Iwasawa theory for false Tate extensions of totally real number fields

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Contents

Preface

For an odd prime p, the Classical Iwasawa Main Conjecture for the extension $\mathbb{Q}^{cyc}/\mathbb{Q}$ is a statement on formal power series over the p-adic integers: On one hand, divisibility properties of zeta values are interpreted as existence of an 'analytic p-adic zeta function'. To do this, one introduces p-adic (pseudo-)measures which are evaluated at a character by integrating over it. On the other hand, the structure theory of Galois modules for the cyclotomic extension associates to a class group tower a characteristic ideal in $\mathbb{Z}_p[[T]]$. The Main Conjecture (MC) asserts that this ideal is indeed generated by the p-adic zeta function.

The Non-commutative Main Conjecture of Iwasawa theory (NMC), as the name implies, is a conjecture concerning more general (in particular, non abelian) 'admissible' extensions of totally real number fields, F_∞/F. Like MC, it also asserts existence (and uniqueness) of p-adic zeta functions. These interpolate complex L-values on negative integers and at the same time are linked via K-theory to characteristic Galois modules. The role of $\mathbb{Z}_p[[T]]$ and its fraction field from the classical theory is now played by K_1-groups.

To be more precise, for $G = \mathrm{Gal}(F_\infty/F) \cong H \rtimes \mathbb{Z}_p$ define the canonical Ore set

$$S = \{f \in \Lambda(G) \mid \Lambda(G)/\Lambda(G)f \text{ is finitely generated } \Lambda(H)\text{-module}\}.$$

With it comes the exact localisation sequence

$$K_1(\Lambda(G)) \to K_1(\Lambda(G)_S) \xrightarrow{\partial} K_0(\Lambda(G), \Lambda(G)_S).$$

The p-adic zeta funktion ζ_p is asserted to exist in the middle group and map to a characteristic complex under ∂. Consequently, integration over characters has to be replaced by a method to evaluate elements of K_1 at (non necessarily one-dimensional) Artin representations.

David Burns and Kazuya Kato proposed a strategy to prove NMC: First describe the K_1 group wherein the p-adic zeta function is supposed to exist by K_1 groups of abelian subquotients. Then show relations between classical p-adic zeta functions in these quotients. The first part is often refered to as the algebraic part of the strategy, the second as analytic. In fact, Hilbert modular forms and the q-expansion principle of Deligne and Ribet play a crucial role in the latter. An account of this is not part of this work, but see for example [Kak10], [DW08].

We are concerned with the algebraic part of the strategy, given that G is isomorphic to $L := \mathbb{Z}_p \rtimes \mathbb{Z}_p^\times$, where the second factor acts on the first via multiplication of ring elements. This is a compact p-adic Lie group with non-commutative completed ring $\Lambda(L) := \mathbb{Z}_p[\![L]\!]$. The problem is to describe $K_1(\Lambda(L))$ and $K_1(\Lambda(L)_S)$ by subsets in $\prod_{i\in\mathcal{I}} \Lambda(U_i^{ab})^\times$ and $\prod_{i\in\mathcal{I}} \Lambda(U_i^{ab})_S^\times$, respectively, the product ranging over a suitable family of open subgroups of L.

To this end we make several reduction steps that are also employed in the much more general work of Kakde, [Kak10], and are dictated by representation theory: First reduce to one-dimensional admissible quotients $L_c, c \in \mathbb{N}$, of L using a result of Kato and Fukaya on projective limits of K-groups, then to hyperelementary subgroups $\mathcal{H}$ of L_c. If $\mathcal{H}$ is l-hyperelementary for some $l \neq p$ then group theoretic arguments show that NMC holds. So ultimately we have reduced the algebraic part for $L = \mathrm{Gal}(F_\infty/F)$ to the case where $\mathcal{H} = \mathrm{Gal}(F'_\infty/F')$ is a p-hyperelementary subgroup of $L_c, c \in \mathbb{N}$. Then a simple group theoretic argument shows that $\mathcal{H}$ is already p-elementary: $\mathcal{H} = \Delta \times P$, P pro-p, Δ cyclic with $p \nmid |\Delta|$.

We denote the dual group of Δ by $\widehat{\Delta}$. One can now proceed in two ways: Either use the canonical isomorphism $\Lambda(\mathcal{H}) \cong \bigoplus_{\chi\in\widehat{\Delta}} \Lambda_\chi(P)$ and the integral logarithm $\mathcal{L}$ of Oliver and Taylor for the pro-p group P or make use of the fact that $\mathcal{L}$ is defined for all pro-finite groups and describe kernel and cokernel of $\mathcal{L}$. The latter involves group homology computations, of which we give some examples to hint at techniques necessary in more general cases, but then will follow the first approach.

The integral logarithm $\mathcal{L}_L$ takes its image in the $\mathbb{Z}_p$-module $\mathbb{Z}_p[\![\mathrm{Conj}(L)]\!]$. We define a suitable family of subgroups $U_i \leq L$ and trace maps τ_i to give an explicit description

$$\tau = (\tau_i)\colon \mathbb{Z}_p[\![\mathrm{Conj}(L)]\!] \xrightarrow{\sim} \Omega \subseteq \prod_i \mathbb{Z}_p[\![U_i^{ab}]\!].$$

In the multiplicative world of K_1 the role of Ω is played by $\Psi \subseteq \prod_i \Lambda(U_i^{ab})^\times$ and τ is replaced by a norm map θ. It is the definition of this Ψ that really poses problems when one deals with a non-p group: $\varphi\colon G \to G, x \mapsto x^p$ induces the transfer map for abelian quotients only if G is a pro-p group. This is exactly the reason why we will give a description of Ψ only for p-elementary quotients of L.

By the following crucial diagram, cf. 4.41, $\theta\colon K_1(\Lambda(G)) \to \Psi$ is an isomorphism

$$\begin{tikzcd}
1 \arrow[r] & \mu \times G^{ab} \arrow[r] \arrow[d, "id"] & K_1(\Lambda(G)) \arrow[r, "\mathcal{L}_G"] \arrow[d, "\theta"] & \mathbb{Z}_p[\![\mathrm{Conj}\, G]\!] \arrow[r, "\omega"] \arrow[d, "\tau", "\wr"'] & G^{ab} \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & \mu \times G^{ab} \arrow[r, "\tilde{\theta}"] & \Psi \arrow[r, "\widetilde{\mathcal{L}}"] & \Omega \arrow[r, "\tilde{\omega}"] & U_0/V_0 \arrow[r] & 1.
\end{tikzcd}$$

To finish the algebraic part of the strategy we define an analogous set Ψ_S and norm map $\theta_S\colon K_1(\Lambda(G)_S) \to \Psi_S$ for the localized Iwasawa algebras, such that $\Psi_S \cap \prod_i \Lambda(U)_i^{ab} = \Psi$. This will conclude the paper. To proof the main conjecture one would have to show properties (i_S) and (ii_S) from section 4.5 relating pairs of abelian p-adic zeta functions.

Kato computed K_1 for open subgroups of L in [Kat05]. These are precisely the groups $p^m \cdot \mathbb{Z}_p \rtimes (\Delta \times U^{(n)})$ with $n, m \geq 0, \Delta \leq \mu_{p-1}$ and $U^{(n)} := \ker(\mathbb{Z}_p^\times \to (\mathbb{Z}/(p^n))^\times)$. His congruences relate not only pairs of p-adic zeta functions, but take the form, cf. *loc.cit.* 8.12.1,

$$\prod_{0<i\leq n} \mathcal{N}^{n-i}(c_i)^{p^i} \equiv 1 \bmod p^{2n}, \text{ for all } n \geq 1,$$

where each c_i contains information about the 0-th as well as the i-th p-adic zeta function in the tower of subquotients.

Some of these were proved in the thesis of Thomas Ward [DW08] by using Hilbert modular forms associated to elliptic curves. It seems to me that given the description obtained in section 4.5 one could expect a similar argument as in [Lee09], Chapter 4, to work. This is considerably easier, since the transfer map ver is naturally defined on Λ-adic Hilbert modular forms and corresponds to the p-power map $\varphi\colon \Lambda(U_n^{ab}) \to \Lambda(U_{n+1}^{ab})$. If one skips the reductions in chapter 3 then the first filtration step $U_0 \supseteq U_1$ were of index $p-1$ leading to the above mentioned problem.

The paper is structured as follows:

In chapter 1 we fix notation and recall the necessary algebraic K-theory.

In chapter 2 we explain the arithmetic setting and what is associated to it: A canonical Ore set $S \subseteq \Lambda(G)$, evaluation at Artin representations and a characteristic complex $C(F_\infty/F)$.

We will then state the conjecture precisely in the language of Coates et. al., cf. 2.18, and in the formulation of Ritter and Weiss, cf. 2.26. Both formulations are in fact equivalent: A proof of this is given in the case where the Galois group contains no element of order p. It is a well known result of Serre, that this implies regularity of the completed group ring $\Lambda(G)$.

We conclude the chapter by outlining the general strategy of Burns and Kato of which the final proposition 4.42 will be a variant.

In the short chapter 3 we follow the reduction steps usually used for Mackey functors to reduce to $\mathrm{Gal}(F_\infty/F)$ being p-elementary.

The last and main chapter 4 finally gives necessary conditions for the NMC to hold for certain extensions. Following Kato's sketch [Kat07] of the same strategy for the p-adic Heisenberg group, which was greatly expanded by A. Leesch in [Lee09], we define an additive map τ, a multiplicative map θ and sets Ω and Ψ as in the five lemma diagramm above. This crucially involves the integral logarithm $\mathcal{L}_G$ of Oliver and Taylor. We will just use it as a tool, since there are already several references on its construction, convergence etc.

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Chapter 1

Preliminaries

In this chapter we fix notation and basic facts that will be used throughout the article. Most of it is standard and as a result we keep this as short as possible.

1.1 Notation

- We assume all rings to be associative and unital. We denote by $R\,[[T]]$ the ring of formal power series over a ring R in one indeterminate T. The term R-module always means *left* R-module if not otherwise specified, likewise the term ideal will be used for *left* ideal.
- We denote the Jacobson radical of a ring R, i.e. the intersection of all its maximal left (or right) ideals, by $Jac(R)$. A ring R is called *local* if R has exactly one maximal (left or right) ideal. R is called *semi-local* if $R/Jac(R)$ is a semisimple Artinian ring.
- Let G_1, G_2 be topological groups. We write $G_1 \trianglelefteq_o G_2$, $G_1 \trianglelefteq_{cl} G_2$, $G_1 \leq_o G_2$ or $G_1 \leq_{cl} G_2$ if G_1 is open normal, closed normal, open or closed subgroup in G_2, respectively.
- As usual, for a group G and subsets $U, V \subseteq G$, the subgroup of G generated by the commutators $uvu^{-1}v^{-1}, u \in U, v \in V$ is denoted by $[U, V]$.
- In contrast, for a ring R, an R-algebra A and subsets $S, T \subseteq A$, the R-subalgebra of A generated by elements $st - ts$, $s \in S, t \in T$ is also denoted by $[S, T]$.

- For a commutative topological ring R and a profinite group G we define the *completed group ring*

$$R[\![G]\!] := \varprojlim_{U \trianglelefteq_o G} R\,[G/U]\,.$$

For $\mathcal{O}$ the ring of integers in a finite extension of $\mathbb{Q}_p$ we set $\Lambda_{\mathcal{O}}(G) := \mathcal{O}[\![G]\!]$, the *Iwasawa algebra* of G with coefficients in $\mathcal{O}$. We simply write $\Lambda(G)$ for $\mathbb{Z}_p[\![G]\!]$ and $\Omega(G)$ for $\mathbb{F}_p[\![G]\!]$.

- For a normal subgroup $U \lhd G$ the kernel of the R-module map $R[\![G]\!] \to R[\![G/U]\!]$ is denoted by $I(U)$. It is the ideal generated by elements $(1-u), u \in U$. In the special case $U = G$ the *augmentation ideal* $\ker(R[\![G]\!] \to R)$ is denoted by I_G.

- For a profinite group G the G-homology groups of a compact G-module A are the left derived functors of the G-coinvariants functor $A_G := A/I_G \cdot A$. Denoting the Pontryagin dual by $\cdot^\vee$, we have functorial isomorphisms $H_i(G, A)^\vee \cong H^i(G, A^\vee)$, cf. (2.6.9) in [NSW08].

Let R be a ring. A subset $S \subset R$ is called multiplicatively closed if $1 \in S$ and $x, y \in S$ implies $xy \in S$. Such S is called a right (resp. left) Ore subset for R if for each $r \in R, s \in S$ there exists $r' \in R, s' \in S$ with $s'r = r's$ (resp. $rs' = sr'$). If S is in addition right (left) invertible[1] Asano proved, building on work of Ore, the

Lemma 1.1. *For R as above and a right Ore and right invertible subset $S \subseteq R$ the right ring of fractions RS^{-1} exists in the sense that there is a ring homomorphism $\varepsilon\colon R \to RS^{-1}$ with*

- *$\varepsilon(s) \in (RS^{-1})^\times$,*
- *every element in RS^{-1} is of the form $\varepsilon(r)\varepsilon(s)^{-1}$,*
- *and $\ker \varepsilon = \{r \in R \mid \exists s \in S : rs = 0\}$, i.e. ε is injective if S contains no zero divisors.*

An analogous assertion holds for 'right' replaced by 'left'.

An element $x \in R$ is called right (resp. left) regular if $xr = 0$ (resp. $rx = 0$) for any $r \in R$ implies $r = 0$. Left and right regular elements are simply called regular. The subset X of all regular elements of a ring R is multiplicatively

[1] a condition which is trivially satisfied if S contains no right zero divisors of R

closed. In a commutative ring X is obviously Ore. We denote the localization of R by X with $Q(R) := RX^{-1}$ and call it the *total ring of fractions* of R.

If R is a ring and R_S its localization at an Ore subset $S \subseteq R$ a left R-module M is called S-*torsion* if $R_S \otimes_R M = 0$.

For more on this subject we refer to §10 of [Lam99].

1.2 p-adic Lie groups

A p-*adic analytic manifold* of dimension d is a topological space X with an (equivalence class of an) atlas $\{(U_i, \varphi_i)_{i \in \mathcal{I}}\}$ of open subsets $U_i \subseteq X$, the $\varphi_i : U_i \to V_i$ being compatible homeomorphisms onto open subsets $V_i \subseteq \mathbb{Q}_p^d$ for some $d \in \mathbb{N}$. Here 'compatible' means that $\varphi_i \circ \varphi_j^{-1} : \mathbb{Q}_p^d \to \mathbb{Q}_p^d$ are p-adic analytical maps, when defined (i.e., when $U_i \cap U_j \neq \emptyset$). For details see [DdSMS99a], Ch. 8, in particular definitions 8.2, 8.6 and 8.8.

A p-*adic Lie group* is a group object in the category of p-adic manifolds with p-adic analytic maps as morphisms.

The interplay between algebraic and analytical structure of p-adic Lie groups (or p-adic analytical groups as they are often called) is very strong and often surprising:

Proposition 1.2. *([DdSMS99b], corollary 8.34) A topological group P is a compact p-adic Lie group if and only if G is profinite containing an open normal uniform*[2] *pro-p subgroup of finite rank.*

Lemma 1.3. *Let G be a compact p-adic Lie group. Denote the radical of the Iwasawa algebra $\Lambda_{\mathcal{O}}(G)$ by J.*

1. *The J-adic topology (where $J^n, n \in \mathbb{N}$ is a fundamental system of neighbourhoods of 0) is the same as the canonical topology of $\Lambda_{\mathcal{O}}(G)$, the latter having $\mathfrak{m}_{\mathcal{O}}^n + I(U), U \subset G$ open, normal as fundamental system.*

2. *$\Lambda_{\mathcal{O}}(G)$ is a semi-local ring and local if and only if G is pro-p.*

3. *If G is pro-p, the maximal ideal of $\Lambda_{\mathcal{O}}(G)$ equals $\mathfrak{m}\Lambda_{\mathcal{O}}(G) + I_G$.*

[2] a condition on certain subgroup series, which we are not going to explain

Proof. For the second and third assertion combine Proposition 1.2 with [NSW08], Proposition 5.2.16. It is proved in *loc.cit.* that the J-adic topology is always finer than the canonical topology of $\Lambda_{\mathcal{O}}(G)$. To see the first assertion note that a compact p-adic Lie group always has an open pro-p subgroup by prop. 1.2. So the index $(G : G_p)$ of G by a p-Sylow subgroup is finite and then J is open in the canonical topology. $\square$

1.3 Lower Algebraic *K*-Theory

All definitions are standard and collected here for convenience. We associate abelian groups K_0 and K_1 to an exact category in terms of generators and relations and denote their group operation additively for K_0, resp. multiplicatively for K_1.

Let $\mathcal{C}$ be a full additive subcategory of an abelian category having a small skeleton, i.e. $\mathcal{C}$ has a full subcategory $\mathcal{C}_{sk}$ which is small and for which the inclusion $\mathcal{C}_{sk} \hookrightarrow \mathcal{C}$ is an equivalence of categories. Let $\mathcal{C}$ be closed under extensions. Such a $\mathcal{C}$ is called an *exact* category.

Definition 1.4. $K_0(\mathcal{C})$ is the abelian group generated by isomorphism classes $[M]$ of objects M of $\mathcal{C}_{sk}$ devided by the relation $[M] = [M'] + [M'']$ if there is an exact sequence[3] $0 \to M' \to M \to M'' \to 0$.

Remark 1.5. Most important to us is the category of finitely generated projective (left-) modules over a ring R, denoted by $\mathcal{P}(R)$. The collections of direct summands $U \oplus V \cong R^n$ for all natural n are sets and their union constitutes a small, skeletal subcategory of $\mathcal{P}(R)$. We denote $K_i(\mathcal{P}(R))$ simply by $K_i(R)$ for i=0,1. Note that $K_0(R)$ is the Grothendieck group of the abelian semigroup $\mathcal{P}_{sk}(R)$ with the direct sum operation and the 0-module as unit element. This is why K_0 of a ring can be thought of as the group where a universal dimension function takes its values. Indeed, every additive function from $\mathcal{P}_{sk}(R)$ to an abelian group must factor through K_0.

Definition 1.6. $K_1(\mathcal{C})$ is the free abelian group on pairs $[M, f]$ with M an object of $\mathcal{C}_{sk}$ and f an automorphism of M. The relations we mod out are

- $[M, f \circ g] = [(M, f)] \cdot [M, g]$ and

[3]'exact' always means exact in the surrounding abelian category

- $[M, f] = [M', f'] \cdot [M'', f'']$ if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & M' \arrow[r] & M \arrow[r] & M'' \arrow[r] & 0. \end{tikzcd}$$

Remark 1.7. There is the following, well known (cf. [Ros96], Thm. 3.1.7) equivalent description of $K_1(R)$: For $a \in R$, define the *elementary matrix* $e_{ij}(a) \in \mathrm{GL}_n(R)$ as the matrix with 1's on the diagonal, with a as the (i, j)-entry and 0's elsewhere. Let $E_n(R) := \langle e_{ij}(a) \mid 1 \le i, j \le n, i \ne j, a \in R \rangle$ be the subgroup of $\mathrm{GL}_n(R)$ generated by these elements. Via the canonical inclusions

$$\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R),\ g \mapsto \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

we define $\mathrm{GL}(R) := \varinjlim_n \mathrm{GL}_n(R)$ and $E(R) := \varinjlim_n E_n(R)$.

Then $K_1(R) \cong \mathrm{GL}(R)/E(R) \cong \mathrm{GL}(R)^{ab} := \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$.

In the case of semi-local rings Vaserstein (cf. [Vas05]) showed that the isomorphism in Remark 1.7 is drastically simpler:

Proposition 1.8. *Let R be a (non-commutative) associative ring with unit such that $R/Jac(R)$ is a product of full matrix rings over division algebras, none of these rings is isomorphic to $M_2(\mathbb{Z}/2\mathbb{Z})$ and not more than one of these rings has order 2. Then $K_1(R) \cong R^\times/[R^\times, R^\times]$, the abelianization of the units of R.*

Definition 1.9. The *relative K_0 group*, $K_0(R, R')$, for a ring homomorphism $\varphi : R \to R'$ is defined as follows: It is the (additive) abelian group generated by triples (M, N, f) with $M, N \in \mathcal{P}(R)$, such that $f\colon R' \otimes_\varphi M \xrightarrow{\sim} R' \otimes_\varphi N$ is an isomorphism of R'-modules. The relations are generated by $[(M, N, fg)] = [(M, N, f)] + [(M, N, g)]$ and $[(M, N, f)] = [(M_1, N_1, f_1)] + [(M_2, N_2, f_2)]$ for every commutative diagramm

$$\begin{tikzcd} 0 \arrow[r] & R' \otimes M_1 \arrow[r] \arrow[d, "f_1"] & R' \otimes M \arrow[r] \arrow[d, "f"] & R' \otimes M_2 \arrow[r] \arrow[d, "f_2"] & 0 \\ 0 \arrow[r] & R' \otimes N_1 \arrow[r] & R' \otimes N \arrow[r] & R' \otimes N_2 \arrow[r] & 0. \end{tikzcd}$$

Of course, the K_i are functors on **Rings**: A homomorphism of rings $\varphi\colon R \to R'$ gives rise to maps

$$K_0(\varphi)\colon [M] \mapsto [R' \otimes_\varphi M]$$
$$K_1(\varphi)\colon [M, f] \mapsto [R' \otimes_\varphi M, id \otimes f]$$

These are well defined homomorphisms since if $M \in \mathcal{P}(R)$ then there exists $N \in \mathcal{P}(R)$ with $M \oplus N \cong R^n$, hence $(R' \otimes M) \oplus (R' \otimes N) \cong R' \otimes (M \oplus N) \cong R'^n$. The tensor product commutes with finite direct sums so φ induces a semigroup morphism $\mathcal{P}_{sk}(R) \to \mathcal{P}_{sk}(R')$. From Remark 1.5 then follows the homomorphism property of the induced map of Grothendieck groups.

