}\hat{\mathfrak{r}}/\mathfrak{a}^{-1}\hat{\mathfrak{r}} & = N_0^{-1}\mathfrak{a}^{-1}/\mathfrak{a}^{-1} \\ & \Cup & & \Cup & \\ & x & \longmapsto & a^{-1}x. & \end{array}$$

When $\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})$ is a unit in A, we write $f(a)$ for the q-expansion of f in $\mathcal{M}_k(\Gamma_{00}(N_0), \mathfrak{c}; A)$ at $(a\mathfrak{r}, a\mathfrak{c}^{-1}, \varepsilon_a, j_{\mathrm{can}})$. Here are the q-expansion principles (cf. [19, (1.2.15–16)], [3, (5.4–5)]):

(4.12a) *If* $f(\mathfrak{a}, \mathfrak{b}, \varepsilon, j) = 0$ *for* $f \in \mathcal{M}_k(\Gamma_{00}(N_0), \mathfrak{c}; A)$ *at a cusp* $(\mathfrak{a}, \mathfrak{b}, \varepsilon, j)$, *then* $f = 0$;

(4.12b) *For each* $\mathfrak{r}(v)$*-subalgebra* A_0 *of* A, *if* $(\mathfrak{a}, \mathfrak{b}, \varepsilon, j)$ *is in fact a cusp over* A_0 *and if* $f(\mathfrak{a}, \mathfrak{b}, \varepsilon, j) \in A_0[[q]]_{\mathfrak{a}\mathfrak{b}}$ *for* $f \in \mathcal{M}_k(\Gamma_{00}(N_0), \mathfrak{c}; A)$, *then*

$$f \in \mathcal{M}_k(\Gamma_{00}(N_0), \mathfrak{c}; A_0).$$

Now we shall clarify the relation between the classical space $S_{k,\hat{w},I}(\Gamma_E^i(N); A)$ and newly defined $\mathcal{M}_k(\Gamma_{00}(N), \mathfrak{c}; A)$. To do this, for each integral ideal $\mathfrak{a}$, put

$$\Gamma_{00}(N_0; \mathfrak{a}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F) \,\middle|\, a, d \in \mathfrak{r},\ c \in N_0\mathfrak{a},\ a-1, d-1 \in N_0\mathfrak{r},\ b \in \mathfrak{a}^{-1} \right\}.$$

Over the complex number field $\mathbf{C}$, to give a quadruple (X, λ, ω, i) is equivalent to giving a triple $(\mathcal{L}, \lambda, i)$ consisting of an $\mathfrak{r}$-lattice $\mathcal{L}$ in $F \otimes_{\mathbf{Q}} \mathbf{C} = F_{\mathbf{C}}$ $(\cong \mathbf{C}^I)$, an $\mathfrak{r}$-linear embedding $i\colon N_0^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} \to N_0^{-1}\mathcal{L}/\mathcal{L}$ and an isomorphism $\lambda\colon \mathscr{L}\wedge_{\mathfrak{r}}\mathscr{L}\cong \mathfrak{d}^{-1}\mathfrak{c}^{-1}$ which is induced by an alternating form $\langle\ ,\ \rangle\colon \mathscr{L}\times\mathscr{L}\to\mathfrak{d}^{-1}\mathfrak{c}^{-1}$ such that

$$\langle x, y\rangle = a\,\mathrm{Im}(xy) \qquad \text{for some } a\in F_{+}^{\times}.$$

(Here $\mathrm{Im}(xy)$ means the imaginary part of $xy\in F_{\mathbf{C}}$ over $F_{\infty}=F\otimes_{\mathbf{Q}}\mathbf{R}$.) In fact, as for the differential ω on $X(\mathbf{C})=F_{\mathbf{C}}/\mathscr{L}$, we take the usual one induced by $du=\sum_{\sigma\in I}du_{\sigma}$ by identifying $F_{\mathbf{C}}$ with $\mathbf{C}^{I}$. The alternating form λ gives the polarization of the complex torus $X(\mathbf{C})$ which turns $X(\mathbf{C})$ into an abelian variety over $\mathbf{C}$. Thus we have the quadruple $(X,\lambda,\omega,i)/\mathbf{C}$ out of $(\mathscr{L},\lambda,i)$ which exhausts all the isomorphism classes of (X,λ,ω,i) over $\mathbf{C}$. Then, by (4.9b), we have

$$f\big(a_{\mathscr{L}},(a\bar a)^{-1}\lambda, ai\big)=a^{-k}f(\mathscr{L},\lambda,i) \qquad \text{for } a\in F_{\mathbf{C}}^{\times}\ \big([3,(5.6)]\big).$$

For each $z\in H^{I}\subset F_{\mathbf{C}}$, we consider a lattice $\mathscr{L}_z=2\pi i(\mathfrak{d}^{-1}\mathfrak{a}^{-1}+\mathfrak{b}z)$, where we regard $\mathfrak{a}$ and $\mathfrak{b}$ as $\mathfrak{r}$-submodules of $F_{\mathbf{C}}$ naturally. Define $\lambda_{\mathrm{can}}\colon \mathscr{L}_z\times\mathscr{L}_z\to\mathfrak{d}^{-1}\mathfrak{c}^{-1}$ by $\lambda_{\mathrm{can}}((2\pi i)(a+bz),(2\pi i)(c+dz))=ad-bc$ and

$$i_{\mathrm{can}}^{\varepsilon}\colon N_0^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}\to X_z=F_{\mathbf{C}}/\mathscr{L}_z$$

by the composition:

$$\begin{aligned}\big(N_0^{-1}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}\big)&=\big(N_0^{-1}\mathfrak{r}/\mathfrak{r}\big)\otimes_{\mathfrak{r}}\mathfrak{d}^{-1}\overset{\varepsilon\otimes\mathrm{id}}{\cong}\big(N_0^{-1}\mathfrak{a}/\mathfrak{a}\big)\otimes_{\mathfrak{r}}\mathfrak{d}^{-1}\\&=N_0^{-1}\mathfrak{a}^{-1}\mathfrak{d}^{-1}/\mathfrak{a}^{-1}\mathfrak{d}^{-1}\hookrightarrow N_0^{-1}\mathscr{L}_z/\mathscr{L}_z\subset X_z.\end{aligned}$$

The linear fractional transformation $z\mapsto\gamma(z)$ induces an isomorphism:

$$\Big(j_I(\gamma,z)\mathscr{L}_{\gamma(z)},\big(j_I(\gamma,z)\overline{j_I(\gamma,z)}\big)^{-1}\lambda_{\mathrm{can}},\,j_I(\gamma,z)i_{\mathrm{can}}^{\varepsilon}\Big)\cong\big(\mathscr{L}_z,\lambda_{\mathrm{can}},i_{\mathrm{can}}^{\varepsilon}\big)$$

if and only if $\gamma\in\Gamma_{00}(N_0;\,\mathfrak{a}^2\mathfrak{c}^{-1}\mathfrak{d})$. Therefore the function on H^I

$$f_{\mathfrak{a},\mathfrak{b},\varepsilon}(z)=f(\mathscr{L}_z,\lambda_{\mathrm{can}},i_{\mathrm{can}}^{\varepsilon}) \qquad \text{for } f\in\mathscr{M}_k(\Gamma_{00}(N_0),\mathfrak{c};\mathbf{C})$$

gives a classical modular form on $\Gamma=\Gamma_{00}(N_0;\,\mathfrak{a}^2\mathfrak{c}^{-1}\mathfrak{d})$ satisfying

(i) $f_{\mathfrak{a},\mathfrak{b},\varepsilon}\colon H^I\to\mathbf{C}$ *is holomorphic*,

(ii) $f_{\mathfrak{a},\mathfrak{b},\varepsilon}|_{k,w}\gamma=f_{\mathfrak{a},\mathfrak{b},\varepsilon}$ for $\gamma\in\Gamma$.

We denote by $\mathscr{M}_k(\Gamma_{00}(N_0;\,\mathfrak{a}))$ the space of functions satisfying the above two conditions for $\Gamma=\Gamma_{00}(N_0;\,\mathfrak{a})$. Moreover, we have:

(4.13a) *The correspondence*: $f\mapsto f_{\mathfrak{a},\mathfrak{b},\varepsilon_a}$ *gives an isomorphism*:

$$\mathscr{M}_k(\Gamma_{00}(N_0),\mathfrak{c};\mathbf{C})\cong\mathscr{M}_k\big(\Gamma_{00}\big(N_0;\,\mathfrak{a}^2\mathfrak{c}^{-1}\mathfrak{d}\big)\big)$$

such that the Fourier expansion of $f_{\mathfrak{a},\mathfrak{b},\varepsilon_a}$ *at* ∞ *gives the* q*-expansion of* f *at* $(\mathfrak{a},\mathfrak{b},\varepsilon_a,j_{\mathrm{can}})$ *for* $a\in F_f^{\times}$ *with* $\mathfrak{a}=a\mathfrak{r}$ (by replacing $\mathbf{e}_F(\xi z)$ by q^{ξ}).

This can be strengthened as follows: Let A be an $\mathfrak{r}(v)$-subalgebra of $\mathbf{C}$ and suppose that $\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})$ is invertible in A. Then for any A-algebra A' in $\mathbf{C}$,

(4.13b) *The correspondence*: $f \mapsto f_{\mathfrak{a}, \ell, \varepsilon_a}$ *induces a q-expansion preserving isomorphism*:

$$\mathcal{M}_k(\Gamma_{00}(N_0), \mathfrak{c}; A') \cong \mathcal{M}_k\left(\Gamma_{00}\left(N_0; \mathfrak{a}^2\mathfrak{c}^{-1}\mathfrak{d}\right); A'\right)$$
$$= \mathcal{M}_k\left(\Gamma_{00}\left(N_0; \mathfrak{a}^2\mathfrak{c}^{-1}\mathfrak{d}\right)\right) \cap A'[[q]]_{\mathfrak{a}\ell}.$$

Let $\Gamma = \Gamma_{00}(N_0; \mathfrak{c})$. By the strong approximation theorem, if we write $\hat{\Gamma}$ for the closure of Γ in $SL_2(F_f)$, then we know that $SL_2(F_f) = SL_2(F) \cdot \hat{\Gamma} = \hat{\Gamma} \cdot SL_2(F)$. Thus for each $x \in SL_2(F_f)$, we may choose $u \in \hat{\Gamma}$ and $\alpha \in SL_2(F)$ such that $x = u\alpha$. For f in $\mathcal{M}_k(\Gamma)$, put $f|_k x = f|_{k,w}\alpha$. This is independent of the choice of α and w (since $\det(\alpha) = 1$) and is determined only by x. We shall quote a theorem of Deligne and Ribet [3, 5.8]:

THEOREM 4.9. *Suppose that A is a discrete valuation ring of a finite extension of $\Phi(v)$. Write $\mathfrak{a} = a\mathfrak{r}$ for $a \in F_f^{\times}$, and suppose that $\mathcal{N}_{F/\mathbf{Q}}(\mathfrak{a})$ is invertible in A. If $f \in \mathcal{M}_k(\Gamma_{00}(N_0), \mathfrak{c}; A')$ for an A-algebra A' inside* $\mathbf{C}$, *then*

$$f(1)|_k\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = f(\mathfrak{a}) \qquad \text{in } A'[[q]]_{(\mathfrak{a}^2\mathfrak{c}^{-1})}.$$

Note that $f(1) \in \mathcal{M}_k(\Gamma_{00}(N_0; \mathfrak{c}^{-1}\mathfrak{d}))$ and $f(\mathfrak{a}) \in \mathcal{M}_k(\Gamma_{00}(N_0; \mathfrak{a}^2\mathfrak{c}^{-1}\mathfrak{d}))$.

THEOREM 4.10. *For any extension $K/\Phi(v)$, if an $\mathfrak{r}(v)$-subalgebra A of K satisfies* (3.1), *we have a natural isomorphism*: $S^*_{k,\hat{w},I}(N; K) \cong S^*_{k,\hat{w},I}(N; A) \otimes_A K$.

This result follows from [32, Th. 1] and [33, p. 683] when $k \in \mathbf{Z} \cdot t$ (i.e., $v = 0$) in view of Corollary 4.5. The general case can be derived from a result of Rapoport [27] (see also [3, p. 258]) by algebra-geometric means. We shall give a proof of this fact in Section 7 by cohomological means.

THEOREM 4.11. *Let A be an integrally closed domain containing $\mathfrak{r}(v)$. Suppose that A is finite flat over either of $\mathfrak{r}(v)$ or $\mathbf{Z}_p$ and satisfies* (3.1) *for $v \in \mathbf{Z}[I]$. Then $S^*_{k,w,I}(N; A)$ is stable under $T_0(\mathfrak{n})$ for all integral ideals $\mathfrak{n}$.*

Proof. By Corollary 4.2, what we have to show is the stability of $S^*_{k,w,I}(N; A)$ under $\mathcal{N}_{F/\mathbf{Q}}(\ell)T_0(\ell, \ell)$ for all ideals ℓ prime to N. We fix a map: $\mathfrak{a} \mapsto \{\mathfrak{a}^v\} \in A$ as in Section 3 and define $T_0(\ell, \ell)$ with respect to this map. For each prime ideal $\mathfrak{p}$ of A, let $A_{\mathfrak{p}}$ be the localization of A at $\mathfrak{p}$. Then $A_{\mathfrak{p}}$ is a valuation ring and $A = \bigcap_{\mathfrak{p}} A_{\mathfrak{p}}$ in the quotient field K of A. We now by definition that

$$\mathrm{S}^*_{k,w,I}(N;A)=\bigcap_{\not{p}}\mathrm{S}^*_{k,w,I}(N;A_{\not{p}})\qquad\text{inside }\mathrm{S}^*_{k,w,I}(N;K).$$

Then the stability of $\mathrm{S}^*_{k,w,I}(N;A)$ under $\mathcal{N}_{F/\mathbf{Q}}(\ell)T_0(\ell,\ell)$ follows from that of $\mathrm{S}^*_{k,w,I}(N;A_{\not{p}})$. Thus we may assume that A is a discrete valuation ring of residual characteristic ℓ. If A is finite flat over $\mathbf{Z}_\ell$, then we can find a finite extension $K_0/\Phi(v)$ inside $\mathbf{C}$ such that the quotient field K of A is the closure of K_0 in $\overline{\mathbf{Q}}_l$. Then

$$\mathrm{S}^*_{k,w,I}(N;A)=\bigoplus_i c_{v,i}\mathrm{S}_{k,\hat{w},I}(\Gamma^i_E(N);A)$$

and

$$\mathrm{S}_{k,\hat{w},I}(\Gamma^i_E(N);A)=\mathrm{S}_{k,\hat{w},I}(\Gamma^i_E(N);K)\cap A[[q]]_i.$$

We put, for each positive integer N_0,

$$\mathcal{M}_k(\Gamma_{00}(N_0;\mathfrak{a}_i^{-1});K)=\mathcal{M}_k(\Gamma_{00}(N_0;\mathfrak{a}_i^{-1});K_0)\otimes_{K_0}K\quad\text{and}$$
$$\mathcal{M}_k(\Gamma_{00}(N_0;\mathfrak{a}_i^{-1});A)=\mathcal{M}_k(\Gamma_{00}(N_0;\mathfrak{a}_i^{-1});K)\cap A[[q]]_i.$$

Now we take a positive integer N_0 contained in the ideal N. By taking $\mathfrak{c}=\mathfrak{a}_i\mathfrak{d}$, we have the inclusion:

$$\begin{array}{lccc} & \mathrm{S}_{k,\hat{w},I}(\Gamma^i_E(N);A)\hookrightarrow\mathcal{M}_k(\Gamma_{00}(N_0;\mathfrak{a}_i^{-1});A) & \cong & \mathcal{M}_k(\Gamma_{00}(N_0),\mathfrak{c};A). \\ (4.14) & \Cup & & \Cup \\ & f(1)=f_{\mathfrak{r},\mathfrak{a}_i^*,\varepsilon_1} & \longleftrightarrow & f\end{array}$$

Note that, for $\mathfrak{a}=a\mathfrak{r}$ $(a\in F_f^\times)$ with $(\mathfrak{a},N_0l)=1$,

$$f|T(\mathfrak{a},\mathfrak{a})(x)=f(xa)=f\left(x\begin{pmatrix}a^2&0\\0&1\end{pmatrix}\begin{pmatrix}a^{-1}&0\\0&a\end{pmatrix}\right)\qquad(f\in\mathrm{S}^*_{k,w,I}(N;\mathbf{C})).$$

Write $s=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}$ and $r=\begin{pmatrix}a^{-2}&0\\0&1\end{pmatrix}$. Then $f|T(a,a)=f|_{k,\hat{w}}sr$. Define

$$V(M)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in V_1(M)\,\middle|\,b\in M\hat{\mathfrak{r}}\right\}\qquad\text{for each ideal }M\text{ of }\mathfrak{r}.$$

For the above s and r, we can find a sufficiently small ideal M so that

$$sV(M)s^{-1}\subset V_1(N_0)\quad\text{and}\quad rV(M)r^{-1}\subset V_1(N_0).$$

We can also decompose by using the same t_i as in (4.3)

$$G_{\mathbf{A}}=\coprod_{i=1}^h GL_2(F)t_iV(M)G_{\infty+}.$$

Note that $\det(s)=1$. Then we can write $t_jsr=\gamma t_iu_1$, $\alpha t_ju_2=t_js$ and $\beta t_iu_3=$

$t_j r$ for $u_1, u_2, u_3 \in V(M)$ and $\alpha, \beta, \gamma \in \mathrm{GL}_2^+(F)$. Then we have that

$$t_j sr = \alpha t_j u_2 r = \alpha t_j r r^{-1} u_2 r = \alpha\beta t_i u_3 r^{-1} u_2 r.$$

Since $u_3 r^{-1} u_2 r \in V_1(N_0)$, by writing f_i for f_{t_i}, we see from (2.6c) that

$$\left(f|_{k,\hat{w}} sr\right)_i = f_j|_{k,\hat{w}}\gamma = \left(f_j|_{k,\hat{w}}\alpha\right)\Big|_{k,\hat{w}}\beta.$$

We may assume that $s \equiv \alpha \bmod MM_2(\hat{\imath})$ and $\det(\alpha) = 1$ by the strong approximation theorem, since $\det(s) = 1$. Then, we see that

$$f_j|_{k,\hat{w}}\alpha = f_j|_k\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix} = f_j(a) \text{ by Theorem 4.9.}$$

On the other hand, we can write $\beta_0 a_i u_0 = a_j a^{-2}$ for $\beta_0 \in F^\times$ and $u_0 \in \hat{\imath}^\times F_{\infty+}^\times$. Then, we can choose $\begin{pmatrix} \beta_0 & 0 \\ 0 & 1\end{pmatrix}$ as β. Thus we see that if $f_j(a) = \sum_\xi a(\xi) q^\xi$,

$$(f|T(a, a))_i = \beta_0^{k-w}\sum_\xi a(\xi) q^{\beta_0\xi}.$$

If $f \in S^*_{k,w,I}(N; A)$, then $f_j \in c_{v,j}\mathscr{M}_k(\Gamma_{00}(N_0), a_j d; A)$ by the inclusion (4.14). Then by Theorem 4.9, we know that

$$(f|T(a, a))_i \in \beta_0^{k-w} c_{v,j} S_{k,\hat{w},I}\left(\Gamma_E^i(N); A\right).$$

Note that $\beta_0 a_i = a_j a^{-2}$ and $c_{v,j} = \mathscr{N}_{F/\mathbf{Q}}(a_j)^{-1}\{(a_j d)^v\}$. Since $\mathscr{N}_{F/\mathbf{Q}}(a)$ is invertible in A (i.e. a is prime to ℓ), we know from a straightforward calculation that

$$\mathscr{N}_{F/\mathbf{Q}}(a)\{a^{-2v}\}\beta_0^{k-w} c_{v,j} A = c_{v,i} A.$$

This shows that if a is prime to ℓN_0, then

$$S^*_{k,w,I}(N; A) \textit{ is stable under } \mathscr{N}_{F/\mathbf{Q}}(a)(a)T_0(a, a).$$

Now write simply m for $[t + n + 2v]$. Since we know from (3.9) that

$$\mathscr{N}_{F/\mathbf{Q}}(a)T_0(a, a) = \mathscr{N}_{F/\mathbf{Q}}(a)^m\{a^{-2v}\}\langle a\rangle_n,$$

and $\langle a\rangle_n$ only depends on the class of a in $\mathrm{Cl}_F(N)$ as shown in Section 3, we can take ideals a, ℓ of $\imath$ such that $a\ell$ is prime to $N_0\ell$, a and ℓ are in the same class in $\mathrm{Cl}_F(N)$ and $\mathscr{N}_{F/\mathbf{Q}}(a)^m\{a^{-2v}\}A + \mathscr{N}_{F/\mathbf{Q}}(\ell)^m\{\ell^{-2v}\}A = A$. Then $T_0(a, a) + T_0(\ell, \ell)$ is a unit multiple of $\langle a\rangle_n$. This shows that under the action of $\mathrm{Cl}_F(N)$, $S^*_{k,w,I}(N; A)$ is stable. For a general ideal a prime to N but not necessarily prime to ℓN_0, we know that

$$\mathscr{N}_{F/\mathbf{Q}}(a)T_0(a, a) = \mathscr{N}_{F/\mathbf{Q}}(a)^m\{a^{-2v}\}\langle a\rangle_n.$$

Since $\mathscr{N}_{F/\mathbf{Q}}(a)^m\{a^{-2v}\}$ is an element in A (even when $-t < n \le 0$; i.e., $0 < k \le 2t$), $S^*_{k,w,I}(N; A)$ is stable under $\mathscr{N}_{F/\mathbf{Q}}(a)T_0(a, a)$, which finishes the proof.

Here we record a byproduct of the proof of Theorem 4.11.

COROLLARY 4.12. *Let A be an $\mathfrak{r}(v)$-algebra as in Theorem* 4.11. *Then* $S^*_{k,w,I}(N; A)$ *is stable under the action of the finite group* $\mathrm{Cl}_F(N)$ *via the operator* $\langle \mathfrak{a} \rangle_n$ *for* $\mathfrak{a} \in \mathrm{Il}(N)$.

For the stability of $S^*_{k,w,I}(N; A)$ under $T_0(\mathfrak{n})$, the assumption that v (and w) is integral and $n \sim -2v$ (and $k \sim 2w$) is absolutely necessary as otherwise one can construct counterexamples [34, Remark 2.9].

5. Duality theorems between Hecke algebras and spaces of cusp forms

Let A be a Dedekind domain containing $\mathfrak{r}(v)$ (inside $\mathbf{C}$ or $\overline{\mathbf{Q}}_p$) satisfying (3.1). Then by Theorem 4.1, $S^*_{k,w,I}(N; A)$ is stable under $T_0(\mathfrak{n})$ for all ideals $\mathfrak{n} \subset \mathfrak{r}$ and hence is stable under $h_{k,w}(N; A)$. We shall define a pairing $\langle\ ,\ \rangle\colon h_{k,w}(N; A) \times S^*_{k,w,I}(N; A) \to A$ by

$$(5.1) \qquad \langle h, f \rangle = a(\mathfrak{r}, f|h).$$

THEOREM 5.1. *The pairing* (5.1) *induces isomorphisms*:

$$\mathrm{Hom}_A\big(h_{k,w}(N; A), A\big) \cong S^*_{k,w,I}\big(N; M_2(F); A\big),$$

$$\mathrm{Hom}_A\big(S^*_{k,w,I}(N; M_2(F); A), A\big) \cong h_{k,w}(N; A).$$

Proof. Firstly, we shall assume A to be a field. Since $S^*_{k,w,I}(N; A)$ and $h_{k,w}(N; A)$ are of finite dimension by Corollary 4.5 or Theorem 3.1, we shall prove the nondegeneracy of the pairing. Suppose $\langle h, f \rangle = 0$ for all h; then from Corollary 4.2,

$$(5.2) \qquad a(\mathfrak{n}, f) = a\big(\mathfrak{r}, f|T_0(\mathfrak{n})\big) = \big\langle T_0(\mathfrak{n}), f\big\rangle = 0 \qquad \text{for all ideals } \mathfrak{n}.$$

Now $f = 0$ by Proposition 4.1. Conversely if $\langle h, f \rangle = 0$ for all f, then for all $\mathfrak{n} \in \mathrm{Il}(1)$,

$$a(\mathfrak{n}, f|h) = a\big(\mathfrak{r}, f|hT_0(\mathfrak{n})\big) = a\big(\mathfrak{r}, f|T_0(\mathfrak{n})h\big) = \big\langle h, f|T_0(\mathfrak{n})\big\rangle = 0.$$

This shows that $f|h = 0$ for all f and hence $h = 0$ as an operator. This finishes the proof in the case where A is a field. For the general case, we may assume that A is a valuation ring by localizing A at prime ideals if necessary. Let L be the quotient field of A. What we have to show is the isomorphism:

$$S^*_{k,w,I}(N; A) \cong \mathrm{Hom}_A\big(h_{k,w}(N; A), A\big).$$

By definition, $h_{k,w}(N; A)$ is an A-subalgebra of $h_{k,w}(N; L)$. Since $h_{k,w}(N; A)$ is finite over A, $h_{k,w}(N; A) \otimes_A L$ is a subalgebra of $h_{k,w}(N; L)$. Thus we can extend any $\phi \in \mathrm{Hom}_A(h_{k,w}(N; A), A)$ to an L-linear map $\phi\colon h_{k,w}(N; L) \to L$.

Then by the duality for L already proved, we can find $f \in S^*_{k,w,I}(N; L)$ such that $\phi(h) = \langle h, f \rangle$ for all $h \in h_{k,w}(N; A)$. Then we know that for all ideals $n \in \mathrm{Il}(1)$,

$$a(n, f) = a(\imath, f|T_0(n)) = \langle T_0(n), f \rangle = \phi(T_0(n)) \in A,$$

since $T_0(n) \in h_{k,w}(N; A)$. This shows that $f \in S^*_{k,w,I}(N; A)$, and the proof is completed.

In the same manner as in [15, §0], we get from Th. 5.1:

COROLLARY 5.2. *Let C denote $\mathbf{C}$ or $\overline{\mathbf{Q}}_p$ according as $A \subset \mathbf{C}$ or $A \subset \overline{\mathbf{Q}}_p$. Then we have bijections*:

$$\begin{aligned}\mathrm{Spec}\big(h_{k,w}(N; A)\big)_{/A}(C) &= \mathrm{Hom}_{A\text{-alg}}\big(h_{k,w}(N; A), C\big)\\ &\cong \big\{f \in S^*_{k,w,I}(N; \mathbf{C}) \big| f|T_0(n) = a(n, f)f \ \ \textit{for all } n\big\},\\ \mathrm{Spec}\big(h^{\mathrm{ord}}_{k,w}(N; A)\big)_{/A}\big(\overline{\mathbf{Q}}_p\big) &= \mathrm{Hom}_{A\text{-alg}}\big(h^{\mathrm{ord}}_{k,w}(N; A), \overline{\mathbf{Q}}_p\big)\\ &\cong \big\{f \in S^*_{k,w,I}(N; \mathbf{C}) \big| f|T_0(n) = a(n, f)f \ \ \textit{with}\\ &\qquad\qquad |a(p, f)|_p = 1\big\}.\end{aligned}$$

Now we fix a finite extension K of $\mathbf{Q}_p$ inside $\overline{\mathbf{Q}}_p$ containing $\Phi(v)$, and let $\mathcal{O}$ denote the p-adic integer ring of K. We fix an ideal N prime to p, and define, for $A = \mathcal{O}$ or K

$$S^*_{k,w,I}(Np^\infty; A) = \varinjlim_{\alpha} S^*_{k,w,I}(Np^\alpha; A).$$

For each element $f \in S^*_{k,w,I}(Np^\infty; K)$, we shall define a p-adic norm by

$$|f|_p = \operatorname*{Sup}_{n} |a(n, f)|_p. \tag{5.3}$$

By Theorem 4.10, $|f|_p$ is a well defined real number. Let $\bar{S}^*_{k,w,I}(Np^\infty; A)$ for $A = \mathcal{O}$ or K be the completion of $S^*_{k,w,I}(Np^\infty; A)$ under this norm. By definition, the function

$$a\colon \mathrm{Il}(1) \times S^*_{k,w,I}(Np^\infty; A) \to A \text{ given by } (a, f) \mapsto a(a, f)$$

is extended by continuity to $\mathrm{Il}(1) \times \bar{S}^*_{k,w,I}(Np^\infty; A)$, and the norm of each element f of $\bar{S}^*_{k,w,I}(Np^\infty; A)$ is again given by (5.3). By the commutativity of (3.5) and by Th. 4.11, $h_{k,w}(Np^\infty; \mathcal{O})$ acts naturally and faithfully on $S^*_{k,w,I}(Np^\infty; \mathcal{O})$ and by continuity, its action extends to $\bar{S}^*_{k,w,I}(Np^\infty; \mathcal{O})$. Thus we can define a pairing

$$\langle\ ,\ \rangle\colon h_{k,w}(Np^\infty; \mathcal{O}) \times \bar{S}^*_{k,w,I}(Np^\infty; \mathcal{O}) \to \mathcal{O} \text{ again by (5.1)}.$$

THEOREM 5.3. *The pairing* $\langle\ ,\ \rangle$ *induces isomorphisms*:

$$h_{k,w}(Np^{\infty};\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}\big(\bar{S}^{*}_{k,w,I}(Np^{\infty};\mathcal{O}),\mathcal{O}\big),$$
$$\bar{S}^{*}_{k,w,I}(Np^{\infty};\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}\big(h_{k,w}(Np^{\infty};\mathcal{O}),\mathcal{O}\big).$$

One can derive this theorem from Theorem 5.1 in exactly the same manner as in [11, II, Th. 1.3]; so, we omit the proof.

COROLLARY 5.4. *If* $v = 0$ *and* $k \geq 2t$ $(k \sim 0)$, *we have a natural isomorphism*:

$$\bar{S}^{*}_{k,w,I}(Np^{\infty};\mathcal{O}) \cong \bar{S}^{*}_{2t,t,I}(Np^{\infty};\mathcal{O})$$

which preserves the map: $(\mathfrak{a}, f) \mapsto a(\mathfrak{a}, f)$. *Moreover, for the ordinary part, we have a similar isomorphism for all pairs* (k, k') *with* $k \sim k'$ *and* $k \geq k' \geq 2t$:

$$e\bar{S}^{*}_{k,w,I}(Np^{\infty};\mathcal{O}) \cong e\bar{S}^{*}_{k',w',I}(Np^{\infty};\mathcal{O}) \qquad (w = v + k - t,\ w' = v + k' - t),$$

which preserves the map: $(\mathfrak{a}, f) \mapsto a(\mathfrak{a}, f)$.

