p-divisible Groups **June 2019**
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3. Exercise Sheet

Exercise 1:

Let k be a perfect field of characteristic $p > 0$. Prove that elements of the Dieudonné ring $D(k)$ admit unique expansions

$$c_0 + \sum_{j>0} c_j F^j + \sum_{j>0} c'_j V^j$$

as finite sums with $c_j, c'_j \in W(k)$. Deduce that $D(k)$ has center $W(\mathbb{F}_p) = \mathbb{Z}_p$ if k is infinite and center $\mathbb{Z}_p[F^f, V^f] \cong \mathbb{Z}_p[x, y]/(xy - p^f)$ if k has finite size $q = p^f$. For any extension of perfect fields k'/k define a natural ring map $W(k') \otimes_{W(k)} D(k) \to D(k')$, and prove it is an isomorphism.

Exercise 2:

Let k be a perfect field of char $p > 0$ and assume k is algebraically closed.

(a) Write down all Dieudonné modules of length 1 up to isomorphism. You should end up with three isomorphism classes. To what group schemes do they correspond?

(b) Let G be a group scheme over k of order p^2. Assume that G is connected, self-dual and that $[p] : G \to G$ is the zero map. Assume furthermore that the kernel of relative Frobenius $F : G \to G^{(p)}$ has order p. (The p-torsion $E[p]$ in a supersingular elliptic curve E over k satisfies all these assumptions.) Determine the Dieudonné module of G.

Exercise 3:

Let $\mathcal{O}_K$ be the ring of integers in $K = \mathbb{Q}_p(\zeta_p)$ where ζ_p is a p-th root of unity. Show that the map

$$\mathcal{O}_K[t]/(t^p - 1) \to \mathrm{Maps}(\mathbb{Z}/p\mathbb{Z}, \mathcal{O}_K),\ t \mapsto \sum_{i=1}^{p} \zeta_p^i \delta_i$$

where for $i \in \mathbb{Z}/p\mathbb{Z}$, δ_i is the characteristic function, defines a non-zero morphism

$$\varphi : (\underline{\mathbb{Z}/p\mathbb{Z}})_{\mathcal{O}_K} \to \mu_{p,\mathcal{O}_K}.$$

Show that on special fibres φ is the zero morphism, and that on generic fibres it is an isomorphism.