Furthermore let R, R', S, S' be rings with homomorphisms, such that the following diagram commutes

$$\begin{tikzcd} R' \arrow[r, "\varphi'"] & S' \\ R \arrow[u] \arrow[r, "\varphi"] & S \arrow[u] \end{tikzcd}$$

and define the map

$$(\varphi, \varphi')_*\colon K_0(R, R') \to K_0(S, S'), \quad (M, N, f) \mapsto (S \otimes_\varphi M, S \otimes_\varphi N, id_{S'} \otimes_{\varphi'} f).$$

Under some conditions there are maps going in the reverse direction:

Definition 1.10. Let $\varphi\colon R \to S$ be a homomorphism of rings and for any S-module M let $r_R M$ be the R module obtained by restriction of scalars, i.e. $r \cdot m = \varphi(r) \cdot m$ for $r \in R, m \in M$. If $R' \in Ob(\mathcal{P}(R))$ and $S' \in Ob(\mathcal{P}(S))$, as above, are in addition finitely generated projective modules themselves then define homomorphisms

$$\begin{aligned} \mathrm{Tr} := \text{Trace}\colon K_0(S) \to K_0(R), &\quad [M] \mapsto [r_R M] \\ \mathrm{N} := \text{Norm}\colon K_1(S) \to K_1(R), &\quad [(M, f)] \mapsto [(r_R M, f)] \\ \mathrm{Tr} := \text{Trace}\colon K_0(S, S') \to K_0(R, R'), &\quad [(M, N, f)] \mapsto [(r_R M, r_R N, f)]. \end{aligned}$$

We recall some key facts about this K-groups:

Proposition 1.11. *(Localisation sequence, cf. [Ven05], §3) For the inclusion morphism $\iota\colon R \to R_S$ of a ring R into the localisation at a suitable*[4] *multiplicative subset S, there are maps*

$$\partial\colon [(R_S \otimes_R M, F)] \mapsto [(M, M, f)],$$

[4] S Ore without zero-divisors is sufficient by work of Berrick-Keating

$$\lambda\colon [(M,N,f)] \mapsto [M]-[N]$$

yielding an exact sequence of abelian groups

$$K_1(R) \xrightarrow{K_1(\iota)} K_1(R_S) \xrightarrow{\partial} K_0(R,R_S) \xrightarrow{\lambda} K_0(R) \xrightarrow{K_0(\iota)} K_0(R_S). \tag{1.1}$$

The following diagram commutes with R, R', S, S' as in Definition 1.10

$$\begin{tikzcd}
K_1(S) \arrow[r] \arrow[d, "N"] & K_1(S') \arrow[r, "\partial"] \arrow[d, "N"] & K_0(S,S') \arrow[r, "\lambda"] \arrow[d, "Tr"] & K_0(S) \arrow[r] \arrow[d, "Tr"] & K_0(S') \arrow[d, "Tr"] \\
K_1(R) \arrow[r] & K_1(R') \arrow[r, "\partial"] & K_0(R,R') \arrow[r, "\lambda"] & K_0(R) \arrow[r] & K_0(R').
\end{tikzcd}$$

Proposition 1.12. *(Morita equivalence, cf. [Ros96], 1.2.4, 2.1.8) For a ring R and the ring of $n \times n$ matrices $\mathrm{M}_n(R)$ over R there are canonical isomorphisms*

$$K_0(R) \cong K_0(\mathrm{M}_n(R)), [M] \mapsto [R^n \otimes_R M],$$
$$K_1(R) \cong K_1(\mathrm{M}_n(R)), [(M,f)] \mapsto [(R^n \otimes_R M, id_{R^n} \otimes_R f)].$$

Now let R be a Noetherian ring and R_S the localisation of R at a left and right Ore subset $S \subseteq R$. Denote by $\mathcal{H}_S^R$ the category of finitely generated S-torsion modules that have a finite resolution by objects in $\mathcal{P}(R)$. It is an exact category by the horseshoe lemma. Further let $\mathcal{C}_S^R$ denote the category of bounded complexes $C^\bullet$ of modules in $\mathcal{P}(R)$, such that $R_S \otimes C^\bullet$ is exact. This obviously is an exact category. For the K_0 groups of these we have the technically important

Proposition 1.13. *If ∂ in diagram 1.1 is surjective we have isomorphisms $K_0(R,R_S) \cong K_0(\mathcal{H}_S^R) \cong K_0(\mathcal{C}_S^R)$*

Proof. Since ∂ is surjective, $K_0(R,R_S)$ is generated by elements of the form $[M,M,gs^{-1}]$ with M a f.g. free R-bimodule, $g \in Hom_R(M,M), s \in S$. By tensoring the exact sequence $Rs \to R \to R/Rs \to 0$ from left with R_S we see that R/Rs and with it M/Ms is a (left) S-torsion module. Therefore

$$K_0(R,R_S) \to K_0(\mathcal{H}_S^R), \quad [M,M,f] \mapsto [M/g(M)] + [M/Ms]$$

is a well defined map.

Let $0 \to P^\bullet \to H \to 0$ be a defining resolution of $H \in \mathcal{H}_S^R$, then obviously

$$K_0(\mathcal{H}_S^R) \to K_0(\mathcal{C}_S^R), [H] \mapsto [P^\bullet]$$

is well defined. Finally, for an object $(D^\bullet, d^\bullet) \in \mathcal{C}_S^R$ the isomorphisms $R_S \otimes D^i \cong \ker d'^i \oplus \operatorname{im} d'^i \cong \operatorname{im} d'^{i-1} \oplus \operatorname{im} d'^i$, where d' denote the induced coboundary maps on $R_S \otimes D^\bullet$, give an isomorphism f fitting in the diagramm

$$\begin{tikzcd}
\bigoplus_i R_S \otimes D^{2i} \arrow[r, "f"] \arrow[d, "\wr"] & \bigoplus_i R_S \otimes D^{2i+1} \arrow[d, "\wr"] \\
\bigoplus_i (\operatorname{im} d'^{2i-1} \oplus \operatorname{im} d'^{2i}) \arrow[r, "id"] & \bigoplus_i (\operatorname{im} d'^{2i+1} \oplus \operatorname{im} d'^{2i+2}).
\end{tikzcd}$$

This gives rise to a map

$$K_0\mathcal{C}_S^R \to K_0(R, R_S), [D^\bullet] \mapsto [\bigoplus_i D^{2i}, \bigoplus_i D^{2i+1}, f].$$

We want to apply the five lemma to sequences analogue to 1.1. The maps corresponding to ∂ and λ are

$$\begin{aligned}
K_1(R_S) \to K_0(\mathcal{H}_S^R), &\quad [R_S \otimes M, gs^{-1}] \mapsto [M/g(M)] + [M/Ms], \\
K_1(R_S) \to K_0(\mathcal{C}_S^R), &\quad [R_S \otimes M, gs^{-1}] \mapsto [M \xrightarrow{g}] + [M \xrightarrow{\cdot s} M], \\
K_0(\mathcal{H}_S^R) \to K_0(R), &\quad [H] \mapsto \sum_i (-1)^i [P^i], \\
K_0(\mathcal{C}_S^R) \to K_0(R), &\quad [D^\bullet] \mapsto \sum_i (-1)^i [D^i].
\end{aligned}$$

One checks easily that these give commutative squares with the above defined group homomorphisms. Three applications of the five lemma finish the proof. $\square$

If $R = \Lambda(G)$, the Iwasawa algebra of a profinite group, and $S = \Lambda(G)_T$ for some Ore subset T we will write $\mathcal{H}_T^{\Lambda(G)} := \mathcal{H}_{\Lambda(G)_T}^{\Lambda(G)}$ and $\mathcal{C}_T^{\Lambda(G)} := \mathcal{C}_{\Lambda(G)_T}^{\Lambda(G)}$.

Lemma 1.14. *If $\Lambda(G)$ is regular and $D^\bullet \in \mathcal{C}_S^{\Lambda(G)}$ then the image of $[D^\bullet]$ in $K_0(\mathcal{H}_S^{\Lambda(G)})$ is $\sum_i (-1)^i [H^i(D^\bullet)]$.*

Proof. For an object $M \in \mathcal{H}_S^{\Lambda(G)}$ choose a projective resolution $P^\bullet \to M \to 0$. We just proved in the proposition above, that $\phi\colon K_0(\mathcal{H}_S^{\Lambda(G)}) \to K_0(\mathcal{C}_S^{\Lambda(G)}), [M] \mapsto [P^\bullet]$ is an isomorphism. Set

$$\psi\colon K_0(\mathcal{C}_S^{\Lambda(G)}) \to K_0(\mathcal{H}_S^{\Lambda(G)}), [D^\bullet] \mapsto \sum_i (-1)^i [H^i(D^\bullet)].$$

By the regularity assumption $H^i(D^\bullet) \in \mathcal{H}_S^{\Lambda(G)}, \forall i$. We compute $\psi \circ \phi([H]) = \sum_i (-1)^i [H^i(P^\bullet)] = [H^0(P^\bullet)] = [H]$, hence $\psi \circ \phi = id$ and $\psi = \phi^{-1}$ because we already know that ϕ is an isomorphism. $\square$

Definition 1.15. For a finite extension L of $\mathbb{Q}_p$ let $\mathcal{O}$ be the ring of integers of L and let Δ be a finite group. We define $SK_1(\mathcal{O}[\Delta]) = ker(K_1(\mathcal{O}[\Delta] \to K_1(L[\Delta]))$. For a profinite group $G = \varprojlim \Delta$ we define $SK_1(\Lambda_{\mathcal{O}}(G)) = \varprojlim SK_1(\mathcal{O}[\Delta])$. For the ring homomorphisms $\Lambda_{\mathcal{O}}(G) \to \Lambda_{\mathcal{O}}(G)_S$ and $\Lambda_{\mathcal{O}}(G) \to \widehat{\Lambda_{\mathcal{O}}(G)}_S$ associated to an Ore subset S, define $SK_1(\Lambda_{\mathcal{O}}(G)_S)$ and $SK_1(\widehat{\Lambda_{\mathcal{O}}(G)}_S)$ to be the image of $SK_1(\Lambda_{\mathcal{O}}(G))$ under the induced maps of K_1 groups. Here $\widehat{\Lambda_{\mathcal{O}}(G)}_S$ is the p-adic completion of $\Lambda_{\mathcal{O}}(G)_S$.

Chapter 2

The Main Conjecture of Non-commutative Iwasawa Theory

After a short review of some algebraic properties of Iwasawa algebras and their localisations we introduce the protagonists of the theory: we define *admissible* extensions of number fields and attach two objects to them, one of algebraic nature, the other one analytic and then formulate the Main Conjecture for Non-commutative Iwasawa Theory (MC for short) in the terms of Coates et al. linking those two objects. In this formulation it was proven by Mahesh Kakde (up to uniqueness of the p-adic zeta function) in [Kak10]. Jürgen Ritter and Alfred Weiss have also come forward with a similar conjecture, the Equivariant Main Conjecture (EMC for short). Its formulation doesn't generalize as well as for MC but in its scope it is equivalent, a fact that is surely known to the expert but will be shown here in detail in the easier regular case. Ritter and Weiss have also shown their Equivariant Main Conjecture to hold [RW10], also up to a uniqueness assertion.

In what follows let p always be a fixed odd prime number.

2.1 A canonical Ore set

Suppose we are given a compact p-adic Lie group G with a closed normal subgroup H such that $G/H \cong \Gamma \cong \mathbb{Z}_p$.

Definition 2.1. Let $S \subseteq \Lambda(G)$ be the set consisting of all $f \in \Lambda(G)$ such that $\Lambda(G)/\Lambda(G)f$ is a finitely generated $\Lambda(H)$-module. S will be called the *canonical Ore subset* of $\Lambda(G)$.

The set S was first described in [Ven05]. Equivalent definitions and the following properties of it are given in [CFK+05].

Proposition 2.2. *(i) If J is any open pro-p subgroup of H, which is normal in G, then $f \in \Lambda(G)$ is in S iff right multiplication by the image of f in $\Omega(G/J)$ is injective.*

(ii) S is a multiplicatively closed left and right Ore subset and contains no zero divisors.

Proof. (i) is part 3 of lemma 2.1 in [CFK+05] and (ii) is theorem 2.4 in *loc.cit.*. □

We conclude that the left and right localisations w.r.t. to S exist (cf. [GW04], theorem 10.3, propositions 10.6 and 10.7). We identify them and denote them by $\Lambda(G)_S$. It follows from the second part of the proposition that the natural map $\Lambda(G) \to \Lambda(G)_S$ is injective.

The notation S instead of the seemingly more appropriate $S(G)$ is justified by the following
Lemma 2.3. *Let $U \subseteq G$ be an open subgroup, then $\Lambda(G)_{S(G)} \cong \Lambda(G)_{S(U)}$.*

Proof. This is proposition 2.10 in [Lee09]. □

If M is a $\Lambda(G)$-module sucht that for each $m \in M$ there is some $s \in S$ with $sm = 0$ then M is called S*-torsion*, in other words we have $\Lambda(G)_S \otimes_{\Lambda(G)} M = 0$.

Proposition 2.4. *([CFK+05], prop. 2.3) A $\Lambda(G)$-module M is S-torsion if and only if it is finitely generated over $\Lambda(H)$.*

Proposition 2.5. *Suppose that G is a compact p-adic Lie group.*

(i) The localisation of the Iwasawa algebra $\Lambda(G)_S$ is semi-local.

(ii) The ∂ homomorphism in the exact localisation sequence 1.1 coming from the injection $\Lambda(G) \hookrightarrow \Lambda(G)_S$ is surjective.

Proof. (i) is [CFK+05], Proposition 4.2. (ii) was proven in *loc. cit.* under the condition that G contains no element of order p. M. Kakde observed in [Kak08], Lemma 1.5, that this condition can be removed. □

2.2 Admissible extensions

The arithmetic situation is shown in the figure below and the fields involved are supposed to satisfy the following conditions:

$$\begin{tikzcd}
 & M_\Sigma \arrow[dl, dash, "X"'] \\
F_\infty \arrow[dr, dash, "H"] \arrow[dd, dash, "G"] & \\
 & F^{cyc} \arrow[dl, dash, "\Gamma"'] \\
F \arrow[d, dash] & \\
\mathbb{Q} &
\end{tikzcd}$$

Assumption 2.6.

- *F is finite over $\mathbb{Q}$ and totally real.*
- *$F_\infty|F$ is Galois and unramified outside a finite set Σ of primes of F.*
- *F_∞ is totally real.*
- *$G := \mathrm{Gal}(F_\infty|F)$ is a p-adic Lie group.*
- *F_∞ contains F^{cyc}, the cyclotomic pro-p extension of F, with $\mathrm{Gal}(F^{cyc}|F) \cong \mathbb{Z}_p$.*

If this is satisfied we call the extension F_∞/F *admissible* and denote the maximal abelian, pro-p and outside of Σ unramified extension of F_∞ by M_Σ. Hence $X := \mathrm{Gal}(M_\Sigma/F_\infty)$ is a natural $\mathbb{Z}_p$-module.

From 2.6 follows that G has a closed normal subgroup H with $G/H \cong \Gamma$. By the previous section we can associate to $\Lambda(G)$ a canonical Ore set S. From now on fix an isomorphism $\Gamma \cong \mathbb{Z}_p$ by choosing a topological generator $\gamma \in \Gamma$. Note though, that the group law in Γ will be denoted multiplicatively. Let M_Σ be the Galois group of the maximal abelian pro-p extension of F_∞, unramified outside the primes above Σ. We set $X = \mathrm{Gal}(M_\Sigma|F_\infty)$. As abelian pro-$p$ group X automatically is a $\mathbb{Z}_p$-module. For $g \in G$ let $\hat{g}$ be an inverse image of g under the natural surjection $\mathrm{Gal}(M_\Sigma|F) \twoheadrightarrow G$ and define $g \cdot x = \hat{g}x\hat{g}^{-1}$; this extends to make X an $\Lambda(G)$-module (for details see Appendix A.1 in [CS06]).

Definition 2.7. We say that F_∞/F satisfies the condition $\boldsymbol{\mu = 0}$ if there is a pro-p open subgroup H' of H such that the Galois group of the maximal unramified abelian p-extension, $L_{F_\infty^{H'}}$, over $F_\infty^{H'}$ is a finitely generated $\mathbb{Z}_p$-module.

Remark 2.8. Iwasawa conjectured that for a finite extension F of $\mathbb{Q}$ the cyclotomic extension F^{cyc} always satisfies $\mu = 0$. Ferrero and Washington proved this conjecture for $F/\mathbb{Q}$ abelian. The following proposition makes clear why we need this arithmetic ingredient in the formulation of the Main Conjecture.

Proposition 2.9. *X is S-torsion if and only if F_∞/F^{cyc} satisfies the hypothesis $\boldsymbol{\mu = 0}$.*

Proof. See Lemma 9 in [Kak10]. □

We want to see X as an element in the relative $K_0(\Lambda(G), \Lambda(G)_S)$. In view of 1.13 this amounts to X having a finite resolution by finitely generated, projective $\Lambda(G)$-modules. In general, however, X fails to have such a resolution. The infinite p-cohomological dimension of $G = \mathbb{F}_p$ hints to the problem: $\Lambda(G)$ is regular if G has no elements of order p. To circumvent this problem Kato proposed the use of étale cohomolgy while Ritter and Weiss used group homology via a translation functor to get a certain 2-extension, cf. section 2.4.3

Kato and Fukaya introduced in [FK06] a complex closely linked to X:

Definition 2.10.

$$C^\bullet := C^\bullet_{F_\infty/F} := RHom_{\mathbb{Z}_p}\left(R\Gamma_{\text{ét}}(Spec(\mathcal{O}_{F_\infty}[1/\Sigma]), \mathbb{Q}_p/\mathbb{Z}_p), (\mathbb{Q}_p/\mathbb{Z}_p)\right).$$

This is an object in the derived category of projective $\Lambda(G)$-modules.

If $U \subseteq G$ is an open subgroup and $V \leq U$ is normal, denote by F_V the fixed field F_∞^V and define

$$C^\bullet_{(U,V)} := RHom_{\mathbb{Z}_p}\left(R\Gamma_{\text{ét}}(Spec(\mathcal{O}_{F_V}[1/\Sigma]), \mathbb{Q}_p/\mathbb{Z}_p), (\mathbb{Q}_p/\mathbb{Z}_p)\right).$$

This is an object in the derived category of projective $\Lambda(U/V)$-modules.

The following facts on $C^\bullet$ are proved in *loc.cit.*:

Lemma 2.11.

1. *$C^\bullet$ is quasi-isomorphic to a bounded complex of finitely generated $\Lambda(G)$-modules.*
2. *For any Galois extension $F \subset K \subset F_\infty$ of F we have*

$$\Lambda(\mathrm{Gal}(K/F)) \otimes^{\mathbb{L}}_{\Lambda(G)} C^\bullet_{F_\infty/F} \cong C^\bullet_{K/F}.$$

   Here the natural surjection $G \to \mathrm{Gal}(K/F)$ induces the right module action of $\Lambda(G)$ on $\Lambda(\mathrm{Gal}(K/F))$.

Lemma 2.12. *One has* $\Lambda(U/V)^{\bullet} \otimes^{\mathbb{L}}_{\Lambda(U)} C^{\bullet} = C^{\bullet}_{(U/V)}$.

Proof. Well known. A detailed proof is in [Lee09], lemma 2.19. □

In view of this lemma and the prop. 1.13 we regard $C^{\bullet}$ as an element of the relative K-group $K_0(\Lambda(G), \Lambda(G)_S)$. When we do this, this element will be called $[C(F_\infty/F)]$.

Lemma 2.13 (cf. lemma 2.19, [Lee09]). *The cohomology groups of* $C^{\bullet}$ *are*

$$\begin{aligned} H^0(C^{\bullet}) &= \mathbb{Z}_p, \\ H^{-1}(C^{\bullet}) &= X_\Sigma, \\ \textit{and } H^i(C^{\bullet}) &= 0, i \neq -1, 0. \end{aligned}$$

A *characteristic element* for an admissible extension F_∞/F is a $\xi \in K_1(\Lambda(G)_S)$ with $\partial(\xi) = -[C(F_\infty/F)]$. By Proposition 2.5 there is always a (non necessary unique!) characteristic element for F_∞/F.

2.3 Evaluation of L-functions and the Main Conjecture

First we recall the usual (complex) Artin L-function of a finite field extension.

Let K/k be a Galois extension of number fields with finite Galois group G. For each prime $\mathfrak{p}$ of k we choose a prime $\mathfrak{P}$ of K lying over $\mathfrak{p}$. Let $G_{\mathfrak{P}}$ be the decomposition group and $I_{\mathfrak{P}}$ be the inertia group. Then $\mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p})) \cong G_{\mathfrak{P}}/I_{\mathfrak{P}}$ is generated by the Frobenius $\varphi_{\mathfrak{P}} \colon x \mapsto x^q = x^{\mathfrak{N}(\mathfrak{p})}$, where $\kappa(\mathfrak{P}) := \mathcal{O}_K/\mathfrak{P}, \kappa(\mathfrak{p}) := \mathcal{O}_k/\mathfrak{p}$ and $\mathfrak{N}(\mathfrak{p}) = \#\kappa(\mathfrak{p})$. Note that for any complex representation $\rho \colon G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V)$ we obtain an operation of $\varphi_{\mathfrak{P}}$ on the $I_{\mathfrak{P}}$ fixed module, $V^{I_{\mathfrak{P}}}$.

For $s \in \mathbb{C}, Re(s) > 1$ we define the complex Artin L-function as

$$\mathcal{L}(K/k, \rho, s) = \prod \frac{1}{\det(\mathbf{1} - \varphi_{\mathfrak{P}} \mathfrak{N}(\mathfrak{p})^{-s} | V^{I_{\mathfrak{P}}})} \tag{2.1}$$

This series does not depend on the choice of $\mathfrak{P}$ over $\mathfrak{p}$ since all ambiguity is by conjugation with elements of G which does not affect $\det$. It is also not affected by the choice of ρ within an equivalence class of representations given by a character χ. Consequently we often write $\mathcal{L}(K/k, \chi, s)$ for the Artin L-function.