This follows from Theorems 5.3, 3.2 and 3.3. Hereafter identifying $e\bar{S}^{*}_{k,w,I}(Np^{\infty};\mathcal{O})$ for all $n \geq 0$ $(k = n + 2t,\ w = v + k - t = n + v + t)$, we write $S^{\mathrm{ord}}_{v}(N;\mathcal{O})$ for $e\bar{S}^{*}_{k,w,I}(Np^{\infty};\mathcal{O})$ and put $S^{\mathrm{ord}}_{v}(N;\overline{\mathbf{Q}}_p) = S^{\mathrm{ord}}_{v}(N;\mathcal{O}) \otimes_{\mathcal{O}} \overline{\mathbf{Q}}_p$.

COROLLARY 5.5. *We have a bijection*:

$$\begin{aligned}\mathrm{Spec}\big(\mathbf{h}^{\mathrm{ord}}_{v}(N;\mathcal{O})\big)_{/\mathcal{O}}\big(\overline{\mathbf{Q}}_p\big) &= \mathrm{Hom}_{\mathcal{O}\text{-alg}}\big(\mathbf{h}^{\mathrm{ord}}_{v}(N;\mathcal{O}),\overline{\mathbf{Q}}_p\big)\\ &\cong \Big\{f \in S^{\mathrm{ord}}_{v}\big(N;\overline{\mathbf{Q}}_p\big)\Big|\, f|T_0(\mathfrak{n}) = a(\mathfrak{n}, f)f\\ &\qquad \textit{for all } n \in \mathrm{Il}(1)\Big\}.\end{aligned}$$

This follows from Theorem 5.3. For the proof, see the proof of the next theorem whose assertion is a little stronger than Corollary 3.5. We shall prove this by assuming Theorem 3.4 which will be in turn proved in Section 12:

THEOREM 5.6. *Let* $\lambda\colon \mathbf{h}^{\mathrm{ord}}_{v}(N;\mathcal{O}) \to \bar{\mathcal{L}}$ *be a* Λ*-algebra homomorphism*, $\mathcal{K}$ *be the quotient field of the image of* λ *and* $\mathcal{I}$ *be the integral closure of* Λ *in* $\mathcal{K}$. *For each* $P \in \mathcal{X}(\mathcal{I}) = \mathrm{Hom}_{\mathcal{O}\text{-alg}}(\mathcal{I},\overline{\mathbf{Q}}_p)$, *we define* $\lambda_P\colon \mathbf{h}^{\mathrm{ord}}_{v}(N;\mathcal{O}) \to \overline{\mathbf{Q}}_p$ *by* $P \circ \lambda$. *Then there exists a unique p-adic cusp form* $f_P \in S^{\mathrm{ord}}_{v}(N;\overline{\mathbf{Q}}_p)$ *such that* $f_P|T_0(\mathfrak{n}) = \lambda_P(T_0(\mathfrak{n}))f_P$ *and* $a(\mathfrak{n}, f_P) = \lambda_P(T_0(\mathfrak{n}))$ *for all* $\mathfrak{r}$*-ideals* $\mathfrak{n}$. *If* $P \in \mathcal{X}_{\mathrm{alg}}(\mathcal{I})$ *and* $n(P) \geq 2v$, *then* $\lambda_P(T_0(\mathfrak{n}))$ *is an algebraic number in* $\overline{\mathbf{Q}}$ *for all* $\mathfrak{n}$, *and when* $a(\mathfrak{n}, f_P) = \lambda_P(T_0(\mathfrak{n}))$ *as a complex number by the fixed embeddings*: $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$ *and* $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$, f_P *coincides with a complex cusp form in* $S^{*}_{k,w,I}(Np^{\alpha(P)}, \varepsilon_P;\mathbf{C})$ *for* $k = n(P) - 2v + 2t$, $w = n(P) - v + t$. *Conversely, suppose there is a non-zero common eigenform* f *in* $S^{\mathrm{ord}}_{v}(N;\mathcal{O})$ *of all* $T_0(\mathfrak{n})$. *Then*, *there exist a Λ-algebra homomorphism λ: $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \to \bar{\mathcal{L}}$ and a point $P \in \mathrm{Spec}(\mathcal{I})(\mathcal{O})$ such that f is a constant multiple of f_P. If f is a complex cusp form of weight $k \geq 2t$, then P as above belongs to $\mathcal{X}_{\mathrm{alg}}(\mathcal{I})$.*

Proof. By definition, λ_P has values in the p-adic integer ring $\mathcal{O}'$ of a finite extension K'/K. Since $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}') = \mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}'$ by Theorems 3.1 and 3.3, we can extend λ to λ': $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}') \to \bar{\mathcal{L}}$ by the combination:

$$\begin{array}{ccccc} \mathbf{h}_v^{\mathrm{ord}}(\mathrm{N};\mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}' & \xrightarrow{\lambda\otimes \mathrm{id}} & \bar{\mathcal{L}} \otimes_{\mathcal{O}} \mathcal{O}' & \longrightarrow & \bar{\mathcal{L}} \\ & & \Cup & & \Cup \\ & & \mathrm{a}\otimes\mathrm{b} & \longmapsto & \mathrm{ab} \end{array}$$

Let $\mathcal{K}'$ be the subfield of $\bar{\mathcal{L}}$ generated by $\mathcal{O}'$ and $\mathcal{K}$, and let $\mathcal{I}'$ denote the integral closure of Λ in $\mathcal{K}'$. Replacing λ and $\mathcal{I}$ by λ' and $\mathcal{I}'$, we may suppose that λ_P has values in $\mathcal{O}$. Then by Theorem 5.3, we can find $f_P \in S_v^{\mathrm{ord}}(N;\mathcal{O})$ such that $\lambda_P(h) = \langle h, f_P\rangle$ for all $h \in \mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O})$. Then we see that $a(\mathfrak{n}, f_P) = \langle T_0(\mathfrak{n}), f_P\rangle = \lambda_P(T_0(\mathfrak{n}))$ for each ideal $\mathfrak{n}$, and

$$\begin{aligned} a(\mathfrak{m}, f_P|T_0(\mathfrak{n})) &= \langle T_0(\mathfrak{m}), f_P|T_0(\mathfrak{n})\rangle = \langle T_0(\mathfrak{m})T_0(\mathfrak{n}), f_P\rangle = \lambda_P(T_0(\mathfrak{m})T_0(\mathfrak{n})) \\ &= \lambda_P(T_0(\mathfrak{n}))a(\mathfrak{m}, f_P) \end{aligned}$$

for ideals $\mathfrak{m}$ and $\mathfrak{n}$. This shows that $f_P|T_0(\mathfrak{n}) = \lambda_P(T_0(\mathfrak{n}))f_P$. If $P \in \mathcal{X}_{\mathrm{alg}}$ and $n(P) \geq 2v$, then λ_P factors through $h_{k,w}(Np^{\alpha(P)}, \varepsilon_P; K)$ for $k = n(P) - 2v + 2t$ and $w = n(P) - v + t$ by Theorem 3.4 and hence factors through $h_{k,w}(Np^{\alpha(P)}; K)$. Since

$$h_{k,w}(Np^{\alpha(P)}; K) = h_{k,w}(Np^{\alpha(P)}; K_0) \otimes_{K_0} K$$

for a suitable finite extension $K_0/\Phi(v)$ inside K and $h_{k,w}(Np^{\alpha(P)}; K_0)$ is of finite dimension over $\mathbf{Q}$ by Theorem 3.1, the restriction of λ_P to $h_{k,w}(Np^{\alpha(P)}; K_0)$ has values in $\overline{\mathbf{Q}}$; in particular, $\lambda_P(T_0(\mathfrak{n})) \in \overline{\mathbf{Q}}$ for all $\mathfrak{n}$. Then by Corollary 5.2, f_P is contained in $S^*_{k,w,I}(Np^{\alpha(P)}, \varepsilon_P; \mathbf{C})$.

Now we shall show the converse: If $f \in S_v^{\mathrm{ord}}(N;\mathcal{O})$ is a common eigenform of all Hecke operators $T_0(\mathfrak{n})$, then we can define an $\mathcal{O}$-algebra homomorphism μ: $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \to \mathcal{O}$ by $f|h = \mu(h)f$. Then, $\mathrm{Ker}(\mu)$ contains at least one minimal prime ideal $\mathfrak{p}$ of $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O})$; namely, there exist a Λ-algebra homomorphism λ: $\mathbf{h}_v^{\mathrm{ord}}(N;\mathcal{O}) \to \bar{\mathcal{L}}$ with kernel $\mathfrak{p}$ and a point $P \in \mathrm{Spec}(\mathcal{I})(\mathcal{O})$ such that $\mu = P \circ \lambda = \lambda_P$. Then

$$\begin{aligned} a(\mathfrak{n}, f) &= \langle T_0(\mathfrak{n}), f\rangle + \langle T_0(\mathfrak{r}), f|T_0(\mathfrak{n})\rangle = \lambda_P(T_0(\mathfrak{n}))a(\mathfrak{r}, f) \\ &= a(\mathfrak{n}, f_P)a(\mathfrak{r}, f) \qquad \text{for all } \mathfrak{n}. \end{aligned}$$

This shows that $f = a(\mathfrak{r}, f)f_P$, which finishes the proof.

6. A theorem of Matsushima and Shimura

In this section, we shall give an exposition of a result of Matsushima and Shimura [22] which relates the space of cusp forms to the cohomology groups of certain sheaves on modular varieties. In this and the following Sections 7, 8, 9 and 10, we treat general quaternion algebras B/F which may ramify at some places of F. Fix a maximal order R of B as in Section 1. For each place σ of F outside Σ^B, we identify $R_\sigma = R \otimes_{\mathfrak{r}} \mathfrak{r}_\sigma$ with $M_2(\mathfrak{r}_\sigma)$ if σ if finite and with $M_2(\mathbf{R})$ if σ is infinite. Let U be an open compact subgroup of $\hat{R}^\times$. We define a subgroup U_p of G_p^B by $U_p = \{x_p \in G_p^B | x \in U\}$ and $U^p = \{x \in U | x_p = 1\}$. We shall decompose

$$G_{\mathbf{A}}^B = \coprod_{i=1}^{h} G_{\mathbf{Q}}^B t_i U G_{\infty+}^B = \coprod_{i=1}^{h} G_{\mathbf{Q}+}^B t_i U G_\infty^B \qquad \left(G_{\mathbf{Q}+}^B = G_{\mathbf{Q}}^B \cap G_f^B G_{\infty+}^B\right)$$

for $t_i \in G_{\mathbf{A}}^B$ such that $t_{i,\infty} \in G_{\infty+}^B$ and $t_{i,p} \in U_p$. As in Section 2, we put

$$\Gamma^i(U) = t_i U G_{\infty+} t_i^{-1} \cap G_{\mathbf{Q}} = t_i U G_\infty t_i^{-1} \cap G_{\mathbf{Q}+},$$
$$\overline{\Gamma}^i(U) = \Gamma^i(U)/F^\times \cap \Gamma^i(U)$$

which are discrete subgroups of $G_{\infty+}$ and $G_{\infty+}/F_\infty^\times$. We define a complex analytic space $X_i(U) = \Gamma^i(U) \backslash \mathscr{X}_B$ ($\mathscr{X}_B = H^{I_B}$), which is a manifold if $\overline{\Gamma}^i(U)$ is without torsion and is compact if B is a division algebra. We shall construct sheaves on the modular variety $X(U) = G_{\mathbf{Q}+} \backslash G_{\mathbf{A}+}/UC_{\infty+} = G_{\mathbf{Q}} \backslash G_{\mathbf{A}}/UC_{\infty+}$ ($G_{\mathbf{A}} = G_f G_{\infty+}$) out of a right module M of U_p or $G_{\infty+}$. The case where we consider right $G_{\infty+}$-modules M (resp. right U_p-modules M) will be referred to as Case ∞ (resp. Case p). We suppose that

(6.1) *M is a finite dimensional real vector spaces in Case ∞, and M or its Pontryagin dual module is a $\mathbf{Z}_p$-module of finite type in Case p.*

We let $G_{\mathbf{Q}}$ act on $G_{\mathbf{A}} \times M$ from the left by $\alpha \cdot (g, m) = (\alpha g, m)$ and let $UC_{\infty+}$ act on it from the right by

$$(g, m) \cdot u = \begin{cases} (gu, mu_\infty) & \text{in Case } \infty, \\ (gu, mu_p) & \text{in Case } p. \end{cases}$$

Giving M the discrete topology, we consider the covering space of $X(U)$:

$$\mathscr{M}(U) = G_{\mathbf{Q}} \backslash G_{\mathbf{A}} \times M/UC_{\infty+}.$$

We denote by ${}^tM$ the left U_p or $G_{\infty+}$ module whose underlying module is M but whose left action is given by $u \cdot m = m \cdot u^{-1}$. Let $\Gamma^i(U)$ act on ${}^tM$ through the natural inclusion: $\Gamma^i(U) \hookrightarrow U_p$ in Case p and $\Gamma^i(U) \hookrightarrow G_{\infty+}$ in Case ∞. Giving ${}^tM$ the discrete topology, we consider the covering space $\mathscr{M}_i(U) =$

$\Gamma^i(U)\backslash \mathscr{X}_B \times {}^tM$, where we have let $\Gamma^i(U)$ act on $\mathscr{X}_B \times {}^tm$ from the left by the diagonal action.

PROPOSITION 6.1. *Put* $z_0 = (\sqrt{-1}, \dots, \sqrt{-1}) \in \mathscr{X}_B$. *Then we have isomorphisms*

(i) $X(U) \cong \coprod_{j=1}^h X_j(U)$ *induced by* $G_{\mathbf{Q}} t_j U G_{\infty+} \ni \alpha t_j u \mapsto t_{j,\infty} u_\infty(z_0) \in \mathscr{X}_B$,

(ii) $\mathscr{M}(U) \cong \coprod_{j=1}^h \mathscr{M}_j(U)$ *induced by*

$$G_{\mathbf{Q}} t_j U G_{\infty+} \times M \ni (\alpha t_j u, m) \mapsto (t_{j,\infty} u_\infty(z_0), t_{j,\sigma} u_\sigma m) \in \mathscr{X}_B \times {}^tM,$$

where $\sigma = p$ *or* ∞ *according as we are in Case* p *or in Case* ∞.

Proof. The assertion (i) is well-known. We shall prove (ii) only in Case p because Case ∞ can be treated similarly. Define ϕ: $G_{\mathbf{Q}} t_j U G_{\infty+} \times M \to \mathscr{X}_B \times {}^tM$ as in the second assertion. Then, for $\beta \in G_{\mathbf{Q}}$ and $s \in UC_{\infty+}$, we know that

$$\phi(\beta \alpha t_j u s, m_0 \cdot s_p) = (t_{j,\infty} u_\infty(z_0), t_{j,p} u_p s_p(s_p^{-1} m_0)) = \phi(t_j u, m_0).$$

If $\alpha t_j u = t_j u'$ for $u, u' \in UG_{\infty+}$ and $\alpha \in G_{\mathbf{Q}}$, then $\alpha = t_j u' u^{-1} t_j^{-1}$ and thus $\alpha \in \Gamma^j(U)$. Writing $m = t'_{j,p} u_p m_0$ and $z = t_{j,\infty} u_\infty(z_0)$, we have

$$\phi(t_j u', m_0) = (t_{j,\infty} u'_\infty(z_0), t_{j,p} u'_p m_0) = (\alpha(z), \alpha m)$$

$$\text{since } \alpha = t_j u'(t_j u)^{-1}.$$

This shows that ϕ induces the isomorphism in (ii).

If $\bar{\Gamma}^i(U)$ is without torsion and if $\Gamma^i(U) \cap F^\times$ acts trivially on ${}^tM$, $\mathscr{M}_i(U)$ is locally isomorphic to the manifold $X_i(U)$, and we can consider the sheaf of continuous sections of $\mathscr{M}_i(U)$ on $X_i(U)$. This sheaf will be denoted by the same symbol $\mathscr{M}_i(U)$. Let $K/\mathbf{Q}_p$ be a finite extension in $\overline{\mathbf{Q}}_p$ containing K_0 as in (1.1). Let $\mathscr{O}$ denote the p-adic integer ring of K. Then, we consider the sheaves on $X(U)$ or $X_i(U)$ corresponding to the module $L(n, v; A)$ for any $\mathscr{O}$-algebra A in Case p and any $\mathbf{R}$-algebra A containing K_0 in Case ∞. The corresponding sheaf will be denoted by

$$\mathscr{L}(n, v; A)_{/X(U)} \quad \text{and} \quad \mathscr{L}_i(n, v; A)_{/X_i(U)}$$

(under the condition: $n + 2v \sim 0$, $\Gamma^i(U) \cap F^\times$ acts trivially on $L(n, v; A)$ either if $U \subset U_1(p)$ for odd prime p or if $U \subset U_1(4)$). For simplicity, we shall assume that $t_{i,\infty} = 1$ for all i, and for $f \in S_{k,w,J}(U; \mathbf{C})$ ($J \subset I_B$), we write f_i for f_{t_i} as in (2.4b). Then, we know that $f_i \in S_{k,w,J}(\Gamma^i(U); \mathbf{C})$. For each subset J of I_B, we put $\bar{J} = \{\tau \in I_B | \tau \notin J\}$ and $dz_J = (\wedge_{\tau \in J} dz_\tau) \wedge (\wedge_{\tau \in \bar{J}} d\bar{z}_\tau)$ as a differential on $\mathscr{X}_B$. For $\gamma \in G_{\infty+}$,

$$\text{(6.2a)} \qquad dz_J \circ \gamma = j_J(\gamma, z)^{-2t_B} \nu(\gamma)^{t_B} dz_J.$$

We define for $n = \sum_\tau n_\tau \cdot \tau \in \mathbf{Z}[I]$,

$$\eta_n^J(z) = \prod_{\tau \in J} \eta_{n_\tau}(z_\tau) \cdot \prod_{\tau \in \bar{J}} \eta_{n_\tau}(\bar{z}_\tau) \in {}^tL(n_B, v_B; \mathbf{C}) \qquad \text{for } z \in \mathscr{Z}_B \text{ by}$$

$$\eta_{n_\tau}(z_\tau) = \sum_{i=0}^{n_\tau} (-z_\tau)^i X^{n_\tau - i} Y^i = {}^t\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} z_\tau \\ 1 \end{pmatrix}\right)^{n_\tau} \cdot \begin{pmatrix} X_\tau \\ Y_\tau \end{pmatrix}^{n_\tau},$$

where we put $\begin{pmatrix} X \\ Y \end{pmatrix}^m = {}^t(X^m, X^{m-1}Y, \dots, Y^m)$ for $0 \le m \in \mathbf{Z}$. Then we have

(6.2b) $\quad \gamma \cdot \eta_n^{J^\gamma}(z) = \eta_n^J(\gamma(z))\nu(\gamma)^{-n_B} j_{J^\gamma}(\gamma, z)^{n_B} \qquad$ *for all* $\gamma \in G_\infty$.

For each $f \in S_{k,w,J}(\Gamma^i(U); B; \mathbf{C})$, we put

$$\omega(f) = f(z) \cdot \eta_n^J(z)\, dz_J,$$

which is a harmonic r-differential form on $\mathscr{Z}_B$ with values in ${}^tL(n, v; \mathbf{C})$ in the sense of [22]. Since $n = k - 2t$ and $v = w - k + t$, we see easily from (6.2a, b) the $\Gamma^i(U)$-invariancy of $\omega(f)$:

(6.3) $\qquad \omega(f) \circ \gamma = \gamma \cdot \omega(f) \qquad$ *for* $\gamma \in \Gamma^i(U)$.

Thus if $\bar{\Gamma}^i(U)$ is without torsion, we can regard $\omega(f)$ as an r-differential form with values in the sheaf $\mathscr{L}_i(n, v; \mathbf{C})_{/X_i(U)}$. Thus by assigning the de Rham cohomology class of $\omega(f)$ to $f \in S_{k,w,J}(\Gamma^i(U); \mathbf{C})$, we have a morphism:

$$S_{k,w,J}(\Gamma^i(U); \mathbf{C}) \to H^r(X_i(U), \mathscr{L}_i(n, v; \mathbf{C})).$$

When $\bar{\Gamma}^i(U)$ has non-trivial torsion elements, by choosing a sufficiently small normal subgroup Γ of $\bar{\Gamma}^i(U)$ of finite index without torsion, with a slight abuse of symbols, we write, for a field K of characteristic 0,

$$H^r(X_i(U), \mathscr{L}_i(n, v; K)) \qquad \text{for } H^r(\Gamma \backslash \mathscr{Z}_B, \mathscr{L}(n, v; K))^{\Gamma^i(U)},$$

where $\mathscr{L}(n, v; K)_{/(\Gamma \backslash \mathscr{Z}_B)}$ is the sheaf on $\Gamma \backslash \mathscr{Z}_B$ defined in exactly the same manner as $\mathscr{L}_i(n, v; K)_{/X_i(U)}$. This space is determined independently of the choice of Γ. Then, the above map induces a morphism: $S_{k,w,J}(U; \mathbf{C}) \to H^r(X(U), \mathscr{L}(n, v; \mathbf{C}))$. If $r = 2s$ ($r = |I_B| = \dim_{\mathbf{C}} \mathscr{Z}_B$) with $0 < s \in \mathbf{Z}$, we put, for each $J \subset I_B$ with $|J| = s$,

$$\omega_J = \bigwedge_{\tau \in J} \operatorname{Im}(z_\tau)^{-2}\, dz_\tau \wedge d\bar{z}_\tau,$$

$$\operatorname{Inv}(X_j(U)) = \sum_{|J|=s} \mathbf{C}\omega_{J/X_j(U)} \in H^r(X_j(U), \mathbf{C}),$$

$$\operatorname{Inv}(U) = \bigoplus_j \operatorname{Inv}(X_j(U)).$$

When $r = 0$ (i.e., B is totally definite), we denote by $\mathrm{Inv}(U)$ the space defined in (2.4d) under the same symbol. Then we have the following result of Matsushima and Shimura:

THEOREM 6.2 ([22, §4]). *Suppose that $n + 2v \sim 0$ and $n \geq 0$ and that B is a division algebra. Put $k = n + 2t$ and $w = v + k - t$. Then, we have a canonical isomorphism induced by $f \mapsto \omega(f)$:*

$$H^r(X(U), \mathcal{L}(n, v; \mathbf{C}))$$
$$\cong \begin{cases} \mathrm{Inv}(U) \oplus \left(\bigoplus_{J \subset I_B} S_{k,w,J}(U; \mathbf{C}) \right) & \text{if } r \text{ is even and } n = 0, \\ \bigoplus_{J \subset I_B} S_{k,w,J}(U; \mathbf{C}) & \text{if either } r \text{ is odd or } n > 0. \end{cases}$$

In fact, in [22, §4], the case where B is indefinite is studied, but the totally definite case follows from (2.6b). This result has already been generalized by Harder ([5], [6]; see also [7]) even for $B = M_2(F)$ with an appropriate modification, but we will not need this general fact later.

We shall now give a definition of sheaves $\mathcal{L}(n, v; A)$ for global $\imath(v)$-algebras A. Let L be a finite extension of the field K_0 as in Section 1, and let A be the integer ring of L. Since G_f^B naturally acts on $L(n, v; L_f)$ and hence on ${}^tL(n, v; L_f)$ (here L_f is the finite part of the adele ring $L_{\mathbf{A}}$ of L). We can consider $t_i \cdot {}^tL(n, v; \hat{A})$ and

$${}^tL_i(n, v; A) = t_i \cdot {}^tL(n, v; \hat{A}) \cap {}^tL(n, v; L),$$

where $\hat{A}$ denotes the compact ring $A \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$. Then naturally, $\Gamma^i(U) = t_i U G_{\infty+} t_i^{-1} \cap G_{\mathbf{Q}}$ acts on ${}^tL_i(n, v; A)$. If $\overline{\Gamma}^i(U)$ is without torsion, then we can consider the sheaf

$$\mathcal{L}_i(n, v; A) = \overline{\Gamma}^i(U) \backslash \mathbf{Z}_B \times {}^tL_i(n, v; A) \text{ over } X_i(U),$$

$$\mathcal{L}(n, v; A) = \coprod_{i=1}^{h} \mathcal{L}_i(n, v; A) \text{ over } X(U).$$

There is a canonical and functorial isomorphism

$$H^r(X(U), \mathcal{L}(n, v; A)) \cong \bigoplus_{i=1}^{h} H^r\left(\overline{\Gamma}^i(U), {}^tL_i(n, v; A)\right), \tag{6.4}$$

where the right-hand side is the group cohomology group for the $\overline{\Gamma}^i(U)$-module ${}^tL_i(n, v; A)$. Note that the right-hand side of (6.4) is defined without the assumption of torsion-freeness of $\overline{\Gamma}^i(U)$.

For each A-algebra D, we define the sheaf $\mathscr{L}(n, v; D)$ on $X(U)$ by

$$\mathscr{L}(n, v; D)_{/X(U)} = \mathscr{L}(n, v; A) \otimes_A D_{/X(U)}.$$

THEOREM 6.3. *Suppose that $\overline{\Gamma}^i(U)$ is without torsion for all $i = 1, \dots, h$. Let D be an A-algebra. Suppose one of the following conditions holds:*

(i) *D is the integer ring of a finite extension of L;*

(ii) *D is a field extension of L;*

(iii) *D is a localization of A.*

Then, we have a canonical isomorphism:

$$H_c^r(X(U), \mathscr{L}(n, v; D)) \cong H_c^r(X(U), \mathscr{L}(n, v; A)) \otimes_A D,$$

where H_c^r means the cohomology group with compact support.

By this theorem, the natural image of $H_c^r(X(U), \mathscr{L}(n, v; A))$ gives an A-integral structure on $H_c^r(X(U), \mathscr{L}(n, v; \mathbf{C}))$ and on $H_c^r(X(U), \mathscr{L}(n, v; \overline{\mathbf{Q}}_p))$.

Proof. We firstly suppose that D is the localization of A by a multiplicative set $S \subset A$. For each $0 \neq a \in S$, we consider an A-module $a^{-1}A \subset L$. Then we have an exact sequence of A-modules:

$$0 \to L_i(n, v; A) \to L_i(n, v; a^{-1}A) \to L_i(n, v; a^{-1}A/A) \to 0.$$

This gives another exact sequence:

$$\begin{aligned} H_c^{r-1}(X(U), \mathscr{L}(n, v; a^{-1}A/A)) &\to H_c^r(X(U), \mathscr{L}(n, v; A)) \\ &\to H_c^r(X(U), \mathscr{L}(n, v; a^{-1}A)) \\ &\to H_c^r(X(U), \mathscr{L}(n, v; a^{-1}A/A)) \\ &\to \cdots. \end{aligned}$$

By taking the injective limit relative to $a \in S$, we have another exact sequence:

$$\begin{aligned} &\varinjlim_a H_c^{r-1}(X(U), \mathscr{L}(n, v; a^{-1}A/A)) \\ &\quad \to H_c^r(X(U), \mathscr{L}(n, v; A)) \\ &\quad \to H_c^r(X(U), \mathscr{L}(n, v; D)) \\ &\quad \to \varinjlim_a H_c^r(X(U), \mathscr{L}(n, v; a^{-1}A/A)). \end{aligned}$$

Any element in the modules at the extreme right and left of the above sequence is killed by some element in S. Therefore, their tensor product with D becomes trivial. Thus by tensoring D to the above sequence, we have

$$\begin{aligned} H_c^r(X(U), \mathscr{L}(n, v; A)) \otimes_A D &\cong H_c^r(X(U), \mathscr{L}(n, v; D)) \otimes_A D \\ &\cong H_c^r(X(U), \mathscr{L}(n, v; D)), \end{aligned}$$

since D is A-flat. Secondly, let $\wp$ be a prime ideal of A, and let $A_\wp$ be the localization of A at $\wp$. As we have already seen,

$$H_c^r(X(U), \mathscr{L}(n, v; A_\wp)) \cong H_c^r(X(U), \mathscr{L}(n, v; A)) \otimes_A A_\wp.$$

On the other hand, by the universal coefficient theorem in [2, II, Th. 18.3], for any flat $A_\wp$-algebra D, we have

$$\begin{aligned} H_c^r(X(U), \mathscr{L}(n, v; A)) \otimes_A D &\cong H_c^r(X(U), \mathscr{L}(n, v; A_\wp)) \otimes_{A_\wp} D \\ &\cong H_c^r(X(U), \mathscr{L}(n, v; D)). \end{aligned}$$

This result includes the Case (ii). Finally, assuming the first condition (i), we have a natural map: $H_c^r(X(U), \mathscr{L}(n, v; A)) \otimes_A D \to H_c^r(X(U), \mathscr{L}(n, v; D))$. After localizing this map at each prime ideal $\wp$, we get an isomorphism by the results already proved. Therefore the original map must be an isomorphism since these cohomology groups are of finite type as modules over A or D (see e.g. [28]).

The type of sheaves over $X(U)$ discussed here was first considered by Langlands (see e.g. [21]), but the definition of these sheaves using local action of U_p or $G_{\infty+}$ is newly adopted here under some influence of Harder [6, p. 131] and the work of Matsushima and Murakami which precedes [22] and is very well suited to p-adic arguments. We may also note that the definition in Case p reminds us of a work of Weil [37] in the case of GL(1).

