AG-Seminar on **G**-Shtukas
Dozenten: Dr. Ludwig und Prof. Böckle
Wintersemester 18/19
Freitag 9-11 Uhr, INF 205, SR3

In this seminar we want to study the theory of local $\mathbf{G}$-shtukas.
Local $\mathbf{G}$-shtukas are an analogue over local function fields of p-divisible groups with additional structure. One can construct moduli spaces of local $\mathbf{G}$-shtukas – the analogues of Rapoport–Zink-spaces.
There are many more results that parallel the theory in mixed characteristics: The generic fibers of the moduli spaces allow a period morphism. Their special fibers have interesting connections with affine Deligne–Lusztig varieties. Via an analogue of the Serre–Tate theorem one can relate local $\mathbf{G}$-shtukas to global $\mathfrak{G}$-shtukas. Moduli spaces of global $\mathfrak{G}$-shtukas are the function field analogues of Shimura varieties and have been used in the work of V. Lafforgue to construct L-parameters of automorphic representations. Through uniformization morphisms, the cohomology of moduli spaces of global $\mathfrak{G}$-shtukas can be linked to the cohomology of the moduli spaces of local $\mathbf{G}$-shtukas.
We will start the seminar with some background material on reductive groups, loop groups, affine Grassmannians and local GL_n-shtukas. Then we will study the general theory of local $\mathbf{G}$-shtukas following the material from [V] and the references therein. When preparing one of the later talks please start from [V] and consult further references for details and proofs.

Talks

1) Parahoric subgroups and the Kottwitz map

References: [HR],[RR]. Define parahoric subgroups ([HR]). Explain the Kottwitz map $B(G) \to \pi_1(G)_\Gamma$ as in [RR, Section 1].

2) Loop groups and affine Grassmannians

References: [G, Section 2], [Zhu, Lecture 1]. Recall the notion of ind-schemes. Define the loop group $L\mathbf{G}$ and the positive loop group $L^+\mathbf{G}$ of a reductive group $\mathbf{G}$ over a field k. Show that $L\mathbf{G}$ has the structure of an ind-scheme, and that $L^+\mathbf{G}$ is a scheme. Define the affine Grassmannian as well as the affine flag variety and show that both are ind-schemes over k. Explain the Cartan decomposition of the affine Grassmannian and the Iwahori-Bruhat decomposition of the affine flag variety.

3) Torsors for loop groups

References: [HV1, Section 2]. Explain the background on torsors for loop groups from [HV1, Section 2], in particular Proposition 2.2 showing the equivalence of categories for $L^+\mathbf{G}$ torsors defined using different topologies.

4) Local shtukas
References: [HS], [P]. Define local shtukas and z-divisible local Anderson modules as in [HS]. Explain the equivalence between effective local shtukas and z-divisible local Anderson modules ([HS, Theorem 8.3]). For that explain the analogue of Dieudonné theory ([P], cf. [HS, Theorem 5.2]). (As a black box you may use that there is an equivalence of *finite locally free strict* $\mathbb{F}_q$*-module schemes* over an $\mathbb{F}_q$-scheme S, and *balanced finite locally free* $\mathbb{F}_q$*-module schemes* over S that can locally on S be embedded into $\mathbb{G}_a^N$ for some set N.)

5) Local G-shtukas
References: [V, Section 2], [HV1]. Define local **G**-shtukas. Explain the equivalence of categories of local GL_n-shtukas and local shtukas ([HV1, Lemma 4.2]). Introduce the Newton point. Explain the notion of a bound of local **G**-shtukas. Discuss the important class of examples of bounds given by Schubert varieties ([V, Example 2.6]).

6) Deformations
References: [V, Section 3] and [HV1, Section 5]. Define deformations of local **G**-shtukas and show that the formal deformation functor is pro-representable ([V, Theorem 3.2]). Explain the explicit description of the deformation space from [HV1, Section 5] for split **G** and bounds given by Schubert varieties. Discuss [HV1, Example 5.10] showing non-smoothness of the deformation space.

7) Moduli spaces of local G-shtukas
References: [V, Section 4], [AH, Section 4]. Introduce moduli spaces of local **G**-shtukas. For that define the functors $\breve{\mathcal{M}}$ and show that they are representable [V, Theorem 4.3].

8) Generic fibers of moduli spaces and level structures
References: [HV2, Sections 5 and 7]. Define étale local **G**-shtukas over analytic spaces and introduce their dual Tate module. Define level structures and construct a corresponding tower of coverings of the analytified moduli spaces of local **G**-shtukas.

9) Affine Deligne-Lusztig varieties
References: [He], [G, Section 4]. Briefly recall usual Deligne-Lusztig varieties (cf. [G, Section 4.1]). Then give an overview of the study of affine Deligne–Lusztig varieties, focussing on whatever you like.

10) The geometry of the special fiber
References: [V, Section 5]. Show that the special fibers of moduli spaces of local **G**-shtukas are given by certain affine Deligne–Lusztig varieties ([V, Theorem 5.3]).

11) Global $\mathfrak{G}$-shtukas
References: [V, Sections 6.1 and 6.2]. Define global $\mathfrak{G}$-shtukas and their moduli spaces. Prove the analogue of the Serre–Tate theorem ([V, Theorem 6.5]).

References

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