Now we let K/k be possibly infinite but unramified almost everywhere, i.e. we are given a finite set Σ of primes of k such that $I_{\mathfrak{P}} \neq 1$ at most for $\mathfrak{P}$ over Σ. In addition we require Σ to contain all infinite primes. Now fix an embedding $\alpha\colon \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. A representation of G is called an *Artin representation* if it is continuous with open kernel. For a p-adic Artin representation $\rho\colon G \longrightarrow \mathrm{Aut}(V)$, V a finite dimensional $\overline{\mathbb{Q}}_p$-vector space we define the complex Artin L-function associated to ρ (with respect to Σ) $L_\Sigma(\rho, s) = L_\Sigma(K/k, \rho, s) := \mathcal{L}_\Sigma(K/k, \alpha \circ \rho, s)$ where the Euler factors of primes $\mathfrak{p} \in \Sigma$ are omittet, i.e. if $\rho\colon \mathrm{Gal}(K/k) \longrightarrow \mathrm{Aut}_{\overline{\mathbb{Q}}_p}(V) \cong \mathrm{GL}_n(\overline{\mathbb{Q}}_p) \overset{\alpha}{\hookrightarrow} \mathrm{GL}_n(\mathbb{C})$

$$L_\Sigma(\rho, s) = \prod_{\mathfrak{p} \notin \Sigma} \frac{1}{\det(\mathbf{1} - \rho(\varphi_{\mathfrak{P}})\mathfrak{N}(\mathfrak{p})^{-s})}. \tag{2.2}$$

Remark 2.14. The polynomial $\det(\mathbf{Id} - \rho(\varphi_{\mathfrak{P}})t) \in \overline{\mathbb{Q}}_p[t]$ has its zeros in roots of unity since $\varphi_{\mathfrak{P}}$ is of finite order. Apart from that, its values depend on the chosen embedding α. See section 1.2 in [CL73] for a description of a canonical choice of α. Klingen and Siegel showed that the values at negative integers $s \in \mathbb{Z} \setminus \mathbb{N}_0$ are algebraic integers. The idea to interpolate this integers p-adically lies at the heart of Iwasawa theory.

The above series converges uniformly on some right half plane (cf. [Neu99], 8.1) and has functorial properties in the argument 'ρ':

Proposition 2.15. *1. If χ, χ' are two characters of G, then*

$$L_\Sigma(\chi + \chi', s) = L_\Sigma(\chi, s) + L_\Sigma(\chi', s).$$

2. If $K' \supseteq K \supseteq k$ are Galois extensions, χ is a character of $G = \mathrm{Gal}(K/k)$ and $\inf_G^{G'}(\chi)$ is the character of $G' = \mathrm{Gal}(K'/k)$ factoring through χ, then

$$L_\Sigma(K/k, \chi, s) = L_\Sigma(K'/k, \inf_G^{G'}(\chi), s).$$

3. For an open normal subgroup $U \subseteq G$ and ρ an Artin representation of U with character χ we have

$$L_\Sigma(K/K^U, \chi, s) = L_\Sigma(K/k, \mathrm{ind}_U^G(\chi), s).$$

Proof. [Neu99], 10.4 translates directly to our situation. □

The p-adic L-function will live in $K_1(\Lambda(G)_S)$ and consequently we want to evaluate elements of this group at representations of G. For that fix an embedding $\alpha\colon \overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. Suppose now we are given an Artin representation $\rho\colon G \to \mathrm{GL}_n(\overline{\mathbb{Q}})$. The image of ρ is finite and adjoining the entries of all its values to $\mathbb{Q}_p$ we obtain a finite extension L of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$. Clearing denominators we see that ρ is isomorphic to a representation $\rho\colon G \to \mathrm{GL}_n(\mathcal{O})$. This continuous homomorphism induces a map $\Lambda(G) \to \mathrm{M}_n(\mathcal{O})$. In addition a surjection $\omega\colon G \to \Gamma \cong \mathbb{Z}_p$ induces a map $\Lambda(G) \to \Lambda(\Gamma)$. Combining both we obtain

$$\begin{aligned} \Lambda(G) &\to \mathrm{M}_n(\mathcal{O}) \otimes_{\mathbb{Z}_p} \Lambda(\Gamma) \cong \mathrm{M}_n(\Lambda_{\mathcal{O}}(\Gamma)) \\ g &\mapsto \rho(g) \otimes \omega(g). \end{aligned}$$

By [CFK+05], lemma 3.3, this extends to a ring homomorphism on the localisation

$$\Phi_\rho\colon \Lambda(G)_S \to \mathrm{M}_n(\mathcal{Q}_{\mathcal{O}}(\Gamma)).$$

Denote by $I := I_\Gamma$ the augmentation ideal $\ker(\varepsilon\colon \Lambda_{\mathcal{O}}(\Gamma) \to \mathcal{O})$ and as usual by $\Lambda_{\mathcal{O}}(\Gamma)_I$ the localisation at I. The augmentation extends to $\varepsilon\colon \Lambda_{\mathcal{O}}(\Gamma)_I \to L$.

Definition 2.16. We define the *evaluation* of an element $f \in K_1(\Lambda(G)_S)$ at an Artin representation ρ by the composition

$$K_1(\Lambda(G)_S) \xrightarrow{K_1(\Phi_\rho)} K_1(\mathrm{M}_n(\mathcal{Q}_{\mathcal{O}}(\Gamma))) \cong (\mathcal{Q}_{\mathcal{O}}(\Gamma)))^\times \to L \cup \{\infty\}, \tag{2.3}$$

where the second map is by Morita equivalence and the third map is $x \mapsto \varepsilon(x)$ for $x \in \Lambda_{\mathcal{O}}(\Gamma)_I$ and $x \mapsto \infty$, else.

As we would expect from seeing the Zeta-function in K_1, the evaluation map behaves naturally with respect to different representations of G:

Proposition 2.17. *Let f be an element of $K_1(\Lambda(G)_S)$ and let $\mathcal{O}$ be the ring of integers in a finite extension of $\mathbb{Q}_p$.*

(i) For an open subgroup $U \subseteq G$ and a continuous character χ of U we have

$$N(f(\chi)) = f(\mathrm{ind}_U^G(\chi))$$

with the norm map $N\colon K_1(\Lambda(G)_S) \to K_1(\Lambda(U)_S)$.

(ii) *For a subgroup $U \leq H = ker\omega_G$, U normal in G, and an Artin representation $\rho\colon G/U \to \mathrm{GL}_n(\mathcal{O})$ let $\inf_G^{G/U}$ be the composition $G \to G/U \xrightarrow{\rho} \mathrm{GL}_n(\mathcal{O})$. Then*

$$f(\inf_G^{G/U}(\rho)) = p_*(f)(\rho).$$

(iii) *Let $\rho\colon G \to \mathrm{GL}_n(L), \rho'\colon G \to \mathrm{GL}_m(L)$ be continuous representations of G. Then $\rho \oplus \rho'\colon G \to \mathrm{GL}_{n+m}(L)$ is continuous with*

$$f(\rho \oplus \rho') = f(\rho) \cdot f(\rho').$$

Proof. For (i) let $\alpha := K_1(\Phi_{\mathrm{ind}_U^G(\chi)})$ and $\beta := K_1(\Phi_\chi)$. Then by the defining equation (2.3) it suffices to show commutativity of

$$\begin{tikzcd} K_1(\Lambda(G)_S) \arrow[r, "\alpha"] \arrow[d, "N"] & K_1(\mathcal{Q}_{\mathcal{O}}(\Gamma)) \\ K_1(\Lambda(U)_S) \arrow[r, "\beta"] & K_1(\mathcal{Q}_{\mathcal{O}}(\Gamma')) \arrow[u, hook] \end{tikzcd}$$

where $\mathbb{Z}_p \cong \Gamma'$ is a quotient of U and the right hand is induced from the natural inclusion $\Gamma' \subseteq \Gamma$. To prove the commutativity let $\iota_B\colon \Lambda(G) \to M_r(\Lambda(U))$ be the ring homomorphism that assigns to $x \in \Lambda(G)$ the coefficient matrix of right multiplication with x on the free left $\Lambda(U)$-module $\Lambda(G)$ with respect to some basis B with cardinality $r := (G : U)$. Choose for B a system of left coset representatives for G/U and set $\sigma_{ji} \in U$, s.t. $\sigma x_i = x_j \sigma_{ji}$ for $\sigma \in G$. By definition of the induced representation, we have

$$\mathrm{ind}_U^G\colon G \to \mathrm{Aut}_{\mathcal{O}}\Big(\bigoplus_{x_i \in B} x_i\mathcal{O}\Big), \quad \sigma \mapsto (x_i\mu \mapsto x_j\chi(\sigma_{ji})\mu) \,\text{with}\mu \in \mathcal{O}.$$

Then the following diagram commutes

$$\begin{tikzcd} \Lambda(G) \arrow[r, "\mathrm{ind}_U^G(\chi)\otimes\omega_G"] \arrow[d, "\iota_B"] & M_r(\mathcal{O}) \otimes_{\mathbb{Z}_p} \Lambda_{\mathcal{O}}(\Gamma) \\ M_r(\Lambda(U)) \arrow[r, "M_r(\chi)\otimes\omega_U"] & M_r(\mathcal{O}) \otimes_{\mathbb{Z}_p} \Lambda_{\mathcal{O}}(\Gamma') \arrow[u, hook] \end{tikzcd}$$

and therefore

$$\begin{tikzcd}
\Lambda(G)_S \arrow[r, "\Phi_{\mathrm{ind}_U^G(\chi)}"] \arrow[d, "\iota_B"] & M_r(\mathcal{Q}_{\mathcal{O}}(\Gamma)) \\
M_r(\Lambda(U)_S) \arrow[r, "M_r(\Phi_\chi)"] & M_r(\mathcal{Q}_{\mathcal{O}}(\Gamma')) \arrow[u, hook]
\end{tikzcd}$$

commutes. Application of K_1 finishes the proof of (i).

For (ii) apply K_1 to the commutative diagram

$$\begin{tikzcd}
\Lambda(G)_S \arrow[r, "\Phi_{\mathrm{inf}_G^{G/U}(\rho)}"] \arrow[d] & M_n(\mathcal{Q}_{\mathcal{O}}(\Gamma)) \arrow[d, equal] \\
\Lambda(G)/U_S \arrow[r, "(\Phi_\rho)"] & M_n(\mathcal{Q}_{\mathcal{O}}(\Gamma)).
\end{tikzcd}$$

Finally, (iii) is clear from definition of the evaluation. □

We now state the

Main Conjecture 2.18. *There is a unique $\zeta = \zeta(F_\infty|F) \in K_1(\Lambda(G)_S)$, such that*

$$\partial(\zeta) = -[C^\bullet] \tag{2.4}$$

and for every Artin character $\rho : G \to \mathcal{O}^\times$ and every r divisible by $p-1$ we have

$$\zeta(\rho\kappa^r) = L_\Sigma(\rho, 1-r). \tag{2.5}$$

Here $\kappa\colon G(F^{\mathrm{cyc}}/F) \to \mathbb{Z}_p^\times$ is the cyclotomic character. An element ζ satisfying equation 2.4 will be called a *characteristic element* for the extension F_∞/F. Lemma 2.5 shows that there always exist characteristic elements.

Remark 2.19. One should think of equation 2.5 as 'ζ interpolates the complex Artin L-function at certain integer points'. See O. Venjakob's Habilitation speech [Ven06] or K. Kato's ICM 2006 speech [Kat06] for more on this 'interpolation philosophy' dating back to Kummer's results on the Riemannian ζ-function. In the abelian case this was first formulated by Iwasawa (for $F = \mathbb{Q}$) and Coates and Greenberg (for general totally real F). In the case $F = \mathbb{Q}$ this was first proven by Mazur-Wiles (again after strong results by Iwasawa) and later by Rubin using Kolyvagin's Euler Systems. The totally real case was finally settled by Wiles in [Wil90].

2.4 The Main Conjecture of Ritter and Weiss

As before let $G = \mathrm{Gal}(F_\infty/F)$ be admissible and in particular $G \cong H \rtimes \Gamma$ with a closed $H \lhd G$, $\Gamma \cong \mathbb{Z}_p$. For this section let G be one-dimensional. For the canonical Ore set we get $S = \{f \mid \Lambda(G)/\Lambda(G)f$ is a fin. gen. $\mathbb{Z}_p$-module$\}$. The action of Γ on H factors through a finite quotient of Γ and we fix a central open $\Gamma' := \Gamma^{p^e}$ in G.

Definition 2.20. Let $T = \Lambda(\Gamma') \setminus p\Lambda(\Gamma')$. Since $\Lambda(\Gamma')$ is in the center of $\Lambda(G)$ this is a central, multiplicatively closed subset and hence left and right Ore. $\Lambda(\Gamma')$ is a domain and therefore right multiplication with $f \in T$ will be injective, which implies $f \in S$ by lemma 2.1 in [CFK+05]. Consequently we have the inclusion $\Lambda(G)_T \subseteq \Lambda(G)_S$.

Proposition 2.21. *This inclusion is an isomorphism.*

Proof. (cf. [Kak08], Lemma 2.1) First note that $\Lambda(G)_T = \Lambda(\Gamma')_T \otimes_{\Lambda(\Gamma')} \Lambda(G)$. We claim that

$$\mathcal{Q}(\Lambda(\Gamma')) \otimes_{\Lambda(\Gamma')} \Lambda(G) = \mathcal{Q}(\Lambda(G)) \tag{2.6}$$

by the map $\frac{s}{t} \otimes x \mapsto \frac{sx}{t}$. This map is surely injective and the lhs of 2.6 is an Artinian ring as it is finite (of dimension $\#G/\Gamma'$) over the field $\mathcal{Q}(\Lambda(\Gamma'))$. In an Artinian ring every regular element x is invertible, or else (x^n) would be an ever decreasing series of ideals. As $\Lambda(G)$ is contained in the lhs every regular element of it is a unit. By the universal property of the total ring of quotients 2.6 must then be surjective. Now since S contains no zero divisors, every element $x \in \Lambda(G)_S \subseteq \mathcal{Q}(\Lambda(G))$ can be written as $x = \frac{a}{t}$ with $a \in \Lambda(G), t \in \Lambda(\Gamma')$. We want to show, that t can be choosen in T, i.e. $t \notin p\Lambda(\Gamma')$. If $t \in p^n\Lambda(\Gamma')$ then $a = tx \in p^n\Lambda(G)_S \cap \Lambda(G) = p^n\Lambda(G)$ so one can divide t and a by p^n. $\square$

Let $\mathcal{Q}(\Lambda(G))$ be the total ring of quotients of $\Lambda(G)$, i.e. the localisation at the set of all (left and right) regular elements. It is $\Lambda(G)_{S^*} = \mathcal{Q}(\Lambda(G)) = \left\{\frac{a}{b} \mid a \in \Lambda(G), b \in \Lambda(\Gamma')\right\}$ (see the first part of Proposition 2.21 above), in particular this is a finite dimensional Artinian algebra over the field $\mathcal{Q}(\Lambda(\Gamma'))$. Changing the coefficient ring from $\mathbb{Z}_p$ to a finite extension $\mathcal{O}$ is well behaved, as we will see in the following lemma. For $\mathcal{Q}^c(G) := \overline{\mathbb{Q}}_p \otimes_{\mathbb{Q}_p} \mathcal{Q}(\Lambda(G))$ however the situation is more subtle:

A character χ of an irreducible representation V_ρ of G is called *of type W* if χ factors through Γ or, equivalently, $\mathrm{res}^H_G \chi = 1$. If χ is an irreducible character of

G then associate to it a primitive central idempotent in $\mathcal{Q}^c(G)$ as follows: Take any irreducible character η contained in $\mathrm{res}_G^H \chi$, let $e(\eta) = \frac{\eta(1)}{|H|}\sum_{h\in H}\eta(h^{-1})h$ and finally set $e_\chi := \sum_{\eta | res_G^H(\chi)} e(\eta)$. This gives indeed all primitive central idempotents (cf. [RW04], prop.5).

Lemma 2.22.

i) If F is a finite extension of $\mathbb{Q}_p$ and $\mathcal{O}$ its ring of integers, then $F\otimes_{\mathbb{Q}_p}\mathcal{Q}(\Lambda(G)) = \mathcal{Q}(\Lambda_{\mathcal{O}}(G))$.

ii) $\mathcal{Q}^c(G)$ is a semisimple Artinian ring and the components of its Wedderburn decomposition are the simple rings $\mathcal{Q}^c(G)e_\chi$ where χ runs through a system of irreducible characters of G up to W-twist.

Proof. i) is Lemma 1 in [RW04], ii) is Proposition 6 in [RW04] □

2.4.1 The analytical side of EMC = MC: Burn's lemma

Denote by Nrd the reduced norm $K_1(\Lambda(G)_S) \to K_1(\mathcal{Q}(\Lambda(G))) \to K_1(Z(\mathcal{Q}(\Lambda(G))) \cong Z(\mathcal{Q}(\Lambda(G)))^\times$, cf. §45A in [CR87]. For $x \in \Lambda(G)_S^\times$ we write $[x]$ for the image of x under the homomorphism $\Lambda(G)_S^\times \to K_1(\Lambda(G)_S)$.

To use a Hom-description of $K_1(\Lambda(G)_S)$ Ritter and Weiss introduce a map *Det* analogous to the evaluation defined in 2.16: Let ρ be an Artin representation of G on the $\overline{\mathbb{Q}}_p$ vector space V. For $x \in \Lambda(G)\cap\Lambda(G)_S^\times$ denote by $r_\rho(x)$ the induced endomorphism on $\mathcal{Q}(\Lambda(G))\otimes_{\Lambda(G)}\mathrm{Hom}_{\overline{\mathbb{Q}}_p[H]}(V_\rho, \overline{\mathbb{Q}}_p\otimes\Lambda(G))$. The latter is a finitely generated $\mathcal{Q}^c(\Gamma)$ module by [RW04], lemma 2, and we can therefore take the $\mathcal{Q}^c(\Gamma)$-determinant of $r_\rho(x)$. With $R(G)$ denoting the free abelian group on irreducible Artin characters of G, the map $Det\colon x \mapsto (\rho \mapsto \det_{\mathcal{Q}^c(\Gamma)}(r_\rho(x)))$ is a homomorphism $K_1(\Lambda(G)_S) \to \mathrm{Hom}^*((R(G), \mathcal{Q}^c(\Gamma)^\times)$, cf. [RW04], theorem 8. Note that $K_1(\Lambda(G)_S)$ is indeed generated by elements $[x]$ as above, since $\Lambda(G)_S$ is semi-local.

In [RW04], prop. 6, a homomorphism $j_\rho : Z(\Lambda_{\mathcal{O}}(G)) \twoheadrightarrow Z(\Lambda_{\mathcal{O}}(G)e_\rho) \cong \mathcal{Q}(\Lambda_{\mathcal{O}}(\Gamma_\rho)) \hookrightarrow \mathcal{Q}(\Lambda_{\mathcal{O}}(\Gamma))$ was defined for every irreducible Artin representation $\rho\colon G \to \mathrm{GL}_n(\mathcal{O}_F)$ over the ring of integers $\mathcal{O} \subseteq \mathcal{O}_F$ of a finite extension $\mathbb{Q}_p \subseteq F$. These j_ρ have the following properties:

$$\bigcap_{\rho\in A(G)} ker(j_\rho) = \{1\}. \tag{2.7}$$

For this note that $\mathcal{Q}(\Lambda(G))$ is semisimple and therefore $\mathcal{Q}(\Lambda_{\mathcal{O}}(G))$ is, too. Its primitive central idempotents e_ρ correspond to the respective identities of the Wedderburn components of $\mathcal{Q}(\Lambda_{\mathcal{O}}(G))$. Obviously an element $x \in Z(\mathcal{Q}(\Lambda_{\mathcal{O}}(G))$ is zero if and only if every projection $x \cdot e_\rho$ of x on the different simple subrings is zero. Since all the other maps in the definition of j_ρ are injective the assertion holds.

And theorem 8 of *loc. cite* gives for every $x \in \mathcal{Q}(\Lambda(G))^\times \cap \Lambda(G)$

$$\det_{\mathcal{Q}^c(\Gamma)}(r_\rho(x)) = j_\rho(Nrd([x])). \tag{2.8}$$

The following lemma is a technical excercise in [Bro82] (cf. *loc. cit.* III,5, ex.2).

Lemma 2.23. *Let G be a profinite group and H be a closed subgroup of G. For an $\Lambda_{\mathcal{O}}(H)$ module M and an $\Lambda_{\mathcal{O}}(G)$ module N there is an isomorphism of $\Lambda_{\mathcal{O}}(G)$ modules*

$$N \otimes_{\mathcal{O}} (\Lambda_{\mathcal{O}}(G) \otimes_{\Lambda_{\mathcal{O}}(H)} M) \cong \Lambda_{\mathcal{O}}(G) \otimes_{\Lambda_{\mathcal{O}}(H)} (N \otimes_{\mathcal{O}} M).$$

Here G acts diagonally on the left hand side and by left multiplication on the right hand side, while H acts diagonally on the inner tensor product on the right hand side.

Using this lemma and following Burns we show the

Lemma 2.24. *For $\xi \in K_1(\mathcal{Q}_{\mathcal{O}}(G))$ and for every Artin representation ρ of G one has $\xi(\rho) = j_\rho(Nrd(\xi))$.*

Proof. By Propositions 2.5 (i) and 1.8 there is $x \in \mathcal{Q}_{\mathcal{O}}(G)^\times$ with $\xi = [x]$, so we can use equation 2.8. For an Artin representation $\rho\colon G \to \mathrm{Aut}(V)$ denote the contragredient representation $G \to \mathrm{Aut}(\mathrm{Hom}(V, \overline{\mathbb{Q}_p}))$ by $\check{\rho}$. Now use the isomorphisms of left $\mathcal{Q}^c(G)$-modules

$$\mathrm{Hom}_{\overline{\mathbb{Q}_p}}(V_\rho, \overline{\mathbb{Q}_p} \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}(G)) \cong V_{\check{\rho}} \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}(G) \cong \Lambda_{\mathcal{O}}(G) \otimes_{\Lambda_{\mathcal{O}}(H)} (V_{\check{\rho}} \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}(H)).$$

The first isomorphism comes as follows: An element f of the left hand side is determined by its image on a basis B of V_ρ, say

$$f(b) = \sum_i^{\text{finite}} x_i^{(b)} \otimes \lambda_i^{(b)}.$$

Since B is finite we can assume that the λ_i don't depend on b. Hence we obtain a map $f \mapsto \sum_i (b \mapsto x_i^{(b)}) \otimes \lambda_i$. One immediately verifies that this is an isomorphism. The second isomorphism is Lemma 2.23 when setting $M = \Lambda_{\mathcal{O}}(H)$, $N = V_{\check{\rho}}$.