7. Hecke operators on cohomology groups and proof of Theorems 3.1 and 4.10

We shall firstly define Hecke operators on $H^r(X(U), \mathscr{L}(n, v; A))$ and on $\oplus_{i=1} H^r(\overline{\Gamma}^i(U), {}^tL_i(n, v; A))$. After that, we shall give a proof of Theorems 3.1 and 4.10. Let σ be either the place p or ∞ of $\mathbf{Q}$. Let Δ be a multiplicative semi-group inside $R_p = R \otimes_{\mathbf{Z}} \mathbf{Z}_p$, and let U and U' be two open compact subgroups of G_f^B such that $U_p \subset \Delta$ and $U_p' \subset \Delta$ if $\sigma = p$. Suppose that $\overline{\Gamma}^i(U)$ and $\overline{\Gamma}^i(U')$ are torsion-free for all i when we consider sheaf cohomology groups. Let M be a right Δ-module or a right G_∞-module according as $\sigma = p$ or ∞ satisfying (6.1). We suppose that E acts trivially on M in order to have the non-trivial sheaf $\mathscr{M}(U)$. Let $x \in G_f C_\infty$ such that $x_p \in \Delta$ if $\sigma = p$. Write $U^x = x_f^{-1} U x_f$ in G_f^B, and put $V = U \cap (U')^{x^{-1}}$. Then $V^x = U^x \cap U'$. Now we shall define a morphism

$$[x]: G_{\mathbf{A}} \times M \to G_{\mathbf{A}} \times M \qquad \text{by } [x](g, m) = (gx, m \cdot x_\sigma).$$

Then we see that, for $\alpha \in G_{\mathbf{Q}}$ and $u \in VC_{\infty+}$,

$$[x](\alpha g u, m u_\sigma) = (\alpha g u x, m u_\sigma x_\sigma) = (\alpha g x u^x, m x_\sigma u_\sigma^x),$$

where $u^x = x^{-1}ux$. Thus the map $[x]$ induces a morphism of sheaves $[x]$: $\mathscr{M}(V) \to \mathscr{M}(V^x)$, and we then have the induced morphism:

$$[x]: H^q(X(V), \mathscr{M}(V)) \to H^q(X(V^x), \mathscr{M}(V^x)). \tag{7.1a}$$

Let pr: $X(V^x) \to X(U')$ be the natural projection, and consider the sheaf $\mathscr{F} = (\mathrm{pr})_*(\mathrm{pr})^*(\mathscr{M}(U')) = (\mathrm{pr})_*(\mathscr{M}(V^x))$. We take a Galois (étale) covering π: $Y \to X(U')$ which factors $X(V^x)$. Put

$$\mathscr{F}' = \pi_*\pi^*(\mathscr{M}(U')),$$
$$\mathscr{G} = \mathrm{Gal}(Y/X(U'))$$

and

$$\mathscr{H} = \mathrm{Gal}(Y/X(V^x)).$$

Since pr is an open map (in fact, it is a local isomorphism), for each open set $O \subset X(U')$, we have that $\Gamma(O, \mathscr{F}) = \Gamma(\mathrm{pr}^{-1}(O), \mathscr{M}(V^x))$. If O is sufficiently small, then there are disjoint open subsets $\{O_\tau\}_{\tau \in \mathscr{G}}$ of Y such that π induces $O_\tau \cong O$ for each τ. Then

$$\mathscr{F}'(O) = \oplus_{\tau \in \mathscr{G}}\pi^*(\mathscr{M}(U'))(O_\tau) \cong \oplus_{\tau \in \mathscr{G}}\Gamma(\mathscr{O}, \mathscr{M}(U''))$$

and

$$H^0(\mathscr{G}, \mathscr{F}'(O)) = \Gamma(O, \mathscr{M}(U')).$$

On the other hand, we have a natural morphism: $\mathscr{F} \to \mathscr{F}'$. For $s \in \mathscr{F}(O)$, we define $\mathrm{Tr}(s) = \sum_{\tau \in \mathscr{G}/\mathscr{H}}\sigma(s) \in H^0(\mathscr{G}, \mathscr{F}'(O))$. This extends to a morphism of sheaves Tr: $\mathscr{F} \to \mathscr{M}(U')$. By definition, $H^q(X(U'), \mathscr{F}) = H^q(X(V^x), \mathscr{M}(V^x))$, and we thus obtain the trace map

$$\mathrm{Tr}_{U'/V^x}: H^q(X(V^x), \mathscr{M}(V^x)) \to H^q(X(U'), \mathscr{M}(U')). \tag{7.1b}$$

We also have the restriction morphism:

$$\mathrm{res}_{U/V}: H^q(X(U), \mathscr{M}(U)) \to H^q(X(V), \mathscr{M}(V)). \tag{7.1c}$$

We shall define $[UxU']$: $H^q(X(U), \mathscr{M}(U)) \to H^q(X(U'), \mathscr{M}(U'))$ by

$$[UxU'] = \mathrm{Tr}_{U'/V^x} \circ [x] \circ \mathrm{res}_{U/V}. \tag{7.2a}$$

Now we shall extend a little the definition of the module $L(n, v; A)$. Let $K/\mathbf{Q}_p$ be a finite extension containing K_0 as in Section 1 and $\mathscr{O}$ be its p-adic integer ring. Suppose that N is prime to p and that every prime factor of Np is unramified in B. Put for $\beta \geq \alpha \geq 0$,

$$\Delta_\beta^\alpha(N) = \left\{x \in \hat{R} \,\middle|\, x_{Np} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } a - 1 \in p^\alpha N\mathfrak{r}_{Np},\ a \in \mathfrak{r}_{Np}^\times,\right.$$
$$\left. c \in p^\beta N\mathfrak{r}_{Np}, \text{ and } x \in G_f^B \right\},$$

$$U_\beta^\alpha(N) = \hat{R}^\times \cap \Delta_\beta^\alpha(N), \qquad V_\beta^\alpha(N) = \hat{R}^\times \cap \Delta_\beta^\alpha(N)^\iota, \qquad \Delta_\beta^\alpha = \Delta_\beta^\alpha(1),$$

where $\Delta^\alpha_\beta(N)^\iota = \{x^\iota | x \in \Delta^\alpha_\beta(N)\}$. Let A be an $\mathscr{O}$-algebra and λ: $Z_\alpha/Z_\beta \to A^\times$ be a character such that $\lambda\chi_{n+2v}$ induces a character of $\overline{Z}_\alpha \to A$. (This is trivially true if α is sufficiently large.) We shall now twist the action of Δ^α_β on $L(n, v; A)$ by the character λ: We let $u \in \Delta^\alpha_\beta$ with $u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ act on $m \in L(n, v; A)$ by

$$m \cdot u = \lambda(a)(m \cdot u_p),$$

where the action of u_p in the parentheses of the right-hand side is the usual action of u_p on $L(n, v; A)$. Similarly, we can let $u \in (\Delta^\alpha_\beta)^\iota$ with $u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ act on $L(n, v; A)$ by $m \cdot u = \lambda(d)(m \cdot u_p)$. The Δ^α_β-module (resp. $(\Delta^\alpha_\beta)^\iota$-module) $L(n, v; A)$ with this twisted action will be denoted by $L(n, v, \lambda; A)$ (resp. $L^*(n, v, \lambda; A)$). Thus we can define the action of $[U^\alpha_\beta(N)xU^\alpha_\beta(N)]$ for $x \in \Delta^\alpha_\beta$ or $[V^\alpha_\beta(N)xV^\alpha_\beta(N)]$ for $x \in (\Delta^\alpha_\beta)^\iota$ on the corresponding cohomology groups. The corresponding sheaves to $L(n, v, \lambda; A)$ and $L^*(n, v, \lambda; A)$ on $X(U)$ ($U \subset U^\alpha_\beta(N)$ or $U \subset V^\alpha_\beta(N)$) will be denoted by $\mathscr{L}(n, v, \lambda; A)$ and $\mathscr{L}^*(n, v, \lambda; A)$.

Let U and U' be two open compact subgroups of Δ^α_β. We shall now modify the operators $[UxU']$ ($x \in \Delta^\alpha_\beta$) analogously to the definition of $T_0(\mathfrak{n})$ out of $T(\mathfrak{n})$. To define the morphism $[x]$ for $x \in \Delta^\alpha_\beta$, we have used the action of x_p on $L(n, v, \lambda; A)$, which will be written as $m \mapsto m \cdot x_p$. Since the underlying A-modules of $L(n, v, \lambda; A)$ and $L(n, 0, \lambda; A)$ are the same, we may use the action of x_p on $L(n, 0, \lambda; A)$, which will be written as $m \mapsto m \circ x_p$, to define a map similar to $[x]$:

$$(x)\colon H^q(X(V), \mathscr{L}(n, v, \lambda; A)) \to H^q(X(V^x), \mathscr{L}(n, v, \lambda; A)).$$

Note that for $x \in \Delta^\alpha_\beta$, $m \cdot x_p = \det(x_p)^v m \circ x_p$. The $\mathscr{O}$-algebra A always satisfies the condition (3.1) since $\mathscr{O}$ does. We fix a character: $F_f^\times \ni a \mapsto \{a^v\} \in A$ as in Section 3. Then $\{\nu(x)^{-v}\}\det(x_p)^v$ is a unit in A, and thus we can define a morphism of sheaves

$$(x)\colon \mathscr{L}(n, v, \lambda; A)_{/X(V)} \to \mathscr{L}(n, v, \lambda; A)_{/X(V^x)}$$

by the correspondence: $(g, m) \mapsto (gx, (\{\nu(x)^{-v}\}\det(x_p)^v)(m \circ x_p))$ for $g \in G^B_{\mathbf{A}}$ and $m \in L(n, v, \lambda; A)$. This induces a linear map

$$(x)\colon H^q(X(V), \mathscr{L}(n, v, \lambda; A)) \to H^q(X(V), \mathscr{L}(n, v, \lambda; A)).$$

We then define (UxU'): $H^q(X(U), \mathscr{L}(n, v, \lambda; A)) \to H^q(X(U'), \mathscr{L}(n, v, \lambda; A))$ by

$$(UxU') = \mathrm{Tr}_{U'/V^x} \circ (x) \circ \mathrm{res}_{U/V}. \tag{7.2b}$$

One can formulate the above definitions for $L^*(n, v, \lambda; A)$ and $L^*(n, v, \lambda; A)$ for $x \in (\Delta^\alpha_\beta)^\iota$ and $U, U' \subset (\Delta^\alpha_\beta)^\iota$ in exactly the same manner as above:

$$(UxU')\colon H^q(X(U), \mathscr{L}^*(n, v, \lambda; A)) \to H^q(X(U'), \mathscr{L}^*(n, v, \lambda; A)).$$

Then we have the relation

(7.3) $\quad [UxU'] = \{\nu(x)^v\}(UxU') \qquad \text{for all } x \in \Delta^\alpha_\beta \quad \text{or} \quad x \in (\Delta^\alpha_\beta)^\iota.$

The operators $[UxU']$ and (UxU') depend only on the double coset $UC_{\infty+}xU'C_{\infty+}$ (and the choice of the map: $a \mapsto \{a^v\}$) and are independent of the choice of x in $UC_{\infty+}xU'C_{\infty+}$. Furthermore, one can verify

$$[U'xU'']\circ[UyU'] = [UyU' \cdot U'xU''], \qquad (U'xU'')\circ(UyU') = (UyU' \cdot U'xU''),$$

where the product on the right-hand side is taken in the abstract Hecke ring as in [36, III]. By decomposing

$$\{x \in \Delta^\alpha_\beta(N) | \nu(x)\mathfrak{r} = \mathfrak{n}\} = \coprod_i U^\alpha_\beta(N)x_iU^\alpha_\beta(N),$$

we can define Hecke operators $T(\mathfrak{n})$ and $T_0(\mathfrak{n})$ by

$$T(\mathfrak{n}) = \sum_i [U^\alpha_\beta(N)x_iU^\alpha_\beta(N)], \qquad T_0(\mathfrak{n}) = \sum_i (U^\alpha_\beta(N)x_iU^\alpha_\beta(N))$$

$$\text{on } H^q(X(U^\alpha_\beta(N)), \mathscr{L}(n, v, \lambda; A)).$$

So far, we have only considered sheaves defined locally at σ. Now we shall extend our definitions of operators (UxU') to global rings A. Let L be a finite extension of K_0, and let A be the integer ring of L. We suppose the condition (3.1) for A. Decompose $G^B_{\mathbf{A}} = \coprod_j G^B_{\mathbf{Q}} t_j V G^B_{\infty+}$ and $G^B_{\mathbf{A}} = \coprod_i G^B_{\mathbf{Q}} t_i' V^x G^B_{\infty+}$. Suppose that $t_{j,\infty} = t'_{i,\infty} = 1$. We put ${}^tL_j(n, v; A) = t_j \cdot {}^tL_j(n, v; \hat{A}) \cap {}^tL(n, v; L)$ and ${}^tL_i'(n, v; A) = t_i' \cdot {}^tL(n, v; \hat{A}) \cap {}^tL(n, v; L)$, where the intersection is taken in ${}^tL(n, v; L_f)$ and $\hat{A} = A \otimes_{\mathbf{Z}} \hat{\mathbf{Z}} \subset L_f$. Expressing $t_jx = \gamma_j t_i' u^x$ for $x \in (G_f \cap \hat{R})C_\infty$ with $u \in VG_{\infty+}$ and $\gamma_j \in G_{\mathbf{Q}}$, we have that $\gamma_j = t_j u^{-1} x t_i'^{-1} \in t_j \hat{R} t_i'^{-1}$, and thus γ_j induces a linear map:

$$\begin{array}{ccc} {}^tL_j(n, v; A) & \longrightarrow & {}^tL_i'(n, v; A) \\ \Cup & & \Cup \\ m & \longrightarrow & \gamma_j^{-1}m = m \cdot \gamma_j. \end{array}$$

Note that

$$\Gamma^i(V^x) = t_i' V^x G_{\infty+} t_i'^{-1} \cap G_{\mathbf{Q}} = \gamma_j^{-1} t_j V G_{\infty+} t_j^{-1} \gamma_j \cap G_{\mathbf{Q}} = \gamma_j^{-1}\Gamma^j(V)\gamma_j.$$

Then we have morphisms of sheaves:

$$[x]_j \colon \mathscr{L}_j(n, v; A)_{/X_j(V)} \to \mathscr{L}_i'(n, v; A)_{/X_i(V^x)}$$

$$\text{induced by } (z, m) \mapsto (\gamma_j^{-1}(z), \gamma_j^{-1}m),$$

$$(x)_j \colon \mathscr{L}_j(n, v; A)_{/X_j(V)} \to \mathscr{L}_i'(n, v; A)_{/X_i(V^x)}$$

$$\text{induced by } (z, m) \mapsto (\gamma_j^{-1}(z), \{\nu(x)^{-v}\}(\gamma_j^{-1}m)).$$

Here, actually, $\{\nu(x)^{-v}\}(\gamma_j^{-1}m) = (\{\nu(x)^{-v}\}\nu(\gamma_j)^v(m \circ \gamma_j))$ is a well-defined element of ${}^tL'_i(n, v; A)$ since $(\{\nu(x)^{-v}\}\nu(\gamma_j)^v)$ is a unit in A. We also have morphisms of group cohomology:

$$[x]_j\colon H^q\big(\overline{\Gamma}^j(V), {}^tL_j(n, v; A)\big) \to H^q\big(\overline{\Gamma}^i(V^x), {}^tL'_i(n, v; A)\big)$$

$$(x)_j\colon H^q\big(\overline{\Gamma}^j(V), {}^tL_j(n, v; A)\big) \to H^q\big(\overline{\Gamma}^i(V^x), {}^tL'_i(n, v; A)\big)$$

which are given by

$$\xi\big|[x]_j\big(\alpha_0, \ldots, \alpha_q\big) = \gamma_j^{-1}\xi\big(\gamma_j\alpha_0\gamma_j^{-1}, \ldots, \gamma_j\alpha_q\gamma_j^{-1}\big),$$

$$\xi\big|(x)_j\big(\alpha_0, \ldots, \alpha_q\big) = \big(\{\nu(x)^{-v}\}\nu(\gamma_j)^v\big)\xi\big(\gamma_j\alpha_0\gamma_j^{-1}, \ldots, \gamma_j\alpha_q\gamma_j^{-1}\big) \circ \gamma_j$$

for each q-homogeneous cocycle $\xi\colon \overline{\Gamma}^j(U) \to {}^tL_j(n, v; A)$.

In order to compare this definition of $[x]_j$ and $(x)_j$ with those of $[x]$ and (x), we denote by A_σ the complex field $\mathbf{C}$ or the closure of A in $\overline{\mathbf{Q}}_p$ according as $\sigma = \infty$ or $\sigma = p$. We can define the sheaf $\mathscr{L}(n, v; A_\sigma)_{/X(U)}$ locally at σ as in the beginning of this section. By Proposition 6.1, we have natural isomorphisms:

$$\begin{array}{ccc} \phi\colon\ G_{\mathbf{Q}} \backslash G_{\mathbf{Q}} t_j V G_{\infty+} \times L(n, v; A_\sigma)/VC_{\infty+} & \cong & \Gamma^j(V) \backslash \mathscr{Z}_B \times {}^tL(n, v; A_\sigma) \\ \Cup & & \Cup \\ (\alpha t_j u, m_0) & \longmapsto & \big(t_{j,\infty}u_\infty(z_0), t_{j,\sigma}u_\sigma m_0\big), \end{array}$$

$$\begin{array}{ccc} \phi'\colon G_{\mathbf{Q}} \backslash G_{\mathbf{Q}} t'_i V^x G_{\infty+} \times L(n, v; A_\sigma)/V^x C_{\infty+} & \cong & \Gamma^i(V^x) \backslash \mathscr{Z}_B \times {}^tL(n, v; A_\sigma) \\ \Cup & & \Cup \\ (\alpha t'_i u^x, m_0) & \longmapsto & \big(t'_{i,\infty}u^x_\infty(z_0), t'_{i,\sigma}u^x_\sigma m_0\big). \end{array}$$

Since $t_j x = \gamma_j t'_i u^x$, we have for any $u' \in VG_{\infty+}$, $t_j u' x = \gamma_j t'_i (uu')^x$ and thus $G_{\mathbf{Q}} t_j V G_{\infty+} x = G_{\mathbf{Q}} t'_i V^x G_{\infty+}$. This means that the map

$$[x]\colon \mathscr{L}(n, v; A_\sigma)_{/X(V)} \to \mathscr{L}(n, v; A_\sigma)_{/X(V^x)}$$

induces a morphism $[x]_j\colon \mathscr{L}_j(n, v; A_\sigma)_{/X_j(V)} \to \mathscr{L}_i(n, v; A_\sigma)_{/X_i(V^x)}$. We shall compute $\phi' \circ [x] \circ \phi^{-1}$. To do this, write

$$z = t_{j,\infty}u'_\infty(z_0)$$

and

$$m = t_{j,\sigma}u'_\sigma m_0.$$

Then

$$\phi(g, m_0) = (z, m) \qquad \text{for } g = \alpha t_j u' \ (\alpha \in G_{\mathbf{Q}} \text{ and } u' \in VG_{\infty+}).$$

On the other hand, we see that

$$\begin{aligned}\phi'([x](g, m_0)) &= \phi'(\alpha t_j u'x, m_0 x_\sigma)\\ &= \phi'\big(\alpha\gamma_j t_i'(uu')^x, m_0 x_\sigma\big)\\ &= \Big(\big(t_i'(uu')^x\big)_\infty(z_0), \big(t_i'(uu')^x\big)_\sigma \cdot x_\sigma^{-1} m_0\Big) = \big(\gamma_j^{-1}(z), \gamma_j^{-1} m\big),\end{aligned}$$

since $t_j u'x = \gamma_j t_i'(uu')^x$. This shows

(7.4a) $$\phi' \circ [x] = [x]_j \circ \phi \quad \textit{and} \quad \phi' \circ (x) = (x)_j \circ \phi.$$

Similarly, when $\bar{\Gamma}^j(U)$ and $\bar{\Gamma}^i(U')$ are torsion-free, we have a commutative diagram:

(7.4b)
$$\begin{array}{ccc} H^q(X_j(V), \mathscr{L}_j(n, v; A)) & \cong & H^q(\bar{\Gamma}^j(V), {}^tL(n, v; A)) \\ \Big\downarrow{\scriptstyle [x]_j \text{ (resp. } (x)_j)} & & \Big\downarrow{\scriptstyle [x]_j \text{ (resp. } (x)_j)} \\ H^q(X_i(V^x), \mathscr{L}_i'(n, v; A)) & \cong & H^q(\bar{\Gamma}^i(V^x), {}^tL_i'(n, v; A)), \end{array}$$

where the horizontal isomorphisms are the canonical ones (see e.g. [28, §2]).

Since the covering π_x: $X(V^x) \to X(U')$ and π: $X(V) \to X(U)$ are étale finite and $\pi_x^*(\mathscr{L}(n, v; A)_{/X(U')}) = \mathscr{L}(n, v; A)_{/X(V^x)}$, we have

$$\mathrm{Tr}_{U'/V^x}: H^q(X(V^x), \mathscr{L}(n, v; A)) \to H^q(X(U'), \mathscr{L}(n, v; A)),$$
$$\mathrm{res}_{U/V}: H^q(X(U), \mathscr{L}(n, v; A)) \to H^q(X(V), \mathscr{L}(n, v; A)).$$

Define $[x]$ and (x): $H^q(X(V), \mathscr{L}(n, v; A)) \to H^q(X(V^x), \mathscr{L}(n, v; A))$ by $[x] = \oplus_j [x]_j$ and $(x) = \oplus_j (x)_j$, and put

$$[UxU'] = \mathrm{Tr}_{U'/V^x} \circ [x] \circ \mathrm{res}_{U/V}, \qquad (UxU') = \mathrm{Tr}_{U'/V^x} \circ (x) \circ \mathrm{res}_{U/V}.$$

Then we see from (7.4) the compatibility between the previous definition and the new one.

Without assuming the torsion-freeness of $\bar{\Gamma}^j(U)$ and $\bar{\Gamma}^i(U')$, we can define

$$[UxU'] \quad \text{and} \quad (UxU'): \bigoplus_j H^q\big(\bar{\Gamma}^j(U), {}^tL_j(n, v; A)\big)$$
$$\to \bigoplus_i H^q\big(\bar{\Gamma}^i(U'), {}^tL_i'(n, v; A)\big).$$

When B is totally definite, the non-trivial cohomology group is obtained only when $q = 0$, and the space of cohomology groups as above is nothing but the space of functions on $G_{\mathbf{A}}^B$ with values in $L(n, v; A)$ on G_f^B satisfying (2.4a) by (2.6b), and the operators $[UxU']$ and (UxU') can be defined as in Sections 2 and

3. Thus we may suppose that B is indefinite. We further suppose that $\nu(V) = \nu(U) = \nu(U')$. Then we can choose t_i so that

$$G_{\mathbf{A}}^B = \coprod_i G_{\mathbf{Q}} t_i V^x G_{\infty+} = \coprod_i G_{\mathbf{Q}} t_i V G_{\infty+} = \coprod_i G_{\mathbf{Q}} t_i U G_{\infty+} = \coprod_i G_{\mathbf{Q}} t_i U' G_{\infty+},$$

simultaneously. Then $X_i(V^x)$ covers $X_i(U')$, and $\overline{\Gamma}^i(V^x)$ is of finite index in $\overline{\Gamma}^i(U')$. Thus we have the transfer map and the restriction map:

$$T_{\overline{\Gamma}^i(U')/\overline{\Gamma}^i(V^x)}\colon H^q\big(\overline{\Gamma}^i(V^x), {}^tL_i(n,v;A)\big) \to H^q\big(\overline{\Gamma}^i(U'), {}^tL_i(n,v;A)\big),$$

$$\mathrm{res}_{\overline{\Gamma}^i(U)/\overline{\Gamma}^i(V)}\colon H^q\big(\overline{\Gamma}^i(U), {}^tL_i(n,v;A)\big) \to H^q\big(\overline{\Gamma}^i(V), {}^tL_i(n,v;A)\big).$$

We define

$$T_{U'/V^x} = \bigoplus_j T_{\overline{\Gamma}^j(U')/\overline{\Gamma}^j(V^x)} \quad \text{and} \quad \mathrm{res}_{U/V} = \bigoplus_i \mathrm{res}_{\overline{\Gamma}^i(U)/\overline{\Gamma}^i(V)}$$

and

$$[UxU'] = T_{U'/V^x} \circ [x] \circ \mathrm{res}_{U/V}, \qquad (UxU') = T_{U'/V^x} \circ (x) \circ \mathrm{res}_{U/V}.$$

Then by (7.4b), we have a commutative diagram:

$$\begin{array}{ccc} H^q(X(U), \mathscr{L}(n,v;A)) & \cong & \bigoplus_i H^q(\overline{\Gamma}^i(U), {}^tL_i(n,v;A)) \\ \Big\downarrow {\scriptstyle [UxU'] \text{ (resp. } (UxU'))} & & \Big\downarrow {\scriptstyle [UxU'] \text{ (resp. } (UxU'))} \\ H^q(X(U'), \mathscr{L}(n,v;A)) & \cong & \bigoplus_i H^q(\overline{\Gamma}^i(U'), {}^tL_i(n,v;A)). \end{array} \tag{7.4c}$$

To prove Theorems 3.1 and 4.10, we need several lemmas.

LEMMA 7.1. *Put $U(N) = \{x \in U_1(N)$ with $x_N = \begin{pmatrix} a & b \\ c & d \end{pmatrix} | d - 1 \in N\mathfrak{r}_N\}$ for each ideal N of $\mathfrak{r}$ outside Σ^B. Let ℓ be a prime ideal of $\mathfrak{r}$ with residual characteristic ℓ outside Σ^B, and let e be the ramification index of ℓ over $\mathbf{Q}$. Then we have*

(i) *If $s > 2e/(\ell - 1)$, then $\overline{\Gamma}^i(U(\ell^s))$ is torsion-free for all i.*

(ii) *For each ideal N as above, if there exists a prime ideal ℓ such that ℓ^s exactly divides N for $s > 2e/(\ell - 1)$, then the order of every torsion element in $\overline{\Gamma}_1^i(N)$ is a divisor of $\mathscr{N}_{F/\mathbf{Q}}(\ell)^{s-1}(\mathscr{N}_{F/\mathbf{Q}}(\ell) - 1)$, where $\overline{\Gamma}_1^i(N)$ stands for $\overline{\Gamma}^i(U_1(N))$.*

(iii) *There are infinitely many square-free ideal N outside Σ^B, such that $\overline{\Gamma}_0^i(N) = \overline{\Gamma}^i(U_0(N))$ is torsion-free for all i. We can choose N so that the residual characteristic of each prime factors of N is arbitrarily large.*

Proof. We first prove the third assertion. We shall identify R_ℓ with $M_2(\mathfrak{r}_\ell)$. If the image of $\gamma \in \Gamma_0^i(1)$ in $\overline{\Gamma}_0^i(1) = \Gamma_0^i(1)/\Gamma_0^i(1) \cap F^\times$ is of order n, then

$\delta = \nu(\gamma)^{-1}\gamma^2$ is of order n or $n/2$ in $\Gamma_0^i(1)$ and $\nu(\delta) = 1$. We suppose for the moment that $\delta \neq \pm 1$. Then $F(\delta) \subset B$ is a quadratic extension of F. We shall choose a prime ideal ℓ which remains prime in $F(\delta)$ and so that the ℓ-adic integer ring of $F(\delta)$ is generated by $\mathfrak{r}_\ell$ and δ. This condition on ℓ depends only on the order n but is independent of the choice of the element δ of order n, and there are infinitely many ℓ with this property. If $\delta \in \Gamma_0^i(\ell)$, then as an element of $M_2(\mathfrak{r}_\ell) = R_\ell$, δ leaves the subspace $\{{}^t(x,0) | x \in \mathfrak{r}/\ell\} \subset (\mathfrak{r}/\ell)^2$ stable. This is impossible because $\mathfrak{r}_\ell[\delta]/\ell$ is a quadratic extension of $\mathfrak{r}/\ell$. Hence $\Gamma_0^i(\ell) \not\ni \delta$. Here we have implicitly chosen t_i so that $t_{i,\ell} = 1$. This is always possible. For each root of unity ζ such that $[F(\zeta) : F] = 2$, we choose distinct prime ideals ℓ_ζ such that ℓ_ζ remains prime in $F(\zeta)$, $\ell_\zeta \notin \Sigma^B$ and ζ and $\mathfrak{r}_{\ell_\zeta}$ generate the ℓ_ζ-adic integer ring of $F(\zeta)$. Since the number of roots of unity ζ with $[F(\zeta) : F] = 2$ is finite, we may put $N_0 = \prod_\zeta \ell_\zeta$, where ζ runs over all such roots of unity. If $\gamma \in \Gamma_0^i(N_0)$ satisfies $\gamma^n = \varepsilon \in \mathfrak{r}^\times$, then the above argument shows that we may assume that $n = 2$. If $\varepsilon = \eta^2$ with $\eta \in \mathfrak{r}^\times$, then $\delta = \eta^{-1}\gamma$ is of order 2. Thus δ is conjugate to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in $M_2(\mathbf{C})$ if one embeds B into $M_2(\mathbf{C})$. Since $\nu(\delta) = 1$, we know that $\delta = -1$ and δ becomes trivial in $\overline{\Gamma}_0^i(N_0)$, and γ also becomes trivial. If $\varepsilon \notin (\mathfrak{r}^\times)^2$, then $F(\gamma) \subset B$ is a quadratic extension of F. The number of distinct quadratic extensions of the form $F(\sqrt{\varepsilon})$ for $\varepsilon \in \mathfrak{r}^\times$ is equal to $|\mathfrak{r}^\times/(\mathfrak{r}^\times)^2|$. For each $\varepsilon \in \mathfrak{r}^\times/(\mathfrak{r}^\times)^2$, we choose a prime ideal ℓ_ε such that $\ell_\varepsilon \notin \Sigma^B$, ℓ_ε remains prime in $F(\sqrt{\varepsilon})$, ℓ_ε is prime to N_0 and the ℓ_ε-adic integer ring of $F(\sqrt{\varepsilon})$ is generated by $\mathfrak{r}_{\ell_\varepsilon}$ and $\sqrt{\varepsilon}$. Put $N = N_0 \cdot \prod_\varepsilon \ell_\varepsilon$. We can choose ℓ_ζ and ℓ_ε so that their residual characteristics are arbitrarily large. If $\varepsilon' = \eta^2\varepsilon$ with $\eta \in \mathfrak{r}^\times$, then $\mathfrak{r}_{\ell_\varepsilon} + \sqrt{\varepsilon'}\,\mathfrak{r}_{\ell_\varepsilon} = \eta(\mathfrak{r}_{\ell_\varepsilon} + \sqrt{\varepsilon}\,\mathfrak{r}_{\ell_\varepsilon}) = \mathfrak{r}_{\ell_\varepsilon} + \sqrt{\varepsilon}\,\mathfrak{r}_{\ell_\varepsilon}$. Thus the condition on ℓ_ε depends only on the class of ε in $(\mathfrak{r}^\times/(\mathfrak{r}^\times)^2)$. Then an argument similar to the case ℓ_ζ prohibits the existence of a non-trivial torsion element in $\overline{\Gamma}_0^i(N)$. This proves the third assertion.