Taking H-coinvariants gives isomorphisms

$$\mathcal{Q}_{\mathcal{O}}(\Gamma) \otimes_{\Lambda_{\mathcal{O}}(\Gamma)} \mathrm{Hom}_{\overline{\mathbb{Q}_p}[H]}(V_\rho, \overline{\mathbb{Q}_p} \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}(G)) \cong \mathcal{Q}(\Gamma) \otimes_{\overline{\mathbb{Q}_p}} V_{\check{\rho}}.$$

Here $r_\rho(x)$ maps to $\delta_\rho(x)\colon \Lambda_{\mathcal{O}}(G) \to \mathrm{End}_{\overline{\mathbb{Q}_p}}(\mathcal{Q}^c(\Gamma) \otimes V_{\check{\rho}})$ with $\delta_\rho(g)(\lambda \otimes v) = \lambda\pi(g) \otimes g^{-1}(v)$. By definition of $V_{\check{\rho}}$ and Morita equivalence we are done. $\square$

This finishes the 'interpolation side' of the main conjectures.

2.4.2 The translation functor

The translation functor $\mathfrak{t}$ as described by Ritter and Weiss uses group homology to give a substitute for the Iwasawa module X in case of non regular $\Lambda(G)$. It provides an equivalence between the categories $\underline{Gr_p}$ of certain group extensions and $\underline{GrMod_p}$, exact sequences of $\Lambda(G)$-modules.

The objects of $\underline{Gr_p}$ are short exact sequences of profinite groups $1 \to N \to G \to Q \to 1$ where N is pro-p and abelian. A morphism between two exact sequences is the obvious commutative diagram. The objects of $\underline{GrMod_p}$ are pairs $(G, 0 \to M \to N \to I_G \to 0)$ where G is a profinite group and the diagram is an exact sequence of compact $\Lambda(G)$-modules. Remember that I_G always denotes the augmentation ideal of $\Lambda(G)$. The morphisms in $\underline{GrMod_p}$ between $(G, 0 \to M \to N \to I_G \to 0)$ and $(G', 0 \to M' \to N' \to I_{G'} \to 0)$ are pairs

$$\begin{array}{ccc} G, & & 0 \longrightarrow M \longrightarrow N \longrightarrow I_G \longrightarrow 0 \\ \downarrow{\scriptstyle\varphi} & & \quad\quad\;\; \downarrow \qquad\quad \downarrow \qquad\quad \downarrow{\scriptstyle\varphi'} \\ G', & & 0 \longrightarrow M' \longrightarrow N' \longrightarrow I_{G'} \longrightarrow 0, \end{array}$$

where φ is a homomorphism of profinite groups, φ' is induced by φ and the diagramm is commutative in $\Lambda(G)-$Mod. To describe the translation functor first suppose $0 \to N \to G \overset{\pi}{\to} Q \to 0$ is an object of $\underline{Gr}_p$ with N, G, Q all finite. Then the snake lemma diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & I(N) \arrow[r, "incl"] \arrow[d] & I_G \arrow[r] \arrow[d] & I_Q \arrow[d] & \\
0 \arrow[r] & I(N) \arrow[r, "incl"] \arrow[d] & \Lambda(G) \arrow[r, "\pi"] \arrow[d, "aug_G"] & \Lambda(Q) \arrow[r] \arrow[d, "aug_Q"] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z}_p \arrow[r, "id"] \arrow[d] & \mathbb{Z}_p \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

gives exactness of the sequence $0 \to I(N) \to I_G \to I_Q \to 0$ of $\mathbb{Z}_p$-modules and consequently of $\frac{I(N)}{I(N) \cdot I_G} \to \frac{I_G}{I(N) \cdot I_G} \to I_Q \to 0$. This last sequence is a sequence of $\mathbb{Z}_p[Q]$-modules: let $q \in Q$ act through left multiplication by any preimage $g \overset{\pi}{\mapsto} q$. Ritter and Weiss then prove ad hoc, that the left term in this sequence is isomorphic to N for abelian, pro-p N, the left map is injective, $\mathfrak{t}$ is compatible with inverse limits and that $\mathfrak{t}$ is in fact an equivalence of categories.

All this can also be derived from the following abstract argument, cf. [NSW08], §V.6:

For an arbitrary sequence of pro-$\mathfrak{c}$ groups[1] $1 \to \mathcal{H} \to \mathcal{G} \to G \to 1$ we take $\mathcal{H}$-coinvariants of $0 \to I_{\mathcal{G}} \to \Lambda(\mathcal{G}) \to \mathbb{Z}_p \to 0$ yielding the exact

$$H_1(\mathcal{H}, \Lambda(\mathcal{G})) \to H_1(\mathcal{H}, \mathbb{Z}_p) \to (I_{\mathcal{G}})_{\mathcal{H}} \to (\Lambda(\mathcal{G}))_{\mathcal{H}} \to (\mathbb{Z}_p)_{\mathcal{H}} \to 0.$$

Note that $\Lambda(\mathcal{G})^\vee = \mathrm{Hom}_{cts}(\Lambda(\mathcal{G}), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Map}_{cts}(\mathcal{G}, \mathbb{Q}_p/\mathbb{Z}_p) = \mathrm{Coind}_1^{\mathcal{G}}(\mathbb{Q}_p/\mathbb{Z}_p)$ is cohomologically trivial and by (2.6.9) in [NSW08] we conclude that $H_1(\mathcal{H}, \Lambda(\mathcal{G})) = 0$. With $H_1(\mathcal{H}, \mathbb{Z}_p) = \mathcal{H}(p)^{\mathrm{ab}}$ the sequence above amounts to

$$0 \to \mathcal{H}(p)^{\mathrm{ab}} \to \frac{I_{\mathcal{G}}}{I_{\mathcal{H}} \cdot I_{\mathcal{G}}} \to I_G \to 0, \tag{2.9}$$

[1] with $\mathfrak{c}$ a class of finite groups closed under kernels, images, extensions and containing $\mathbb{Z}/p\mathbb{Z}$

which is the t image of $\mathcal{H} \hookrightarrow \mathcal{G} \twoheadrightarrow G$ for $\mathcal{H}$ pro-p, abelian.

2.4.3 The Equivariant Main Conjecture

The Equivariant Main Conjecture is formulated in [RW04] as follows:

Lemma 2.25. *For an admissible one-dimensional extension F_∞/F and a finite set Σ of primes of F_∞ containing all ramified primes and those above ∞ and p, let M_Σ be the maximal abelian, pro-p, outside of Σ unramified extension of F_∞ and set $G := \mathrm{Gal}(F_\infty/F), X_\Sigma := \mathrm{Gal}(M_\Sigma/F_\infty)$. Then for each $\Lambda(G)$-monomorphism $\psi\colon \Lambda(G) \to I_G$ there is a commutative diagram of $\Lambda(G)$-modules with rows and colums all exact*

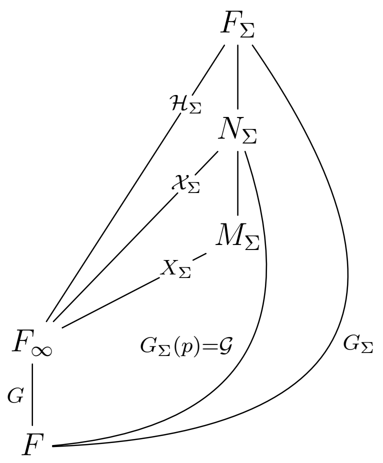
$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \Lambda(G) \arrow[r, "id"] \arrow[d, "\Psi"] & \Lambda(G) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & X_\Sigma \arrow[r] \arrow[d, "id"] & Y \arrow[r] \arrow[d] & I_G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_\Sigma \arrow[r] \arrow[d] & \operatorname{coker}\Psi \arrow[r] \arrow[d] & \operatorname{coker}\psi \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2.10}$$

and Y finitely generated and of projective dimension ≤ 1 over $\Lambda(G)$.

Proof. The only thing to prove here is existence of such an Y and exactness of the middle row. Consider the arrangement of field extensions depicted in the figure below. Let F_Σ, resp. N_Σ, be the maximal outside of Σ unramified extension of F, resp. its maximal subfield with pro-p Galois group. With the notation from the diagram we have $X_\Sigma = \mathcal{X}_\Sigma^{ab}$ and $\mathcal{X}_\Sigma \hookrightarrow \mathcal{G} = \mathrm{Gal}(N_\Sigma/F) \twoheadrightarrow G$ is exact. The middle row in diagramm 2.10 is the right column in prop. 5.6.7. in [NSW08]. Corollary 10.4.9 and prop. 8.3.18 in *loc.cit.* give $\mathrm{cd}_p\mathcal{G} \leq \mathrm{cd}_p G_\Sigma \leq 2$. So the additional assumptions in prop. 5.6.7. of *loc.cit.* hold and the lower row gives an exact sequence

$$0 \to H_2(\mathcal{X}_\Sigma, \mathbb{Z}_p) \to N^{\mathrm{ab}}_{\mathcal{X}_\Sigma}(p) \to \Lambda(G)^d \to Y \to 0,$$

where $N^{\mathrm{ab}}_{\mathcal{X}_\Sigma}(p)$ is a finitely generated projective $\Lambda(G)$-module. Finally $H_2(\mathcal{X}_\Sigma, \mathbb{Z}_p) = 0$ by the weak Leopoldt conjecture (thm. 10.3.22 in [NSW08]). $\square$



2.4.4 The regular case

If $\Lambda(G)$ is regular, i.e. G has no elements of order p, then a finitely generated $\Lambda(G)$-module has a finite resolution by finitely generated projective modules.

Given a $\Lambda(G)$-monomorphism ψ like in diagram 2.10 we have the following commutative diagrams of $\Lambda(G)$-modules

$$\begin{tikzcd}
 & & \Lambda(G) \arrow[r, equal] \arrow[d, "\Psi"] & \Lambda(G) \arrow[d, "\psi"] & \\
0 \arrow[r] & X_\Sigma \arrow[r] & Y \arrow[r] & I_G \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \Lambda(G) \arrow[r, equal] \arrow[d, "\psi"] & \Lambda(G) \arrow[d, "\tilde{\psi}"] & & \\
0 \arrow[r] & I_G \arrow[r] & \Lambda(G) \arrow[r] & \mathbb{Z}_p \arrow[r] & 0,
\end{tikzcd}$$

from which we get by taking cokernels $X_\Sigma \hookrightarrow \mathrm{coker}\Psi \twoheadrightarrow \mathrm{coker}\psi$ and $\mathrm{coker}\psi \hookrightarrow \mathrm{coker}\tilde{\psi} \twoheadrightarrow \mathbb{Z}_p$.

Since ψ is injective, $\psi(1)$ is a non-zero divisor and since $\mathrm{coker}\psi$ is annihilated by $\psi(1)$, it is S^*-torsion. Also I_G is finitely generated ($\Lambda(G)$ is Noetherian) and so is $\mathrm{coker}\psi$. X_Σ is S^*-torsion by prop. 2.9. Since this implies X_Σ is finitely generated over $\Lambda(H)$, it is clearly f.g. over $\Lambda(G)$.

If in a short exact sequence $A \hookrightarrow B \overset{\phi}{\twoheadrightarrow} C$ of $\Lambda(G)$-modules A and C are S^*-torsion then so is B: $x \in B \Rightarrow s\phi(x) = 0$ for some $s \in S^* \Rightarrow sx \in \ker\phi = A \Rightarrow tsx = 0$ for some $t \in S^*$. Now use that S^* is multiplicative. Consequently cokerΨ is S^*-torsion.

If $\psi(1)$ is central in $\Lambda(G)$, then coker$\widetilde{\psi}$ is annulated by $\psi(1)$: $x \in \Lambda(G) \Rightarrow \widetilde{\psi}(1)x = \widetilde{\psi}(x) \in \mathrm{im}\widetilde{\psi}$. $\mathbb{Z}_p$ is obviously a finitely generated $\Lambda(G)$-module. It is also annulated by $\psi(1)$.

So we have cokerΨ, coker$\widetilde{\psi} \in \mathcal{H}_{S^*}^{\Lambda(G)}$ and by prop. 1.13 can regard their images in $K_0T(\Lambda(G))$.

Conjecture 2.26 (Equivariant Main Conjecture)**.** *Set* $\widetilde{\mho}_\Sigma := [coker\Psi] - [coker\widetilde{\psi}] \in K_0T(\mathcal{Q}\Lambda(G)) = K_0(\Lambda(G), \mathcal{Q}(\Lambda(G)))$. *Ritter and Weiss formulate in [RW04], p.14, the Conjecture that there is a unique element* $\widetilde{\Theta}_S \in K_1(\mathcal{Q}(\Lambda(G)))$*, s.t.*

$$\partial(\widetilde{\Theta}_\Sigma) = \widetilde{\mho}_\Sigma, \tag{2.11}$$

$$j_\rho(Nrd(\widetilde{\Theta}_\Sigma)) = L_{k,\Sigma}(\rho), \forall\rho \in R(G). \tag{2.12}$$

As we have seen in lemma 2.24 the second part is equivalent to $Det(\widetilde{\Theta}_\Sigma) = L_{k,\Sigma}$.

The first, algebraic part of this translates to the Non-commutative Main Conjecture as follows:

$\widetilde{\mho}_\Sigma = [\mathrm{coker}\Psi] - [\mathrm{coker}\widetilde{\psi}] = [\mathrm{coker}\psi] + [X_\Sigma] - [\mathrm{coker}\psi] - [\mathbb{Z}_p] = [X_\Sigma] - [\mathbb{Z}_p] = [H^0(C^\bullet)] - [H^-1(C^\bullet)] = \sum_i (-1)^i [H^i(C^\bullet)] = -[C^\bullet]$ where the fourth equality is because of lemma 2.13 and the last is due to lemma 1.14.

2.4.5 The non-regular case

In the non-regular case, i.e. when G has an element of order p, two strategies were pointed out by O. Venjakob to show the correspondence between $-[C^\bullet]$ and $\widetilde{\mho}_\Sigma$.

The first relies on the observation (by K. Kato) that there is a totally real field extension F'_∞ over F_∞, unramified almost everywhere, with its Galois group $G' := \mathrm{Gal}(F'_\infty/F)$ a p-adic Lie group containing no element of order p. Consequently the regular case above applies and we have $-[C^\bullet(F'_\infty/F] = \widetilde{\mho'}_\Sigma$. The extension F'_∞/F_∞ is in general an infinite pro-p abelian one (cf. [BV05], lemma

6.1). Its construction uses Kummer theory and supposes, that the p-th roots of unity are in F, so it may not be applicable to every situation. To descent back down to G, one uses a deflation map. If the extension F'_∞/F_∞ is finite this was described by Ritter and Weiss in their paper [RW02].

The second strategy uses sequences of 2-extensions. An account of this strategy can be seen in the recent course notes by Venjakob, [Ven11].

2.5 The strategy of Burns and Kato

Burns and Kato have proposed a strategy for proving the Non-commutative Main Conjecture if G is a p-adic Lie group with a surjection $\omega\colon G \to \Gamma \cong \mathbb{Z}_p$. It makes use of the Commutative Main Conjecture known to hold by the afforementioned results of Wiles et al.

Let $\mathcal{I}$ be a set of pairs (U, V) of subgroups of G, such that $U \subseteq G$ is open, V is a closed subgroup of ker ω and V normal in U with U/V abelian. For such a pair define

$$\theta_{(U,V)} : K_1(\Lambda(G)) \xrightarrow{Norm} K_1(\Lambda(U)) \xrightarrow{\text{proj}_*} K_1(\Lambda(U/V)) = \Lambda(U/V)^\times$$

and analogous for the localisations

$$\theta_{S,(U,V)} : K_1(\Lambda(G)_S) \xrightarrow{Norm} K_1(\Lambda(U)_S) \xrightarrow{\text{proj}_*} K_1(\Lambda(U/V)_S) = \Lambda(U/V)_S^\times.$$

For a given set $\mathcal{I}$ as above we combine them in the homomorphisms

$$\theta\colon K_1(G) \to \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times,$$
$$\theta_S\colon K_1(G) \to \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)_S^\times.$$

Now let $\Psi \leq \prod\limits_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times$ and $\Psi_S \leq \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)_S^\times$ be subgroups, such that the following holds:

Assumption 2.27. *1. $\theta\colon K_1(\Lambda(G)) \to \Psi$ is an isomorphism,*

2. im$(\theta_S) \subseteq \Psi_S$,

3. $\Psi = \Psi_S \cap \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times$ *and*

4. *for any Artin representation ρ of G there is a finite family (U_i, V_i) in $\mathcal{I}$ and one-dimensional Artin characters χ_i of U_i/V_i , s.t. ρ is a $\mathbb{Z}$-linear combination of* $\mathrm{ind}_{U_i}^G \chi_i$.

The following theorem is due to D. Burns and K. Kato. For a closed subgroup $G' \leq G$ let F'_G be the fixed field $F_\infty^{G'}$ of G' . We denote the p-adic zeta function (with relation to Σ) for the (abelian) extension F_V/F_U by $\zeta_{(U,V)}$.

Theorem 2.28. *(cf. [Kat07]) Let Ψ, Ψ_S be subgroups as above and suppose we have $(\zeta_{U,V})_{(U,V)\in\mathcal{I}} \in \Psi_S$. Then the p-adic zeta function ξ for F_∞/F with relation to Σ exists and $\partial(\xi) = -[C^\bullet]$.*

Proof. Let f be a characteristic element for F_∞/F and for $(U,V) \in \mathcal{I}$ define

$$\begin{aligned}(f_{U,V}) &:= \theta_S(f) \in \Lambda(U/V)_S^\times, \\ (u_{U,V}) &:= (\zeta_{U,V})(f_{U,V})^{-1}.\end{aligned}$$

Consider the following diagram

$$\begin{tikzcd} K_1(\Lambda(G)_S) \arrow[r, "\mathrm{Norm}"] \arrow[d, "\partial"] & K_1(\Lambda(U)) \arrow[r, "proj_*"] \arrow[d, "\partial"] & \Lambda(U/V)_S^\times \arrow[d, "\partial"] \\ K_0[\Lambda(G), \Lambda(G)_S] \arrow[r, "\mathrm{Tr}"] & K_0[\Lambda(U), \Lambda(U)_S] \arrow[r, "proj_*"] & \Lambda(U/V)_S^\times/\Lambda(U/V)^\times. \end{tikzcd}$$

It is commutative by the definition of restriction of scalars and the map ∂ given in section 1.3.

By lemma 2.12, $proj_* \circ Tr(-[C(F_\infty/F)]) = -[C_{(U,V)}]$. By Commutative Iwasawa theory $\partial(\zeta_{U,V}) = -[C_{(U,V)}]$ and consequently $u_{U,V} \in \ker\partial = \Lambda(U/V)^\times$. The image of f under θ_S is in Ψ_S by assumption 2.27 (2). By the assumption on $\zeta_{U,V}$ in the theorem $(u_{U,V}) \in \Psi_S \cap \prod_{(U,V)\in\mathcal{I}} \Lambda(U/V)^\times = \Psi$ by 2.27 (3).

By assumption 2.27 (1) the unique preimage of $(u_{U,V})$, say u, under θ exists in $K_1(\Lambda(G))$. Identifying $K_1(\Lambda(G))$ with its image in $K_1(\Lambda(G)_S)$ we set $\xi := uf \in K_1(\Lambda(G)_S)$. This is by definition a characteristic element for F_∞/F and has $\theta_S(\xi) = (\zeta_{U,V})$.

It remains to show the interpolation property. For this let ρ be an Artin representation of G and $\rho = \sum_{i=0}^{m} r_i \operatorname{ind}_{U_i}^{G}(\chi_i)$ with χ_i according to assumption 2.27 (4). Then by proposition 2.15

$$L_\Sigma(1-r,\rho) = \prod_i L_\sigma(1-r, \operatorname{ind}_{U_i}^{G}(\chi_i))^{r_i} = \prod_i L_\Sigma(1-r,\chi_i)^{r_i}.$$

We have $\xi(\rho\kappa^r) = \prod_i^m \xi(\operatorname{ind}_{G}^{U_i}(\chi_i)\kappa^r)^{r_i}$ by additivity of evaluation. If κ_U denotes the cyclotomic character for the fixed field $F_U = F_\infty^U$ then

$$\xi(\operatorname{ind}_{G}^{U_i}(\chi_i)\kappa^r) = \theta_{S,U,V}(\xi)(\chi_i\kappa_U^r) = \zeta_{U,V}(\chi_i,\kappa_U^r) = L_\Sigma(1-r,\chi_i).$$

This shows existence of the p-adic ζ-function for F_∞/F. Uniqueness follows from an easy diagram chase using the injectivity of θ. $\square$

Chapter 3

Reduction arguments

Mahesh Kakde [Kak10] recently reduced the proof of the main conjecture for general admissible (see 2.6) p-adic Lie extensions to the pro-p case. Another account of this is in the lecture notes [Suj11] by R. Sujatha.

Each reduction step has two parts: One K-theoretic, giving an isomorphism $K_1'(\Lambda(\mathcal{G})) \cong \varprojlim K_1'(\Lambda(G))$ where the limit ranges over a suitable family of quotients or subgroups of $\mathcal{G}$ with norm maps and one representation theoretic using the compatibility of evaluation and complex Artin L-functions with tensor products of representations to glue the main conjectures on each level G together in one large commutative diagram. Kakde uses the uniqueness assertion in the main conjecture in every reduction step. This is possible due to his formulation using the quotient group $K_1' = K_1/SK_1$ rather then the whole K_1.

Conjecture 3.1. ***(Kakde's Main Conjecture)***

There is a unique $\zeta(F_\infty/F)$ in $K_1'(\Lambda(\mathcal{G})_S)$ such that for any Artin representation ρ of $\mathcal{G}$ and any integer $r \equiv 0$ mod $p-1$

$$\begin{aligned} \partial(\zeta(F_\infty/F)) &= -[C(F_\infty/F)] \text{ and} \\ \zeta(F_\infty/F)(\rho\kappa_F^r) &= L_\Sigma(\rho, 1-r). \end{aligned}$$

3.1 Reduction to the rank 1 case

Lemma 3.2. *For a compact p-adic Lie group P we have an isomorphism*

$$K_1'(\Lambda_{\mathcal{O}}(P)) \xrightarrow{\sim} \varprojlim_{\Delta} K_1'(\mathcal{O}[\Delta])$$

where Δ runs over the finite quotients of P and the inverse limit is taken w.r.t. the norm maps.

Proof. Denote by J_P the Jacobson radical of $\Lambda_{\mathcal{O}}(P)$. Then Fukaya and Kato show in [FK06], prop. 1.5.1, that $K_1(\Lambda(P)) \xrightarrow{\sim} \varprojlim_n K_1(\Lambda_{\mathcal{O}}(P)/J_P^n)$. So

$$\begin{aligned}
K_1(\Lambda_{\mathcal{O}}(P)) &\cong \varprojlim_{n} K_1(\Lambda_{\mathcal{O}}(P)/J_P^n) \\
&\cong \varprojlim_{(U,r)} K_1(\mathcal{O}[P/U]/\mathfrak{m}^r\mathcal{O}[P/U]) \\
&\cong \varprojlim_{U}\varprojlim_{r} K_1(\mathcal{O}[P/U]/\mathfrak{m}^r\mathcal{O}[P/U]) \\
&\cong \varprojlim_{U}\varprojlim_{n} K_1(\mathcal{O}[P/U]/J_{P/U}^n) \\
&\cong \varprojlim_{U} K_1(\mathcal{O}[P/U]).
\end{aligned}$$

Here U runs over the open normal subgroups of P and $r \in \mathbb{N}$. The first and last isomorphisms come from the aforementioned result of Fukaya and Kato. For the second isomorphism note that the J_P-topology on $\Lambda_{\mathcal{O}}(P)$ is the same as the canonical one by lemma 1.3, i.e. for any $n \in \mathbb{N}$ there are $r \in \mathbb{N}$ and $U \lhd_o P$ with $\mathfrak{m}^r\Lambda_{\mathcal{O}}(P) + I(U) \leq J_P^n$ and the map $((U,r) \mapsto m$ minimal, s.t.$J_P^m \leq \mathfrak{m}^r\Lambda_{\mathcal{O}}(P) + I(U))$ gives a cofinal system. Then we use the usual categorial argument, [Mac98] IX.3 theorem 1. For the third isomorphism, note that $\mathrm{GL}_2(R)$ surjects onto $K_1(R)$ for semilocal rings R. Consequently the projective limits are taken in the category of profinite groups and we use the corollary in [Mac98] §IX.8. For fixed $U \lhd_o P$ the $\mathfrak{m}^r, r \in \mathbb{N}$, constitute a fundamental system of neighbourhoods of $0 \in \mathcal{O}[P/U]$. By the same reasoning as in step two we conclude that they are cofinal in the powers of the radical and get the fourth isomorphism.

Finally, $SK_1(\mathcal{O}[P/U])$ is finite by a result of Higman, [Hig40]. By definition $SK_1(\Lambda_{\mathcal{O}}(P)) = \varprojlim_U SK_1(\mathcal{O}[P/U])$ and by the exactness of $\varprojlim$ on sequences of compacta we have

$$K_1'(\Lambda_{\mathcal{O}}(P)) \cong \varprojlim_U \big(K_1(\mathcal{O}[P/U])/SK_1(\mathcal{O}[P/U])\big).$$

$\square$

As immediate consequence we get the

Corollary 3.3. *Let $\mathcal{Q}_1(G) = \{G/U; U \text{ is open in } H \text{ and normal in } G\}$. Then*

$$K_1'(\Lambda_{\mathcal{O}}(G)) \cong \varprojlim_{G' \in \mathcal{Q}_1(G)} K_1'(\Lambda_{\mathcal{O}}(G')),$$

where the projective limit is taken w.r.t the norm maps.

Proposition 3.4. *Let F_∞/F be an admissible extension with $\mathcal{G} = \mathrm{Gal}(F_\infty/F)$. Then Kakde's Main Conjecture is true if and only if it is true for every $G' \in \mathcal{Q}_1(\mathcal{G})$.*

Proof. The 'only if' part is clear from Fukaya/Kato (functoriality of $C(F'/F)$) and the functoriality of the L-functions. For the 'if' part consider the commutative diagram

$$\begin{tikzcd}
K_1'(\Lambda(\mathcal{G})) \arrow[r] \arrow[d, "\simeq"] & K_1'(\Lambda(\mathcal{G})_S) \arrow[r] \arrow[d, "N"] & K_0(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})_S) \arrow[r] \arrow[d] & 0 \\
\varprojlim_{G'} K_1'(\Lambda(G')) \arrow[r] & \varprojlim_{G'} K_1'(\Lambda(G')_S) \arrow[r] & \varprojlim_{G'} K_0(\Lambda(G)', \Lambda(G)'_S).
\end{tikzcd}$$

Let $f \in K_1'(\Lambda(\mathcal{G})_S)$ be a characteristic element for F_∞/F and denote the image $N(f)$ by $(f_{G'})$. Let $\zeta_{G'} \in K_1'(\Lambda(G')_S)$ be the unique element satisfying the Main Conjecture over G'. Set $u_{G'} := \zeta_{G'} f_{G'}^{-1}$. Then by commutativity of the right square $\partial_{G'}(u_{G'}) = 0$ and by exactness of the bottom row $(u_{G'}) \in \varprojlim K_1'(\Lambda(G'))$. The left vertical is an isomorphism by the corollary above. Let u denote a preimage of $(u_{G'})$ in $K_1'(\Lambda(\mathcal{G}))$ under this map. Then $u \cdot f$ is a characteristic element of F_∞/F, too. Now let ρ be an Artin representation of G factoring over a fixed G'. Then for $r \equiv 0(p-1)$ we have $uf(\rho\kappa^r) = \zeta_{G'}(\rho\kappa^r) = L_\Sigma(\rho, 1-r)$. $\square$

3.2 Further reductions

Suppose $\mathcal{G}$ is one-dimensional. Then by fixing a section of $\mathcal{G} \to \Gamma$ we get an isomorphism $\mathcal{G} \cong H \rtimes \Gamma$ for some finite normal subgroup H of $\mathcal{G}$. The action of Γ on H must factor through a finite quotient and therefore we can and will fix an open central $\Gamma' := \Gamma^{p^n}$, where n is chosen minimal. For subgroups P of the finite group $G \cong \mathcal{G}/(\Gamma'^p)$ denote the inverse image of P in $\mathcal{G}$ by U_P.

Definition 3.5. A finite group P is called *l-hyperelementary* for a prime l if $P \cong C_m \rtimes \pi$ with π a finite l-group, C_m cyclic of order m with $l \nmid m$. It is called *hyperelementary* if it is l-hyperelementary for some l.

Theorem 3.6 ([Kak10], thm. 27)**.** *The main conjecture is valid for an admissible one-dimensional p-adic Lie extension F_∞/F with Galois group $\mathcal{G}$ if and only if the following holds: for every hyperelementary subgroup P of $G = \mathcal{G}/(\Gamma'^p)$, the main conjecture is valid for $F_\infty/F_\infty^{U_P}$.*

Proof. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
K_1'(\Lambda(\mathcal{G})) \arrow[r] \arrow[d, "N", "\sim"'] & K_1'(\Lambda(\mathcal{G})_S) \arrow[r] \arrow[d, "N_S"] & K_0(\Lambda(\mathcal{G}), \Lambda(\mathcal{G})_S) \arrow[r] \arrow[d] & 0 \\
\varprojlim_P K_1'(\Lambda(U_P)) \arrow[r] & \varprojlim_P K_1'(\Lambda(U_P)_S) \arrow[r, "(\partial_P)"] & \varprojlim_P K_0(\Lambda(U_P), \Lambda(U_P)_S),
\end{tikzcd}$$

where the projective limits in the lower row are taken with respect to the maps induced by Norms and conjugation.

The 'only if' part follows by the same argument as in Prop. 3.4. Now let f be a characteristic element for $C(F_\infty/F)$ and let (f_P) be its image under N_S. The collection of p-adic zeta functions ζ_P satisfying the main conjectures for $F_\infty/F_\infty^{U_P}$ is an element in the projective limit $\varprojlim_P K_1'(\Lambda(U_P)_S)$ by their respective uniqueness. By the commutativity of the right square $\partial_P(f_P\zeta_P^{-1}) = 0, \forall P$ and from left exactness of $\varprojlim_P$ follows the existence of $u_P \in K_1'(\Lambda(U_P))$ with $u_P \mapsto f_P\zeta_P^{-1}$. By their definition (u_P) is in the projective limit $\varprojlim_P K_1'(\Lambda(U_P))$. N is an isomorphism by [Kak10], lemma 26 [1], and we conclude that there is $u \in K_1'(\Lambda(\mathcal{G}))$ with $u \mapsto (u_P)$. We let $\zeta := fu$. Then ζ is a characteristic element for F_∞/F. Now we have to show that ζ interpolates the complex Artin L-function for F_∞/F.

Let $\rho : \mathcal{G} \to \overline{\mathbb{Q}}_p$ be an Artin character of $\mathcal{G}$. Then by Theorem 19 in [Ser77] there are (non-necessarily one-dimensional) Artin representations ρ_P of U_P and

[1] this is ultimately relying on work of C.T.C. Wall and A. Dress in the finite case

integers n_P with $\rho = \sum_P n_P \operatorname{ind}_{U_P}^{\mathcal{G}} \rho_P$. For any integer $r \equiv 0 \pmod{p-1}$, we have

$$\begin{aligned}
\zeta(\rho\kappa_F^r) &= \prod_P \zeta(\operatorname{ind}_{U_P}^{\mathcal{G}} \rho_P \kappa_F^r)^{n_P} \\
&= \prod_P \zeta_P(\rho_P \kappa_{F_P}^r)^{n_P} \\
&= \prod_P L_\Sigma(\rho_P, 1-r)^{n_P} = L_\Sigma(\rho, 1-r).
\end{aligned}$$

□

Remark 3.7. The l-hyperelementary case splits into two completely different cases: $l \neq p$ and $l = p$. The former is easier, the latter is the main problem: Kakde in [Kak10], as well as Ritter and Weiss in their sequence of papers, reduce the p-hyperelementary case to p-elementary groups, i.e. groups of the form $P \times \Delta$,P pro-p and Δ a finite cyclic group of order prime to p. The non-p part Δ can then be transferred to the coefficient ring by the isomorphism

$$\Lambda(P \times \Delta) \cong \prod_{\chi \in \hat{\Delta}} \Lambda_{\mathcal{O}_\chi}(P).$$

Here $\hat{\Delta}$ is the group of characters of Δ and $\mathcal{O}_\chi$ is the unramified extension of $\mathbb{Z}_p$ obtained by adjoining the values of χ. So using Burns and Kato strategy one is finally down to the question of describing $K_1(\Lambda_{\mathcal{O}}(P))$ for a pro-p group P and a finite unramified extension $\mathcal{O}$ of $\mathbb{Z}_p$.

Case 1:

Theorem 3.8. *Let F_∞/F be admissible and satisfy $\mu = 0$ and lets assume $\mathcal{G}$ is such that $\mathcal{G}/\Gamma'$ is l-hyperelementary for $l \neq p$. Then the Main Conjecture 3.1 for F_∞/F is true.*

Proof. Theorem 31 in [Kak10]. □

Case 2:

Assume that $\mathcal{G}$ is a rank one quotient of the false Tate group L. Consequently it is of the form $\mathcal{G} = C \rtimes (\Gamma \times \Delta)$ with $C = \langle \bar{\varepsilon} \rangle$ cyclic of p-power order, say $|C| = p^n$. Then $\mathcal{G}$ has a central open subgroup $\Gamma' := \Gamma^{p^{n-1}} \cong \mathbb{Z}_p$ and $G := \mathcal{G}/\Gamma' = C \rtimes (D \times \mu_{p-1})$ is finite with D cyclic of order p^{n-1}.

Lemma 3.9. *Suppose $H \leq G$ is a p-hyperelementary subgroup of G then H is already p-elementary.*

Proof. Let $H = C_k \rtimes \pi$ with π a p-group and C_k cyclic of order $p \nmid k$. Since $H \leq G$ and $ord(G) = p^{n\cdot(n-1)} \cdot (p-1)$ it is $k \mid p-1$. Then π acts on C_k trivially, since the image of $\pi \to Aut(C_k) \subseteq S_k$ must be a p-group and hence is trivial. $\square$

Lemma 3.10. *If $\mathcal{G}$ is isomorphic to the false Tate group $\mathbb{Z}_p \rtimes \mathbb{Z}_p^\times$ then $SK_1(\Lambda(\mathcal{G}))$ is trivial, hence Kakde's Main Conjecture 3.1 implies the Non-Commutative Main Conjecture 2.18.*

Proof. If G is a finite quotient of the false Tate group L, then its p-Sylow subgroup is of the form $N \rtimes C$ with N, C cyclic p-groups. Proposition 12.7 in [Oli88] applies to give $SK_1(\Lambda(G)) = 1$. $SK_1(\Lambda(L)_S)$ is by definition the image of $SK_1(\Lambda(L))$ in the localisation sequence. We just saw that the latter is trivial as projective limit of trivial groups. $\square$

Chapter 4

K_1 of certain completed p-adic group rings

In order to proof the Main Conjecture 2.18 for a specific group $\mathcal{G}$ via the strategy in section 2.5, we have to describe $K_1(\Lambda(\mathcal{G}))$ and $K_1(\Lambda(\mathcal{G})_S)$ by sets Ψ_G and $\Psi_{G,S}$ satifying certain assumptions. In particular we prove $\Psi_G \cong K_1(\Lambda(\mathcal{G}))$. To this end Kato proposed the use of a logarithm map. This 'integral logarithm' has values in $\mathbb{Z}_p[\![\mathrm{Conj}(\mathcal{G})]\!]$ and takes norm maps between K_1-groups to trace maps between $\mathbb{Z}_p$-modules.

Suppose $\mathcal{G}$ satisfies assumption 4.1 below, i.e. $\mathcal{G} \cong \mathbb{Z}_p \rtimes \mathbb{Z}_p^\times$, then in view of the reduction steps in Chapter 3 it suffices to pursue the above program for $K_1(\Lambda(G))$ (and its localization) when

a) $\mathcal{G}_1 = H \rtimes \Gamma$ is a one-dimensional quotient of $\mathcal{G}$ and

b) $G = U_P$ the preimage of a finite p-elementary subgroup of $\mathcal{G}_1/(Z(\mathcal{G}_1) \cap \Gamma)$.

Since the additive side is considerably easier we will give a description of the trace image of $\mathbb{Z}_p[\![\mathrm{Conj}(\mathcal{G})]\!]$ for the whole false Tate group $\mathcal{G}$ in section 4.2. For the multiplicative part we will use the reductions a) and b) above and describe $K_1(\Lambda(\Delta \times P))$ and $K_1(\Lambda(\Delta \times P)_S)$ for p-elementary subgroups of $\mathcal{G}$.

4.1 The false Tate group

We will be concerned with groups G satisfying the

Assumption 4.1. *Let p be an odd prime. There is an isomorphism of topological groups $s\colon L \xrightarrow{\sim} G$ of topological groups from the 'false Tate group'*

$$L := \mathbb{Z}_p \rtimes \mathbb{Z}_p^\times \cong \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

to G.

Remark 4.2. L is a compact p-adic Lie group: An atlas is given by the collection of charts with disjoint images $\{\psi_i\colon \mathbb{Z}_p \times \mathbb{Z}_p \to G, (a,b) \mapsto t(i + p \cdot a, b), 1 \le i \le p-1\}$.

A description of $K_1(\Lambda(G))$ will only be done for groups G satisfying the following

Assumption 4.3. *There is an isomorphism $s\colon P \to G$ of topological groups from the maximal pro-p subgroup of the false Tate group*

$$P := \mathbb{Z}_p \rtimes U^{(1)}$$

to G. Here, as usual, $U^{(n)} := \ker\left(\mathbb{Z}_p^\times \to (\mathbb{Z}/p^n\mathbb{Z}_p)^\times\right)$.

From now on suppose G satisfies assumption 4.1 or 4.3. Denote by $\langle\cdot\rangle$ the closed subgroup generated by "$\cdot$".

We fix the following important **notation:** Let z be a primitive $(p-1)$-th root of unity in $\mathbb{Z}_p$ and $d = (1+p)$, a topological generator of $U^{(1)} \cong \mathbb{Z}_p$, s.t. $\mathbb{Z}_p^\times \cong \langle z\rangle \times \langle d\rangle$. We define elements $\varepsilon, \delta, \zeta \in L$:

$$\varepsilon := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \delta := \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}, \quad \zeta = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Then L, resp. P, has a surjection ω to $\Gamma \cong \mathbb{Z}_p$ with kernel isomorphic to $\langle \varepsilon, \zeta\rangle$, resp. $\langle\varepsilon\rangle$.

If G falls under assumption 4.1 we omit reference to the isomorphism $s\colon L \to G$ and denote the elements $s(\varepsilon)$, resp. $s(\delta), s(\zeta)$, by ε, resp. δ, ζ. Any element g in G can be uniquely written as $g = \varepsilon^i \delta^j \zeta^k$, with i, j, k all minimal. For such g define $t_g := t := d^j z^k \in \mathbb{Z}_p^\times$. With this notation the following relations in G are verified by simple computations in L :

- $\delta\zeta = \zeta\delta$
- $\delta^n \varepsilon^k = \varepsilon^{d^n k} \delta^n, \zeta^n \varepsilon^k = \varepsilon^{z^n k} \zeta^n$
- $[\delta^n, \varepsilon^k] = \varepsilon^{(d^n - 1)k}$.

Let $U_0 := G = \langle \zeta, \delta, \varepsilon \rangle$ and for $n \geq 1$ set

$$U_n := \langle \delta^{p^{n-1}}, \varepsilon \rangle, V_n := [U_n, U_n] \overset{(1)}{=} \langle \varepsilon^{p^n} \rangle.$$

Identity (1) follows from $[\varepsilon^i \delta^{p^n \cdot k}, \varepsilon^j \delta^{p^n \cdot l}] = \varepsilon^{(1-d^{p^n \cdot k})i + (d^{p^n \cdot l} - 1)j} \in \langle \varepsilon^{p^{n+1}} \rangle$ and $\varepsilon^{p^{n+1}} \equiv [\varepsilon, \delta^{p^n}] \bmod [\varepsilon, \delta^{p^{n+1}}]$.

Similarly, if G satisfies assumption 4.3: Let $U_0' := G = \langle \delta, \varepsilon \rangle$ and for $n \geq 1$ set

$$U_n' := \langle \delta^{p^n}, \varepsilon \rangle, V_n' := [U_n', U_n'] = \langle \varepsilon^{p^{n+1}} \rangle.$$

Lemma 4.4. *If $G \cong L$ then U_n is an open subgroup of G for all n, V_n is a closed subgroup in the kernel of $\omega \colon G \to \Gamma$, normal in U_n and U_n/V_n is abelian and analogous if $G \cong P$.*

Proof. If $G \cong L$ the first assertion is clear, since U_n has index $\#(\langle \zeta \rangle) \cdot \# \left(\langle \delta \rangle / \langle \delta^{p^{n-1}} \rangle \right)$ in G. Since ω has abelian image the other assertions follow by definition of V_n; the same arguments work for $G \cong P$. $\square$

For all $n \in \mathbb{N}$, the family $C_n := \{z, z^2, \ldots, z^{p-1}, dz, dz^2, \ldots, d^{p^{n-1}} z^{p-2}\}$ is a system of representatives for $(\mathbb{Z}_p/(p^n))^\times$.

Lemma 4.5. *(i) The collections $\{\delta^j \zeta^k \mid 0 \leq k \leq p-2, 1 \leq j \leq p^{n-1}\}$ comprise coset representatives for L/U_n.*

(ii) The collection $\{\varepsilon^i \delta^j \zeta^k \mid 0 \leq k \leq p-2; j \in \mathbb{Z}_p; i = 0, 1, p, p^2, \ldots, p^{v_p(1-d^j z^k)-1} = p^{v_p(1-t)-1}\}$ is a system of representatives for the conjugacy classes of L.

(iii) The collections $\{\delta^j \mid 1 \leq j \leq p^n\}$ comprise coset representatives for P/U_n'.

(iv) The collection $\{\varepsilon^i \delta^j \mid j \in \mathbb{Z}_p; i = 0$ or $i = \mu p^k, \mu = 1, 2, \ldots, p-1, 0 \leq k \leq v_p(j)\}$ is a system of representatives for the conjugacy classes of P.

Proof. (i) and (iii) are obvious: For $g = \left(\begin{smallmatrix} x & a \\ 0 & 1 \end{smallmatrix}\right), h = \left(\begin{smallmatrix} y & b \\ 0 & 1 \end{smallmatrix}\right)$ it is $gh = \left(\begin{smallmatrix} xy & xb+a \\ 0 & 1 \end{smallmatrix}\right)$ and $xb + a = 0$ for a suitable choice of b.

For (ii) note that $hgh^{-1} = \left(\begin{smallmatrix} x & ya+(1-x)b \\ 0 & 1 \end{smallmatrix}\right)$, with g, h as above. Choosing y appropriate one has $ya = p^{v_p(a)}$. Then choose b such as to cancel ya if $v_p(1-x) \leq v_p(a)$.

Lastly, for (iv) use the same notation as for (ii) and write

$$a = a_0 p^{v_p(a)} + a_1 p^{v_p(a)+1} + \cdots$$

with $1 \leq a_0 \leq p-1$. There is $y \in U^{(1)}$ such that $ya = a_0 p^{v_p(a)}$. We can cancel ya by $(1-x)b$ iff $v_p(1-x) \leq v_p(a)$, but $v_p(1-x) = v_p(1-d^j) = v_p(j) + 1$. $\square$

Definition 4.6.

- Let $c \in \mathbb{N}$ be minimal such that $\varepsilon^{p^c} = 1$ and $c := \infty$ if ε has no finite order. For $c \in \mathbb{N} \cup \{\infty\}$ define $\underline{c} := \{0, 1, \ldots, c\} \subseteq \mathbb{N}$.
- The indexing sets in Lemma 4.5 will be of frequent use. We define

$$C := \{(i,j,k) \mid i,j,k \text{ as in Lemma 4.5 (ii)}\} \text{ and}$$

$$C_n := \{(i,j,k) \in C \mid 0 \leq j \leq p^{n-1} - 1\}.$$

- Similarly we define

$$C' := \{(i,j) \mid i,j \text{ as in Lemma 4.5 (iv)}\} \text{ and}$$

$$C'_n := \{(i,j) \in C \mid 0 \leq j \leq p^n - 1\}.$$

Remark 4.7.