Now we shall prove the first assertion. Put

$$X = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathfrak{r}_\ell) \middle| c \in \ell^s,\ a \equiv d \equiv 1 \bmod \ell^s \right\},$$

$$\mathscr{X} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathfrak{r}_\ell) \middle| a, c, d \in \ell^s \right\}.$$

We consider the exponential map exp: $\mathscr{X} \to X$ and the logarithm log: $X \to \mathscr{X}$ defined by $\exp(T) = \sum_{n=0}^\infty T^n/n!$ and $\log(1 + T) = \sum_{n=1}^\infty (-1)^{n+1}T^n/n$. Let us firstly check the ℓ-adic convergence of these series. We fix a prime element π of $\mathfrak{r}_\ell$ and take a quadratic extension $A = \mathfrak{r}_\ell[\sqrt{\pi}]$, which is a complete valuation ring. Put $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\pi}^{\,s} \end{pmatrix}$ as an element of $M_2(A)$. Then we see that

$\alpha^{-1}\mathscr{X}\alpha \subset \sqrt{\pi}^{\,s}M_2(A)$. Thus if $s > 2e/\ell - 1$ ($2e$ is the ramification index of $A/\mathbf{Z}_\ell$), the convergence of exp: $\alpha^{-1}\mathscr{X}\alpha \to \alpha^{-1}X\alpha$ and log: $\alpha^{-1}X\alpha \to \alpha^{-1}\mathscr{X}\alpha$ follows from a standard argument (see e.g. [17, §3, Lemma 3]). Since

$$\exp(\alpha^{-1}x\alpha) = \alpha^{-1}\exp(x)\alpha \text{ and } \log(\alpha^{-1}x\alpha) = \alpha^{-1}\log(x)\alpha,$$

we have well-defined exp: $\mathscr{X} \to X$ and log: $X \to \mathscr{X}$. Since exp and log are mutually inverse, we know that $\mathscr{X} \cong X$. Since $\log(x^n) = n\log(x)$ for each integer n, we know that X is torsion-free. One can always decompose

$$G_{\mathbf{A}}^B = \coprod_i G_{\mathbf{Q}} t_i U G_{\infty +} \qquad \textit{for } U = U(\ell^s) \quad \textit{with } t_i \textit{ satisfying } t_{i,\ell} = 1;$$

thus, $\Gamma^i(U)$ can be embedded into X and hence $\Gamma^i(U)$ is torsion-free. Note that $\overline{X} = X/X \cap \mathfrak{r}_\ell^\times$ is also an ℓ-adic Lie group and is torsion-free. Since $\overline{\Gamma}^i(U)$ is isomorphic to the image of $\Gamma^i(U)$ in $\overline{X}$, $\overline{\Gamma}^i(U)$ is also torsion-free. By Proposition 6.1, the torsion-freeness of $\overline{\Gamma}^i(U)$ for all i does not depend on the choice of t_i, and hence the first assertion follows.

Finally we shall prove the second assertion. We have an exact sequence for $U = U(\ell^s)$: $1 \to U_\ell/U_\ell \cap \mathfrak{r}_\ell^\times \to U_1(\ell^s)_\ell/U_\ell \cap \mathfrak{r}_\ell^\times \to (\mathfrak{r}/\ell^s\mathfrak{r})^\times \to 1.$

$$\begin{array}{ccc} \Psi & & \Psi \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & & ad^{-1} \bmod \ell^s \end{array}$$

If $T \subset \overline{\Gamma}_1^i(N)$ is a finite subgroup, T injects into $(\mathfrak{r}/\ell^s\mathfrak{r})^\times$ since $U_\ell/U_\ell \cap \mathfrak{r}_\ell^\times$ is torsion-free. Thus we see that T is a divisor of $\mathscr{N}(\ell)^{s-1}(\mathscr{N}(\ell) - 1)$ which is the order of $(\mathfrak{r}/\ell^s\mathfrak{r})^\times$.

Proof of Theorems 3.1 *and* 4.10. We shall prove the two theorems simultaneously. By (2.6c), (6.3) and (7.4a), the isomorphism of Theorem 6.2 is equivariant under the abstract Hecke ring $R(U_1(N), \Delta_1(N))$. When $F = \mathbf{Q}$, the assertions of Theorems 3.1 and 4.10 are well-known (see e.g. [36, III]). Thus we may assume that $F \neq \mathbf{Q}$. Then we can find a division quaternion algebra B/F which is unramified everywhere at finite places of F. We fix such a B. Let K_0 be the finite extension of $\mathbf{Q}$ as in (1.1) for this B. Firstly, we suppose that the algebra A as in Theorem 3.1 contains the integer ring $\mathfrak{r}_0$ of K_0. Let L be the quotient field of A. We put for each open subgroup U of $\mathrm{GL}_2(\hat{\mathfrak{r}})$,

$$S(U) = \bigoplus_{J \subset I_B} S_{k,w,J}(U; B; \mathbf{C})$$

and identify $S(U)$ with the subspace of $H^r(X(U), \mathscr{L}(n, v; \mathbf{C}))$. We take a normal open compact subgroup U of $U_1(N)$ such that $\overline{\Gamma}^i(U)$ is torsion-free for all

i. Then, by using Hochschild-Serre spectral sequence [16], we know

$$(7.5)\quad H^r(X(U), \mathscr{L}(n, v; L))^{U_1(N)} \cong \Big(\bigoplus_i H^r\big(\overline{\Gamma}^i(U), {}^tL_i(n, v; L)\big)\Big)^{U_1(N)}$$
$$\cong \Big(\bigoplus_j H^r\big(\overline{\Gamma}_1^j(N), {}^tL_j(n, v; L)\big)\Big),$$

where we have written $\overline{\Gamma}_1^j(N)$ for $\overline{\Gamma}^j(U_1(N))$. Since on

$$\bigoplus_j H^r\big(\overline{\Gamma}_1^j(N), {}^tL_j(n, v; L)\big),$$

$T_{U_1(N)/U} \circ \mathrm{res}_{U_1(N)/U}$ is a multiplication of the degree $(X(U): X(U_1(N)))$, Theorem 6.3 combined with (7.5), shows that

$$\Big(\bigoplus_i H^r\big(\overline{\Gamma}_1^i(N), {}^tL_i(n, v; A)\big)\Big) \otimes_A \mathbf{C} \cong H^r\big(X(U_1(N)), \mathscr{L}(n, v; \mathbf{C})\big).$$

Let $H = H(A)$ be the image of $\oplus_i H^r(\overline{\Gamma}_1^i(N), {}^tL_i(n, v; A))$ in $S = S(U_1(N))$ under the isomorphism of Theorem 6.2. In the same manner as in the proof of [7, 4.6], we can show that $H \otimes_A \mathbf{C} = S$ even when r is even and $n = 0$. Since H is stable under the action of $R(U_1(N), \Delta_1(N))$, we know that $h_{k,w}(N; A) \hookrightarrow \mathrm{End}_A(H)$. Thus $h_{k,w}(N; A)$ is a flat A-module of finite type. For any A-algebra D inside $\mathbf{C}$, $H \otimes_A D$ is a D-submodule of S stable under $h_{k,w}(N; D)$ and hence

$$h_{k,w}(N; D) \hookrightarrow \mathrm{End}_A(H \otimes_A D) \cong \mathrm{End}_A(H) \otimes_A D$$

since D is A-flat and H is A-projective. Since $h_{k,w}(N; D)$ is generated over D by $T_0(n)$ for all n, we know from this fact

$$h_{k,w}(N; D) \cong h_{k,w}(N; A) \otimes_A D.$$

Before proving Theorem 3.1 in general, we shall prove Theorem 4.10. By the duality in Theorem 5.1, applying the above identity to $D = K_0$ and with A as in Theorem 4.10 for $K = K_0$, we know that

$$S^*_{k,w,I}(N; M_2(F); K_0) \cong S^*_{k,w,I}(N; M_2(F); A) \otimes_A K_0.$$

On the other hand, by Corollary 4.5, we know that

$$S^*_{k,w,I}(N; M_2(F); K_0) \cong S^*_{k,w,i}(N; M_2(F); \Phi(v)) \otimes_{\Phi(v)} K_0.$$

In order to treat the general case where A is an arbitrary $\imath(v)$-subalgebra of a finite extension $K/\Phi(v)$, we change the notation and denote by L the quotient field of A; so, $L \subset K$. If there is an element $f \in S^*_{k,w,I}(N; L)$, we can find, by the above facts, $a \in L^\times$ such that $af \in S^*_{k,w,I}(N; A)$. This shows that

$$S^*_{k,w,I}(N; M_2(F); K) \cong S^*_{k,w,I}(N; M_2(F); L) \otimes_L K$$
$$= S^*_{k,w,I}(N; M_2(F); A) \otimes_A K,$$

and thus Theorem 4.10 follows. Replacing H and S by $S^*_{k,w,I}(N;A)$ and $S^*_{k,w,I}(N;\mathbf{C})$, we see that the reasoning which proves Theorem 3.1 in the case: $A \supset \mathfrak{r}_0$ still works well in the general case of $A \supset \mathfrak{r}(v)$; so, we now obtain the theorem.

Now we shall determine the structure of cohomology groups $H^r(X(U_1(N)), \mathscr{L}(n, v; K))$ as modules over the Hecke algebra when K is a field.

THEOREM 7.2. *Let B be a division quaternion algebra over F unramified at every finite place, and put $r = |I_B| = \dim_{\mathbf{C}} \mathscr{X}_B$. For each field extension K/K_0 for the field K_0 as in (1.1), $H^r(X(U_1(N)), \mathscr{L}(n, v; K))$ is free of rank 2^r over $h_{k,w}(N;K)$ if either $k > 2t$ or r is odd. (Here we have used the notation of Theorem 6.2 and $H^r(X(U_1(N)), \mathscr{L}(n, v; K))$ means $\oplus_i H^r(\overline{\Gamma}_1^i(N), {}^tL(n, v; K))$ when $\overline{\Gamma}_1^i(N)$ has non-trivial torsion elements.)*

Proof. Let $H = H(A)$ be as in the proof of Theorems 3.1 and 4.10. Then $H(K_0)$ is stable under the action of $[U_1(N)xU_1(N)]$ for all $x \in G_f^B C_\infty$. Put $C = C_\infty / C_{\infty+}$, which is isomorphic to $\{\pm 1\}^{I_B}$ as a group. Then C acts on $H(K_0)$ via $[U_1(N)cU_1(N)]$ for $c \in C_\infty$. For each character $\varepsilon\colon C \to \{\pm 1\}$ and for each subalgebra A of $\mathbf{C}$, we define

$$H_\varepsilon(A) = \{m \in H(A) | m|c = \varepsilon(c)m \text{ for all } c \in C\}.$$

Since C acts on the subsets of I_B transitively via $J \to J^c$ as in (2.2a) and since the action of $c \in C$ induces an isomorphism: $S_{k,w,J}(U) \cong S_{k,w,J^c}(U)$ by Theorem 2.2, we know that

$$S_{k,w,I_B}(N;B;\mathbf{C}) \cong H_\varepsilon(\mathbf{C}) \cong H_\varepsilon(K_0) \otimes_{K_0} \mathbf{C} \text{ as } h_{k,w}(N;\mathbf{C})\text{-modules}$$

via $f \mapsto \sum_{s \in C} \varepsilon(s)\omega(f)|s$. We can define a non-degenerate pairing

$$[\ ,\ ]\colon L(n^B, v^B; \mathbf{C}) \times L(n^B, v^B; \mathbf{C}) \to \mathbf{C}$$

such that $[x\gamma, y\gamma] = [x, y]$ for $\gamma \in \overline{\Gamma}_1^i(N)$ (cf. [29, II], [36, 8.2], [35, (1.2a, b)]). Then for $f \in S_{k,w,I_B}(N;B;\mathbf{C})$ and $g \in S_{k,w,\phi}(N;B;\mathbf{C})$, we define

$$(f, g) = \sum_i \int_{X_i(U_1(N))} [f_1(z), g_i(z)] \cdot \mathrm{Im}(z)^{k_B}\, d\mu(z),$$

where $f_i = f_{t_i}$ and $g_i = g_{t_i}$ as in (2.4b) and

$$d\mu(z) = \mathrm{Im}(z)^{-2t_B} \cdot \textstyle\prod_{\sigma \in I_B} |dz_\sigma \wedge d\bar{z}_\sigma|.$$

Now let ω be an element of G_f^B such that $\omega_N = \begin{pmatrix} 0 & -1 \\ \nu & 0 \end{pmatrix}$ for $\nu \in \mathfrak{r}_N$ with $\nu\mathfrak{r}_N = N\mathfrak{r}_N$ and $\omega_\sigma = 1$ for all places σ outside N. We decompose

$$S_{k,w,J}(N;B;\mathbf{C}) = \bigoplus_\psi S_{k,w,J}(N, \psi; \mathbf{C}),$$

where ψ runs over characters of $\mathrm{Cl}_F(N)$ and

$$S_{k,w,J}(N, \psi; \mathbf{C}) = \left\{f \in S_{k,w,J}(N; B; \mathbf{C}) \middle| f|\langle a\rangle_n = \psi(a)f \text{ for } a \in \mathrm{Il}(N)\right\}.$$

For each $f \in S_{k,w,J}(N, \psi; \mathbf{C})$, regarding ψ as an idele character, we define

$$(f|W)(x) = \bar{\psi}(\nu(x))f(x\omega).$$

Then it is known that W gives an automorhpism of $S_{k,w,J}(N; B; \mathbf{C})$ (see e.g. [35, Lemma 1.3], [8, 3.9]) and if we put $\langle f, g\rangle = (f, g|W)$, then it satisfies $\langle f|h, g\rangle = \langle f, g|h\rangle$ for $h \in h_{k,w}(N; \mathbf{C})$ (e.g. [14, Lemma 6.4], [15, §3]). This shows that

$$S_{k,w,I_B}(N; B; \mathbf{C}) \cong \mathrm{Hom}_{\mathbf{C}}\big(S_{k,w,\phi}(N; B; \mathbf{C}), \mathbf{C}\big) \text{ as an } h_{k,w}(N; \mathbf{C})\text{-module.}$$

On the other hand, $S_{k,w,\phi}(N; B; \mathbf{C}) \cong S^*_{k,w,I}(N; M_2(F); \mathbf{C})$ as an $h_{k,w}(N; \mathbf{C})$-module by Theorems 2.1 and 2.2. and Proposition 2.3. By Theorem 5.1, we know

$$S^*_{k,w,I}(N; M_2(F); \mathbf{C}) \cong \mathrm{Hom}_{\mathbf{C}}\big(h_{k,w}(N; \mathbf{C}), \mathbf{C}\big) \text{ as an } h_{k,w}(N; \mathbf{C})\text{-module.}$$

These facts show that, as $h_{k,w}(N; K_0)$-modules,

$$H_\epsilon(K_0) \otimes_{K_0} \mathbf{C} \cong S_{k,w,I_B}(N; \mathbf{C}) \cong h_{k,w}(N; \mathbf{C}) \cong h_{k,w}(N; K_0) \otimes_{K_0} \mathbf{C}.$$

Hence we know

(7.6) $\qquad H_\epsilon(K_0) \cong h_{k,w}(N; K_0)$ *as* $h_{k,w}(N; K_0)$*-modules.*

Then we conclude the assertion of the theorem by extending the scalar field to K from K_0.

Remark 7.3. We have proved actually a little stronger result than the statement of Theorem 7.2. Namely, *$H(K)$ for any field extension K/K_0 is free of rank 2^r over $h_{k,w}(N; K)$, even when r is even and $k = 2t$.*

We shall record a byproduct of the proof of Theorem 7.2 (cf. [14, Lemma 6.4]):

COROLLARY 7.4. *$h_{k,w}(N; \Phi(v))$ is a Frobenius algebra over $\Phi(v)$.*

8. Comparison between cohomology groups of different weights

Let U be an open compact subgroup of $\hat{R}^\times$ and suppose:

(8.1a) $\qquad$ *Every prime factor of p is unramified in B.*

We suppose the following condition when we consider sheaf cohomology groups:

(8.1b) $\qquad$ *$\bar{\Gamma}^i(U)$ is torsion-free for all i.*

Now we shall define several morphisms between cohomology groups. Almost all results here will be formulated for the sheaf cohomology groups $H^q(X(U), \mathscr{L}(n, v, \lambda; A))$ under the assumptions (8.1a, b) but can be naturally reformulated in a standard manner in terms of group cohomology: $\oplus_i H^q(\overline{\Gamma}^i(U), {}^tL_i(n, v, \lambda; A))$, and then the result will be valid without the assumption (8.1b). The interpretation is automatic and is left to the reader.

By definition, $L(n, v, \lambda; A)$ as in Section 7 is the space of homogeneous polynomials of variables $(X_\sigma, Y_\sigma)_{\sigma \in I}$. We evaluate each polynomial $P(X_\sigma, Y_\sigma) \in L(n, v, \lambda; A)$ at $(X_\sigma, Y_\sigma) = (1, 0)$ for all $\sigma \in I$, and we then have a morphism of an A-module:

$$i\colon L(n, v, \lambda; A) \to A.$$

Let K be a finite extension of $\mathbf{Q}_p$ containing K_0 as in (1.1), and let $\mathscr{O}$ be the p-adic integer ring of K. Let $\lambda\colon Z_\alpha/Z_\beta \to \mathscr{O}^\times$ be a character for integers $\beta \geq \alpha \geq 0$. For each $k \in \mathbf{Z}[I]$, we occasionally denote by χ_k the character:

$$\mathfrak{r}_p^\times \ni x \mapsto x^k = \prod_{\sigma \in I} (x^\sigma)^{k_\sigma} \in \mathscr{O}.$$

This notation is consistent with the character $\chi_k\colon Z(N) \to \mathbf{Z}_p^\times$ already defined in Section 3 when $k \sim 0$. Since $u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (for $u \in U_0(p^\beta)$) satisfies the congruence: $u_p \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \bmod p^\beta M_2(\mathfrak{r}_p)$ and its action on $L(n, v, \lambda; A)$ for $A = \mathscr{O}/p^\beta\mathscr{O}$ or $p^{-\beta}\mathscr{O}/\mathscr{O}$ factors through the matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathfrak{r}/p^\beta\mathfrak{r})$, we know that for $u \in U_\beta^\alpha(1)$,

$$i(P \cdot u_p) = \lambda\chi_n(a)\nu(u_p)^v i(P) \qquad \text{for } P \in L(n, v, \lambda; A).$$

Then i induces a morphism of sheaves for all $U \subset U_\beta^\alpha(1)$:

$$i\colon \mathscr{L}(n, v, \lambda; A)_{/X(U)} \to \mathscr{L}(0, v, \lambda\chi_n; A)_{/X(U)}$$

and hence we have, for $A = \mathscr{O}/p^\beta\mathscr{O}$ or $p^{-\beta}\mathscr{O}/\mathscr{O}$,

(8.2a) $i_*\colon H^q(X(U), \mathscr{L}(n, v, \lambda; A)) \to H^q(X(U), \mathscr{L}(0, v, \lambda\chi_n; A))$.

Here we have abused symbols slightly. In fact, $\lambda\chi_n$ is not necessarily a character of Z_α/Z_β with values in $\mathscr{O}/p^\beta\mathscr{O}$, but we can let $u \in \Delta_\beta^\alpha$ with $u_p = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ act on $L(0, v; A)$ by $P \cdot u = \lambda\chi_n(a)(P \cdot u_p)$, where $P \cdot u_p$ on the right-hand side denotes the original action of u_p on $L(0, v; A)$. Each element in the center $a \in \mathfrak{r}_p^\times$ with $a \equiv 1 \bmod p^\alpha\mathfrak{r}_p$ acts then on $L(0, v, \lambda\chi_n; A)$ by $\lambda(a)\chi_{n+2v}(a)$; and hence E acts trivially on this module. Thus if the sheaf $\mathscr{L}(n, v, \lambda; A)$ is well-defined on $X(U)$, then $\mathscr{L}(0, v, \lambda\chi_n; A)_{/X(U)}$ is also well-defined. As with i above, identifying $L^*(0, v, \lambda\chi_n; A)$ with A (for $A = \mathscr{O}/p^\beta\mathscr{O}$ or $p^{-\beta}\mathscr{O}/\mathscr{O}$), one can define another $V_\beta^\alpha(1)$-morphism

$$j\colon L^*(0, v, \lambda\chi_n; A) \to L^*(n, v, \lambda; A)$$

by $j(a) = aY^n = a \cdot \prod_{\sigma \in I} Y_\sigma^{n_\sigma}$. This map induces, for $A = \mathcal{O}/p^\beta\mathcal{O}$ or $p^{-\beta}\mathcal{O}/\mathcal{O}$

(8.2b) $j_*\colon H^q(X(U), \mathcal{L}^*(0, v, \lambda\chi_n; A)) \to H^q(X(U), \mathcal{L}^*(n, v, \lambda; A))$

if $U \subset V_\beta^\alpha(1)$.

Now we take $\omega = \omega_\beta \in G_f^B$ such that $\omega_p = \begin{pmatrix} 0 & 1 \\ -p^\beta & 0 \end{pmatrix}$ and $\omega_\sigma = 1$ for all places σ outside p. We note the identity inside $M_2(\imath_p)$:

$$\begin{pmatrix} a & b \\ p^\beta c & d \end{pmatrix} \omega_p = \omega_p \begin{pmatrix} d & -c \\ -p^\beta b & a \end{pmatrix}.$$

Let $\phi\colon L(0, v, \lambda\chi_n; A) \to L^*(0, v, \lambda\chi_n; A)$ be the identity map of the underlying space A. Then if $A = \mathcal{O}/p^\beta\mathcal{O}$ or $p^{-\beta}\mathcal{O}/\mathcal{O}$, $\phi(m \cdot u_p) = \phi(m)(\omega^{-1}u\omega)_p$ for $u \in U_\beta^\alpha(1)$ and $m \in L(0, v, \lambda\chi_n; A)$. Therefore, the map

$$\begin{array}{ccc} [\omega]\colon G_{\mathbf{A}} \times L(0, v, \lambda\chi_n; A) & \to & G_{\mathbf{A}} \times L^*(0, v, \lambda\chi_n; A) \\ \Downarrow & & \Downarrow \\ (g, m) & \longmapsto & (g\omega, \phi(m)) \end{array}$$

induces a morphism of sheaves

$$[\omega]\colon \mathcal{L}(0, v, \lambda\chi_n; A)_{/X(U)} \to \mathcal{L}^*(0, v, \lambda\chi_n; A)_{/X(U^\omega)}$$

if $U \subset U_\beta^\alpha(1)$. Thus, if $U \subset U_\beta^\alpha(1)$ and $A = \mathcal{O}/p^\beta\mathcal{O}$ or $p^{-\beta}\mathcal{O}/\mathcal{O}$,

(8.2c) $$W = [\omega]\colon H^q(X(U), \mathcal{L}(0, v, \lambda\chi_n; A)) \to H^q(X(U^\omega), \mathcal{L}^*(0, v, \lambda\chi_n; A)).$$

Let δ be an element of G_f^B such that $\delta_p = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\delta_\sigma = 1$ for all other places σ. Then for $V = U^\omega \cap \delta U \delta^{-1}$, if $U \subset \Delta_\beta^\alpha$, then $V \subset (\Delta_\beta^\alpha)^\iota$ and $V^\delta = \delta^{-1}U^\omega\delta \cap U \subset \Delta_\beta^\alpha$. We consider the map

$$[\delta]\colon L^*(n, v, \lambda; A) \to L(n, v, \lambda; A)$$

defined by $m \mapsto \phi(m \cdot \delta)$, where δ acts on $L(n, v, \lambda; A)$ through the identification of the underlying space $L(n, v, \lambda; A) = L(n, v; A)$ and ϕ is induced by the identity map of the underlying space $L(n, v; A)$. Then we see that $[\delta](m \cdot u) = ([\delta](m)) \cdot u^\delta$ for $u \in V$ ($u^\delta = \delta^{-1}u\delta$). Thus this induces a morphism of sheaves

$$[\delta]\colon \mathcal{L}^*(n, v, \lambda; A)_{/X(V)} \to \mathcal{L}(n, v, \lambda; A)_{/X(V^\delta)}.$$

Therefore, we can define

$$[U^\omega\delta U]\colon H^q(X(U^\omega), \mathcal{L}^*(n, v, \lambda; A)) \to H^q(X(U), \mathcal{L}(n, v, \lambda; A))$$

by

(8.2d) $$[U^\omega\delta U] = \mathrm{Tr}_{U/V^\delta} \circ [\delta] \circ \mathrm{res}_{U^\omega/V}.$$

We now suppose that

$$(8.3)\quad U_p = R_p^\times \quad and \quad U = U_p \times U^p \qquad for\ U_p = \{x_p | x \in U\} \quad and$$
$$U^p = \{x \in U | x_p = 1\}.$$

We put $U_\beta^\alpha = U \cap U_\beta^\alpha(1)$ for integers $\beta \geq \alpha \geq 0$. For each $a \in \imath_p$, we define $\delta_a \in G_f^B$ by $(\delta_a)_p = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ and $(\delta_a)_\sigma = 1$ for σ outside p. Then we see

$$(8.4)\qquad (U_\beta^\alpha)^\omega \delta U_\beta^\alpha = \coprod_a (U_\beta^\alpha)^\omega \delta\delta_a \qquad if\ \beta > 0,$$

where a runs over a complete representative set for $\imath_p / p^\beta \imath_p$. Then we shall define morphisms (for $A = \mathcal{O}/p^\beta\mathcal{O}$ or $p^{-\beta}\mathcal{O}/\mathcal{O}$):

$$\pi = \pi_\beta\colon H^q\big(X(U_\beta^\alpha), \mathscr{L}(0, v, \lambda\chi_n; A)\big) \to H^q\big(X(U_\beta^\alpha), \mathscr{L}(n, v, \lambda; A)\big),$$
$$\iota = \iota_\beta\colon H^q\big(X(U_\beta^\alpha), \mathscr{L}(n, v, \lambda; A)\big) \to H^q\big(X(U_\beta^\alpha), \mathscr{L}(0, v, \lambda\chi_n; A)\big)$$

by $\iota = i_*$ and $\pi = [(U_\beta^\alpha)^\omega \delta U_\beta^\alpha] \circ j_* \circ W$. We fix a character of semi-group: $\text{Il}(1) \ni a \mapsto \{a^v\} \in \mathcal{O}$ as in Section 3 and define operators $(U_\beta^\alpha x U_\beta^\alpha)$ for $x \in \Delta_\beta^\alpha$ as in Section 7. Then the importance of the morphisms ι and π comes from the following result:

THEOREM 8.1. *Let $A = \mathcal{O}/p^\beta\mathcal{O}$ or $p^{-\beta}\mathcal{O}/\mathcal{O}$. For each pair of integers (α, β) with $\beta \geq \alpha \geq 0$ and $\beta > 0$,*

$$\pi \circ \iota = p^{-\beta v}\{p^{\beta v}\} T_0(p^\beta)\ on\ H^q\big(X(U_\beta^\alpha), \mathscr{L}(n, v, \lambda; A)\big),$$
$$\iota \circ \pi = p^{-\beta v}\{p^{\beta v}\} T_0(p^\beta)\ on\ H^q\big(X(U_\beta^\alpha), \mathscr{L}(0, v, \lambda\chi_n; A)\big),$$

where $T_0(p^\beta) = (U_\beta^\alpha x U_\beta^\alpha)$ for x with $x_p = \begin{pmatrix} 1 & 0 \\ 0 & p^\beta \end{pmatrix}$ and $x_\sigma = 1$ for σ outside p. Furthermore ι is equivariant under the operators $(U_\beta^\alpha y U_\beta^\alpha)$ for $y \in \Delta_\beta^\alpha$ on both the cohomology groups. These assertions are also valid for cohomology groups with compact support.

We note that $p^{-\beta v}\{p^{\beta v}\}$ is a unit in $\mathcal{O}$, and therefore, $\iota \circ \pi$ and $\pi \circ \iota$ are unit multiplies of $T_0(p^\beta)$.

Proof. We shall prove the assertions only for the usual cohomology groups since the case of compact support can be handled in exactly the same manner. Let $\phi\colon L(n, v, \lambda; A) \to L^*(n, v, \lambda; A)$ be the identity map of the underlying space $L(n, v; A)$. Let $m \mapsto m \circ \omega$ denote the action of ω on $L(n, 0; A)$. Identifying $L(n, v, \lambda; A)$ and $L^*(n, v, \lambda; A)$ with $L(n, 0; A)$ as A-modules naturally, we define

$$(\omega)_0\colon G_{\mathbf{A}} \times L(n, v, \lambda; A) \to G_{\mathbf{A}} \times L^*(n, v, \lambda; A)$$
$$\text{by } (\omega)_0(g, m) = (g\omega, \phi(m \circ \omega)).$$

Then we can easily check that $(\omega)_0$ induces a morphism of sheaves

$$(\omega)_0\colon \mathscr{L}(n, v, \lambda; A)_{/X(U^\alpha_\beta)} \to L^*(n, v, \lambda; A)_{/X((U^\alpha_\beta)^\omega)}$$

and a morphism of cohomology groups

$$(\omega)_0\colon H^q\Big(X\big(U^\alpha_\beta\big), \mathscr{L}(n, v, \lambda; A)\Big) \to H^q\Big(X\Big(\big(U^\alpha_\beta\big)^\omega\Big), \mathscr{L}^*(n, v, \lambda; A)\Big).$$

Since $\omega_p\binom{x}{y} = \binom{y}{0}$ on $(A^I)^2$, we have $j(i(m)) = m \circ \omega_p$ for $m \in L(n, v, \lambda; A)$. Then we see that

$$j_* \circ W \circ i_* = (\omega)_0\colon H^q\Big(X\big(U^\alpha_\beta\big), \mathscr{L}(n, v, \lambda; A)\Big)$$
$$\to H^q\Big(X\Big(\big(U^\alpha_\beta\big)^\omega\Big), \mathscr{L}^*(n, v, \lambda; A)\Big).$$

This shows that

$$\pi \circ \iota = \Big[\big(U^\alpha_\beta\big)^\omega \delta U^\alpha_\beta\Big] \circ (\omega)_0 = \big(p^{-\beta v}\{p^{\beta v}\}\big)\big(U^\alpha_\beta \omega\delta U^\alpha_\beta\big) = \big(p^{-\beta v}\{p^{\beta v}\}\big)T_0(p^\beta),$$

because $x = \omega\delta \in \Delta^\alpha_\beta$. Now we shall prove the second identity about $\iota \circ \pi$. We have by definition that $\iota \circ \pi = i_* \circ [(U^\alpha_\beta)^\omega \delta U^\alpha_\beta] \circ j_* \circ W$. Put $V = U^\alpha_\beta \cap xU^\alpha_\beta x^{-1}$ for x as in the theorem. Then we see that $V^x = x^{-1}Vx = \delta^{-1}(U^\alpha_\beta)^\omega\delta \cap U^\alpha_\beta$. We write simply S for U^α_β. Then by (8.2d), the above expression of $\iota \circ \pi$ shows that

$$\iota \circ \pi = i_* \circ \mathrm{Tr}_{S/V^x} \circ [\delta] \circ \mathrm{res}_{S^\omega/V^{x\delta}} \circ j_* \circ [\omega]$$
$$= \mathrm{Tr}_{S/V^x} \circ i_* \circ [\delta] \circ j_* \circ \mathrm{res}_{S^\omega/V^{x\delta}} \circ [\omega].$$

The commutativity of i_* and Tr follows from the fact that $\Delta^\alpha_\beta \supset S \supset V^x$ and that i is a morphism of Δ^α_β-modules. Since $x = \omega\delta$, we see that

$$V = S \cap xSx^{-1} = S \cap \omega\delta S\delta^{-1}\omega^{-1} = \omega(S^\omega \cap \delta S\delta^{-1})\omega^{-1} = \omega V^{x\delta}\omega^{-1}.$$

By definition, we have that $\mathrm{res}_{S^\omega/V^{x\delta}} \circ [\omega] = [\omega] \circ \mathrm{res}_{S/V}$. Thus we have that

$$\iota \circ \pi = \mathrm{Tr}_{S/V^x} \circ i_* \circ [\delta] \circ j_* \circ [\omega] \circ \mathrm{res}_{S/V}.$$

Note that $i(j(m) \circ \delta) = i(mY^n \circ \delta) = i(mX^n) = m$ for $m \in L(0, v, \lambda\chi_n; A)$. Thus, as a morphism of sheaves, we see $i \circ [\delta] \circ j$ is given by

$$\mathbf{A} \times L(0, v, \lambda\chi_n; A) \to G_{\mathbf{A}} \times L^*(0, v, \lambda\chi_n; A)$$
$$(g, m) \to (g\delta, \phi(m)).$$

This shows that $\{p^{-\beta v}\}p^{\beta v}(\iota \circ \pi) = \mathrm{Tr}_{S/V^x} \circ (\omega\delta) \circ \mathrm{res}_{S/V} = T_0(p^\beta)$. As for the last assertion, for $y \in \Delta^\alpha_\beta$, we see $\iota \circ (SyS) = (SyS) \circ \iota$ because i is a morphism of Δ^α_β-modules.