- The center of G is defined to be $Z(G) = \{g \in G \mid gh = hg, \forall h \in G\} = \{g \mid [g,h] = 1, \forall h\}$. To determine the center we note that every $g \in G$ is of the form $g = \varepsilon^i \delta^j \zeta^k$ and g is in the center if and only if it commutes with all generators:

$$[\varepsilon^i \delta^j \zeta^k, \varepsilon] = \varepsilon^{d^j z^k - 1} = 1 \Leftrightarrow k = p-1, j \geq c-1 \text{ if } c \in \mathbb{N} \text{ and } j = 0 \text{ else,}$$

$$[\varepsilon^i \delta^j \zeta^k, \delta] = \varepsilon^{i(1-d)} = 1 \Leftrightarrow i = 0 \text{ and}$$
$$[\varepsilon^i \delta^j \zeta^k, \zeta] = \varepsilon^{i(1-z)} = 1 \Leftrightarrow i = 0.$$

We conclude that $Z(G) = \langle \delta^{p^{c-1}} \rangle$ if $ord(\varepsilon) = p^c$ is finite and in this case $G/Z(G)$ is finite and every element in the factor group has a unique representation as $g = \varepsilon^i \delta^j \zeta^k$ with $1 \leq i \leq p^c, 1 \leq j \leq p^{c-1}$ and $0 \leq k \leq p-2$ if $G \cong L$ or $g = \varepsilon^i \delta^j$ with $1 \leq i \leq p^c, 1 \leq j \leq p^{c-1}$ if $G \cong P$, respectively.

- A group G satisfying assumption 4.1 is the projective limit of groups with open center, as can be seen directly:

$$G \cong \varprojlim_n \begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}/(p^n) \\ 0 & 1 \end{pmatrix}.$$

  In general a compact p-adic Lie G with $H \hookrightarrow G \twoheadrightarrow \Gamma$ is the projective limit of its rank 1 quotients $\mathcal{G}$ with a similar sequence. Γ is free, so $\mathcal{H} \rtimes \Gamma \cong \mathcal{G}$ with finite $\mathcal{H}$ and the center of this contains Γ^{p^n} for $n \gg 1$ and thus is open.

4.2 Trace maps and the additive side

For an arbitrary group G and a commutative ring R, define $R[\mathrm{Conj}(G)]$ to be the free R-module on the set of conjugacy classes of G, i.e. $R[\mathrm{Conj}(G)] := \bigoplus_{C \in \mathrm{Conj}(G)} RC$. For $g \in G$ let $\mathrm{class}_G(g) := \{\sigma g \sigma^{-1} \mid \sigma \in G\}$. Note that

$$\varphi : R[\mathrm{Conj}(G)] \to R[\mathrm{Conj}(G)], \quad \mathrm{class}_G(\sigma) \mapsto \mathrm{class}_G(\sigma^p)$$

is a well defined map, although the multiplication of G does in general not extend to $R[\mathrm{Conj}(G)]$.

For a group G, a subgroup $H \leq G$ and a quotient G' of G, we define the natural maps

$$\begin{aligned} \iota\colon\ & R[\mathrm{Conj}(H)] \to R[\mathrm{Conj}(G)], \quad \mathrm{class}_H(\sigma) \mapsto \mathrm{class}_G(\sigma) \\ \pi\colon\ & R[\mathrm{Conj}(G)] \to R[\mathrm{Conj}(G')], \quad \mathrm{class}_G(\sigma) \mapsto \mathrm{class}_{G'}(\overline{\sigma}) \\ p_{conj}\colon\ & R[G] \to R[\mathrm{Conj}(G)], \quad \sigma \mapsto \mathrm{class}_G(\sigma). \end{aligned}$$

Suppose H has finite index in G and let $C(H,G)$ denote a system of representatives for the left cosets of H in G. We define the R-linear map

$$\begin{aligned} \mathrm{Tr}_{G|H} : R[\mathrm{Conj}(G)] &\to R[\mathrm{Conj}(H)], \\ \mathrm{class}_G(\sigma) &\mapsto \sum_{\substack{\nu \in C(H,G) \\ \nu\sigma\nu^{-1} \in H}} \mathrm{class}_H(\nu\sigma\nu^{-1}). \end{aligned}$$

This is independent of the choice of $C(H,G)$ and well defined, i.e. not depending on the choice of σ within a conjugation class.

Lemma 4.8. *(cf. lemmma 3.4 in [Lee09]) p_{conj} induces an isomorphism*

$$H_0(G, R[G]) \cong R[G]/[R[G], R[G]] \cong R[\mathrm{Conj}(G)].$$

For a profinite group G we define $R[\![\mathrm{Conj}(G)]\!] := \varprojlim_{U \lhd_o G} R[\mathrm{Conj}(G/U)]$ using the maps

$$\pi\colon R[\mathrm{Conj}(G/U)] \to R[\mathrm{Conj}(G/V)], \quad \text{for } U, V \lhd_o G, U \subseteq V.$$

Lemma 4.9. *(cf. lemmma 3.5 in [Lee09]) The map $p_{conj}\colon \Lambda(G) \to \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]$ (induced from the maps p_{conj} above) gives an isomorphism*

$$\Lambda(G)/\overline{[\Lambda(G), \Lambda(G)]} \cong \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!].$$

The following lemma is key in understanding the image of the trace map.

Lemma 4.10. *Let G be a p-adic Lie group and $Z(G)$ its center. $\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]$ is a $\Lambda(Z(G))$-module and the $\Lambda(Z(G))$-module homomorphism*

$$\Lambda(Z(G))[\![G/Z(G)]\!] \to \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]$$

is surjective.

Furthermore, if G falls under assumption 4.1 and $Z(G)$ is open in G, i.e. $c < \infty$, then

$$\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \cong \bigoplus_{(i,j,k) \in C_c} \Lambda(Z(G))\mathit{class}_G(\varepsilon^i \delta^j \zeta^k).$$

If G falls under assumption 4.3 and $Z(G)$ is open in G, i.e. $c < \infty$, then

$$\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \cong \bigoplus_{(i,j) \in C'_c} \Lambda(Z(G))\mathit{class}_G(\varepsilon^i \delta^j).$$

On the right hand side, $\Lambda(Z(G))\mathit{class}_G(g)$ means the $\Lambda(Z(G))$-submodule of $\mathbb{Z}_p[\![\mathrm{Conj}\, G]\!]$ generated by $\mathit{class}_G(g)$.

Proof. The first part is in [Lee09], lemma 3.7: For $z \in Z(G), g \in G$ we have $\text{class}_G(zg) = z \cdot \text{class}_G(g)$. Any continuous section of $G \twoheadrightarrow G/Z(G)$ will yield a $\Lambda(Z(G))$-linear surjection

$$\Lambda(Z(G))[\![G/Z(G)]\!] \to \Lambda(G).$$

Note that the second part of *loc.cite* cannot be applied here, since $[G,G] = \langle \varepsilon \rangle \nsubseteq Z(G)$. Rather, we us the ad hoc description of $\mathrm{Conj}(G)$ in lemma 4.5:

$$\mathrm{Conj}(G) = \coprod_x Z(G)\text{class}_G(x),$$

the disjoint union ranging over $x = \varepsilon^i \delta^j \zeta^k$, resp. $x = \varepsilon^i \delta^j$, with (i,j,k), resp. (i,j), in the purported index set. $\square$

Definition 4.11.

- The p-adic completion of a $\mathbb{Z}_p$-module M is, as usual, defined by

$$M^\wedge := \varprojlim_n M/p^n M.$$

- Assume that G is a one-dimensional p-adic Lie group with a surjection to $\mathbb{Z}_p$. Then the center $Z(G) \subseteq G$ is open and we define

$$\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]_S := \Lambda(Z(G))_{S(Z(G))} \otimes_{\Lambda(Z(G))} \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!].$$

  By lemma 4.10 this tensor product is well defined.

- For such G define

$$\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]_S^\wedge := (\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]_S)^\wedge.$$

Definition 4.12. Suppose G satisfies assumption 4.1. For $n \in \underline{c}$ define the $\Lambda(Z(G))$-module homomorphisms

$$\begin{aligned} \tau_n = \pi \circ \mathrm{Tr}_{G|U_n} &\colon \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \to \mathbb{Z}_p[\![\mathrm{Conj}(U_n)]\!] \to \Lambda(U_n/V_n), \\ \tau = (\tau_n)_n &\colon \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \to \prod_{n \in \underline{c}} \Lambda(U_n/V_n). \end{aligned}$$

If in addition c is finite then define

$$\begin{aligned} \mathrm{Tr}_{G|U_n,S} &\colon \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]_S \to \mathbb{Z}_p[\![\mathrm{Conj}(U_n)]\!]_S, \\ z \otimes \text{class}_G(x) &\mapsto \begin{cases} z \cdot \sum_{g \in G/U_n} \text{class}_{U_n}(gxg^{-1}), & \text{if } x \in U_n \\ 0, & \text{else} \end{cases} \end{aligned}$$

and by functoriality of localisation and completion

$$\begin{aligned}\tau_{n,S} = \pi_S \circ \mathrm{Tr}_{G|U_n,S}\colon \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]_S^\wedge \to \mathbb{Z}_p[\![\mathrm{Conj}(U_n)]\!]_S^\wedge \to \Lambda(U_n/V_n)_S^\wedge,\\ \tau_S = (\tau_{n,S})\colon \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]_S^\wedge \to \prod_{n\in\underline{c}} \Lambda(U_n/V_n)_S^\wedge.\end{aligned}$$

Analogous maps $\tau_n', \tau_{n,S}'$ and τ', τ_S' are defined for G satisfying 4.3.

Ultimately we want to see that τ and τ' are injective. We start with describing $\mathrm{im}(\tau_n)$ in $\Lambda(U_n/V_n)$ and turn to τ_n' later.

Let $U_{0,0} := U_0$ and for $n \in \underline{c}, n \geq 1$ define a family of subgroups of $U_n = \langle \delta^{p^{n-1}}, \varepsilon\rangle$ by

$$U_{n,k} := \langle \delta^{p^{n-1}}, \varepsilon^{p^k}\rangle, 0 \leq k \leq n.$$

Then $U_n = U_{n,0} \supseteq U_{n,1} \supseteq \cdots \supseteq U_{n,n}$, each $U_{n,k}$ is normal in G and $U_{n,n}/V_n = Z(G/V_n)$ by remark 4.7.

For $n \in \underline{c}, 1 \leq i < p^n$ define the element

$$h_{n,i} := \sum_{\substack{t=1\\(t,p)=1}}^{p^{n-v_p(i)}} \varepsilon^{p^{v_p(i)}\cdot t}.$$

Lemma 4.13. *Let φ denote the Euler function. For $n \in \underline{c}$, the image of τ_n is*

$$I_n := \left\langle \varphi(p^n), p^{v_p(i)} h_{n,i} \mid 1 \leq i \leq p^{n-1} \right\rangle_{\Lambda(U_{n,n}/V_n)}$$

and the image of $\tau_{n,S}$ in $\Lambda(U_n/V_n)_S$ is

$$I_{n,S} := \left\langle \varphi(p^n), p^{v_p(i)} h_{n,i} \mid 1 \leq i \leq p^{n-1} \right\rangle_{\Lambda(U_{n,n}/V_n)_S}.$$

Proof. It suffices to describe the image of τ_n, since by definition of $\tau_{n,S}$ its mapping properties are entirely analogous.

First, let $n = 0$, then $\mathrm{im}(\tau_0) = \mathrm{im}(\pi \circ \mathrm{Tr}_{G|G}) = im(\pi) = \Lambda(U_0/V_0)$. On the other hand $\varphi(p^0) = 1$.

So from now on let $n \geq 1$. The set $\{\mathrm{class}_G(\varepsilon^i\delta^j\zeta^k) \mid i = 0, 1, 2, \ldots, p^c;\ j \leq p^{c-1};\ k \leq p-2\} = \{\mathrm{class}_G(\varepsilon^i\delta^j\zeta^k) \mid i = 0, 1, p \ldots, p^c;\ j \leq p^{c-1};\ k \leq p-2\}$ generates the $\Lambda(Z(G))$-module $\mathbb{Z}_p[\![Conj(G)]\!]$ topologically, see lemma 4.10.

Note that, depending on the order of ε, this set may be infinite. Since $\delta^{p^{n-1}} \in U_{n,n}/V_n = Z(G/V_n)$ we have

$$\begin{aligned} im\ \tau_n &= \langle \tau_n(\text{class}_G(\varepsilon^i \delta^j \zeta^k)) \mid i, j, k \text{ as before} \rangle_{\Lambda(Z(G/V_n))} \\ &= \langle \tau_n(\text{class}_G(\varepsilon^i \delta^j \zeta^k)) \mid i = 1, p, \dots, p^n;\ 0 \le j \le p^{n-1} - 1;\ 0 \le k \le p-2 \rangle_{\Lambda(U_{n,n}/V_n)}. \end{aligned}$$

For the second equality note that

$$\tau_n(\text{class}_G(g)) = \pi\Big(\sum_{x \in G/U_n} \text{class}_{U_n}(xgx^{-1})\Big) = \sum_x class_{U_n/V_n}(\overline{xgx^{-1}}) = \sum_x \overline{x} \cdot \overline{g} \cdot \overline{x^{-1}}$$

and therefore $\tau_n(\text{class}_G(\varepsilon^{p^n})) = \tau_n(\text{class}_G(\varepsilon^0))$.

If $0 \ne j < p^{n-1}$, then $\delta^j \notin U_n = \langle \delta^{p^{n-1}}, \varepsilon \rangle$ and consequently $\tau_n(\text{class}_G(\varepsilon^i \delta^j)) = 0$ by definition. An analogous argument holds for $0 \ne k < p-1 : \varepsilon^i \delta^j \zeta^k \notin U_n$ for $n \ge 1$.

Finally we deal with $j = 0, k = 0$, i.e. we compute

$$\begin{aligned} \tau_n(\text{class}_G(\varepsilon^i)) &= \pi\Big(\sum_{\substack{0 \le l \le p-2 \\ 1 \le m \le p^{n-1}}} \text{class}_{U_n}(\delta^m \zeta^k \varepsilon^i \zeta^{-k} \delta^{-m})\Big) \\ &= \pi\Big(\sum_{l,m} \text{class}_{U_n}(\varepsilon^{i \cdot d^m \cdot z^l})\Big) = \sum_{\substack{1 \le t \le p^n \\ (p,t)=1}} \overline{\varepsilon}^{i \cdot t}. \end{aligned}$$

The last equation holds since $d^m z^l$ runs through a system of representatives of $(\mathbb{Z}_p/p^n)^\times \cong G/U_n$.

These sums are 'truncated' by the relation $\overline{\varepsilon}^{p^n} = 1$. If $i = p^n$ this evaluates to $\varphi(p^n)$. To determine the corresponding coefficients for $i \ne p^n$, note that all elements $i \cdot t$ have the p-valuation of i, of which there are $(p-1)p^{n - v_p(i) - 1}$ in $\mathbb{Z}/p^n$. $(\mathbb{Z}/p^n)^\times$ is acting transitively on these elements and the order of the stabilizer is given by $\frac{\#(\mathbb{Z}/p^n)^\times}{(p-1)p^{n-v_p(i)-1}} = p^{v_p(i)}$. The assertion follows. $\square$

To further describe the image of τ in $\prod_n I_n$ we need a relative trace map:

Definition 4.14. Let S be a finitely generated free left R-algebra. Define the natural antihomomorphism[1] from the left R-module S to the right R-module R

$$\text{Tr}\colon S \longrightarrow End_R(S) \cong \text{M}_n(R) \stackrel{Trace}{\longrightarrow} R.$$

[1]meaning $f(\lambda m) = f(m) \cdot \lambda, \lambda \in R$

It induces a homomorphism of left R-modules

$$\mathrm{Tr}\colon S/[S,S] \to R/[R,R]$$

and, if R, S are topological rings, a continuos homomoprhism

$$\mathrm{Tr}\colon S/\overline{[S,S]} \to R/\overline{[R,R]}.$$

Remark 4.15. Assume there is a basis $\{x_i\}_{i=1,\dots,n} \subseteq S^\times$ and for $s \in S, j = 1, \dots, n$ set $x_{ij} \in R$, such that

$$x_j s = \sum_i x_{ij} x_i.$$

Finally let $\pi_i\colon \bigoplus_j Rx_j \twoheadrightarrow Rx_i \cong R$ denote the projection onto the i-th direct summand. Then x_{ij} and Tr can be computed with $x_{ij} = \pi_i(x_j s) = \pi_0(x_j s x_i^{-1})$ and

$$\mathrm{Tr}(s) = \sum_i x_{ii} = \sum_i \pi_0(x_i s x_i^{-1}). \tag{4.1}$$

Lemma 4.16. *For $m \le n \in \underline{c}$, let*

$$\begin{aligned} \mathrm{Tr}_{m,n} &= \mathrm{Tr}\colon \Lambda(U_m/V_m) \to \Lambda(U_n/V_m) \\ \mathrm{Tr}_{m,n,S} &= (\mathrm{Tr})_S\colon \Lambda(U_m/V_m)_S \to \Lambda(U_n/V_m)_S, \end{aligned}$$

and

$$\begin{aligned} p_{n,m} &= p_*\colon \Lambda(U_n/V_n) \to \Lambda(U_n/V_m), \\ p_{n,m,S} &= (p_*)_S\colon \Lambda(U_n/V_n)_S \to \Lambda(U_n/V_m)_S \end{aligned}$$

be the Trace, resp. projection, homomorphisms. We define the $\Lambda(Z(G))$-modules

$$\Omega := \Omega_G := \left\{ (x_n)_n \in \prod_{n \in \underline{c}} I_n \mid \mathrm{Tr}_{m,n}(x_m) = p_{n,m}(x_n) \textit{ for } m \le n \right\},$$

$$\Omega_S := \Omega_{G,S} := \left\{ (x_n)_n \in \prod_{n \in \underline{c}} I_{n,S} \mid \mathrm{Tr}_{m,n,S}(x_m) = p_{n,m,S}(x_n) \textit{ for } m \le n \right\}.$$

Assume that c is finite. Then $\mathrm{im}(\tau) \subseteq \Omega$ and $im(\tau_S) \subseteq \Omega_S$.

Proof. Clearly it sufficess to prove the assertion for τ.

To this end we define elements $c_{itn} \in \Lambda(G), i \in \mathbb{Z}_p, t \in \mathbb{Z}_p^\times, n \in \mathbb{N}$ by

$$c_{itn} := \begin{cases} 0, & v_p(1-t) < n \\ \sum\limits_{\substack{s=1 \\ (p,s)=1}}^{p^n} \varepsilon^{i\cdot s} =: \sum\limits_{s=1}^{p^n}{}' \varepsilon^{i\cdot s}, & v_p(1-t) \geq n. \end{cases}$$

The proof of lem. 4.13 shows $\tau_n(\mathrm{class}_G(\varepsilon^i\delta^j\zeta^k)) = \overline{c_{itn}\delta^j\zeta^k} \in \Lambda(U_n/V_n)$ with $t = d^j \cdot z^k$, as usual.

If $m \geq 1$ the set $\{\delta^{p^m\cdot l} \mid l = 1, \ldots, p^{n-m}\}$ is a basis of the $\Lambda(U_n/V_m)$-module $\Lambda(U_m/V_m)$. If $v_p(1-t) \geq n$ then

$$\begin{aligned} \mathrm{Tr}_{m,n}(c_{itn}\delta^j\zeta^k) &\overset{(4.1)}{=} \sum_{l=1}^{p^{n-m}} \delta^{p^m\cdot t} c_{itn} \delta^{-p^m\cdot t}\delta^j\zeta^k \\ &= \sum_{t=1}^{p^{n-m}} \sum_{s=1}^{p^m}{}' \varepsilon^{i\cdot s\cdot d^{p^m\cdot l}}\delta^j\zeta^k = p^{n-m}\sum_{s=1}^{p^m}{}' \varepsilon^{i\cdot s\cdot d^{p^m\cdot l}}\delta^j\zeta^k. \end{aligned}$$

The last equation holds since $d^{p^m\cdot l} \equiv 1 \bmod p^m$ for all t and hence $\varepsilon^{i\cdot s\cdot d^{p^m\cdot l}} \equiv \varepsilon^{i\cdot s} \bmod V_m$. On the other hand

$$\begin{aligned} p_{n,m}(c_{itn}\delta^j\zeta^k) =& \sum_{s=1}^{p^n}{}' \varepsilon^{i\cdot s}\delta^j\zeta^k \\ =& p^{n-m}\sum_{s=1}^{p^m}{}' \varepsilon^{i\cdot s}\delta^j\zeta^k. \end{aligned}$$

If $v_p(1-t) < n$ then $\delta^j\zeta^k \notin U_n$, consequently $\delta^{p^m\cdot l}\delta^j\zeta^k \notin U_n$ and $\mathrm{Tr}_{m,n}(c_{itn}\delta^j\zeta^k) = 0$ by (4.1). On the other hand $p_{n,m}(c_{itn}\delta^j\zeta^k) = p_{n,m}(0) = 0$. So the assertion holds in case $m \geq 1$.

If $m = 0, m < n$, similarly, the set $\{\delta^l\zeta^k \mid l = 1, \ldots, p^n, k = 0, \ldots, p-2\}$ is a basis of the $\Lambda(U_n/V_m)$-module $\Lambda(U_m/V_m)$. An analogous argument as above finishes the proof. $\square$

Theorem 4.17. *The homomorphism*

$$\tau\colon \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \to \Omega$$

is an isomorphism of $\Lambda(Z(G))$-modules.

Proof. We remind ourselves of the definition of c and C_c in 4.6. Define a possible inverse map

$$\widetilde{\tau}\colon \Omega \to \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!], \quad (x_n)_n \mapsto \sum_n^c \widetilde{\tau}_n(x_n),$$

where

$$\widetilde{\tau}\colon I_n \to \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!], \quad c_{itn}\delta^j\zeta^k \mapsto \begin{cases} 0 & \text{if } n \neq v_p(1-t) \\ \mathrm{class}_G(\varepsilon^i\delta^j\zeta^k) & \text{if } n = v_p(1-t). \end{cases}$$

First suppose $c < \infty$. Then by Lemma 4.5 (ii) the elements $\mathrm{class}_G(\varepsilon^i\delta^j\zeta^k), (i,j,k) \in C_c$ generate $\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]$ as $\Lambda(Z)$ module. Then $\widetilde{\tau} \circ \tau = id$ and τ is injective.

We now show that $\widetilde{\tau}$ is injective. Let $x = (x_n)_n$ be in the kernel of $\widetilde{\tau}$. Write

$$x_n = \sum_{(i,j,k)\in C_c} c_{itn}\delta^j\zeta^k z_{itn}$$

where $z_{itn} \in \Lambda(Z(G))$. Then

$$\begin{aligned} 0 = \widetilde{\tau}(x) &= \sum_{n=0}^{c} \widetilde{\tau}_n(x_n) = \sum_n \sum_{i,j,k} \widetilde{\tau}_n(c_i t n \delta^j\zeta^k) z_{itn} \\ &= \sum_{i,j,k} \mathrm{class}_G(\varepsilon^i\delta^j\zeta^k) z_{it\tilde{t}}, \text{ with } \tilde{t} := v_p(1-t). \end{aligned}$$

From the direct sum composition in Lemma 4.5 (ii) we see that $\mathrm{class}_G(\varepsilon^i\delta^j\zeta^k) \cdot z_{it\tilde{t}} = 0, \forall (i,j,k) \in C_c$. Applying the $\Lambda(Z(G))$-linear map $\tau_{\tilde{t}}$ to this we get

$$c_{it\tilde{t}} \cdot z_{it\tilde{t}} = 0. \tag{4.2}$$

It remains to show that $c_{itn} \cdot z_{itn} = 0$, for all $(i,j,k) \in C_c, n \leq c$. Note that we have not used the condition that $x \in \Omega$, yet.

In the previous lemma we saw that for $0 \neq n \leq q \leq c, z \in \Lambda(Z(G))$

$$p_{q,n}(c_{itq}\delta^j\zeta^k z) = \mathrm{Tr}_{n,q}(c_{itn}\delta^j\zeta^k z) = \begin{cases} 0 & \text{if } v_p(1-t) < q, \\ p^{q-n}c_{itn}\delta^j\zeta^k z & \text{if } v_p(1-t) \geq q. \end{cases}$$

Applying this to x_n yields

$$\sum_{\substack{(i,j,k)\in C_c \\ q\leq v_p(1-t)}} p^{q-n}c_{itn}\delta^j\zeta^k \cdot z_{itn} = \mathrm{Tr}_{n,q}(x_n)$$

$$= p_{q,n}(x_q) = \sum_{\substack{(i,j,k)\in C_c \\ q\leq v_p(1-t)}} p^{q-n}c_{itn}\delta^j\zeta^k \cdot z_{itq} \in \Lambda(U_q/V_n).$$

Hence $c_{itn}z_{itn} = c_{itq}z_{itq}, \forall n \leq q$ and by equation 4.2 we see that

$$c_{itn}z_{itn} = c_{itn}z_{it\tilde{t}} = 0, \forall n \leq v_p(1-t).$$

This finishes the proof of injectivity for $\widetilde{\tau}$ if $c < \infty$.

If ε is of infinite order we make the usual projective limit argument:

$$\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \cong \varprojlim_n [\![\mathrm{Conj}\,(G/V_n)]\!] \cong \varprojlim_n \Omega_{G/V_n} \cong \Omega_G.$$

If we set $I_G := \prod_{n\leq c_G} \tau_n(\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]), f_G := \mathrm{Tr}_{j,i} - p_{i,j} \colon I_G \to \prod_{i\geq j} \Lambda(U_i/V_j)$ the last isomorphism is by the following clearly commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega_G \arrow[r, "incl"] \arrow[d, "\wr"] & I_G \arrow[r, "f_G"] \arrow[d, "\wr"] & \prod_{i\geq j} \Lambda(U_i/V_j) \arrow[d, "\wr"] \\
0 \arrow[r] & \varprojlim_n \Omega_{G/V_n} \arrow[r, "incl"] & \varprojlim_n I_{G/V_n} \arrow[r, "\varprojlim f_{G/V_n}"] & \varprojlim_n \prod_{n\geq i\geq j} \Lambda(U_i/V_j).
\end{tikzcd}$$

Note that all maps here are $\Lambda(Z(G))$-module homomorphisms. $\square$

To carry over this description of $H_0(G, \Lambda(G)) = \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]$ to $\mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]_S$ and then to its p-adic completion one uses once more the projective limit over quotients with open center:

Corollary 4.18. *We have an isomorphism of $\Lambda(Z(G))$-modules*

$$\tau_S\colon \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]^\wedge_S \to \widehat{\Omega_S} := \varprojlim_n \Omega^{\langle p\rangle}_{G/V_n,S}$$

Proof. This is Corollary 3.25 in [Lee09]. □

One uses the exact same techniques as above to show the following

Proposition 4.19. *Suppose G satifies assumption 4.3 and suppose $c < \infty$.*

Define for $n \in \underline{c}$, $1 \leq i < p^{n+1}$ the elements

$$h'_{n,i} := \sum_{k=0}^{p^{n-v_p(i)}} \varepsilon^{i\cdot d^k}.$$

Then the image of τ'_n in $\Lambda(U'_n/V'_n)$, resp. $\tau'_{n,S}$ in $\Lambda(U'_n/V'_n)_S$, is

$$\begin{aligned} I'_n &:= \langle p^n, p^{v_p(i)} h'_{n,i}\rangle_{\Lambda(\langle\delta^{p^n}\rangle)}, \\ \textit{resp. } I'_{n,S} &:= \langle p^n, p^{v_p(i)} h'_{n,i}\rangle_{\Lambda(\langle\delta^{p^n}\rangle)_{S(\Lambda(\langle\delta^{p^n}\rangle))}}. \end{aligned}$$

Furthermore

$$\tau'\colon \mathbb{Z}_p[\![\mathrm{Conj}(P)]\!] \to \Omega' \tag{4.3}$$

is an isomorphism of $\Lambda(Z(G))$-modules and $\mathrm{im}(\tau'_S) \subseteq \Omega'_S$, where $\Omega' \subseteq \prod_{n\in\underline{c}} I'_n$ and $\Omega'_S \subseteq \prod_{n\in\underline{c}} I'_{n,S}$ are defined by the same conditions as Ω and Ω_S, i.e. $\mathrm{Tr}_{m,n}(x_m) = p_{n,m}(x_n)$.

Corollary 4.20. *Let $\mathcal{O}$ be the ring of integers in a finite algebraic extension of $\mathbb{Q}_p$, then tensoring 4.3 with $\mathcal{O}$ we obtain an isomorphism $\tau'_{\mathcal{O}}\colon \mathcal{O}[\![\mathrm{Conj}(P)]\!] \to \Omega'_{\mathcal{O}}$, since $\mathcal{O}$ is finite, free over $\mathbb{Z}_p$.*

4.3 The integral logarithm $\mathcal{L}$

The integral logarithm $\mathcal{L}\colon K_1(\Lambda(G)) \to \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!]$ was constructed for finite groups by M. Taylor and R. Oliver simultaneously to be used in the study of the Whitehead group, i.e. the quotient of K_1 by the canonical units. For the construction of the integral logarithm for pro-finite groups we refer to chapter 5 of [SV11]. The calculations omittet there can be found in [Lee09], chapter 3.3.

We also need an integral logarithm for $K_1(\widehat{\Lambda(G)_S})$, compatible with the non localized map from above. If the following assumption holds, the construction of such an extension can be simplified as in [Lee09]. There are results in this direction, cf. [Bur10] lemma 5.1 and remark 5.7.

Assumption 4.21. *Suppose G is a compact p-adic Lie group that is p-elementary with open center Z. Every $x \in (\Lambda\widehat{(G)_{S(G)}})^\times$ can be written as $x = uv$ with $u \in \widehat{\Lambda(Z)_{S(Z)}}^\times, v \in \Lambda(G)^\times$.*

Since we could not show the asserted decomposition, we will instead use the construction of Kakde, relying on

$$\widehat{\Lambda(G)_S}^\times = (1 + Jac(\widehat{\Lambda(G)_S})) \cdot \Lambda\widehat{(\sigma(\Gamma))}_S, \tag{4.4}$$

where σ is a continuous, homomorphic section of the surjection $G \twoheadrightarrow \Gamma$, cf. lemma 5.1 in [SV11].

Proposition 4.22. *Let $\varphi\colon \operatorname{Conj}(P) \to \operatorname{Conj}(P)$ be the map $class_P(g) \mapsto class_P(g^p)$ and denote the induced map on $\mathbb{Z}_P[\![\operatorname{Conj}(P)]\!]$ also by φ. For a compact p-adic Lie group P there is a well-defined group homomorphism*

$$\mathcal{L}_P\colon K_1(\Lambda(P)) \to \mathbb{Z}_p[\![Conj(P)]\!], \quad x \mapsto (1 - p^{-1}\varphi) \circ log(x),$$

where log is defined by the usual power series.

If in addition G is pro-p and there is a surjective homomorphism $P \to \mathbb{Z}_p$ with kernel, say H, and N an open subgroup of H that is normal in P, we can write an element $x \in ((\Lambda(P/N)_S)^{\langle p \rangle})^\times$ as $x = uv$, $u \in 1 + Jac(\Lambda\widehat{(P/N)}_S)$, $v \in (\Lambda(\widehat{Z(P/N)})_S)^\times$ according to 4.4, where $Z(P/N)$ denotes the center. Define

$$\mathcal{L}_{P,S}\colon K_1(\widehat{\Lambda(P)_S}) \to \mathbb{Z}_p[\![Conj(P)]\!]\widehat{_S}$$

to be the composition of the natural homomorphism

$$K_1(\widehat{\Lambda(P)_S}) \to \varprojlim_N K_1((\Lambda(P/N)_S)^{\langle p \rangle})$$

and the homomorphism

$$\varprojlim_N K_1(\Lambda\widehat{(P/N)}_S) \to \varprojlim_N (\mathbb{Z}_p[\![\operatorname{Conj}(P/N)]\!]_S)^{\langle p \rangle}.$$

Here N is as above and the later map is induced from the maps

$$K_1(\widehat{\Lambda(P/N)}_S) \to (\mathbb{Z}_p[\![\mathrm{Conj}(P/N)]\!]_S)^{\langle p\rangle},$$
$$[x]_{\widehat{\Lambda(P/N)}_S} \mapsto \frac{1}{p}log(u^p\varphi(u)^{-1}) + (1-\frac{1}{p}\varphi)\circ log([v]_{\Lambda(P/N)}).$$

Proof. The last definition makes sense, as $\frac{1}{p}log(x^p\varphi(x)^{-1}) = (1-\frac{1}{p}\varphi)\circ log([x]_{\Lambda(P/N)})$ for x in the intersection of the domains. There are various convergence and well definedness issues here, for details consult [SV11], chapter 5. □

By varying the coeffient ring over unramified extensions $\mathcal{O}$ of $\mathbb{Z}_p$ one can easily extend this to p-elementary groups P.

4.4 Kernel and cokernel of $\mathcal{L}$

For a finite group G and a fixed prime number p let G_r denote the p-regular part of G, i.e. the elements of G whose orders are prime to p. If x is arbitrary in G then write $ord(x) = p^r\cdot m, (m,p) = 1 = \mu p^r + \lambda m$. Since $x^1 = x^{\mu p^r+\lambda m} = x^{\mu p^r}\cdot x^{\lambda m}$ we have the decomposition $x = x_r\cdot x_u = x_u\cdot x_r, p \mid ord(x_u), p\nmid ord(x_r)$. Note that G_r is in general not a subgroup of G but closed under conjugation by arbitrary elements of G. The quotient set of this G-action plays an important role in modular representation theory: The number of elements of $G\backslash G_r$ is the number of irreducible characters of G modulo p, cf. [Ser77].

Lemma 4.23. *Let $f\colon G\to H$ be a homomorphism of finite groups. Then*

(i) f restricts to a well defined map $G_r \to H_r$.

(ii) If f is surjective, f induces a surjection $G_r \to H_r$.

Proof. (i) If there is n with $p\nmid n, x^n = 1$ then $(f(x))^n = 1$, too. For (ii), note that with $y\in H_r$ it is $y^k\in H_r, \forall k$ and hence $\langle y\rangle \subseteq H_r$. Now let $x\in G$ with $f(x) = y$ and $ord(x) = p^l\cdot m, p\nmid m$. Then x^{p^l} is p-regular. Since $p\nmid |\langle y\rangle|$ the p^l-power map is an element of the automorphism group of $\langle y\rangle$. Hence there is $y' = y^k$ with $y'^{p^l} = y$. Now x^k is a preimage of y^k and consequently $(x^k)^{p^l} = (x^{p^l})^k$ is a p-regular preimage of y. □

For a ring R, let $R(G_r)$ be the free (left) R-module on the set of p-regular elements of G. G operates on $R(G_r)$ by conjugation on the basis. If $G = \varprojlim G_\lambda$ is the projective limit of finite groups G_λ and π_λ denotes the canonical projection, by Lemma 4.23 we can define the projective limit of the sets $G_{\lambda,r}$ of p-regular elements.

Lemma 4.24. *Denoting by G_r the set $\{x \in G \mid \pi_\lambda(x) \in G_{\lambda,r}, \forall\lambda\}$ we have a bijection $\alpha\colon G_r \to \varprojlim G_{\lambda,r}$.*

Proof. This is obvious since the maps over which we take the projective limits are the same. $\square$

Let $\varphi : R \to R$ denote the Frobenius map for a finite, unramified extension R of $\mathbb{Z}_p$ and consider the map $\Phi : R(G_r) \to R(G_r), \sum r_i g_i \mapsto \sum \varphi(r_i) g_i^p$. If we take homology with $R(G_r)$ coefficients, Φ denotes the induced map on homology, too.

For an abelian group A let $A(p)$ denote the p-primary subgroup. If Ψ is an endomorphism of A, A^Ψ denotes the subgroup of elements fixed by Ψ and A_Ψ denotes the quotient of A by all elements $(\Psi - id)(a), a \in A$. We now state the crucial

Proposition 4.25. *(cf. [Oli88], thm 12.9 iii)*
For a finite group G and a local field F, unramified, of finite degree over $\mathbb{Q}_p$ and with ring of integral elements $R = \mathcal{O}_F$, the following sequence is exact

$$\begin{tikzcd}
& 1 \arrow[r] & H_1(G, R(G_r))^\Phi \oplus H_0(G, R/2(G_r))^\Phi \arrow[r] & K_1'(R[G])(p) \arrow[dll, "\mathcal{L}_G"] \\
& H_0(G, R[G]) \arrow[r] & H_1(G, R(G_r))_\Phi \oplus H_0(G, R/2(G_r))_\Phi \arrow[r] & 1.
\end{tikzcd}$$

Remark 4.26.
- In our situation $R = \mathbb{Z}_p$ so the Frobenius action on coefficients is just by $g \mapsto g^p$.
- Since $p \neq 2, R/2 = 0$ and the corresponding summands vanish.

A result of Wall gives the following splitting of K_1 for finite groups.

Proposition 4.27. *Let G be an arbitrary finite group and let J denote the Jacobson radical of the p-adic group ring $\Lambda = \mathbb{Z}_p[G]$. Then there is an exact sequence of abelian groups*

$$1 \to V \to K_1(\Lambda) \to K_1(\Lambda/J) \to 1,$$

in which V is a pro-p group and $K_1(\Lambda/J)$ is a finite group of order prime to p.

Furthermore this sequence splits canonically to give an isomorphism $K_1(\Lambda) \cong V \times K_1(\Lambda/J)$.

Proof. [CR87] Theorem 45.31 □

Remark 4.28. This combined with the previous proposition shows exactness of the sequence

$$\begin{aligned} 1 &\to (\Lambda/J)^\times \times H_1(G, \mathbb{Z}_p(G_r))^\Phi \times SK_1(\mathbb{Z}_p[G]) \to K_1(\mathbb{Z}_p[G]) \\ &\to H_0(G, \mathbb{Z}_p[G]) \to H_1(G, \mathbb{Z}_p(G_r))_\Phi \to 1 \end{aligned}$$

for a finite group G.

Lemma 4.29. *If G is a profinite group, let A_G denote the projective limit $\varprojlim_{U \lhd_o G} \mathbb{Z}_p((G/U)_r)$. Then taking the projective limit over the sequence in remark 4.28 we conclude exactness of the following sequence*

$$\begin{aligned} 1 &\to H_1(G, A_G)^\Phi \times (\Lambda(G)/Jac)^\times \times SK_1(\Lambda(G)) \to K_1(\Lambda(G)) \\ &\to \mathbb{Z}_p[\![\mathrm{Conj}(G)]\!] \to H_1(G, A_G)_\Phi \to 1. \end{aligned}$$

Proof. The projective limit commutes with taking K_1 and SK_1 by lemma 3.2. Group homology commutes with projective limits by Theorem 2.6.9 and Proposition 1.2.5 in [NSW08]. Since all terms involved are compact abelian groups, the projective limit is exact. For finite G the Φ invariants, resp. coinvariants, of $H_1(G, A_G)$ are compacta as closed subset, resp. quotient set, of a compact set. Hence the projective limit over G is exact on the first and last term of the sequence. □

Remark 4.30. Some remarks on computing $H_1(G, A_G)$:

- Note that $e \in G_r$ for a finite group G and G acts trivially on the submodule $\mathbb{Z}_p e$ of $\mathbb{Z}_p(G_r)$. If $\mathcal{G}$ is a pro-p group this inclusion is an equality, hence kernel and cokernel of our sequence specialize to $H_1(\mathcal{G}, \mathbb{Z}_p) = \mathcal{G}^{ab} = (\mathcal{G}^{ab} \times \mu_{p-1})(p)$. Compare Lemmata 66 and 67 in [Kak10] or Corollary 3.42 in [Lee09].

- If G is finite and a direct product $\Delta \times \pi$, with Δ abelian, $p \nmid |\Delta|$ and an arbitrary (not necessarily abelian) p-group π then $G_r = \Delta$ and
$$H_1(G, A_G) = H_1(G, \mathbb{Z}_p(\Delta)) \cong \bigoplus_{x \in \Delta} H_1(G, \mathbb{Z}_p) \cong \bigoplus_{x} \pi^{ab}.$$
In particular any p-elementary group falls under this assumption.

4.4.1 A computational example

In this section we compute kernel and cokernel for a specific family of non p-groups. The results will not be used in the remainder of this paper, but are supposed to point at the involved difficulties and techniques. In general one may try to use group theoretic reductions and the Hochschild-Serre spectral sequence to generalize these computations.

For this section let p be an odd prime and let $G^n := \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, where the last term acts by inversion on the second. Since $\#G^n = p^{2n} \cdot 2$ the order of any $x \in G^n_r, x \neq e$ is 2. Then $G^n_r = \{e := (0,0,0), e_i := (0,i,1); i \in \mathbb{Z}/p^n\mathbb{Z}\}$. Let $G = \varprojlim_n G^n$ with the natural projection maps. Note that Φ acts trivially on the coefficients, as $(0,i,1)^2 = e$ and p is odd.

Lemma 4.31. *With $R = \mathbb{Z}_p$ the action of G^n on $R(G^n_r)$ introduced above, we have $H_1(G^n, R(G^n_r)) = \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z}, \forall n.$*

Proof. Denoting the Pontryagin dual $Hom_{cts}(X, \mathbb{R}/\mathbb{Z})$ of X by $X^\vee$ one has $H_1(G^n, \mathbb{Z}_p(G^n_r)) \cong H^1(G^n, \mathbb{Z}_p(G^n_r)^\vee)^\vee = H^1(G^n, \mathbb{Q}_p/\mathbb{Z}_p(G^n_r))^\vee$ (cf. [NSW08], 2.6.9) and we therefore deal with cohomology with $A_n := \mathbb{Q}_p/\mathbb{Z}_p(G^n_r)$ coefficients instead. The exact sequence

$$0 \longrightarrow \underbrace{\overbrace{\mathbb{Z}/p^n\mathbb{Z}}^{=:U^n} \times \overbrace{\mathbb{Z}/p^n\mathbb{Z}}^{=:V^n}}_{=:H^n} \longrightarrow G^n \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

yields a Hochschild-Serre spectral sequence. Denoting G^n, H^n, V^n, U^n and A_n by G, H, V, U and A, respectively, the corresponding 5-term exact sequence is

$$0 \longrightarrow H^1(\mathbb{Z}/2\mathbb{Z}, A^H) \xrightarrow{\inf} H^1(G, A) \xrightarrow{res} H^1(H, A)^{\mathbb{Z}/2\mathbb{Z}} \xrightarrow{tg} \dots \tag{4.5}$$

$$\dots \xrightarrow{tg} H^2(\mathbb{Z}/2\mathbb{Z}, A^H) \longrightarrow H^2(G, A).$$

Note that $U = \langle u = (1,0,0)\rangle$ is central in G, hence it operates trivialy on A. The action of $V = \langle v = (0,1,0)\rangle$ on G_r^n gives a decomposition in two orbits: $\{e\}\coprod\{e_i, i = 0, \dots p^n - 1\}$. Indeed, we have $e_i = (0,i,1)$ and $e_i^v = (0,1,0)(0,i,1)(0,p^n-1,0) = (0,i+2,1) = e_{i+2}$. Since $2 \nmid p^n$ the operation of V on the e_i's is transitive. An element $\hat{x} \in A$ can be written as $\sum_{i=0}^{p^n-1} x_i e_i + xe$.