We shall now define the ordinary part of cohomology groups. It is known (cf. [28, Propositions 4, 9 and 18]) that

(8.5) *$H^q(X(U), \mathscr{L}(n, v, \lambda; A))$ is of finite type as an $\mathscr{O}$-module if A is an $\mathscr{O}$-module of finite type.*

When $B = M_2(F)$, the space $X(U)$ is not compact; so, we denote by

$H_P^q(X(U), \mathscr{L}(n, v, \lambda; A))$ the image of $H_c^q(X(U), \mathscr{L}(n, v, \lambda; A))$ inside the usual cohomology group $H^q(X(U), \mathscr{L}(n, v, \lambda; A))$. By (8.5), the parabolic cohomology group $H_P^q(X(U), \mathscr{L}(n, v, \lambda; A))$ is also an $\mathscr{O}$-module of finite type. Let $T_0(p^\beta)$ be the operator as in Th. 8.1. Since the $\mathscr{O}$-linear endomorphism algebra of the cohomology groups H^r or H_P^r over $X(U_\beta^\alpha)$ with coefficients in $\mathscr{L}(n, v, \lambda; A)$ for $A = \mathscr{O}/p^\beta\mathscr{O}$ or $p^{-\beta}\mathscr{O}/\mathscr{O}$ is finite and of p-power torsion over $\mathscr{O}$, the limit

$$E = \lim_{\alpha\to\infty} T_0(p^\beta)^{p^{m\alpha}}$$

exists in the endomorphism algebra for a suitable choice of a positive integer m, and $e = E^{p^m-1}$ becomes an idempotent. This idempotent is determined independently of the choice of m (cf. [10, p. 236]) and depends only on $T_0(p)$.

COROLLARY 8.2. *The morphism ι in Theorem 8.1 induces an isomorphism:*

$$eH^q\big(X\big(U_\beta^\alpha\big), \mathscr{L}(n, v, \lambda; A)\big) \cong eH^q\big(X\big(U_\beta^\alpha\big), \mathscr{L}(0, v, \lambda\chi_n; A)\big)$$

for $A = \mathscr{O}/p^\beta\mathscr{O}$ or $p^{-\beta}\mathscr{O}/\mathscr{O}$.

The same type of assertion is also valid for parabolic cohomology groups.

Proof. We have a commutative diagram:

$$\begin{tikzcd} H^q(X(U_\beta^\alpha),\mathscr{L}(n, v, \lambda; A)) \arrow[r, "\iota"] \arrow[d, "T"] & H^q(X(U_\beta^\alpha),\mathscr{L}(0, v, \lambda\chi_n; A)) \arrow[d, "T"] \arrow[dl, "\pi"] \\ H^q(X(U_\beta^\alpha),\mathscr{L}(n, v, \lambda; A)) \arrow[r, "\iota"] & H^q(X(U_\beta^\alpha),\mathscr{L}(0, v, \lambda\chi_n; A)), \end{tikzcd}$$

where $T = \varepsilon T_0(p^\beta)$ with $\varepsilon = p^{-\beta v}\{p^{\beta v}\}$. Note that ε is a unit in $\mathscr{O}$ by definition. Therefore, if p^m denotes the cardinality of the residue field of $\mathscr{O}$, then we have $\lim_{\alpha\to\infty}\varepsilon^{p^{\alpha m}(p^m-1)} = 1$. Thus we know that for a suitable multiple m' of m,

$$e = \lim_{\alpha\to\infty} T^{p^{m'\alpha}(p^{m'}-1)} = \lim_{\alpha\to\infty} T_0\big(p^\beta\big)^{p^{m'\alpha}(p^{m'}-1)}.$$

Thus T is invertible on

$$eH^q\big(X\big(U_\beta^\alpha\big), \mathscr{L}(n, v, \lambda; A)\big) \text{ and } H^q\big(X\big(U_\beta^\alpha\big), \mathscr{L}(0, v, \lambda\chi_n; A)\big).$$

This combined with the above diagram shows the result.

PROPOSITION 8.3. *If λ is a character of Z_α/Z_β for $\beta \geq \alpha \geq 0$ with $\beta > 0$, then the restriction map induces an isomorphism for each $\gamma \geq \beta$:*

$$eH^q\big(X\big(U_\beta^\alpha\big), \mathscr{L}(n, v, \lambda; A)\big) \cong eH^q\big(X\big(U_\gamma^\alpha\big), \mathscr{L}(n, v, \lambda; A)\big)$$

for $A = \mathscr{O}$, $\mathscr{O}/p^m\mathscr{O}$, $p^{-m}\mathscr{O}/\mathscr{O}$ or $K/\mathscr{O}$.

The same type of assertion also holds for the parabolic cohomology groups.

Proof. Take $x \in G_f^B$ such that $x_p = \begin{pmatrix} 1 & 0 \\ 0 & p^\delta \end{pmatrix}$ with $\delta = \gamma - \beta$ and $x_\sigma = 1$ for σ outside p. Then, we claim that

$$U_\gamma^\alpha x U_\beta^\alpha = U_\beta^\alpha x U_\beta^\alpha. \tag{8.6}$$

We simply write S for U_γ^α and Q for U_β^α and put $V = S^x \cap S$ and $V' = S^x \cap Q$. Then, we see that

$$V_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_p \middle| b \in p^\delta \mathfrak{r}_p \right\},$$

$$V_p' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Q_p \middle| c \in p^\beta \mathfrak{r}_p,\ b \in p^\delta \mathfrak{r}_p \right\}.$$

Thus $Q_p = \coprod_{u \bmod p^\delta \mathfrak{r}_p} V_p' \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ and $S_p = \coprod_{u \bmod p^\delta \mathfrak{r}_p} V_p \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$. This shows that

$$x_p^{-1} S_p x_p Q_p = \coprod_{u \bmod p^\delta \mathfrak{r}_p} x_p^{-1} S_p x_p \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},$$

$$x_p^{-1} S_p x_p S_p = \coprod_{u \bmod p^\delta \mathfrak{r}_p} x_p^{-1} S_p x_p \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},$$

which proves the claim (8.6). We have a commutative diagram by the above proof of (8.6):

$$\begin{tikzcd} H^q(X(V'), \mathcal{L}(n, v, \lambda; A)) \arrow[r, "\mathrm{Tr}"] \arrow[d, "\mathrm{res}"] & H^q(X(Q), \mathcal{L}(n, v, \lambda; A)) \arrow[d, "\mathrm{res}"] \\ H^q(X(V), \mathcal{L}(n, v, \lambda; A)) \arrow[r, "\mathrm{Tr}"] & H^q(X(S), \mathcal{L}(n, v, \lambda; A)). \end{tikzcd} \tag{8.7}$$

This combined with (8.6) shows the commutativity of

$$\begin{tikzcd} H^q(X(Q), \mathcal{L}(n, v, \lambda; A)) \arrow[r, "\mathrm{res}"] \arrow[d, "(QxQ) = T_0(p^\delta)"] & H^q(X(S), \mathcal{L}(n, v, \lambda; A)) \arrow[d, "(QxQ) = T_0(p^\delta)"] \arrow[dl, "(SxQ)"'] \\ H^q(X(Q), \mathcal{L}(n, v, \lambda; A)) \arrow[r, "\mathrm{res}"] & H^q(X(S), \mathcal{L}(n, v, \lambda; A)). \end{tikzcd} \tag{8.8}$$

We verify this as follows:

$$\begin{aligned} \mathrm{res}_{Q/S} \circ (SxQ) &= \mathrm{res}_{Q/S} \circ \mathrm{Tr}_{Q/V'} \circ (x) \circ \mathrm{res}_{S/V'^{x^{-1}}} \\ &= \mathrm{Tr}_{S/V} \circ \mathrm{res}_{V'/V} \circ (x) \circ \mathrm{res}_{S/V'^{x^{-1}}} \quad \text{by (8.7)} \\ &= \mathrm{Tr}_{S/V} \circ (x) \circ \mathrm{res}_{V'^{x^{-1}}/V^{x^{-1}}} \circ \mathrm{res}_{S/V'^{x^{-1}}} = (SxS). \end{aligned}$$

Similarly we can check that $(SxQ) \circ \mathrm{res}_{Q/S} = (QxQ)$. Then an argument similar to that in the proof of Corollary 8.2 derives the result from (8.8).

COROLLARY 8.4. *There is a Hecke operator equivariant isomorphism:*

$$\iota \circ \mathrm{res}\colon eH^q\big(X(U^\alpha_\beta), \mathscr{L}(n, v, \lambda; A)\big) \cong eH^q\big(X(U^\alpha_\gamma), \mathscr{L}(0, v, \lambda\chi_n; A)\big) \qquad (\gamma \geq \beta \geq \alpha > 0)$$

for $A = \mathcal{O}/p^\gamma\mathcal{O}$ *or* $p^{-\gamma}\mathcal{O}/\mathcal{O}$ *if* $Z_\beta \subset \mathrm{Ker}(\lambda)$. *The same type of assertion also holds for the parabolic cohomology groups.*

This is a combination of Proposition 8.3 and Corollary 8.2.

Definition 8.5. Let U be an open subgroup of $\hat{R}^\times$ such that $U = R_p^\times \times U^p$. For each $n \in \mathbf{Z}[I]$ $(n \geq 0)$ with $n \sim -2v$,

$$\text{(8.9)} \qquad \begin{aligned} \mathscr{V}_q(n, v; U) &= \varinjlim_\alpha H^q\big(X(U^\alpha_\alpha), \mathscr{L}(n, v; K/\mathcal{O})\big) \\ &\cong \varinjlim_\alpha \Big(\bigoplus_i H^q\big(\overline{\Gamma}^i(U^\alpha_\alpha), {}^tL(n, v; K/\mathcal{O})\big)\Big), \\ \mathscr{V}_q^{\mathrm{ord}}(n, v; U) &= \varinjlim_\alpha eH^q\big(X(U^\alpha_\alpha), \mathscr{L}(n, v; K/\mathcal{O})\big) \\ &\cong \varinjlim_\alpha e\Big(\bigoplus_i H^q\big(\overline{\Gamma}^i(U^\alpha_\alpha), {}^tL(n, v; K/\mathcal{O})\big)\Big), \\ \mathscr{V}_q(0, v, \chi_n; U) &= \varinjlim_\alpha H^q\big(X(U^\alpha_\alpha), \mathscr{L}(0, v, \chi_n; p^{-\alpha}\mathcal{O}/\mathcal{O})\big) \\ &\cong \varinjlim_\alpha \Big(\bigoplus_i H^q\big(\overline{\Gamma}^i(U^\alpha_\alpha), {}^tL(0, v, \chi_n; p^{-\alpha}\mathcal{O}/\mathcal{O})\big)\Big), \\ \mathscr{V}_q^{\mathrm{ord}}(0, v, \chi_n; U) &= \varinjlim_\alpha eH^q\big(X(U^\alpha_\alpha), \mathscr{L}(0, v, \chi_n; p^{-\alpha}\mathcal{O}/\mathcal{O})\big) \\ &\cong \varinjlim_\alpha e\Big(\bigoplus_i H^q\big(\overline{\Gamma}^i(U^\alpha_\alpha), {}^tL(0, v, \chi_n; p^{-\alpha}\mathcal{O}/\mathcal{O})\big)\Big), \end{aligned}$$

where the injective limit is taken relative to the restriction maps. When $U = U_1(N)$, we write $\mathscr{V}_q(n, v; N)$ and $\mathscr{V}_q^{\mathrm{ord}}(n, v; N)$ etc. for these modules. Since $\nu(U^\alpha_\alpha) = \nu(U)$ for all α, if B is indefinite, we can choose $t_i \in G_f^B$ so that $G_{\mathbf{A}}^B = \coprod_{i=1}^h G_{\mathbf{Q}}^B t_i U^\alpha_\alpha G^B_{\infty+}$ independently of α, and thus in this case, we can interchange $\varinjlim_\alpha$ and $\oplus_i$ in the above definition; but when B is definite, one cannot choose t_i independently of α, and the order of $\varinjlim_\alpha$ and $\oplus_i$ must be as above. The terms on the extreme right of the above definition work well without assuming (8.1b). On these modules $\mathscr{V}_q(n, v; N)$ and $\mathscr{V}_q(0, v, \chi_n; N)$, Hecke operators $T_0(\mathfrak{n})$ and $T_0(\mathfrak{n}, \mathfrak{n})$ naturally act because the restriction maps are compatible with $T_0(\mathfrak{n})$ and $T_0(\mathfrak{n}, \mathfrak{n})$ (cf. (2.9a, b) and (3.5)).

THEOREM 8.6. *Let U be an open subgroup of $\hat{R}^\times$ such that $U = R_p^\times \times U^p$. If $n \sim n' \sim -2v$ and $n \geq n' \geq 0$, then there is a canonical isomorphism*:

$$\mathscr{V}_q^{\mathrm{ord}}(n, v; U) \cong \mathscr{V}_q^{\mathrm{ord}}(0, v, \chi_n; U) \cong \mathscr{V}_q^{\mathrm{ord}}(0, v, \chi_{n'}; U) \cong \mathscr{V}_q^{\mathrm{ord}}(n', v; U).$$

When $U = U_1(N)$ for an ideal N prime to p, this isomorphism is equivariant under Hecke operators $T_0(\mathfrak{n})$ and $T_0(\mathfrak{n}, \mathfrak{n})$ for all $\mathfrak{n}$.

By this theorem, the module $\mathscr{V}_q^{\mathrm{ord}}(n, v; U)$ depends only on $v \bmod \mathbf{Z} \cdot t$, and thus we write it as $\mathscr{V}_q^{\mathrm{ord}}(v; U)$ and $\mathscr{V}_q^{\mathrm{ord}}(v; N)$ when $U = U_1(N)$.

Proof. By definition, we have a commutative diagram: for all $\beta \geq \alpha > 0$,

$$\begin{array}{ccc} H^q(X(U_\alpha^\alpha), \mathscr{L}(n, v; p^{-\alpha}\mathscr{O}/\mathscr{O})) & \xrightarrow{\mathrm{res}} & H^q\big(X\big(U_\beta^\beta\big), \mathscr{L}\big(n, v; p^{-\beta}\mathscr{O}/\mathscr{O}\big)\big) \\ \big\downarrow \iota_\alpha & & \big\downarrow \iota_\beta \\ H^q(X(U_\alpha^\alpha), \mathscr{L}(0, v, \chi_n; p^{-\alpha}\mathscr{O}/\mathscr{O})) & \xrightarrow{\mathrm{res}} & H^q\big(X\big(U_\beta^\beta\big), \mathscr{L}\big(0, v, \chi_n; p^{-\beta}\mathscr{O}/\mathscr{O}\big)\big). \end{array}$$

Thus we see that

$$\begin{aligned} \mathscr{V}_q^{\mathrm{ord}}(n, v; U) &= \varinjlim_\alpha eH^q(X(U_\alpha^\alpha), \mathscr{L}(n, v; K/\mathscr{O})) \\ &= \varinjlim_\alpha \varinjlim_\beta eH^q\big(X(U_\alpha^\alpha), \mathscr{L}\big(n, v; p^{-\beta}\mathscr{O}/\mathscr{O}\big)\big) \\ &= \varinjlim_\alpha eH^q(X(U_\alpha^\alpha), \mathscr{L}(n, v; p^{-\alpha}\mathscr{O}/\mathscr{O})) \\ &\cong \varinjlim_\alpha eH^q(X(U_\alpha^\alpha), \mathscr{L}(0, v, \chi_n; p^{-\alpha}\mathscr{O}/\mathscr{O})) \qquad \text{(by Cor. 8.2)} \\ &= \mathscr{V}_q^{\mathrm{ord}}(0, v, \chi_n; U). \end{aligned}$$

As Δ_α^α-modules, $L(0, v, \chi_n; p^{-\alpha}\mathscr{O}/\mathscr{O})$ and $L(0, v, \chi_{n'}; p^{-\alpha}\mathscr{O}/\mathscr{O})$ are the same and are equal to $L(0, v; p^{-\alpha}\mathscr{O}/\mathscr{O})$. Thus we know that

$$\mathscr{V}_q^{\mathrm{ord}}(0, v, \chi_n; U) = \mathscr{V}_q^{\mathrm{ord}}(0, v, \chi_{n'}; U) \cong \mathscr{V}_q^{\mathrm{ord}}(n', v; U).$$

The equivariance under Hecke operators follows from that of ι_α in Theorem 8.1. The maps

$$\iota_\beta \circ \mathrm{res}\colon H^q(X(U_\alpha^\alpha), \mathscr{L}(n, v; p^{-\beta}\mathscr{O}/\mathscr{O})) \to H^q\big(X\big(U_\beta^\alpha\big), \mathscr{L}\big(0, v, \chi_n; p^{-\beta}\mathscr{O}/\mathscr{O}\big)\big)$$

are compatible with the restriction map $\mathrm{res}_{U_\beta^\alpha/U_\gamma^\gamma}$ for any $\gamma > \beta$. Thus we can take the limit

$$\text{(8.10)} \qquad \iota = \varinjlim_\beta \iota_\beta \circ \mathrm{res}\colon H^r(X(U_\alpha^\alpha), \mathscr{L}(n, v; K/\mathscr{O})) \to \mathscr{V}_r(0, v, \chi_n; U)$$

$$\text{for } r = |I_B| = \dim_{\mathbf{C}} \mathscr{Z}_B.$$

The following result is essentially due to Shimura [30]:

THEOREM 8.7. *Suppose that* $r = |I_B| = 0$ *or* 1. *Then the kernel of* ι *in* (8.10) *has only finitely many elements.*

Proof. Let $\Phi = \Gamma^i(U_\alpha^\alpha)$ in the case of $r = 1$ and $\Phi = (U_\alpha^\alpha)_p$ in the case of $r = 0$. First we shall show that if an $\mathscr{O}$-submodule V in $\mathrm{Ker}(i)$ $(\subset L(n, v; K/\mathscr{O}))$ is stable under the action of Φ, then we can find a positive integer γ so that $p^\gamma V = 0$. In fact, by the strong approximation theorem, the closure of Φ in U_p is a p-adic Lie group containing $\mathrm{SL}_2(\imath_p) \cap U_\alpha^\alpha$ if $r = 1$. Thus the Φ-module $L(n, v; K)$ is absolutely simple. Let M be the subalgebra of $\mathrm{End}_K(L(n, v; K))$ generated over $\mathscr{O}$ by the action of Φ. Then the simplicity of the Φ-module $L(n, v; K)$ shows that $M \otimes_{\mathscr{O}} K$ coincides with $\mathrm{End}_K(L(n, v; K))$. Thus there is an element $E_j \in M$ such that the coefficient of $E_j(\sum_{0 \le i \le n} a_i X^i Y^{n-i})$ in $X^n = \prod_\sigma X_\sigma^{n_\sigma}$ is equal to $p^\gamma a_j$ for all $\sum_i a_i X^i Y^{n-i} \in L(n, v; K)$ for each $0 \le j < n$. If $P = \sum_i a_i X^i Y^{n-i} \in L(n, v; K/\mathscr{O})$ is contained in V, then $E_j \cdot P \in V$ by the stability of V under M. Since $0 = i(E_j \cdot P) = p^\gamma a_j$, we know that $p^\gamma P = 0$ and hence $p^\gamma V = 0$. Especially, the order of V is finite.

Next we shall deal with the proof in the case of $r = 0$. Let S be the space of functions $f\colon G_{\mathbf{A}}^B \to L(n, v; K/\mathscr{O})$ satisfying $f(axu) = f(x)u_p$ for $u \in U_\alpha^\alpha G_{\infty +}^B$ and $a \in G_{\mathbf{Q}}^B$. Then $S \cong H^0(X(U_\alpha^\alpha), \mathscr{L}(n, v; K/\mathscr{O}))$. If $i(f(x)) = 0$ for all $x \in G_{\mathbf{A}}^B$ $(f \in S)$ (this is equivalent to supposing that $f \in \mathrm{Ker}(\iota)$), then f must have values in $\mathrm{Ker}(i)$ in $L(n, v; K/\mathscr{O})$. Since $f(xu) = f(x)u_p$ for $u \in U_\alpha^\alpha$, the subspace in $\mathrm{Ker}(i)$ generated by the values $f(x)$ for all $x \in G_{\mathbf{A}}^B$ is stable under Φ. As already seen, we then know that $p^\gamma f = 0$ for a positive γ independent of f. This shows the result in the case of $r = 0$.

Now we suppose that $r = 1$. In this case, we can take the decomposition $G_{\mathbf{A}}^B = \coprod_{i=1}^h G_{\mathbf{Q}}^B t_i U_\beta^\beta G_{\infty +}$ independently of β. For simplicity, we hereafter write $\overline{\Phi}$ for $\overline{\Gamma}^i(U_\alpha^\alpha)$ for a fixed i. Put

$$\Gamma_\beta = \left\{\gamma \in \Gamma^i(U_\alpha^\alpha) \middle| \gamma \equiv 1 \bmod p^\beta R_p\right\}, \qquad \overline{\Gamma}_\beta = \Gamma_\beta / \Gamma_\beta \cap F^\times.$$

Then $H^1(\overline{\Gamma}_\beta, {}^tL(n, v; p^{-\beta}\mathscr{O}/\mathscr{O})) \cong \mathrm{Hom}(\overline{\Gamma}_\beta, {}^tL(n, v; p^{-\beta}\mathscr{O}/\mathscr{O}))$. By virtue of a theorem of Shimura, which is given in [26, Th. 3.1.3], the injective limit of the restriction maps

$$I\colon H^1\left(\overline{\Phi}, {}^tL(n, v; K/\mathscr{O})\right) \cong \varinjlim_\beta H^1\left(\overline{\Phi}, {}^tL\left(n, v; p^{-\beta}\mathscr{O}/\mathscr{O}\right)\right)$$

$$\to \varinjlim_\beta \mathrm{Hom}_\Phi\left(\overline{\Gamma}_\beta, {}^tL\left(n, v; p^{-\beta}\mathscr{O}/\mathscr{O}\right)\right)$$

is known to have a finite kernel. On Ohta's article [26], he made an assumption that B is a division algebra to assure (in [26, p. 27]) the vanishing of $H^1(\Delta, {}^tL(n, v; \mathbf{C}))$ for a congruence subgroup Δ of $(B \otimes_F F')$ for totally real fields $F' \neq \mathbf{Q}$ by the result of Matsushima and Shimura [22, Th. 7.1]. However, by works of Harder ([5] and [6],) this vanishing has been proved even for $M_2(F')$ for $F' \neq \mathbf{Q}$. Thus, this division assumption can be eliminated without affecting the original proof, and the fact: $|\mathrm{Ker}(I)| < \infty$ is valid even for $B = M_2(\mathbf{Q})$. For $\xi \in \mathrm{Hom}_\Phi(\overline{\Gamma}_\beta, {}^tL(n, v; p^{-\beta}\mathcal{O}/\mathcal{O}))$, we know that $\xi(u\delta u^{-1}) = u\xi(\delta)$ for every $u \in \Phi$ and $\delta \in \overline{\Gamma}_\beta$. If $i \circ \xi = 0$, then for all $u \in \Phi$, $i(u\xi(\delta)) = 0$. Thus the value of ξ is contained in a Φ-submodule of $\mathrm{Ker}(i)$ in $L(n, v; K/\mathcal{O})$. Thus we can find $\gamma > 0$ independently of β so that $p^\gamma\xi = 0$. Especially, if $\xi \in H^1(\overline{\Phi}, L(n, v; K/\mathcal{O}))$ and if $\xi \in \mathrm{Ker}(\iota)$, then $p^\gamma\xi$ is contained in $\mathrm{Ker}(I)$. Therefore $\mathrm{Ker}(\iota) \subset p^{-\gamma}\mathrm{Ker}(I)$ has only finitely many elements.

9. Controllability of $\mathscr{V}^{\mathrm{ord}}(v; U)$

By definition, we can identify for each $\mathcal{O}$-module A, $H^0(X(U), \mathscr{L}(n, v, \lambda; A))$ with the space of functions f: $G_{\mathbf{A}}^B \to L(n, v, \lambda; A)$ satisfying $f(axu) = f(x)u_p$ for all $a \in G_{\mathbf{Q}}^B$ and $u \in UC_{\infty+}$. Thus, if we define for each normal subgroup V of U, the action of U/V on $H^0(X(V), \mathscr{L}(n, v, \lambda; A))$ by

$$f\|[u](x) = f(xu) \cdot u_p^{-1},$$

then we have

$$H^0\big(U/V, H^0(X(V), \mathscr{L}(n, v, \lambda; A))\big) \cong H^0(X(U), \mathscr{L}(n, v, \lambda; A)). \tag{9.1}$$

We shall generalize this controllability to general cohomology groups of dimension 1. Thus we may assume that B is indefinite. We fix an open compact subgroup U of $\hat{R}^\times$ with $U = R_p^\times \times U^p$. Then we choose the decomposition independently of α and β:

$$G_{\mathbf{A}}^B = \coprod_{i=1}^h G_{\mathbf{Q}}^B t_i U_\beta^\alpha G_{\infty+}^B \qquad \text{with } t_i \in G_f^B \quad \text{and} \quad t_{i,p} = 1.$$

Fix integers $\beta \geq \alpha > 0$ and one index i $(= 1, \dots, h)$, and write simply $\Gamma = \Gamma_\beta^\alpha$ for $\overline{\Gamma}^i(U_\beta^\alpha)$ and M for ${}^tL(n, v, \lambda; A)$ (for a fixed $\mathcal{O}$-module A). For a general group Φ and a left Φ-module X, let $C^q(\Phi, X)$ be the space of all functions on Φ^{q+1} with values in X. We shall define as usual the coboundary operator $\delta = \delta_q$: $C^q(\Phi, X) \to C^{q+1}(\Phi, X)$ by

$$\delta f(\gamma_0, \dots, \gamma_{q+1}) = \sum_{i=0}^{q+1} (-1)^i f(\gamma_0, \dots, \hat{\gamma}_i, \dots, \gamma_{q+1}). \tag{9.2}$$

Then

$$H^q(\Phi, X) = \mathrm{Ker}\left(\delta_q\colon C^q(\Phi, X)^\Phi \to C^{q+1}(\Phi, X)^\Phi\right)/$$
$$\mathrm{Im}\left(\delta_{q-1}\colon C^{q-1}(\Phi, X)^\Phi \to C^q(\Phi, X)^\Phi\right),$$

where Φ acts on $f \in C^q(\Phi, X)$ by $(f \cdot \gamma)(\gamma_0, \dots, \gamma_q) = \gamma^{-1} f(\gamma\gamma_0, \dots, \gamma\gamma_q)$. Now we write down explicitly the action of $[U_\beta^\alpha x U_\beta^\alpha]$ and $(U_\beta^\alpha x U_\beta^\alpha)$ on the cohomology group $H^q(\Gamma, M)$ by using homogeneous cochains. For each $x \in G_{\mathbf{Q}}$, we can decompose $\Gamma x \Gamma = \coprod_i \Gamma x_i$ as a disjoint union of finitely many left cosets. Write for each $\gamma \in \Gamma$, $x_i \gamma = \gamma^{(i)} x_{\gamma(i)}$ for some index $\gamma(i)$ $(= 1, \dots, h)$ and $\gamma^{(i)} \in \Gamma$. Then

$$(*) \qquad x_i^{-1}\gamma^{(i)} = \gamma x_{\gamma(i)}^{-1}.$$

Then we shall define the action of $\Gamma x \Gamma$ on $C^q(\Gamma, M)$ by

$$(9.3) \qquad \begin{aligned} f\big|[\Gamma x \Gamma](\gamma_0, \dots, \gamma_q) &= \sum_i x_i^{-1} \cdot f\left(\gamma_0^{(i)}, \dots, \gamma_q^{(i)}\right), \quad \text{and} \\ f\big|(\Gamma x \Gamma)(\gamma_0, \dots, \gamma_q) &= \sum_i \left(\left\{\nu(x_i)^{-v}\right\}\nu(x_i)^v\right)\left(x_i^{-1} \circ f\left(\gamma_0^{(i)}, \dots, \gamma_q^{(i)}\right)\right), \end{aligned}$$

where we have implicitly supposed that x is contained in (Δ_β^α) to have its action on M and in the second formula, we have let x_i^{-1} act on $f(\gamma_0^{(i)}, \dots, \gamma_q^{(i)})$ as an element of ${}^tL(n, 0, \lambda; A)$. Note that $\{\nu(x_i)^{-v}\}\nu(x_i)^v$ is a unit in $\mathcal{O}$, and hence, the operator $(\Gamma x \Gamma)$ is well defined. Actually, the above definition of the action of $\Gamma x \Gamma$ depends on the choice of the representatives $\{x_i\}$. We shall see later that the action induced on the cohomology groups is independent of the choice of $\{x_i\}$; so, for a moment, we fix the decomposition: $\Gamma x \Gamma = \coprod_i \Gamma x_i$. Now we have by the definition:

$$(9.4\mathrm{a}) \qquad \delta(f|[\Gamma x \Gamma]) = (\delta f)|[\Gamma x \Gamma], \qquad \delta(f|(\Gamma x \Gamma)) = (\delta f)|(\Gamma x \Gamma).$$

If $f \in C^q(\Gamma, M)$ is Γ-invariant, then

$$(9.4\mathrm{b}) \qquad \begin{aligned} f\big|[\Gamma x \Gamma](\gamma\gamma_0, \dots, \gamma\gamma_q) &= \sum_i x_i^{-1} f\left(\gamma^{(i)}\gamma_0^{(\gamma(i))}, \dots, \gamma^{(i)}\gamma_q^{(\gamma(i))}\right) \\ &= \sum_i x_i^{-1}\gamma^{(i)} f\left(\gamma_0^{(\gamma(i))}, \dots, \gamma_q^{(\gamma(i))}\right) \\ &= \sum_i \gamma x_{\gamma(i)}^{-1} f\left(\gamma_0^{(\gamma(i))}, \dots, \gamma_q^{(\gamma(i))}\right) \quad \text{by } (*) \\ &= \gamma\left(f\big|[\Gamma x \Gamma](\gamma_0, \dots, \gamma_q)\right). \end{aligned}$$

Therefore, $(\Gamma x \Gamma)$ and $[\Gamma x \Gamma]$ give an operator on $C(\Gamma, M) = \oplus_q C^q(\Gamma, M)$ compatible with the action of Γ and coboundary operator δ, and hence, they act naturally on cohomology groups. We shall now show the independence of the operator $[\Gamma x \Gamma]$ and $(\Gamma x \Gamma)$ on $H^q(\Gamma, M)$ from the choice of the decomposition:

$\Gamma x \Gamma = \coprod_i \Gamma x_i$. Since the proof is the same for $(\Gamma x \Gamma)$ and $[\Gamma x \Gamma]$, we only deal with $[\Gamma x \Gamma]$. Let Γ' be a subgroup of finite index of Γ. We decompose $\Gamma = \coprod_i \Gamma' x_i$ and define, similarly to $\Gamma x \Gamma$, an operator $T_{\Gamma/\Gamma'}$: $C^q(\Gamma', M) \to C^q(\Gamma, M)$ by

$$f\big|T_{\Gamma/\Gamma'}\big(\gamma_0, \dots, \gamma_q\big) = \sum_i x_i^{-1} f\big(\gamma_0^{(i)}, \dots, \gamma_q^{(i)}\big),$$

where for each $\gamma \in \gamma$, $\gamma^{(i)} \in \Gamma'$ is defined by $(*)$. Then it is known by Eckmann [4, Th. 7] that $T_{\Gamma/\Gamma'}$: $H^q(\Gamma', M) \to H^q(\Gamma, M)$ is defined independently of the decomposition $\Gamma = \coprod_i \Gamma' x_i$. Writing Γ' for $\Gamma \cap x^{-1}\Gamma x$ and Γ'' for $x\Gamma' x^{-1}$, we see easily that

(9.4c) $\quad [\Gamma x \Gamma] = \mathrm{res}_{\Gamma/\Gamma''} \circ [x] \circ T_{\Gamma'/\Gamma}, \qquad (\Gamma x \Gamma) = \mathrm{res}_{\Gamma/\Gamma''} \circ (x) \circ T_{\Gamma'/\Gamma},$

where

$$f\big|[x]\big(\gamma_0, \dots, \gamma_q\big) = x^{-1} f\big(x\gamma_0 x^{-1}, \dots, x\gamma_q x^{-1}\big),$$
$$f\big|(x)\big(\gamma_0, \dots, \gamma_q\big) = \big(\{\nu(x)^{-v}\}\nu(x)^v\big)x^{-1} \circ f\big(x\gamma_0 x^{-1}, \dots, x\gamma_q x^{-1}\big).$$

Thus $[\Gamma x \Gamma]$ and $(\Gamma x \Gamma)$ are also independent of the decomposition. Similarly to the above argument, we can define the left action of double coset $\Gamma x \Gamma$ on the homology group $H_q(\Gamma, M)$ by using the left coset decomposition $\Gamma x \Gamma = \coprod_i \Gamma x_i$. Especially, on $H_0(\Gamma, M) = M/DM$ ($DM = \sum_{\gamma \in \Gamma}(\gamma - 1)M$), $\Gamma x \Gamma$ acts by $m \mapsto \sum_i x_i^{-1} \cdot m$.

Now we recall briefly the construction in [16] of the Hochschild-Serre spectral sequence in order to study the action of the double coset on the spectral sequence. We write $\Gamma = \Gamma_\beta^\alpha$ and $\Gamma' = \Gamma_\beta^{\alpha'}$ for $\beta \geq \alpha' \geq \alpha \geq 0$ ($\beta > 0$) (thus Γ' is a normal subgroup of Γ). Put $C = \sum_q C^q(\Gamma, M)$, $C^{\Gamma'} = \sum_q C^q(\Gamma, M)^{\Gamma'}$, $L^{p,q} = C^p(\Pi, C^q(\Gamma, M)^{\Gamma'})$ for $\Pi = \Gamma/\Gamma'$, $L = \sum_{p,q} L^{p,q}$, $L_i^q = \sum_{p \geq i} L^{p,q}$, $L_i = \sum_{q=0}^{\infty} L_i^q$. Then L_i gives a filtration of L. On $L^{p,q}$, we define a differential operator δ_Π: $L^{p,q} \to L^{p+1,q}$ relative to Π as in (9.2), and for $f \in L^{p,q}$ put $\delta_\Gamma(f)(\pi_0, \dots, \pi_p) = \delta_\Gamma(f(\pi_0, \dots, \pi_p))$ for $\pi_i \in \Pi$ by applying the coboundary operator relative to Γ to the value $f(\pi_0, \dots, \pi_p)$. Define $\delta = \delta_\Pi + (-1)^p \delta_\Gamma$ on $L^{p,q}$. Then δ gives a differential operator preserving the filtration (cf. [16, II]).

Let $x \in G_f^B$ be an element such that $x_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$ and $x_\sigma = 1$ for σ outside p. Then, for sufficiently large m divisible by the order of $F^\times \backslash F_{\mathbf{A}}^\times / \nu(U) F_{\infty+}^\times$, we have

$$t_i x^m \in \gamma t_i\big(U_\beta^{\alpha'}\big) G_{\infty+}^B \qquad \textit{with } \gamma \in G_{\mathbf{Q}}^B.$$

This follows from the strong approximation theorem. Next, we can choose the decomposition $\Gamma'\gamma\Gamma' = \coprod_i \Gamma'\gamma_i$. Then we have another decomposition $\Gamma\gamma\Gamma = \coprod_i \Gamma\gamma_i$ for the same representatives γ_i. This fact follows from [31, (2.19.3)] (see also Lemma 4.7 in the text). Then by (9.4b), $(\Gamma\gamma\Gamma)$ preserves $C^q(\Gamma, M)^{\Gamma'}$, the filtration $\{L_i\}_i$ and the double filtration $\{L_i^q\}$. Moreover, by (9.4a), $(\Gamma\gamma\Gamma)$ commutes with δ_Γ and by definition commutes with δ_Π. Thus, by the construction of the spectral sequence of Hochschild and Serre [16, 1.7], the operator $(\Gamma\gamma\Gamma)$ gives an endomorphism of the spectral sequence:

$$H^p(\Gamma/\Gamma', H^q(\Gamma', M)) \Rightarrow H^n(\Gamma, M). \tag{9.5}$$

That is, $(\Gamma\gamma\Gamma)$ is compatible with the filtration of each term of (9.5) and commutes with all the differential maps of (9.5). Especially, $(\Gamma\gamma\Gamma)$ acts on $H^q(\Gamma', M)$ by $(\Gamma'\gamma\Gamma')$, and this action gives an endomorphism of $H^q(\Gamma', M)$ compatible with the action of $\Pi = \Gamma/\Gamma'$; therefore, it induces an endomorphism of $H^p(\Pi, H^q(\Gamma', M))$. By [31, (2.19.3)] (or else, as already seen in §7 in the text), the action of $(\Gamma\gamma\Gamma)$ coincides with $T_0(p^m)$ as in Theorem 8.1. When $M = {}^tL(n, v, \lambda; A)$ with $A = p^{-j}\mathcal{O}/\mathcal{O}$ or $\mathcal{O}/p^j\mathcal{O}$, a finite power of $(\Gamma\gamma\Gamma)$ thus gives the idempotent e as in Section 8 on each $H^p(\Gamma/\Gamma', H^q(\Gamma', M))$ or $H^n(\Gamma, M)$. Therefore the idempotent e gives the endomorphism of the spectral sequence (9.5) for A as above. By taking the injective limit relative to j, this fact is also true for $A = k/\mathcal{O}$. Thus one knows:

THEOREM 9.1. *Suppose that $r = |I_B| > 0$. Then, for the idempotent e associated with $T_0(p)$, (9.5) gives the following spectral sequence for $A = \mathcal{O}/p^\mu\mathcal{O}$, $p^{-\mu}\mathcal{O}/\mathcal{O}$ and $K/\mathcal{O}$:*

$$H^i\Big(\Gamma_\beta^\alpha/\Gamma_\beta^{\alpha'}, eH^j\Big(\Gamma_\beta^{\alpha'}, {}^tL(n, v, \lambda; A)\Big)\Big) \Rightarrow eH^q\Big(\Gamma_\beta^\alpha, {}^tL(n, v, \lambda; A)\Big)$$

for each $\beta \geq \alpha' \geq \alpha \geq 0$ with $\beta > 0$.

LEMMA 9.2. *Suppose that $r = |I_B| > 0$. Then the idempotent e annihilates $H^0(\Gamma, M)$ and $H_0(\Gamma, M)$ for $M = {}^tL(n, v, \lambda; A)$ with $A = \mathcal{O}$, $\mathcal{O}/p^\mu\mathcal{O}$, $p^{-\mu}\mathcal{O}/\mathcal{O}$ and $K/\mathcal{O}$.*

Proof. We shall prove the assertion only for $H^0(\Gamma, M)$ since the other case can be treated similarly. We may assume that $A = \mathcal{O}/p^\mu\mathcal{O}$. For any given $\rho \geq \mu$, by making m large, we can find $\gamma_u \in G_{\mathbf{Q}}^B$ for each $u \in \mathfrak{r}_p$ so that $t_i x^m\left(\begin{smallmatrix} 1 & u \\ 0 & 1\end{smallmatrix}\right) \in \gamma_u t_i U(p^\rho) G_{\infty+}$, where $U(p^\rho) = \{u \in U | u_p \equiv 1 \bmod p^\rho R_p\}$. Then $\gamma_u \equiv \begin{pmatrix} 1 & u \\ 0 & p^m\end{pmatrix} \bmod p^\rho R_p$ and $\Gamma\gamma\Gamma = \coprod_{u \bmod p^m} \Gamma\gamma_u$ by (8.6). Thus, for $P = \sum_{0 \leq j \leq n} a_j X^j Y^{n-j}$ in $L(n, v, \lambda; A)$,

$$P|(\Gamma\gamma\Gamma) = \left(\{\nu(\gamma)^{-v}\}\nu(\gamma)^v\right)a_n \cdot \sum_{0 \leq j \leq n} \binom{n}{j} X^{n-j}Y^j \cdot \sum_{u \in \mathfrak{r}/p^m\mathfrak{r}} u^j$$

$$\left(\binom{n}{j} = \prod_{\sigma\in I}\binom{n_\sigma}{j_\sigma}\right).$$

For sufficiently large m, we always have $\sum_{u \in \imath/p^m \imath} u^j = 0 \bmod p^\mu$, and the lemma follows.

Corollary 9.3. *Let* $M = {}^tL(n, v, \lambda; A)$ *for* $A = \mathcal{O}/p^\mu\mathcal{O}$, $p^{-\mu}\mathcal{O}/\mathcal{O}$ *and* $K/\mathcal{O}$ *and assume that* $r = |I_B| > 0$. *Suppose that* $H^i(\Gamma_\beta^\alpha/\Gamma_\beta^{\alpha'}, eH^j(\Gamma_\beta^{\alpha'}, M)) = 0$ *for* $0 < i \le q$ *and* $j \le q - 1$ ($\beta \ge \alpha' \ge \alpha \ge 0$ *and* $\beta > 0$). *Then the restriction map induces an isomorphism*:

$$eH^q(\Gamma', M)^\Gamma \cong eH^q(\Gamma, M) \qquad \text{for } \Gamma = \Gamma_\beta^\alpha \text{ and } \Gamma' = \Gamma_\beta^{\alpha'}.$$

Proof. Put $E_2^{i,j} = eH^i(\Pi, H^j(\Gamma', M))$ and $E^q = eH^q(\Gamma, M)$. By Theorem 9.1, we have the spectral sequence: $E_2^{i,j} \Rightarrow E^q$. We have a canonical filtration: $E^q = E_0^q \supset \cdots \supset E_q^q \supset 0$ with $E_i^q/E_{i+1}^q \cong E_\infty^{i,q-i}$. We have differentials $d_k^{i,j}$: $E_k^{i,j} \to E_k^{i+k,j-k+1}$, and by definition

$$(*) \qquad E_{k+1}^{i,j} = \operatorname{Ker}\left(d_k^{i,j}\right)/\operatorname{Im}\left(d_k^{i-k,j+k-1}\right).$$

Since $E_2^{i,j} = 0$ for $i < 0$ or $j \le 0$ by Lemma 9.2, we know that $E_k^{i,j} = 0$ for $i < 0$ or $j \le 0$. Thus if $i - k < 0$ and $j - k + 1 \le 0$ (i.e. $k > i$ and $k \ge j + 1$), then $E_{k+1}^{i,j} \cong E_k^{i,j}$. Thus $E_i^q/E_{i+1}^q \cong E_k^{i,q-i}$ for $k \ge i + 1$ and $k \ge q + 1$. If $0 < i \le q$ and $j \le q - 1$, by assumption

$$(**) \qquad E_2^{i,j} = H^i\left(\Pi, eH^j(\Gamma', M)\right) \cong eH^i\left(\Gamma, H^j(\Gamma', M)\right) = 0.$$

By the construction $(*)$, we see by induction on k that $E_k^{i,q-i} = 0$ if $i > 0$. Then we have that $eH^q(\Gamma, M) \cong E_{q+1}^{0,q}$. Now for k with $q + 1 \ge k > 2$, assume that

$$eH^q(\Gamma, M) \cong E_{q+1}^{0,q} \cong E_q^{0,q} \cong \cdots \cong E_k^{0,q}.$$

Then $E_k^{0,q} = \operatorname{Ker}(d_{k-1}^{0,q})$ and $d_{k-1}^{0,q}$: $E_{k-1}^{0,q} \to E_{k-1}^{k-1,q-k+2}$. Since $k > 2$, $E_{k-1}^{k-1,q-k+2} = 0$ by $(**)$, and thus $E_k^{0,q} \cong E_{k-1}^{0,q}$. By induction on k, we conclude that

$$eH^q(\Gamma, M) \cong E_2^{0,q} = eH^0(\Gamma, H^q(\Gamma', M)) \cong H^0(\Gamma, eH^q(\Gamma', M)),$$

which was to be shown.

Let U be either $U_1(N)$ or $U_1(NL) \cap V_1(L)$ for an ideal L outside Σ^B and Np. Note that $U \cap F_{\mathbf{A}}^\times$ is then equal to $U_F(N)$ or $U_F(NL)$. Then for each $\beta \ge \alpha' \ge \alpha \ge 0$ ($\beta > 0$), we have an isomorphism $\overline{\Gamma}_\beta^\alpha/\overline{\Gamma}_\beta^{\alpha'} \cong \imath^\times t_i U_\beta^\alpha t_i^{-1}/\imath^\times t_i U_\beta^{\alpha'} t_i^{-1}$ by the strong approximation theorem. Thus the group $\overline{Z}_\alpha/\overline{Z}_{\alpha'}$ naturally acts on $H^q(\Gamma_\beta^{\alpha'}, M)$ for $M = {}^tL(n, v, \lambda; A)$ if $\operatorname{Ker}(\lambda) \supset Z_\beta$, since

$$\imath^\times t_i U_\beta^\alpha t_i^{-1}/\imath^\times t_i U_\beta^{\alpha'} t_i^{-1} \cong \imath^\times U_\beta^\alpha/\imath^\times U_\beta^{\alpha'} \cong \imath^\times\left(U_\beta^\alpha \cap F_f^\times\right)/\imath^\times\left(U_\beta^{\alpha'} \cap F_f^\times\right) \cong \overline{Z}_\alpha/\overline{Z}_{\alpha'}.$$

(Strictly speaking, when $U = U_1(NL) \cap V_1(L)$, $Z_\alpha(NL)/Z_{\alpha'}(NL)$ acts on the cohomology group, but we know that if $\alpha \geq 0$, $W_\alpha(NL) = W_\alpha(N)$.) If α is sufficiently large so that $\mathrm{Ker}(\lambda) \supset Z_\alpha$, then as U_β^α-module, $L(n, v, \lambda; A) = L(n, v; A)$. Thus we know that

$$\begin{aligned}\mathscr{V}_q^{\mathrm{ord}}(v; U) &\cong \varinjlim_{\alpha} eH^q\big(X(U_\alpha^\alpha), \mathscr{L}(n, v, \lambda; K/\mathcal{O})\big) \\ &\cong \varinjlim_{\alpha} eH^q\big(X(U_\alpha^\alpha), \mathscr{L}(n, v; K/\mathcal{O})\big).\end{aligned}$$

However, the action of $\overline{Z}$ on the extreme right module and that of the middle term is different. We let $\overline{Z}$ act on $\mathscr{V}_q^{\mathrm{ord}}(v; U)$ by identifying it with $\varinjlim eH^q(X(U_\alpha^\alpha), \mathscr{L}(n, v; K/\mathcal{O}))$ via the character $\mathrm{Il}(N) \ni a \mapsto T(a, a)$. This is independent of the choice of n with $n \sim -2v$ by Theorem 8.6. We write this standard action on $\mathscr{V}_q^{\mathrm{ord}}(v; U)$ as $\xi \mapsto \xi|\langle z \rangle$ ($z \in \overline{Z}$), and the action induced by the middle term as $\xi \mapsto \xi|\langle z \rangle_{n, v, \lambda}$. The latter action factors through the finite group $\overline{Z}_\alpha/\overline{Z}_\beta$ on $eH^q(X(U_\beta^\alpha), \mathscr{L}(n, v, \lambda; K/\mathcal{O}))$. Then by definition (cf. (3.9)), we have

(9.6) $\langle z \rangle = \lambda\chi_{n+2v}(z)\langle z \rangle_{n, v, \lambda}$ *for* $z \in \overline{Z}_\alpha$ *if* λ *is a character of* Z_α *such that* $\lambda\chi_{n+2v}$ *factors through* $\overline{Z}_\alpha$.

For each character λ: $\overline{Z}_1 \to \mathcal{O}^\times$, we put

$$\mathscr{V}_q^{\mathrm{ord}}(v; U)[\lambda] = \left\{x \in \mathscr{V}_q^{\mathrm{ord}}(v; U) \,\middle|\, x|\langle z \rangle = \lambda(z)x \text{ for all } z \in \overline{Z}_1\right\}.$$

THEOREM 9.4. *Let U be as above. Suppose that $0 \leq q \leq 1$. Then, for each $\beta > 0$, the restriction map induces an isomorphism:*

$$e\left(\bigoplus_{i=1}^{h} H^q\left(\overline{\Gamma}^i\left(U_\beta^1\right), {}^tL(n, v, \lambda; K/\mathcal{O})\right)\right) \cong \mathscr{V}_q^{\mathrm{ord}}(v; U)[\lambda\chi_{n+2v}]$$

for each finite order character λ: $Z_1/Z_\beta \to \mathcal{O}^\times$ such that $\lambda\chi_{n+2v}$ factors through $\overline{Z}_1$ for each $n \geq 0$ with $n \sim -2v$.

Proof. By Lemma 9.2, the assumption of Corollary 9.3 is satisfied for $q = 1$, and thus we have by (9.6) that

$$\begin{aligned}&eH^q\left(\overline{\Gamma}^i\left(U_\beta^1\right), {}^tL(n, v, \lambda; K/\mathcal{O})\right) \\ &\quad\cong \left\{x \in eH^q\left(\overline{\Gamma}^i\left(U_\beta^\beta\right), {}^tL(n, v; K/\mathcal{O})\right) \,\middle|\, x|\langle z \rangle = \lambda\chi_{n+2v}(z)x \text{ for all } z \in \overline{Z}_1\right\}.\end{aligned}$$

On the other hand, the left-hand side of the above formula is isomorphic to $\mathscr{V}_q^{\mathrm{ord}}(v; U)[\lambda\chi_{n+2v}]$ under the restriction map. The case: $q = 0$ follows from Proposition 8.3 and (9.1).

Remark 9.5. The proof of Theorem 9.4 shows a little more general result for $0 \le q \le 1$ and for each $\beta \ge \alpha > 0$:

$$e\left(\bigoplus_{i=1}^{h} H^q\left(\overline{\Gamma}^i\left(U_\beta^\alpha\right), {}^tL(n, v, \lambda; K/\mathcal{O})\right)\right)$$
$$\cong \left\{x \in \mathscr{V}_q^{\mathrm{ord}}(v; U) | x | \langle z \rangle = \lambda\chi_{n+2v}(z)x \text{ for all } z \in \overline{Z}_\alpha\right\}.$$

10. Co-freeness of $\mathscr{V}_q^{\mathrm{ord}}(v; N)$ over Λ

Let U be either $U_1(N)$ or $U_1(N\ell) \cap V_1(\ell)$ for an ideal N outside p and Σ^B and for a prime ideal ℓ outside Np and Σ^B. For each finite order character ε: $Z_\alpha \to \overline{\mathbf{Q}}^\times$, let $K(\varepsilon)$ denote the subfield of $\overline{\mathbf{Q}}_p$ generated by the values of ε over K, and let $\mathcal{O}(\varepsilon)$ be the p-adic integer ring of $K(\varepsilon)$, and let $\beta(\varepsilon)$ denote the minimal integer such that $\mathrm{Ker}(\varepsilon) \supset Z_\beta$ and $Z_\beta \cong \overline{Z}_\beta$. In this section, we always suppose that B is a division algebra.

THEOREM 10.1. *Let α be a positive integer and suppose that $\overline{\Gamma}^i(U_\alpha^\alpha)$ is torsion-free for all i. Let $q = 0$ or 1. If $eH^q(X(U_\beta^\gamma), \mathscr{L}(n, v, \lambda; K(\lambda)/\mathcal{O}(\lambda)))$ is p-divisible for all finite order characters λ: $Z_\alpha \to \overline{\mathbf{Q}}^\times$ such that $\lambda\chi_{n+2v}$ factors through $\overline{Z}_\alpha$ and for all pair of integers γ, β with $\alpha \le \gamma \le \beta$ and $\beta \ge \beta(\lambda)$, then the Pontryagin dual module $V_q^{\mathrm{ord}}(v; U)$ of $\mathscr{V}_q^{\mathrm{ord}}(v; U)$ is free of finite rank over the continuous group algebra $\mathcal{O}[[\overline{Z}_\alpha]]$ of $\overline{Z}_\alpha$.*

Before proving the theorem, we prepare:

LEMMA 10.2. *Let $\mathscr{G}$ be a topological group isomorphic to a product of finitely many copies of $\mathbf{Z}_p$ and a finite group. Let $\mathcal{O}[[\mathscr{G}]]$ be the continuous group algebra of $\mathscr{G}$. For each finite order character ε: $\mathscr{G} \to \overline{\mathbf{Q}}_p^\times$, let P_ε: $\mathcal{O}[[\mathscr{G}]] \to \overline{\mathbf{Q}}_p$ be the induced $\mathcal{O}$-algebra homomorphism. Then the subset of* $\mathrm{Spec}(\mathcal{O}[[\mathscr{G}]])$ *consisting of the points P_ε for all finite order characters ε of $\mathscr{G}$ is Zariski dense.*

Proof. What we have to show is that $\bigcap_\varepsilon \mathrm{Ker}(P_\varepsilon) = \{0\}$. Let $C(\mathscr{G}, \mathcal{O})$ denote the space of all continuous $\mathcal{O}$-valued functions on $\mathscr{G}$. Let $\mathrm{Meas}(\mathscr{G}, \mathcal{O})$ be the $\mathcal{O}$-linear dual of $C(\mathscr{G}, \mathcal{O})$; i.e., $\mathrm{Meas}(\mathscr{G}, \mathcal{O}) = \mathrm{Hom}_{\mathcal{O}}(C(\mathscr{G}, \mathcal{O}), \mathcal{O})$. We can identify $\mathrm{Meas}(\mathscr{G}, \mathcal{O})$ with the space of bounded p-adic measures on $\mathscr{G}$ with values in $\mathcal{O}$. Let $\mathrm{Meas}(\mathscr{G}, \Omega)$ be the space of all bounded p-adic measures on $\mathscr{G}$ with values in the p-adic completion Ω of $\overline{\mathbf{Q}}_p$. $\mathrm{Meas}(\mathscr{G}, \mathcal{O})$ is an $\mathcal{O}$-algebra under the convolution product and is isomorphic to $\mathcal{O}[[\mathscr{G}]]$ (e.g. [20]). Especially, for each finite order character ε: $\mathscr{G} \to \overline{\mathbf{Q}}_p^\times$, the map: $\mathrm{Meas}(\mathscr{G}, \mathcal{O}) \ni u \mapsto \int_{\mathscr{G}} \varepsilon\, du \in \Omega$ coincides with the algebra homomorphism P_ε: $\mathcal{O}[[\mathscr{G}]] \to \Omega$. Since the subspace of locally constant functions on $\mathscr{G}$ is dense in $C(\mathscr{G}, \mathcal{O})$, if $\int_{\mathscr{G}} \phi\, du = 0$ for all locally constant ϕ, then $u = 0$. Note that every locally constant function ϕ can be written as a linear combination of finite order characters $\varepsilon\colon \mathscr{G} \to \overline{\mathbf{Q}}_p^\times$ over $\overline{\mathbf{Q}}_p$. Thus, if $u \in \bigcap_\varepsilon \mathrm{Ker}(P_\varepsilon)$, then $\int_{\mathscr{G}} \phi\, du = 0$ for all locally constant ϕ and hence $u = 0$. Q.E.D.

Proof of Theorem 10.1. By Theorem 6.3, without losing generality, we may replace K by its finite extension. Thus we may assume that we have a finite order character $\varepsilon\colon Z_\alpha/Z_\gamma \to \mathscr{O}$ such that $\varepsilon\chi_{n+2v}$ factors through $\overline{Z}_\alpha$. We write $V_q^{\mathrm{ord}}(v; U)$ sometimes as $V/\mathscr{O}$ to indicate its dependence on $\mathscr{O}$. We have an exact sequence

$$(*)\qquad eH^q\big(X\big(U_\gamma^\alpha\big), \mathscr{L}(n, v, \varepsilon; K)\big) \xrightarrow{\theta} eH^q\big(X\big(U_\gamma^\alpha\big), \mathscr{L}(n, v, \varepsilon; K/\mathscr{O})\big) \longrightarrow eH^q\big(X\big(U_\gamma^\alpha\big), \mathscr{L}(n, v, \varepsilon; \mathscr{O})\big).$$

Since the middle term of $(*)$ is p-divisible by assumption and the last term is of finite type as an $\mathscr{O}$-module (cf. [28, §2]), θ must be surjective. Hence the $\mathscr{O}$-rank of the Pontryagin dual of the middle term is finite. We write it as s. Let $\overline{Z}$ act on $\mathscr{V}_q^{\mathrm{ord}}(v; U)$ via the action $\langle z\rangle_{n,v,\varepsilon}$ as in (9.6). We write A for $\mathscr{O}[[\overline{Z}_\alpha]]$. Then by Theorem 9.4, $V \otimes_A A/P_{\mathrm{id}}A$ for the identity character id: $\overline{Z} \to \mathbf{Q}_p^\times$ is isomorphic to the Pontryagin dual module of the middle term of $(*)$. Thus we know that $V/P_{\mathrm{id}}V \cong \mathscr{O}^s$. Thus V is generated over A by s-elements, and we have a surjective morphism of A-modules $\psi\colon A^s \to V$ which induces an isomorphism

$$(10.1\mathrm{a})\qquad (A/mA)^s \cong V \otimes_A A/mA \quad \textit{for the maximal ideal } m \textit{ of } A.$$

Note that $A_\lambda = \mathscr{O}(\lambda)[[\overline{Z}_\alpha]] \cong A \otimes_{\mathscr{O}} \mathscr{O}(\lambda)$. We write $K/\mathscr{O}(\lambda)$ for $K(\lambda)/\mathscr{O}(\lambda)$. By using the same type of exact sequence as $(*)$, we know

$$eH^q\big(X\big(U_\beta^\beta\big), \mathscr{L}(n, v, \varepsilon; K/\mathscr{O}(\lambda))\big) \cong eH^q\big(X\big(U_\beta^\beta\big), \mathscr{L}(n, v, \varepsilon; \mathscr{O}(\lambda))\big) \otimes_{\mathscr{O}(\lambda)} (K/\mathscr{O}(\lambda))$$

and by Theorem 6.3,

$$eH^q\big(X\big(U_\beta^\beta\big), \mathscr{L}(n, v, \varepsilon; K/\mathscr{O}(\lambda))\big) \cong eH^q\big(X\big(U_\beta^\beta\big), \mathscr{L}(n, v, \varepsilon; K/\mathscr{O})\big) \otimes_{\mathscr{O}} \mathscr{O}(\lambda).$$

Therefore, we see

$$(10.1\mathrm{b})\qquad V/\mathscr{O} \otimes_{\mathscr{O}} \mathscr{O}(\lambda) \cong V/\mathscr{O} \otimes_A A_\lambda \cong V/\mathscr{O}(\lambda).$$

Then for any finite order character $\lambda\colon \overline{Z}_\alpha \to \overline{\mathbf{Q}}^\times$, we know from the assumption that

$$V/\mathscr{O} \otimes_A A_\lambda/P_\lambda A_\lambda \cong \mathscr{O}(\lambda)^{s'} \qquad \text{for some integer } s'.$$

Then by (10.1a, b), s' must be equal to s. Thus we have a surjection induced by ψ:

$$\mathscr{O}(\lambda)^s \cong (A_\lambda/P_\lambda A_\lambda)^s \to V/\mathscr{O} \otimes_A A_\lambda/P_\lambda A_\lambda \cong \mathscr{O}(\lambda)^s.$$

From the exact sequence: $0 \to \mathrm{Ker}(\psi) \to A^s \to V/\mathscr{O} \to \mathscr{O}$, we conclude in view of the flatness of $\mathscr{O}(\lambda)/\mathscr{O}$ the exactness of

$$0 \to \mathrm{Ker}(\psi) \otimes_{\mathscr{O}} \mathscr{O}(\lambda) \to A_\lambda^s \to V/\mathscr{O}(\lambda) \to 0.$$

Thus we know that $\mathrm{Ker}(\psi) \otimes_{\mathscr{O}} \mathscr{O}(\lambda) \subset (P_\lambda A_\lambda)^s = P_\lambda(A_\lambda^s)$. Since $\mathscr{O}(\lambda)$ is faithfully flat over $\mathscr{O}$, we conclude that $\mathrm{Ker}(\psi) \subset (P_\lambda \cap A)A^s$ for all finite order characters λ: $\overline{Z}_\alpha \to \overline{\mathbf{Q}}^\times$. Then by Lemma 10.2, we know the vanishing $\mathrm{Ker}(\psi) = 0$, which proves the theorem.

COROLLARY 10.3. *We suppose that* $r = |I_B| = 0$ *or* 1. *Then* $V_r^{\mathrm{ord}}(v; N)$ *is a free module of finite rank over* $\mathscr{O}[[\overline{Z}_1]]$, *if one of the following two conditions is satisfied*:

(10.2a) *For a primitive* p-th *root of unity* ζ_p *in* $\mathbf{C}$, $[F(\zeta_0): F] > 2$,

(10.2b) $\overline{\Gamma}_1^i(Np)$ *is torsion free for all* i.

Proof. Firstly, we suppose that (10.2b) is satisfied. We write U for $U_1(Np)$. When $r = 0$, the triviality of $\overline{\Gamma}_1^i(Np)$ means that

$$H^0\big(X\big(U_\beta^\alpha\big), \mathscr{L}(n, v, \lambda; K/\mathscr{O})\big) \cong L(n, v, \lambda; K/\mathscr{O})^h$$

and thus it is p-divisible for all $0 < \alpha \le \beta$ and λ. Hence we can apply Theorem 10.1 and get the result. When $r = 1$, it is known by [36, Propositions 8.1 and 8.2], if $\overline{\Gamma}_1^i(Np)$ is without torsion, then

$$H^2\big(X\big(U_\beta^\alpha\big), \mathscr{L}(n, v, \lambda; \mathscr{O})\big) \cong \bigoplus_i H_0\big(\overline{\Gamma}^i\big(U_\beta^\alpha\big), {}^tL(n, v, \lambda; \mathscr{O})\big),$$

and hence by Lemma 9.2, $eH^2(X(U_\beta^\alpha), \mathscr{L}(n, v, \lambda; \mathscr{O})) = 0$. From the exact sequence:

$$\begin{aligned} eH^1\big(X\big(U_\beta^\alpha\big), \mathscr{L}(n, v, \lambda; K)\big) &\to eH^1\big(X\big(U_\beta^\alpha\big), \mathscr{L}(n, v, \lambda; K/\mathscr{O})\big) \\ &\to eH^2\big(X\big(U_\beta^\alpha\big), \mathscr{L}(n, v, \lambda; \mathscr{O})\big) = 0, \end{aligned}$$

we know the p-divisibility of the middle term, and hence, the result follows from Theorem 10.1. Now we shall suppose (10.2a). On $F(\zeta_p)$, the Frobenius element at each prime ideal ℓ of $\imath$ prime to p acts by $\zeta_p \to \zeta_p^{\mathscr{N}(\ell)}$ for the norm map $\mathscr{N}$: $F \to \mathbf{Q}$. Since $[F(\zeta_p): F] > 2$, we can find a prime $\ell \nmid N$ unramified over $\mathbf{Z}$ such that $\mathscr{N}(\ell) \not\equiv \pm 1 \bmod p$. Since the degree d of $X(V)/X(U)$ for $V = U_1(N\ell p) \cap V_1(\ell)$ divides $(\mathscr{N}(\ell) + 1)(\mathscr{N}(\ell) - 1)$, d is prime to p. By Lemma 7.1, $\overline{\Gamma}^i(V)$ is torsion-free. The similar argument as in the case where (10.2b) is satisfied shows that $V_r^{\mathrm{ord}}(v; V)$ is free of finite rank over $\mathscr{O}[[\overline{Z}_1(N\ell)]]$. Since $\overline{Z}_1 = \overline{Z}_1(N)$ can be identified with a subgroup of $\overline{Z}_1(N\ell)$ (because the covering degree of $\overline{Z}_1(N\ell)/\overline{Z}_1$ is prime to p), $V_r^{\mathrm{ord}}(v; V)$ is free over $\mathscr{O}[[\overline{Z}_1]]$. For the trace map $\mathrm{Tr}_{U/V} = \mathrm{Tr}$: $\mathscr{V}_r^{\mathrm{ord}}(v; V) \to \mathscr{V}_r^{\mathrm{ord}}(v; N)$ and the restriction map $\mathrm{res}_{U/V} = \mathrm{res}$: $\mathscr{V}_r^{\mathrm{ord}}(v; N) \to \mathscr{V}_r^{\mathrm{ord}}(v; V)$, we know that $\mathrm{Tr} \circ \mathrm{res}$ is the multiplication by d. Thus $V_r^{\mathrm{ord}}(v; N)$ is a direct summand of the $\mathscr{O}[[\overline{Z}_1]]$-free module $\mathscr{V}_r^{\mathrm{ord}}(v; V)$ and is hence free over $\mathscr{O}[[\overline{Z}_1]]$.

As for the structure of the Λ-module, we have:

COROLLARY 10.4. *Suppose that $r = |I_B| = 0$ or 1. Let $\mathbf{P}$ be the product of all distinct prime factors of p in $\mathfrak{r}$. Suppose that $p > 2$ and further assume either* (10.2a) *or the following condition*:

(10.2c) *$\overline{\Gamma}_1^i(N\mathbf{P})$ is torsion free for all i.*

Then $V_r^{\mathrm{ord}}(v; N)$ is Λ-free of finite rank, where $\Lambda = \mathcal{O}[[W]]$ is the continuous group algebra of the torsion-free part W of $\overline{Z}(N)$.

Proof. We shall prove the assertion only in the case $r = 1$, since the other case can be treated more easily. So far, we have worked with the filtration $Z_\alpha \subset Z(N)$, but we can define another filtration of subgroups Z^α of $Z(N)$ given by the kernel of the natural map: $Z \to \mathrm{Cl}_F(N\mathbf{P}^\alpha)$. Since p is odd, even if one defines $\overline{Z}^\alpha$ by the kernel of "$\overline{Z} \to \overline{\mathrm{Cl}}_F(N\mathbf{P}^\alpha)$," we can identify $\overline{Z}^\alpha$ with Z^α naturally. Hereafter we always identify these two groups. These two filtrations are cofinal; i.e., $Z^\alpha \supset Z_\alpha$ and $Z_\alpha \supset Z^{m\alpha}$ if p divides $\mathbf{P}^m$. If p is unramified in $\mathfrak{r}$, then the two filtrations coincide. We can define $\Delta_{(\beta)}^{(\alpha)}$ similarly to Δ_β^α by replacing p by $\mathbf{P}$ in the definition of Δ_β^α in Section 7 and define $U_{(\beta)}^{(\alpha)}$ by $U \cap \Delta_{(\beta)}^{(\alpha)}$ for U as in Theorem 9.4. By scrutinizing every step ascending towards the proof of Theorem 9.4, one can check that the corresponding statement to Theorem 9.4 for $U_{(\beta)}^{(1)}$ is true. Put

$$\begin{aligned}\mathcal{V}(i) &= \varinjlim_{\alpha} eH^1\big(\overline{\Gamma}_1^i(N\mathbf{P}^\alpha), {}^tL(n, v; K/\mathcal{O})\big)\\ &= \varinjlim_{\alpha} eH^1\big(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(n, v; K/\mathcal{O})\big)\end{aligned}$$

by choosing $n \geq 0$ with $n \sim -2v$. Then in exactly the same manner as in the proof of Theorem 10.1 and Corollary 10.3, we can prove that $V_r^{\mathrm{ord}}(v; N) = \oplus_i V(i)$ is $\mathcal{O}[[Z^1]]$-free, where $V(i)$ is the Pontryagin dual of $\mathcal{V}(i)$. Put $W^1 = W \cap Z^1$. Since $[Z^0 : Z^1]$ is prime to p, the natural projection: $W \to \mathrm{Cl}_F(1)$ induces an injection: $W/W^1 \to \mathrm{Cl}_F(1)$. We can identify the set of connected components of $X(U)$ for $U = U_1(Np^\alpha)$ (for any α) with the group $\mathrm{Cl}_F(1)$ via the correspondence: $G_{\mathbf{Q}} t_i U G_{\infty+} \mapsto F^\times \nu(t_i)\hat{\mathfrak{r}}^\times F_{\infty+}^\times$. Thus we can write $\mathcal{V}_r^{\mathrm{ord}}(v; N) = \oplus_{i \in \mathrm{Cl}_F(1)} \mathcal{V}(i)$. Then the action of $w \in W/W^1 \subset \mathrm{Cl}_F(1)$ interchanges the connected components according to the multiplication of $\nu(w) = w^2$ in $\mathrm{Cl}_F(1)$. If $p > 2$, the map: $w \mapsto w^2$ gives an automorphism of the image $\overline{W}$ of W in $\mathrm{Cl}_F(1)$. By taking a coset decomposition $\mathrm{Cl}_F(1) = \coprod_j \overline{W} \cdot j$, put $V_0 = \oplus_j V(j)$. Then V_0 is $\mathcal{O}[[W^1]]$-free, and $V_r^{\mathrm{ord}}(v; N) \cong \mathrm{Ind}_{W^1}^W(V_0) \cong V_0 \otimes_{\mathcal{O}[[W^1]]} \Lambda$ is Λ-free.

11. Proof of Theorems 3.2 and 3.3

When we consider the field $F \neq \mathbf{Q}$, we always fix here and in the following section a quaternion algebra B over F unramified at all finite places and

$r = |I_B| \leq 1$. Therefore B is a division quaternion algebra and $X(U)$ is always compact. Such a quaternion algebra always exists if $[F:\mathbf{Q}] > 1$, and we have a relation: $[F:\mathbf{Q}] \equiv r \bmod 2$. When $F = \mathbf{Q}$, we take $M_2(\mathbf{Q})$ as B.

Proof of Theorem 3.2. Firstly we suppose that $F \neq \mathbf{Q}$. Let K be a finite extension of $\mathbf{Q}_p$ containing K_0 as in Section 1. Let $\mathcal{O}$ be the p-adic integer ring of K. It is sufficient to prove the result over $\mathcal{O}$ since $h_{k,w}(Np^\infty; \mathcal{O}) \cong h_{k,w}(Np^\infty; \hat{\mathcal{O}}(v)) \otimes_{\hat{\mathcal{O}}(v)} \mathcal{O}$ for the p-adic closure $\hat{\mathcal{O}}(v)$ of $\mathcal{O}(v)$ in $\overline{\mathbf{Q}}_p$. We consider

$$\mathcal{V}_r(0; N) = \varinjlim_{\alpha} H^r(X_\alpha, K/\mathcal{O}) \qquad \text{for } X_\alpha = X(U_1(Np^\alpha)).$$

We have an exact sequence: $H^1(X_\alpha, K) \to H^1(X_\alpha, K/\mathcal{O}) \to H^2(X_\alpha, \mathcal{O})$, When $r = 1$, $H^2(X_\alpha, \mathcal{O}) \cong \mathcal{O}$; thus, $H^1(X_\alpha, K/\mathcal{O})$ is p-divisible. Writing $\mathcal{V}_\alpha$ for $H^1(X_\alpha, K/\mathcal{O})$, we can identify $h_{2t,t}(Np^\alpha; \mathcal{O})$ with the subalgebra of $\mathrm{End}_{\mathcal{O}}(\mathcal{V}_\alpha)$ generated over $\mathcal{O}$ by $T(n)$ for all n by Theorem 6.2 (in this case: $v = 0$, $T_0(n) = T(n)$ for all n). Now we consider the case: $r = 0$. In this case, let S_α (resp. T) be the space of functions f: $G_{\mathbf{A}}^B \to K/\mathcal{O}$ (resp. f: $F_{\mathbf{A}}^\times \to K/\mathcal{O}$) such that $f(axu) = f(x)$ for $a \in G_{\mathbf{Q}}^B$ and $u \in U_1(Np^\alpha)G_\infty$ (resp. $a \in F^\times$ and $u \in \hat{\imath}^\times F_{\infty+}^\times$). Since the reduced norm map ν: $G_{\mathbf{A}} \to F_{\mathbf{A}}^\times$ induces an injection ν^*: $T \to S_\alpha$, the quotient $\mathcal{V}_\alpha = S_\alpha/\nu^*(T)$ is p-divisible, and therefore, we can identify, by Theorem 6.2, $h_{2t,t}(Np^\alpha; \mathcal{O})$ with the subalgebra of $\mathrm{End}_{\mathcal{O}}(\mathcal{V}_\alpha)$ generated by $T(n)$. For each $a \in F_+^\times = F^\times \cap F_{\infty+}^\times$, we consider the quadratic extension $F(\sqrt{-a})$, which is totally imaginary. Since B is unramified at all finite places of F, $F(\sqrt{-a})$ can be embedded into B ([36, (9.2.6)]), and hence, $a = \nu(\sqrt{-a}) \in \nu(B^\times)$. That is, ν: $G_{\mathbf{Q}}^B \to F_+^\times$ is surjective. For $a \in F_+^\times$, $y \in F_{\mathbf{A}+}^\times = F_f^\times F_{\infty+}^\times$ and $c \in \hat{\imath}^\times F_{\infty+}^\times$, we choose $b \in G_{\mathbf{Q}}^B$, $x \in G_{\mathbf{A}}^B$ and $u \in U_1(Np^\alpha)G_\infty^B$ such that $\nu(b) = a$, $\nu(x) = y$ and $\nu(u) = c$. If $f \in S_\alpha$ is of the form $\nu^*(\phi)$ for ϕ: $F_{\mathbf{A}+}^\times \to K/\mathcal{O}$, then $\phi(ayc) = \phi(\nu(bxu)) = f(bxu) = f(x) = \phi(y)$ and thus $f \in \nu^*T$. That is if $f \in \mathcal{V}_\alpha$ becomes trivial in $\mathcal{V}_\beta$ for $\infty \geq \beta > \alpha > 0$, then f must be zero in $\mathcal{V}_\alpha$. Therefore the natural map: $\mathcal{V}_\alpha \to \mathcal{V}_\beta$ is injective. When $r = 1$, by virtue of a result of Shimura as in Theorem 8.7, the natural map: $\mathcal{V}_\alpha \to \mathcal{V}_\beta$ $(\alpha < \beta \leq \infty)$ has a finite kernel. Then in either case: $r = 0$ or 1, the natural $\mathcal{O}$-algebra homomorphism ρ_β^α: $h_{2t,t}(Np^\beta; \mathcal{O}) \to h_{2t,t}(Np^\alpha; \mathcal{O})$ $(\alpha < \beta \leq \infty)$ can be defined by the commutative diagram:

$$\begin{tikzcd} \mathcal{V}_\alpha \arrow[r, "\mathrm{res}"] \arrow[d, "\rho_\beta^\alpha(h)"'] & \mathcal{V}_\beta \arrow[d, "h"] \\ \mathcal{V}_\alpha \arrow[r, "\mathrm{res}"] & \mathcal{V}_\beta, \end{tikzcd}$$

where we understand that $\mathscr{V}_\infty = \varinjlim_\alpha \mathscr{V}_\alpha$. Especially $h_{2t,t}(Np^\infty; \mathcal{O})$ acts faithfully on $\mathscr{V}_\infty$ and hence is naturally embedded into $\mathrm{End}_\Lambda(\mathscr{V}_\infty)$. Now we consider the morphism defined in (8.10) for $n \sim 0$ ($n > 0$):

$$\iota\colon \bigoplus_i H^r\big(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(n,0; K/\mathcal{O})\big) \to \mathscr{V}_r(0; N).$$

When $r = 0$ and $n \sim 0$, the space on the left-hand side can be identified with the space $S_{k,k-t,\phi}(Np^\alpha; K/\mathcal{O})$ ($k = n + 2t$) of functions $f\colon G_{\mathbf{A}}^B \to L(n,0; K/\mathcal{O})$ such that

$$f(axu) = f(x) \cdot u_p \quad \text{for } a \in G_{\mathbf{Q}}^B \text{ and } u \in U_1(Np^\alpha)G_{\infty+}.$$

Write $f(x) = \sum_{0 \le i \le n} f_i(x) X^i Y^{n-i}$. Then the map $\iota\colon S_{k,k-t,\phi}(Np^\alpha; K/\mathcal{O}) \to S_\infty = \varinjlim_\alpha S_\alpha$ can be defined by $\iota(f) = f_n$. Since for $u \in U_1(Np^\alpha)G_{\infty+}$ with $u_p = \begin{pmatrix} a & * \\ 0 & b \end{pmatrix}$,

$$f_n(xu) = f_n(x)a^n,$$

we know that if $\iota(f) \in \nu^*(T)$, then the value of f_n is annihilated by $a^n - 1$ for all $a \in \mathfrak{r}_p^\times$ with $a \equiv 1 \bmod p^\alpha \mathfrak{r}_p$. Thus we can choose $\beta > 0$ independently of f so that $p^\beta f \in \mathrm{Ker}(\iota)$ if $\iota(f) \in \nu^*(T)$. Then by Theorem 8.7, the map induced by ι:

$$I_\alpha\colon \bigoplus_i H^r\big(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(n,0; K/\mathcal{O})\big) \to \mathscr{V}_\infty$$

is of finite kernel. This fact is also true by Theorem 8.7 in the case: $r = 1$ since $[F:\mathbf{Q}] \equiv 1 \mod 2$ in this case. Let $\mathscr{V}_\alpha^n$ be the image of $\oplus_i H^r(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(n,0;K))$ in $\oplus_i H^r(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(n,0;K/\mathcal{O}))$. Then $\mathscr{V}_\alpha^n$ is the p-divisible part of the latter module. Then, by Theorem 6.2, for each $n \sim 0$ ($n \ge 0$), $h_{k,k-t}(Np^\alpha; \mathcal{O})$ ($k = n + 2t$) can be identified with the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(\mathscr{V}_\alpha^n)$ generated by $T(\mathfrak{n})$ for all $\mathfrak{n}$. Since $I_\alpha\colon \mathscr{V}_\alpha^n \to I_\alpha(\mathscr{V}_\alpha^n) \subset \mathscr{V}_\infty$ is an isogeny, $h_{k,k-t}(Np^\alpha; \mathcal{O})$ can be also identified with the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(I_\alpha(\mathscr{V}_\alpha^n))$ generated by $T(\mathfrak{n})$ for all $\mathfrak{n}$. The restriction of operators in $h_{2t,t}(Np^\infty; \mathcal{O}) \subset \mathrm{End}_{\mathcal{O}}(\mathscr{V}_\infty)$ to the subspace $I_\alpha(\mathscr{V}_\alpha^n)$ induces a surjective $\mathcal{O}$-algebra homomorphism $I_\alpha^*\colon h_{2t,t}(Np^\infty; \mathcal{O}) \to h_{k,k-t}(Np^\alpha; \mathcal{O})$ which takes $T(\mathfrak{n})$ to $T(\mathfrak{n})$ because of Theorem 8.1. Thus we have a commutative diagram for $0 < \alpha < \beta < \infty$:

$$\begin{tikzcd} h_{2t,t}(Np^\infty; \mathcal{O}) \arrow[rr, "I_\alpha^*"] \arrow[dr, "I_\beta^*"'] & & h_{k,k-t}(Np^\alpha; \mathcal{O}). \\ & h_{k,k-t}(Np^\beta; \mathcal{O}) \arrow[ur, "\rho_\beta^\alpha"'] & \end{tikzcd}$$

where ρ_β^α is the natural projection map given in Section 3. These maps are surjective morphisms in the category of compact rings where the projective limit of surjective morphisms is always surjective. By taking the projective limit of I_α^*, we have a surjective $\mathscr{A}$-algebra homomorphism

$$I_\infty^*\colon h_{2t,t}(Np^\infty;\mathscr{O}) \to h_{k,k-t}(Np^\infty;\mathscr{O}) \qquad \textit{for } k \geq 2t \textit{ and } k \sim 0,$$

which takes $T(n)$ to $T(n)$. When $F = \mathbf{Q}$, we have to replace H^1 by the parabolic cohomology groups H_P^1 defined in [36, Chap. 8] (see also [14, §4]). Then every step in the above argument for $F \neq \mathbf{Q}$ can be checked for H_P^1 and we obtain the result even for $F = \mathbf{Q}$.

Now we shall prove the injectivity of I_∞^* when $f \neq \mathbf{Q}$. Put $\mathscr{V}_\infty^n = \varinjlim_\alpha \mathscr{V}_\alpha^n$. We consider the map $I_\infty = \varinjlim_\alpha I_\alpha\colon \mathscr{V}_\infty^n \to \mathscr{V}_\infty$. By definition, we have a commutative diagram: for all $h \in h_{2t,t}(Np^\infty;\mathscr{O})$,

$$\begin{tikzcd} \mathscr{V}_\infty^n \arrow[r, "I_\infty"] \arrow[d, "I_\infty^*(h)"] & \mathscr{V}_\infty \arrow[d, "h"] \\ \mathscr{V}_\infty^n \arrow[r] & \mathscr{V}_\infty \end{tikzcd} .$$

Thus if I_∞ is surjective, I_∞^* must be injective since $h_{2t,t}(Np^\infty;\mathscr{O}) \subset \mathrm{End}_\Lambda(\mathscr{V}_\infty)$. If $r = 0$, by identifying $\mathscr{V}_\infty$ with $S_\infty/\nu^*(T)$ and $\mathscr{V}_\infty^n$ with the p-divisible subspace of $\varinjlim_\alpha S_{k,k-t,\phi}(Np^\alpha; K/\mathscr{O})$, we see easily the surjectivity of I_∞, and the result follows. Thus we shall prove the surjectivity of I_∞ in the case of $r = 1$. Put $\mathscr{W}_\alpha^n = \oplus_i H^1(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(n,0;K/\mathscr{O}))$. Then we have an exact sequence for $n > 0$:

$$0 \to \mathscr{V}_\alpha^n \to \mathscr{W}_\alpha^n \to H^2\big(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(n,0;\mathscr{O})\big) \to 0.$$

Then, by [36, Prop. 8.1 and 8.2], there is an isogeny

$$f_\alpha\colon H^2\big(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(n,0;\mathscr{O})\big) \to {}^tL(n,0;\mathscr{O})/D_\alpha^i,$$

$$g_\alpha\colon {}^tL(n,0;\mathscr{O})/D_\alpha^i \to H^2\big(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(0,n;\mathscr{O})\big)$$

such that $f_\alpha \circ g_\alpha = g_\alpha \circ f_\alpha = M \cdot \mathrm{id}$, where M is the least common multiple of the order of all torsion elements in $\overline{\Gamma}_1^i(Np^\alpha)$ and

$$D_\alpha^i = \sum_{\gamma \in \Gamma_1^i(Np^\alpha)} (\gamma - 1)\cdot {}^tL(n,0;\mathscr{O}).$$

Thus by taking the injective limit relative to α, we have an exact sequence: $0 \to \mathscr{V}_\infty^n \to \mathscr{W}_\infty^n \to \mathscr{X} \to 0$, where $\mathscr{X} = \varinjlim_\alpha(\oplus_i H^2(\overline{\Gamma}_1^i(Np^\alpha), {}^tL(n,0;\mathscr{O}))$. We shall define a map

$$\phi_\beta^\alpha\colon {}^tL(n,0;\mathscr{O})/D_\alpha^i \to {}^tL(n,0;\mathscr{O})/D_\beta^i \qquad \text{for } \beta > \alpha$$

by the transfer map given by the correspondence: $x \mapsto \sum \gamma^{-1}x$ for a decomposition $\overline{\Gamma}^i(Np^\alpha) = \coprod_\gamma \overline{\Gamma}^i(Np^\beta)\gamma$. Then the injective limit for $\mathscr{X}$ is compatible with the injective limit $\mathscr{X}' = \varinjlim_\alpha(\oplus_i {}^tL(n,0;\mathscr{O})/D^i_\alpha)$ relative to ϕ^α_β under f_α and g_α. (This follows from the proof of the existence of f_α and g_α given in [36]). Thus we have $f\colon \mathscr{X} \to \mathscr{X}'$ and $g\colon \mathscr{X}' \to \mathscr{X}$ such that $f \circ g = g \circ f = M \cdot \mathrm{id}$ for a suitable positive integer M. Let Γ^i_α be the closure of $\Gamma^i_1(Np^\alpha)$ in $G_p = \mathrm{GL}_2(F_p)$. Note that D^i_α is a closed submodule of ${}^tL(n,0,\mathscr{O})$. Therefore for any $\gamma \in \Gamma^i_\alpha$, we know that $(\gamma - 1)^tL(n,0;\mathscr{O}) \subset D^i_\alpha$. Especially, for every $a \in \mathfrak{r}_p^\times$ with $a \equiv 1 \bmod p^\alpha\mathfrak{r}_p$, put $\gamma_a = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \in \mathrm{GL}_2(F_p)$; then, $\gamma_a \in \Gamma^i_\alpha$. Thus $D^i_\alpha \supset D_\alpha = \sum_a(\gamma_a - 1)^tL(n,0;\mathscr{O})$, where a runs over all elements in $\mathfrak{r}_p^\times$ congruent to 1 mod p^α. Note that γ_a acts on the monomial X^n as $\gamma_a X^n = a^n X^n$. Therefore, on the image of X^n in ${}^tL(n,0;\mathscr{O})/D_\alpha$, every element γ of $\Gamma^i_1(Np^\alpha)$ acts trivially since $\gamma \equiv \left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right) \bmod p^\alpha R_p$. Therefore we know that $i(\phi^\alpha_\beta(x)) = [\overline{\Gamma}^i_1(Np^\alpha)\colon \overline{\Gamma}^i_1(Np^\beta)]i(x)$. Since $[\overline{\Gamma}^i_1(Np^\alpha)\colon \overline{\Gamma}^i(Np^\beta)]$ is a p-power whose exponent increases accordingly to β, I_∞ is the zero map on $\mathscr{X}'$. Therefore, we have the commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{V}^n_\infty & \longrightarrow & \mathscr{W}^n_\infty & \longrightarrow & \mathscr{X} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle I_\infty} & & \Big\downarrow{\scriptstyle I_\infty} & & \Big\downarrow{\scriptstyle I_\infty} & & \\
0 & \longrightarrow & I_\infty(\mathscr{V}^n_\infty) & \rightarrow & \mathscr{V}_\infty & \longrightarrow & \mathscr{Y} & \longrightarrow & 0
\end{array}$$

and $\mathscr{Y}$ is annihilated by M. Since $\mathscr{Y}$ is a surjective image of the p-divisible group $\mathscr{V}_\infty$, it must be trivial. This shows the surjectivity of $I_\infty\colon \mathscr{V}^n_\infty \to \mathscr{V}_\infty$, which finishes the proof when $F \neq \mathbf{Q}$. When $F = \mathbf{Q}$, as seen in [14, Lemma 7.2], we already have a surjective inverse map of $I^*_\infty\colon \mathscr{h}_{k,k-t}(Np^\infty;\mathscr{O}) \to \mathscr{h}_{2t,t}(Np^\infty;\mathscr{O})$, and hence I^*_∞ must be an isomorphism (in [14], we made the assumption: $p \geq 5$, but this condition is not necessary for the proof of [14, Lemma 7.2]).

Proof of Theorem 3.3. It is sufficient to prove the assertion over $\mathscr{O}$ since $\mathscr{h}^{\mathrm{ord}}_{k,w}(Np^\infty;\mathscr{O}) \cong \mathscr{h}^{\mathrm{ord}}_{k,w}(Np^\infty;\mathscr{O}(v)) \otimes_{\hat{\mathscr{O}}(v)} \mathscr{O}$. Fix v as in Section 3, and define $k = n + 2t$, $w = v + k - t$ for each $n \geq 0$ with $n \sim -2v$. We firstly suppose that $F \neq \mathbf{Q}$. Let $\mathscr{V}_{\alpha,n} = \mathscr{V}_{\alpha,n}(N)$ be the image in $e(\oplus_i H^r(\overline{\Gamma}^i_1(Np^\alpha), {}^tL(n,v;K/\mathscr{O})))$ of $e(\oplus_i H^r(\overline{\Gamma}^i_1(Np^\alpha), L(n,v;K))$. By definition, the restriction map takes $\mathscr{V}_{\alpha,n}$ into $\mathscr{V}_{\beta,n}$ for $\beta > \alpha$. Put $\mathscr{V}_{\infty,n}(N) = \varinjlim_\alpha \mathscr{V}_{\alpha,n}(N)$. We know that $\mathscr{V}_{\infty,n} \subset \mathscr{V}_r^{\mathrm{ord}}(v;N)$. We choose another ideal M prime to p such that N divides M and $\overline{\Gamma}^i_1(M)$ is torsion-free for all i. Such an ideal M exists by Lemma 7.1. Then $\mathscr{V}_{\infty,n}(M)$ coincides with $\mathscr{V}_r^{\mathrm{ord}}(v;M)$. Note that the trace map of $\oplus_i H^r(\overline{\Gamma}^i_1(Mp^\alpha), {}^tL(n,v;K))$ to $\oplus_i H^r(\overline{\Gamma}^i_1(Np^\alpha), {}^tL(n,v;K))$ is always surjective by Theorems 6.2 and 6.3 and is compatible with the injective systems $\{\mathscr{V}_{\alpha,n}(N)\}_\alpha$ and $\{\mathscr{V}_{\alpha,n}(M)\}_\alpha$. Thus $\mathscr{V}_{\infty,n}(N)$ is the surjective image of $\mathscr{V}_{\infty,n}(M) = \mathscr{V}_r^{\mathrm{ord}}(v; M)$ and is independent of n if $n \sim -2v$ because the trace map is independent of weight n by Theorem 8.1. Since $h_{k,w}^{\mathrm{ord}}(Np^\infty; \mathscr{O})$ can be identified with the subalgebra of $\mathrm{End}_\Lambda(\mathscr{V}_{\infty,n}(N))$ and the restriction map $\mathscr{V}_{\alpha,n}(N) \to \mathscr{V}_{\infty,n}(N)$ is injective by Remark 9.5, $h_{k,w}^{\mathrm{ord}}(Np^\infty; \mathscr{O})$ can be identified with the subalgebra of $\mathrm{End}_\Lambda(\mathscr{V}_{\infty,n}(N))$ generated topologically over Λ by $T_0(n)$ for all $n \in \mathrm{Il}(1)$. Since the Pontryagin dual module $V_r^{\mathrm{ord}}(v; N)$ is a Λ-module of finite type by Theorem 9.4, the Pontryagin dual module $V_{\infty,n}(N)$ of $\mathscr{V}_{\infty,n}(N)$ is also of finite type, and hence $h_{k,w}^{\mathrm{ord}}(Np^\infty; \mathscr{O})$ is a Λ-module of finite type since Λ is noetherian. Since $V_r^{\mathrm{ord}}(v; M)$ is $\mathscr{O}[[W_1]]$-free by Corollary 10.3 and Λ is finite and faithfully flat over $\mathscr{O}[[W_1]]$, $V_r^{\mathrm{ord}}(v; M)$ is Λ-torsion-free. Since $V_{\infty,n}(N)$ can be identified with a Λ-submodule of $V_r^{\mathrm{ord}}(v; M)$, $V_{\infty,n}(N)$ is Λ-torsion-free, and hence $\mathrm{End}_\Lambda(V_{\infty,n}(N))$ is Λ-torsion-free (even, in fact, Λ-reflexive). Therefore $h_{k,w}^{\mathrm{ord}}(Np^\infty; \mathscr{O})$, which is a Λ-submodule of $\mathrm{End}_\Lambda(V_{\infty,n}(N))$, is Λ-torsion-free. Since $V_{\infty,n}(N)$ is independent of n and $h_{k,w}^{\mathrm{ord}}(Np^\infty; \mathscr{O})$ is generated over Λ by $T_0(n)$ for al n, $h_{k,w}^{\mathrm{ord}}(Np^\infty; \mathscr{O})$ is independent of n and only depends on v mod $\mathbf{Z}\cdot t$. This shows the assertion when $F \neq \mathbf{Q}$. The case: $F = \mathbf{Q}$ can be handled in exactly the same manner as above, replacing the usual cohomology groups H^1 by the parabolic ones H_P^1. The case: $F = \mathbf{Q}$ under an additional assumption: $p \geq 5$ has already been treate[14, §1]. However, by carefully analyzing the proof of [14, Th. 3.1], one finds that the assumption: $p \geq 5$ can be removed if $\overline{\Gamma}_1^i(N)$ is torsion-free. Then the above argument in the case where $F \neq \mathbf{Q}$ works well even for $F = \mathbf{Q}$.

12. Proof of Theorems 3.4 and 3.6 and Corollary 3.7

When $r = |I_B| = 1$, $C_\infty/C_{\infty+}$ is an abelian group of order 2. By the result in Section 7, we can let $C_\infty/C_{\infty+}$ act on $\mathscr{V}_1^{\mathrm{ord}}(v; N)$. We write simply $\mathscr{V}$ for $\mathscr{V}_1^{\mathrm{ord}}(v; N)$ and put $\mathscr{V}_\pm = \{x \pm (x|c) | x \in \mathscr{V}\}$ for the generator c of $C_\infty/C_{\infty+}$. If $\mathscr{V}$ is p-divisible, then $\mathscr{V}_\pm$ is also p-divisible and $\mathscr{V} = \mathscr{V}_+ + \mathscr{V}_-$. When $p > 2$, $\mathscr{V} = \mathscr{V}_+ \oplus \mathscr{V}_-$. Before proving Theorem 3.4, we shall show:

THEOREM 12.1. *Let K be a finite extension of $\mathbf{Q}_p$ containing K_0 as in Section 1 and $\mathscr{O}$ be its p-adic integer ring. When $r = 1$, let V denote the Pontryagin dual module of one of $\mathscr{V}_\pm$ and when $r = 0$, let V denote the Pontryagin dual module of $\mathscr{V}_0^{\mathrm{ord}}(v; N)$. Then we have an isomorphism of Hecke modules:*

$$V_P = V \otimes_\Lambda \Lambda_P \cong \mathbf{h}_v^{\mathrm{ord}}(N; \mathscr{O}) \otimes_\Lambda \Lambda_P \qquad \text{for all } P \in \mathscr{X}_{\mathrm{alg}}(\Lambda) \text{ with } n(P) \geq 2v,$$

where Λ_P is the localization of Λ at P. Especially $\mathbf{h}_v^{\mathrm{ord}}(N; \mathscr{O}) \otimes_\Lambda \Lambda_P$ is free of finite rank over Λ_P for all $P \in \mathscr{X}_{\mathrm{alg}}(\Lambda)$ with $n(P) \geq 2v$.

Proof. For simplicity, we write $\mathbf{h}_P$ for $\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \otimes_\Lambda \Lambda_P$. Firstly suppose that $\overline{\Gamma}_1^i(N\mathbf{P})$ is torsion free for all i for $\mathbf{P}$ as in Corollary 10.4. We also suppose that $p > 2$ or $r = 0$. Then V is free of finite rank over Λ by Corollary 10.4. Let $\Lambda' = \mathcal{O}[[W_1]]$ and put $Q = P \cap \Lambda'$. Then Q corresponds to the restriction of the character $\chi_{n(P)}\varepsilon_P$ to W_1. Let Λ'_0 be the localization of Λ' at Q and put $V_Q = V \otimes_{\Lambda'} \Lambda'_Q$. By Theorem 9.4, we know from (7.6) that

$$V_Q/QV_Q \cong \bigoplus_{\varepsilon} h_{k,w}^{\mathrm{ord}}\left(Np^{\alpha(P)}, \varepsilon; K\right)$$
$$\text{for } k = n(P) - 2v + 2t \text{ and } w = v + k - t,$$

where ε runs over all characters of W whose restriction to W_1 coincides with ε_P. This shows that $V_P/PV_P \cong h_{k,w}^{\mathrm{ord}}(Np^{\alpha(P)}, \varepsilon_P; K)$. By definition, we have a natural surjection: $\mathbf{h}_P/P\mathbf{h}_P \to h_{k,w}^{\mathrm{ord}}(Np^{\alpha(P)}, \varepsilon_P; K)$. Let $\bar{x}$ be the element in V_P/PV_P corresponding to the identity of $h_{k,w}^{\mathrm{ord}}(Np^{\alpha(P)}, \varepsilon_P; K)$. Take $x \in V_P$ such that $x \bmod P = \bar{x}$, and define $\phi\colon \mathbf{h}_P \to V_P$ by $\phi(h) = hx$. Then, by Nakayama's lemma, ϕ is surjective. It is clear from the proof of Theorem 3.2 and 3.3 that $\mathbf{h}_P$ acts faithfully on V_P and hence ϕ is an isomorphism. For general N, we take an ideal M prime to p such that $\overline{\Gamma}_1^i(MN)$ is torsion-free for all i. Let us consider $\mathrm{Tr}\colon \mathscr{V}_r^{\mathrm{ord}}(v; MN) \to \mathscr{V}_r^{\mathrm{ord}}(v; N)$ and res: $\mathscr{V}_r^{\mathrm{ord}}(v; N) \to \mathscr{V}_r^{\mathrm{ord}}(v; MN)$. Then $\mathrm{Tr} \circ \mathrm{res}$ coincides with the multiplication of a positive integer d. Therefore, if $p > 2$ or $r = 0$, we conclude from the result for level MN that V_P is Λ_P-free and

$$V_P/PV_P \cong h_{k,w}^{\mathrm{ord}}\left(Np^{\alpha(P)}, \varepsilon_P; K\right).$$

Then the same argument as above shows the assertion. When $p = 2$ and $r = 1$, the kernel of the natural map: $\mathscr{V}_+ \oplus \mathscr{V}_- \to \mathscr{V}$ is annihilated by 2, and thus for the Pontryagin dual modules V^+, V^- and V^* of $\mathscr{V}_+$, $\mathscr{V}_-$ and $\mathscr{V}$, respectively,

$$\left(V^+ \otimes_\Lambda \Lambda_P\right) \oplus \left(V^- \otimes_\Lambda \Lambda_P\right) \cong V^* \otimes_\Lambda \Lambda_P.$$

This combined with the above argument shows the result.

Proof of Theorem 3.4. By Theorem 12.1, if K contains K_0 as in (1.1), we know from Theorem 9.4 and (7.6) that

$$\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \otimes_\Lambda \Lambda_P/P\Lambda_P \cong V_P/PV_P \cong h_{k,w}^{\mathrm{ord}}\left(Np^{\alpha(P)}, \varepsilon_P; K\right)$$

as in the theorem. By Theorems 3.1 and 3.3, we know that $\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \cong \mathbf{h}_v^{\mathrm{ord}}(N; \hat{\mathcal{O}}(v)) \otimes_{\hat{\mathcal{O}}(v)} \mathcal{O}$. Then the general case follows from the result for $\mathcal{O}$.

Before proving Theorem 3.6 and Corollary 3.7, we prepare some lemmas.

LEMMA 12.2. *Let $\mathfrak{p}$ be a prime factor of p in $\mathfrak{r}$ and N be an ideal prime to $\mathfrak{p}$. Let $\lambda\colon h_{k,w}(N\mathfrak{p}^\alpha; \overline{\mathbf{Q}}) \to \overline{\mathbf{Q}}$ be a primitive homomorphism with character $\psi\colon \mathrm{Cl}_F(N\mathfrak{p}^\alpha) \to \overline{\mathbf{Q}}^\times$ and $C(\psi)$ be the conductor of ψ. We write β for the exponent of $\mathfrak{p}$ in $C(\psi)$ and let ψ_0 denote the primitive character*

$\psi_0\colon \mathrm{Cl}_F(C(\psi)) \to \overline{\mathbf{Q}}^\times$ *associated with* ψ. *Then we have*

(12.1a) $|\lambda(T(\mathfrak{p}))|^2 = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{p})^{[n+2v]+1}$ *if* $0 < \alpha = \beta$,

(12.1b) $\lambda(T(\mathfrak{p}))^2 = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{p})^{[n+2v]}\psi_0(\mathfrak{p})$ *if* $\alpha = 1$ *and* $\beta = 0$,

(12.1c) $\lambda(T(\mathfrak{p})) = 0$ *if* $\alpha \geq 2$ *and* $\beta > \alpha$,

where $n + 2v = [n + 2v]t$ *stands for* $[n + 2v] \in \mathbf{Z}$.

This fact is well-known; so, we give a sketch of a proof. We shall give a representation theoretic proof. An elementary proof in the case of $F = \mathbf{Q}$ can be found in [24]. We write f_λ for the unique cusp form in $S^*_{k,w,I}(N\mathfrak{p}^\alpha; M_2(F); \mathbf{C})$ such that $a(\mathfrak{n}, f_\lambda) = \lambda(T_0(\mathfrak{n}))$ for all $\mathfrak{n}$. Let $\pi(\lambda)$ be the automorphic representation attached to f_λ, and let $\pi_{\mathfrak{p}}(\lambda)$ be the $\mathfrak{p}$-component of $\pi(\lambda)$. Suppose firstly that $\pi_{\mathfrak{p}}(\lambda)$ is a principal series representation $\pi(\xi, \eta)$ for quasi-characters ξ, η of $F_{\mathfrak{p}}^\times$. If ξ and η are both ramified, then $\alpha \geq 2$ and $\alpha > \beta$ and $\lambda(T(\mathfrak{p})) = 0$. If one of ξ and η, say ξ, is unramified, then $\xi|_{\mathfrak{r}_{\mathfrak{p}}^\times} = \psi|_{\mathfrak{r}_{\mathfrak{p}}^\times}$, $\alpha = \beta$ and $\lambda(T(\mathfrak{p})) = \xi(\pi)$ for a prime element π of $\mathfrak{r}_{\mathfrak{p}}$. Note that $\pi_{\mathfrak{p}}(\lambda) \otimes \omega^a$ for $a = ([n + 2v] + 1)/2$ is a unitary representation for $\omega(x) = |x|_p$. Then, by the classification of unitary representations of $GL_2(F_{\mathfrak{p}})$, $\xi\omega^a$ is a unitary character, or otherwise, $\eta\omega^a$ is also unramified. The latter case cannot happen when $\alpha > 0$. Then $|\lambda(T(\mathfrak{p}))|^2 = |\xi(\pi)|^2 = \mathcal{N}_{F/\mathbf{Q}}(\mathfrak{p})^{[n+2v]+1}$. Secondly, we suppose that $\pi_{\mathfrak{p}}(\lambda)$ is a special representation $\sigma(\xi, \eta)$. If ξ is ramified, then η is also ramified, $\alpha \geq 2$, $\alpha > \beta$ and $\lambda(T(\mathfrak{p})) = 0$. If ξ is unramified, then as above, we know that $\alpha = 1$, $\beta = 0$ and (12.1b) holds. When $\pi_{\mathfrak{p}}(\lambda)$ is absolutely cuspidal, then $\alpha \geq 2$, $\alpha > \beta$ and $\lambda(T(\mathfrak{p})) = 0$. This shows the result.

For each ideal $\mathfrak{m}$ of $\mathfrak{r}$, we can define a linear map

$$[\mathfrak{m}]\colon S^*_{k,w,I}(N; M_2(F); \mathbf{C}) \to S^*_{k,w,I}(N\mathfrak{m}; M_2(F); \mathbf{C})$$

$$\text{by } a(\mathfrak{n}, f|[\mathfrak{m}]) = a(\mathfrak{n}\mathfrak{m}^{-1}, f) \qquad (\text{cf. [23]}).$$

Then we see from (4.2) that $T_0(\mathfrak{p}) \circ [\mathfrak{p}]$ is the identity map on $S^*_{k,w,I}(N\mathfrak{p}^\alpha; \mathbf{C})$ for $\alpha > 0$. For each $\mathcal{O}$-algebra homomorphism $\lambda\colon h_{k,w}(Np^\alpha; \mathcal{O}) \to \overline{\mathbf{Q}}_p$, we put for any field extension L/K

$$S^{\mathrm{ord}}(Np^\alpha, \lambda; L) = \Big\{ g \in eS^*_{k,w,I}(Np^\alpha; M_2(F); L) \Big| g|T_0(\ell) = \lambda(T_0(\ell))g$$
$$\text{except for finitely many prime ideals } \ell \Big\}$$

and let f_λ denote the cusp form in $S^*_{k,w,I}(Np^\alpha; M_2(F); \mathbf{C})$ satisfying $a(\mathfrak{n}, f_\lambda) =$

$\lambda(T_0(n))$ for all $n \in \mathrm{Il}(1)$. Then the same argument which proves [11, I, Prop. 4.4 and Lemma 3.3] combined with Lemma 12.2 shows:

PROPOSITION 12.3. *Suppose that N is prime to p and $k \geq 2t$. Let λ: $h_{k,w}^{\mathrm{ord}}(Np^{\alpha}; \mathcal{O}) \to \mathcal{O}$ be an $\mathcal{O}$-algebra homomorphism of conductor C and λ_0: $\eta_{k,w}(C; \mathcal{O}) \to \mathcal{O}$ be the primitive homomorphism associated with λ. Decompose $C = C_0C_1$ such that C_1 divides p^{α} and C_0 divides N. Then $f = f_{\lambda_0}|e$ is non-zero and $S^{\mathrm{ord}}(Np^{\alpha}, \lambda; K) = \sum_{m|N/C_0} K \cdot (f|[m])$. Moreover, let $V = \oplus_{\lambda} S^{\mathrm{ord}}(Np^{\alpha}, \lambda; \overline{\mathbf{Q}}_p)$, where the sum is taken over all $\mathcal{O}$-algebra homomorphisms λ: $h_{k,w}^{\mathrm{ord}}(Np^{\alpha}; \mathcal{O}) \to \overline{\mathbf{Q}}_p$ with conductor divisible by N. Then the subalgebra $\mathscr{H}_V$ of $\mathrm{End}_{\overline{\mathbf{Q}}_p}(V)$ generated over K by Hecke operators $T_0(n)$ for all n is semi-simple.*

We say that λ: $h_{k,w}^{\mathrm{ord}}(Np^{\alpha}; \mathcal{O}) \to \overline{\mathbf{Q}}_p$ is p-adically primitive if $S^{\mathrm{ord}}(Np^{\alpha}, \lambda; \overline{\mathbf{Q}}_p)$ is of dimension 1 (or equivalently, the conductor of λ is divisible by N). This notion coincides with the primitivity of complex cusp forms if the conductor of λ is divisible by every prime factor of p. Put $S_v^{\mathrm{ord}}(N; K/\mathcal{O}) = S_v^{\mathrm{ord}}(N; K)/S_v^{\mathrm{ord}}(N; \mathcal{O}) \cong S_v^{\mathrm{ord}}(N; \mathcal{O}) \otimes_{\mathcal{O}} K/\mathcal{O}$. Then the following fact easily follows from Theorem 5.3:

LEMMA 12.4. *The pairing $\langle\ ,\ \rangle$: $\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \times S_v^{\mathrm{ord}}(N; K/\mathcal{O}) \to K/\mathcal{O}$ induced by the pairing of Theorem* 5.3 *gives isomorphisms*:

$$\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}\big(S_v^{\mathrm{ord}}(N; K/\mathcal{O}), K/\mathcal{O}\big),$$
$$S_v^{\mathrm{ord}}(N; K/\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}\big(\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}), K/\mathcal{O}\big).$$

Proof of Theorem 3.6 *and Corollary* 3.7. For each divisor D of N, the map $[D]$: $S_{k,w,I}^{*}(Np^{\infty}/D; \mathcal{O}) \to S_{k,w,I}^{*}(Np^{\infty}; \mathcal{O})$ preserves the norm (5.3), and since D is prime to p, $[D]$ commutes with $T_0(p)$ and hence with e. Therefore, it induces a map $[D]$: $S_v^{\mathrm{ord}}(N/D; \mathcal{O}) \to S_v^{\mathrm{ord}}(N; \mathcal{O})$ by continuity and also an injection $[D]$: $S_v^{\mathrm{ord}}(N/D; K/\mathcal{O}) \to S_v^{\mathrm{ord}}(N; K/\mathcal{O})$. Then, by Lemma 12.4, it gives a surjective morphism of Λ-modules $[D]^*$: $\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \to \mathbf{h}_v^{\mathrm{ord}}(N/D; \mathcal{O})$. The natural inclusion I_D: $S_v^{\mathrm{ord}}(D; K/\mathcal{O}) \to S_v^{\mathrm{ord}}(N; K/\mathcal{O})$ induces a surjective morphism of Λ-modules I_D^*: $\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O}) \to \mathbf{h}_v^{\mathrm{ord}}(D; \mathcal{O})$. Put $\mathbf{P}(N) = \mathbf{P}_v(N; \mathcal{O}) = \bigcap_{\imath \supsetneq D \supset N}(\mathrm{Ker}([D]^*) \cap \mathrm{Ker}(I_{N/D}^*))$, and write $\mathbf{h}(N)$ for $\mathbf{h}_v^{\mathrm{ord}}(N; \mathcal{O})$ for simplicity. Put

$$S(A) = \sum_{\imath \supsetneq D \supset N} \big(S_v^{\mathrm{ord}}(N/D; A)\big|[D] + S_v^{\mathrm{ord}}(N/D; A)\big) \qquad \text{for } A = \mathcal{O} \text{ and } K/\mathcal{O}.$$

Then $S(A)$ is stable under Hecke operators $T_0(n)$ for all n, and therefore, the Λ-module $\mathbf{P}(N)$ is in fact an ideal of $\mathbf{h}(N)$, since $\mathbf{P}(N)$ is the annihilator of $S(K/\mathcal{O})$. Since $\mathbf{P}(N)$ is the Pontryagin dual module of Λ-divisible module

$S_v^{\mathrm{ord}}(N; K/\mathcal{O})/S(K/\mathcal{O})$, it is without Λ-torsion. Suppose that $h \in \mathbf{P}(N)$ is nilpotent. Then the image h_α in $h_{k,w}^{\mathrm{ord}}(Np^\alpha; \mathcal{O})$ for every $\alpha > 0$ and every $n \sim -2v$ $(n \geq 0)$ is also nilpotent. Since h annihilates $S(\mathcal{O})$, h_α annihilates old forms in $S_{k,w,I}^*(Np^\alpha; M_2(F); K)$. Thus h_α must vanish by Proposition 12.3. This shows that $h = 0$ and thus $\mathbf{P}(N)$ has no nilpotent elements. Now consider $\mathcal{Q} = \mathcal{Q}(N) = \mathbf{h}(N) \otimes_\Lambda \mathcal{L}$ for the quotient field $\mathcal{L}$ of Λ, which is a finite dimensional artinian algebra over $\mathcal{L}$. Since $\mathcal{P}(N) = \mathbf{P}(N) \otimes_\Lambda \mathcal{L}$ has no nilpotent elements, it is actually a semi-simple subalgebra which is a direct factor of $\mathcal{Q}$. Thus we can decompose

(12.2a) $\quad \mathcal{Q}(N) = \mathcal{P}(N) \oplus \mathcal{B}(N)$ *as an algebra direct sum.*

Now we choose an $\mathcal{O}$-valued point P of $\mathrm{Spec}(\Lambda)$ in $\mathcal{X}_{\mathrm{alg}}(\Lambda)$ with $n(P) \geq 2v$. Considering P as an ideal of Λ, we write M_P for the localization at P of each Λ-module M and put $M[P] = \{m \in M \mid am = 0 \text{ for all } a \in P\}$. Since we know that

$$\mathbf{h}(N)_P/P\mathbf{h}(N)_P \cong h_{k,w}^{\mathrm{ord}}\left(Np^{\alpha(P)}, \varepsilon_P; K\right)$$
$$(k = n(P) - 2v + 2t,\ w = v + k - t) \quad \text{by Theorem 3.4,}$$

the kernel of the natural surjection ρ: $\mathbf{h}(N)/P\mathbf{h}(N) \to h_{k,w}^{\mathrm{ord}}(Np^{\alpha(P)}, \varepsilon_P; \mathcal{O})$ is annihilated by $Q \in \Lambda - P$. Then $P + Q\Lambda$ contains a power m^β of the maximal ideal m of Λ. Thus p^β annihilates $\mathrm{Ker}(\rho)$. By Theorem 5.3, this implies

(12.2b) $\quad S_v^{\mathrm{ord}}(N; \mathcal{O})[P] = S_{k,w}^{\mathrm{ord}}\left(Np^{\alpha(P)}, \varepsilon_P; \mathcal{O}\right).$

Similarly we know

(12.2c)
$$S(\mathcal{O})[P] = \sum_{\imath \supsetneq D \supset N} \left(S_{k,w}^{\mathrm{ord}}\left(Np^{\alpha(P)}/D, \varepsilon_P; \mathcal{O}\right)[D] + S_{k,w}^{\mathrm{ord}}\left(Np^{\alpha(P)}/D, \varepsilon_P; \mathcal{O}\right)\right).$$

Let $V = \oplus_\lambda S^{\mathrm{ord}}(Np^{\alpha(P)}, \lambda; \overline{\mathbf{Q}}_p)$, where λ runs over all $\mathcal{O}$-algebra homomorphisms: $h_{k,w}^{\mathrm{ord}}(Np^{\alpha(P)}, \varepsilon_P; \mathcal{O}) \to \overline{\mathbf{Q}}_p$ with conductor divisible by N, and let $\mathcal{H}_V$ be the Hecke algebra for V as in Proposition 12.3. Then (12.2c) implies

(12.2d) $\mathcal{H}_V \cong \mathbf{P}(N)_P/P\mathbf{P}(N)_P$, *which is an algebra direct factor of* $h_{k,w}^{\mathrm{ord}}(Np^{\alpha(P)}, \varepsilon_P; K)$.

This implies that the idempotent of $\mathcal{Q}(N)$ is in fact contained in $\mathbf{P}(N)_P$ and thus

(12.2e) $\quad \mathbf{h}(N)_P = \mathbf{P}(N)_P \oplus \mathbf{B}(N)_P$ *for the complementary algebra direct summand* $\mathbf{B}(N)_P$.

We now define two divisors C, D of N with $CD \supset N$, a map

$$i_{C,D}\colon \mathbf{h}(N)_P \longrightarrow \mathbf{P}(D)_P$$

by the combination: $\quad \mathbf{h}(N)_P \xrightarrow{[C]^*} \mathbf{h}(N/C)_P \xrightarrow{I_D^*} \mathbf{h}(D)_P \xrightarrow{\mathrm{pr}} \mathbf{P}(D)_P,$

where pr denote the projection map of the decomposition (12.2e). We consider the map

$$i = \bigoplus_{C,D} i_{C,D}\colon \mathbf{h}(N)_P \to \bigoplus_{D|N} \bigoplus_{C|(N/D)} \mathbf{P}(D)_P,$$

where (C, D) runs over all pairs of divisors of N with $CD \supset N$. By (12.2d), Proposition 12.3 shows that i induces an isomorphism: $\mathbf{h}(N)_P/P\mathbf{h}(N)_P = \oplus_{D|N} \oplus_{C|(N/D)} \mathbf{P}(D)_P/P\mathbf{P}(D)_P$. Therefore i induces an isomorphism: $\mathbf{h}(N)_P = \oplus_D \oplus_C \mathbf{P}(D)_P$. Then the set of p-adically primitive homomorphisms: $h^{\mathrm{ord}}_{k,w}(Np^{\alpha(P)}, \varepsilon_P; \mathcal{O}) \to \overline{\mathbf{Q}}_p$ consists of homomorphisms factoring through $\mathbf{P}(N)_P/P\mathbf{P}(N)_P \cong \mathscr{K}_V$. Thus we have a bijection:

$$\mathrm{Hom}_\Lambda\left(\mathbf{h}(N)_P, \bar{\mathscr{L}}\right) \cong \mathrm{Hom}_K\left(h^{\mathrm{ord}}_{k,w}\left(Np^{\alpha(P)}, \varepsilon_P; K\right), \overline{\mathbf{Q}}_p\right),$$

and the set of primitive homomorphisms: $\mathbf{h}^{\mathrm{ord}}_v(N; \mathcal{O}) \to \bar{\mathscr{L}}$ consists of Λ-algebra homomorphisms factoring through $\mathbf{P}(N)_P$ and corresponding to the set of p-adically primitive homomorphisms of $h^{\mathrm{ord}}_{k,w}(Np^{\alpha(P)}, \varepsilon_P; K)$ into $\overline{\mathbf{Q}}_p$. This shows the theorem in view of [23]. Corollary 3.7 also follows from the above proof (especially (12.2e)).

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REFERENCES

[1] N. BOURBAKI, *Commutative Algebra*, Hermann, Paris, 1972.

[2] G. E. BREDON, *Sheaf Theory*, McGraw-Hill, 1967.

[3] P. DELIGNE and K. A. RIBET, Values of abelian L-functions at negative integers over totally real fields, Invent. Math. **59** (1980), 227–286.

[4] B. ECKMANN, Cohomology of groups and transfer, Ann. of Math. **58** (1953), 481–493.

[5] G. HARDER, On the cohomology of SL(2, $\mathfrak{O}$), Proc. summer school of group rep., Budapest 1971: *Lie Groups and Their Representations*, pp. 139–150.

[6] ______, On the cohomology of discrete arithmetically defined groups, Proc. Int. Coll. on discrete subgroups of Lie groups and applications to moduli, Bombay 1973, pp. 129–160.

[7] H. HIDA, On abelian varieties with complex multiplication as factors of the abelian variety attached to Hilbert modular forms, Japan. J. Math. **5** (new series, 1979), 157–208.

[8] ______, On abelian varieties with complex multiplication as factors of the jacobians of Shimura curves, Amer. J. Math. **103** (1981), 727–776.

[9] ______, Congruences of cusp forms and special values of their zeta functions, Invent. Math. **63** (1981), 225–261.

[10] ______, On congruence divisors of cusp forms as factors of the special values of their zeta functions, Invent. Math. **64** (1981), 221–262.

[11] ______, A p-adic measure attached to the zeta functions associated with two elliptic modular forms I, Invent. Math. **79** (1985), 159–195; II, Ann. l'institut Fourier 38, (1988).

[12] ______, Iwasawa modules attached to congruences of cusp forms, Ann Scient. Ec. Norm. Sup. 4-éme série **19** (1986), 231–273.

[13] ______, Hecke algebras for GL_1 and GL_2, Sem. de Théorie des Nombres, Paris, 1984–1985, Progress in Math. No **63** (1986), 131–163.

[14] H. HIDA, Galois representations into $GL_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms, Invent. Math. **85** (1986), 545–613.
[15] ______, Modules of congruence of Hecke algebras and L-functions associated with cusp forms, Amer. J. Math **110** (1988), 323–382.
[16] G. HOCHSCHILD and J-P. SERRE. Cohomology of group extensions, Trans. A.M.S. **74** (1953), 110–134.
[17] K. IWASAWA, *Lectures on p-Adic L-Functions*, Ann. of Math. Studies 74, Princeton University Press, Princeton, 1972.
[18] H. JACQUET and R. P. LANGLANDS, *Automorphic Forms on* GL(2), Lecture notes in Math. 114, Springer, Berlin-New York-Heidelberg, 1970.
[19] N. M. KATZ, p-adic L-functions for CM-fields, Invent. Math. **49** (1978), 199–297.
[20] S. LANG, *Cyclotomic Fields*, Grad. Texts in Math. 59, Springer, Berlin-Heidelberg-New York, 1978.
[21] R. P. LANGLANDS, On the zeta function of some simple Shimura varieties, Can. J. Math. **31** (1979), 1121–1216.
[22] Y. MATSUSHIMA and G. SHIMURA, On the cohomology groups attached to certain vector valued differential forms on the product of the upper half planes. Ann. of Math. **78** (1963), 417–449.
[23] T. MIYAKE, On automorphic forms on GL_2 and Hecke operators, Ann. of Math. **94** (1971), 174–189.
[24] ______, *Modular Forms*, Springer.
[25] D. MUMFORDS, *Geometric Invariant Theory*, Springer, Berlin-Heidelberg-New York, 1965.
[26] M. OHTA, On l-adic representations attached to automorphic forms, Japan. J. Math. 8 (new series, 1982), 1–47.
[27] M. RAPOPORT, Compactifications de l'espace de modules de Hilbert-Blumenthal, Compositio Math. **36** (1978), 255–335.
[28] J-P. SERRE, Cohomology des groupes discretes, in *Prospect in Mathematics*, pp. 77–169, Ann. of Math. Studies No. 70, Princeton University Press, Princeton.
[29] G. SHIMURA, On Dirichlet series and abelian varieties attached to automorphic forms, Ann. of Math. **76** (1962), 237–294.
[30] ______, An l-adic method in the theory of automorphic forms (unpublished, 1968).
[31] ______, On canonical models of arithmetic quotients of bounded symmetric domains I, Ann. of Math. **91** (1970), 144–222.
[32] ______, On the Fourier coefficients of modular forms of several variables, Göttingen Nachr. Akad. Wiss. (1975), 261–268.
[33] ______, Theta functions with complex multiplication, Duke Math. J., 43 (1976) 673–696.
[34] ______, The special values of the zeta functions associated with Hilbert modular forms, Duke Math. J. **45** (1978), 637–679.
[35] ______, On certain zeta functions attached to two Hilbert modular forms II. The case of automorphic forms on a quaternion algebra, Ann. of Math. **114** (1981), 569–607.
[36] ______, *Introduction to the Arithmetic Theory of Automorphic Functions*, Iwanami Shoten and Princeton University Press, Tokyo-Princeton, 1971.
[37] A. WEIL, On a certain type of characters of the idèle class group of an algebraic number field, Proc. Symp. on Algebraic Number Theory, Tokyo-Nikko, 1955, pp. 1–7.
[38] ______, *Basic Number Theory*, Springer, Berlin-Heidelberg-New York, 1967.
[39] ______, *Dirichlet Series and Automorphic Forms*, Lecture notes in Math. No. 189, Springer, Berlin-Heidelberg-New York, 1970.
[40] A. WILES, On ordinary λ-adic representations associated to modular forms, preprint.

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