From this we deduce $A^H = A^V = R^\vee \oplus R^\vee = \mathbb{Q}_p/\mathbb{Z}_p \oplus \mathbb{Q}_p/\mathbb{Z}_p =: \mathcal{R} \oplus \mathcal{R}$, one summand per orbit. Note also that this module has trivial $\mathbb{Z}/2\mathbb{Z}$-action. So we get $H^1(\mathbb{Z}/2\mathbb{Z}, \mathcal{R}\oplus\mathcal{R}) = \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathcal{R}\oplus\mathcal{R}) = 0$ since $\mathcal{R}$ has no 2-torsion. For the cyclic group $\mathbb{Z}/2\mathbb{Z}$ the cup-product induces (cf. [NSW08], prop 1.7.1) an isomorphism[2]

$$H^2(\mathbb{Z}/2\mathbb{Z}, \mathcal{R}\oplus\mathcal{R}) \cong \widehat{H}^0(\mathbb{Z}/2\mathbb{Z}, \mathcal{R}\oplus\mathcal{R}) =$$
$$(\mathcal{R}\oplus\mathcal{R})^{\mathbb{Z}/2\mathbb{Z}} \big/ \mathcal{N}_{\mathbb{Z}/2\mathbb{Z}}\cdot(\mathcal{R}\oplus\mathcal{R}) = \mathcal{R}\oplus\mathcal{R}\big/ 2\mathcal{R}\oplus 2\mathcal{R} = 0$$

since multiplication by 2 is an isomorphism of $\mathcal{R}$. So we get from (4.5) that $H^1(G,A) \cong H^1(H,A)^{\mathbb{Z}/2\mathbb{Z}}$.

In the same way the exact sequence $0 \longrightarrow U \longrightarrow H \longrightarrow V \longrightarrow 0$ gives rise to

$$0 \longrightarrow H^1(V, A^U) \xrightarrow{\inf} H^1(H,A) \xrightarrow{res} H^1(U,A)^V \xrightarrow{tg} \dots \tag{4.6}$$
$$\dots \xrightarrow{tg} H^2(V, A^U) \longrightarrow H^2(H,A).$$

From what we said above we see that A has trivial U action. So

$$\begin{aligned} H^1(U,A)^V &= \mathrm{Hom}(U,A)^V = \{\hat{x} \in A \mid p^n \cdot \hat{x} = 0\}^V \\ &= \{\hat{x} \in A \mid x_i = x_j, \forall i,j = 0,\dots,p^n-1 \text{ and } p^n\cdot x = p^n \cdot x_i = 0\} \\ &\cong \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z}. \end{aligned}$$

For later use, we point out that $\sigma_n \colon U = \langle u\rangle \to A, u \mapsto 1/p^n e$, resp. $\tau_n \colon U \to A, u \mapsto 1/p^n \sum_l e_l$, are generators of these cyclic groups.

To compute $H^2(V,A)$ we use once more the isomorphism induced by the cup-product on the (Tate-)cohomology of the finite cyclic group $V = \langle v\rangle$:

[2] $\mathcal{N}_G$ for a finite group G as usual denotes the element $\sum_{g\in G} g$ in the integral group ring

$H^2(V,A) \cong \widehat{H}^0(V,A) = A^v\big/_{\mathcal{N}_V \cdot A} \cong \mathcal{R} \oplus \mathcal{R}\big/_{\mathcal{N}_V(A)}$. On a single element $\hat{x} \in A$ it is

$$\begin{aligned}\mathcal{N}_V \cdot \hat{x} &= \sum_{j=0}^{p^n-1} v^j (\sum_{i=0}^{p^n-1} x_i e_i + xe) \\ &= \sum_i \left(\sum_j x_{i+2j} \right) e_i + p^n \cdot xe = \sum_i \left(\sum_j x_j \right) e_i + p^n xe,\end{aligned}$$

so in all e_i components $\mathcal{N}_V \cdot \hat{x}$ has coefficient $\sum_i x_i$. Hence we mod out by $\mathcal{R} \oplus p^n\mathcal{R}$ and so $H^2(V,A) \cong \mathcal{R}\big/p^n\mathcal{R} = 0$.

The computation of $H^1(V, A^U)$ in (4.6) proceeds as follows: Note that $A^U = A$ and let $\tilde{f}\colon V \to A$ denote any (inhomogenous) 1-cocycle in $Z^1(V,A)$. For the moment write $V = \langle v \rangle$ multiplicatively. By the cocycle relation

$$\tilde{f}(x) + x\tilde{f}(y) = \tilde{f}(xy), \forall x, y \in V$$

we see that $\tilde{f}$ is uniquely determined by its image $\hat{f} = fe + \sum_i f_i e_i$ on v. Furthermore, $0 = \tilde{f}(e) = \tilde{f}(v^{p^n}) = \mathcal{N}_V \hat{f} = p^n fe + \sum_i \left(\sum_j v^j f_i e_i \right) = p^n fe + \sum_i S e_i$, with $S = \sum_j f_j$. We conclude $p^n f = 0 = S$.

Since V is cyclic, the 1-coboundaries are of the form $(v-1)\hat{y} = 0e + \sum_i (y_{i-2} - y_i)e_i$, where $\hat{y} = ye + \sum_i y_i e_i \in A$ arbitrary. To decide whether $\hat{f} = (v-1)\hat{y}$ is solvable we are reduced to the system of linear equations

$$\begin{aligned} f_0 &= y_{p^n-2} - y_0, \\ f_1 &= y_{p^n-1} - y_1, \\ f_2 &= y_0 - y_2, \\ &\vdots \\ f_{p^n-1} &= y_{p^n-3} - y_{p^n-1}, \end{aligned}$$

which has a solution, since $\sum_i f_i = 0$ and the linear system on the right hand side has rank $p^n - 1$. We conclude that $H^1(V,A) \cong \{f \in \mathcal{R} \mid p^n \cdot f = 0\} \cong \mathbb{Z}/p^n\mathbb{Z}$, with generator $\omega_n\colon V \to A, v \mapsto 1/p^n e$.

So far, we have shown that the sequence

$$0 \to \langle \omega_n \rangle \xrightarrow{\inf} H^1(H,A) \xrightarrow{res} \langle \sigma_n \rangle \oplus \langle \tau_n \rangle \to 0$$

is exact with $ord(\omega_n) = ord(\sigma_n) = ord(\tau_n) = p^n$. As before write the groups U, resp. V, multiplicatively with generators u, resp. v. To construct a splitting of the sequence define $s_n\colon H \to A, \quad u^iv^j \mapsto i/p^n e$ and $t_n\colon\ : H \to A, \quad u^iv^j \mapsto i/p^n \sum_l e_l$. One verifies that these are cocycles and hence define classes of order p^n in $H^1(H,A)$. Since $res(s_n) = \sigma_n$, $res(t_n) = \tau_n$, we get a splitting $H^1(H,A) = H^1(V,A^U) \oplus H^1(U,A)^V$.

The operation of $\mathbb{Z}/2\mathbb{Z} = \langle\iota\rangle$ on $H^1(H,A)$ is defined by conjugation and commutes with inflation and restriction ([NSW08], prop 1.5.4) so we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(V,A^U) \arrow[r, "\inf"] \arrow[d, "\iota_*"] & H^1(H,A) \arrow[r, "res"] \arrow[d, "\iota_*"] & H^1(U,A)^V \arrow[r] \arrow[d, "\iota_*"] & 0 \\
0 \arrow[r] & H^1(V,A^U) \arrow[r, "\inf"] & H^1(H,A) \arrow[r, "res"] & H^1(U,A)^V \arrow[r] & 0.
\end{tikzcd}$$

Hence we can compute ι_*-invariants on the direct summands of $H^1(H,A) = H^1(V,A) \oplus H^1(U,A)^V$: $\widehat{H}^1(V,A)^{\iota_*} \cong \widehat{H}^{-1}(V,A)^{\iota_*}$ and since $\sigma : \sum_{i=0}^{p^n-1} x_ie_i + xe \mapsto \sum_{i=0}^{p^n-1} x_{-i}e_i + xe$ on ${}_{\mathcal{N}_V}A$ and the e_i-components get factored out, ι_* acts trivial on the first summand. Finally $H^1(U,A)^{V,\iota_*} = H^1(U,A)^V$ because V already acts transitively on the e_i's.

We compile everything together: $H_1(G^n, R(G_r^n)) \cong \left(H^1(H^n, \mathcal{R}(G_r^n))^{\mathbb{Z}/2\mathbb{Z}}\right)^\vee \cong \left(H^1(V^n, \mathcal{R}(G_r^n))^{\iota_*} \oplus H^1(U^n, \mathcal{R}(G_r^n))^{V,\iota_*}\right)^\vee \cong (\mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/p^n\mathbb{Z})^\vee$ and the assertion follows.

□

Lemma 4.32. *Using the notation from the previous proof, the natural maps* $(\mathbb{Z}/p^n\mathbb{Z})^3 \cong H^1(G^n, A^n) \xrightarrow{\pi} H^1(G^{n+1}, A^{n+1}) \cong (\mathbb{Z}/p^{n+1}\mathbb{Z})^3$ *are multiplication by p. Hence* $H_1(G, \mathbb{Z}_p(G_r)) \cong \mathbb{Z}_p^3$ *by duality of homology and cohomology.*

Proof. To avoid confusion we denote the elements of G_r^n and G_r^{n+1} by $\{e, e_l\}$ and $\{e', e_l'\}$, respectively. Similarly, we denote generators of U^n,V^n,U^{n+1}, V^{n+1} by u, v, u', v', respectively. The reduction of elements $i \in \mathbb{Z}/p^{n+1}\mathbb{Z}$ modulo p^n is denoted by $\bar{i}$.

The previous proof shows that $H^1(G^n, A^n)$ is generated by $\inf(\omega_n), s_n, t_n \in H^1(H^n, A^n)$, with $res_U^H(s_n) = \sigma_n$,$res_U^H(t_n) = \tau_n$. Let $\varphi\colon H^{n+1} \to H^n$ and $F\colon A^n \to A^{n+1}$ be the natural maps induced from the projection $G^{n+1} \to G^n$. Then for $f \in Z^1(H^n, A^n)$ the canonical map on cocycles is $\pi(f) = F \circ f \circ \varphi \in Z^1(H^{n+1}, A^{n+1})$. Here F is the Pontryagin dual of the map $\mathbb{Z}_p(G^{n+1}) \to \mathbb{Z}_p(G^n)$.

Viewing A^n as the set of maps from G_r^n to $\mathbb{Q}_p/\mathbb{Z}_p$, F corresponds to the reduction map

$$Maps(G_r^n, \mathbb{Q}_p/ZZ_p) \to Maps(G_r^{n+1}, \mathbb{Q}_p/\mathbb{Z}_p), \quad f \mapsto (e' \mapsto f(e), e'_l \mapsto f(e_{\bar{l}}).$$

This yields

$$\begin{aligned}
\pi(s_n)(u'^i v'^j)(e') &= F \circ s_n \circ \varphi(u'^i v'^j)(e') = F \circ s_n(u^{\bar{i}} v^{\bar{j}})(e') \\
&= s_n(u^{\bar{i}} v^{\bar{j}})(e) = \bar{i}/p^n = pi/p^{n+1} = ps_{n+1}(u'^i v'^j)(e') \\
\pi(s_n)(u'^i v'^j)(e'_l) &= s_n(u^{\bar{i}} v^{\bar{j}})(e_{\bar{l}}) = 0 = ps_{n+1}(u'^i v'^j)(e'_l) \\
\pi(t_n)(u'^i v'^j)(e') &= t_n(u^{\bar{i}} v^{\bar{j}})(e) = 0 = pt_{n+1}(u'^i v'^j)(e') \\
\pi(t_n)(u'^i v'^j)(e') &= t_n(u^{\bar{i}} v^{\bar{j}})(e) = \bar{i}/p^n = pi/p^{n+1} = pt_{n+1}(u'^i v'^j)(e'_l) \\
\pi(\inf(\omega_n))(u'^i v'^j)(e') &= \inf(\omega_n)(u^{\bar{i}} v^{\bar{j}})(e) = \omega_n(v^{\bar{j}})(e) \\
&= \bar{j}/p^n = pj/p^{n+1} = p \inf(\omega_{n+1})(u'^i v'^j)(e') \\
\pi(\inf(\omega_n))(u'^i v'^j)(e'_l) &= \omega_n(v^{\bar{j}})(e_{\bar{l}}) = 0 = p \inf(\omega_{n+1})(u'^i v'^j)(e'_l)
\end{aligned}$$

$\square$

4.5 Norm maps and the multiplicative side

Suppose G is a group satisfying assumption 4.3 and furthermore $c < \infty$.

For the rest of the chapter τ, resp. τ_n, will denote the map τ', resp. τ'_n from section 4.2, i.e. the additive map for the group G.

Let $n \geq 0$ and let $\mathcal{O}$ be the ring of integers in a finite algebraic extension of $\mathbb{Q}_p$. Then we define the maps

$$\begin{aligned}
N\colon K_1(\Lambda_{\mathcal{O}}(G)) &\to K_1(\Lambda_{\mathcal{O}}(U'_n)), \\
p_*\colon K_1(\Lambda_{\mathcal{O}}(U'_n)) &\to K_1(\Lambda_{\mathcal{O}}(U'_n/V'_n)) = \Lambda_{\mathcal{O}}(U'_n/V'_n)^\times, \\
\theta_n = p_* \circ N\colon K_1(\Lambda_{\mathcal{O}}(G)) &\to \Lambda_{\mathcal{O}}(U'_n/V'_n)^\times,
\end{aligned}$$

where N is the norm and p_* is the projection map.

For $m \leq n$ we define the relative norms and projections

$$\begin{aligned}
N_{m,n}\colon \Lambda_{\mathcal{O}}(U'_m/V'_m)^\times &\to \Lambda_{\mathcal{O}}(U'_n/V'_m)^\times, \\
p_{n,m}\colon \Lambda_{\mathcal{O}}(U'_n/V'_n)^\times &\to \Lambda_{\mathcal{O}}(U'_n/V'_m)^\times.
\end{aligned}$$

Analogous maps are defined for the localized Iwasawa algebras and denoted by $N_S, p_{*,S}, \theta_{n,S}, N_{m,n,S}$ and $p_{n,m,S}$, respectively.

Finally, for $n \geq 1$, define

$$\varphi\colon U'_{n-1}/V'_{n-1} \to U'_n/V'_n, \quad g \mapsto g^p.$$

φ is a continuous group homomorphism by [DdSMS99b], 0.2 (iii). It extends to continuous ring homomorphisms

$$\varphi\colon \Lambda_{\mathcal{O}}(U'_{n-1}/V'_{n-1}) \to \Lambda_{\mathcal{O}}(U'_n/V'_n) \text{ and } \varphi\colon \Lambda_{\mathcal{O}}(U'_{n-1}/V'_{n-1})_S \to \Lambda_{\mathcal{O}}(U'_n/V'_n)_S.$$

Indeed, φ restricts to an injection $U'^{ab}_{n-1} \supseteq \Gamma \hookrightarrow \Gamma^p \subseteq U'^{ab}_n$, hence

$$\begin{tikzcd}
\Lambda(U'^{ab}_{n-1}) \arrow[r] \arrow[d, "\psi"] & \Lambda(U'^{ab}_n) \arrow[d, "\psi'"] \\
\mathbb{F}_p[\![\Gamma]\!] \arrow[r, hook] & \mathbb{F}_p[\![\Gamma^p]\!]
\end{tikzcd}$$

commutes. And by [CFK+05], lemma 2.1, f is in $S(U'_{n-1})$ if and only if $\psi(f) \neq 0$. $\varphi(f) \in S(U'_n)$ follows.

Note that φ is not well-defined for the whole group L, as $\varphi(\zeta) = \zeta \notin U_1$. This and the generalization of lemma 4.34 below are the reasons why a direct generalization of this approach from the maximal pro-p subgroup to L fails.

Remark 4.33. Recall that for $n \geq 0$, $U^{(n)} := \ker(\mathbb{Z}_p^\times \to (\mathbb{Z}_p/p^n\mathbb{Z}_p)^\times)$. Kato defined in [Kat05] rings $B_n = \mathbb{Z}_p[\zeta_{p^n}][\![U^{(n)}]\!]$, $A_n = \Lambda_{\mathcal{O}}(U^{(n)}) \subseteq B_n$ and representations $\rho_n\colon G \to \mathrm{GL}_{\varphi(p^n)}(B_n)$. The induced maps $det(\tilde{\rho_n})\colon K_1(\Lambda(G)) \to K_1(B_n) \cong B_n^\times$ correspond to the maps θ_n from above.

Define the sets

$$\Psi := \Psi_{\mathcal{O}} := \{(x_n)_n \in \prod_{n \leq c} \Lambda_{\mathcal{O}}(U'_n/V'_n)^\times | (x_n)_n, \text{ s.t. } (i) \text{ and } (ii) \text{ below hold}\}$$

$$\begin{aligned}
&(i)\ N_{n,m}(x_m) = p_{m,n}(x_n), \text{for } m \leq n \leq c,\\
&(ii)\ x_n\varphi(x_{n-1})^{-1} \in 1 + I_n, \text{for } 1 \leq n \leq c
\end{aligned}$$

and

$$\Psi_S := \Psi_{\mathcal{O},S} := \{(x_n)_n \in \prod_{n \leq c} \Lambda_{\mathcal{O}}(U'_n/V'_n)_S^\times | (x_n)_n, \text{ s.t. } (i_S) \text{ and } (ii_S) \text{ below hold}\}$$

$(i_S)\ N_{n,m}(x_n) = p_{m,n}(x_m)$, for $m \leq n \leq c$,
$(ii_S)\ x_n\varphi(x_{n-1})^{-1} \in 1 + I_{n,S}$, for $1 \leq n \leq c$.

Lemma 4.34. *We have a well-defined map* $\varphi^*\colon G \to \Psi, \quad g \mapsto (\varphi^n(g))_n$.

Proof. By multiplicativity of φ it suffices to show this for the generators ε and δ of G. For condition (i) note that $\{1, \delta^{p^m}, \dots, \delta^{p^m(p^{n-m}-1)}\}$ is a basis for the $\Lambda_{\mathcal{O}}(U'_n/V'_m)$-module $\Lambda_{\mathcal{O}}(U'_m/V'_m)$ so

$$N_{m,n}(\varphi^m(\varepsilon)) = \det \begin{pmatrix} \varepsilon^{p^m} & 0 & \dots \\ 0 & \ddots & 0 \\ \vdots & 0 & \varepsilon^{p^m} \end{pmatrix} = \varepsilon^{p^n} = p_{n,m}(\varphi^n(\varepsilon))$$

$$\text{and } N_{m,n}(\varphi^m(\delta)) = \det \begin{pmatrix} 0 & \cdots & 0 & \delta^{p^n} \\ 1 & \ddots & & 0 \\ & \ddots & 0 & \vdots \\ & & 1 & 0 \end{pmatrix} = (-1)^{p^{n-m}-1}\delta^{p^n} = p_{n,m}(\varphi^n(\delta)).$$

Condition (ii) is obvious from the homomorphism property of φ. □

Lemma 4.35. *Let W_1 is p-adic Lie group and W_2 be an open subgroup of W_1. If W_2 is commutative consider the Trace homomorphism*

$$\mathrm{Tr}\colon \mathbb{Q}_p[\![\mathrm{Conj}(W_1)]\!] \to \mathbb{Q}_p[\![W_2]\!].$$

Then the following diagram commutes

$$\begin{tikzcd} K_1(\Lambda_{\mathcal{O}}(W_1)) \arrow[r, "log"] \arrow[d, "\mathrm{N}"] & \mathbb{Q}_p[\![\mathrm{Conj}(W_1)]\!] \arrow[d, "\mathrm{Tr}"] \\ \Lambda(W_2)^\times \arrow[r, "log"] & \mathbb{Q}_p[\![W_2]\!]. \end{tikzcd} \tag{4.7}$$

From this we get commutativity of the following diagram for every $n \leq c$

$$\begin{tikzcd} K_1(\Lambda_{\mathcal{O}}(G)) \arrow[r, "log"] \arrow[d, "\theta_n"] & \mathbb{Q}_p[\![\mathrm{Conj}(G)]\!] \arrow[d, "\tau"] \\ \Lambda_{\mathcal{O}}(U'_n/V'_n)^\times \arrow[r, "log"] & \mathbb{Q}_p[\![U'_n/V'_n]\!]. \end{tikzcd} \tag{4.8}$$

Proof. cf. [Lee09], Lemma 3.48: Use Theorem 6.2 in [Oli88] to derive $\log(N(x)) = \mathrm{Tr}(log(x))$ for $x \in 1 + Jac(\Lambda_{\mathcal{O}}(G))$ and finite G. Then take the projective limit to generalize this to the profinite case. In a second step observe that $p_*\colon \Lambda_{\mathcal{O}}(G) \to \Lambda_{\mathcal{O}}(G/V_n')$ commutes with $\log$. $\square$

Lemma 4.36. *For $n \geq 1$ the set I_n is multiplicatively closed. As a consequence $1 + I_n$ is a group.*

Proof. It suffices to show $p^{v_p(i)}h_{n,i} \cdot p^{v_p(j)}h_{n,j} \in I_n$ for $v_p(i) \leq v_p(j)$. It is $h_{n,i} = \sum_k \varepsilon^{i+k}$, where $k < p^{n+1}$ ranges over the integers with $k \equiv 0(\mathrm{mod}\ p^{v_p(i)})$. Hence this sum has $p^{n-v_p(i)}$ terms.

First suppose $v_p(i) < v_p(j)$ then multiplication by ε^{j+l} is a permutation of the summands in $h_{n,i}$ and consequently $p^{v_p(i)}h_{n,i} \cdot p^{v_p(j)}h_{n,j} = p^{v_p(i)+v_p(j)}p^{n-v_p(j)}h_{n,i} = p^n p^{v_p(i)}h_{n,i} \in I_n$.

Similar if $v_p(i) = v_p(j)$ with $i + j \not\equiv 0(\mathrm{mod}\, p^{v_p(i)})$: $p^{v_p(i)}h_{n,i} \cdot p^{v_p(j)}h_{n,j} = p^{v_p(i)+v_p(j)}p^{n-v_p(j)} \sum_k \varepsilon^{i+j+k} = p^n p^{v_p(i+j)}h_{n,i+j} \in I_n$.

Finally if $i + j \equiv 0(\mathrm{mod}\ p^{v_p(i)})$,

$$\begin{aligned} p^{v_p(i)}h_{n,i}p^{v_p(j)}h_{n,j} &= p^{n+v_p(i)} \sum_{v_p(k)>v_p(i)} \varepsilon^k = p^{n+v_p(i)} \sum_{n \geq l > v_p(i)} \sum_{v_p(k)=l} \varepsilon^k \\ &= p^{n+v_p(i)} \sum_l \sum_{j=1}^{p-1} \sum_{\substack{k \equiv jp^l \\ mod\ p^{l+1}}} \varepsilon^k = p^{n+v_p(i)} \sum_l \sum_j h_{n,jp^l} \in I_n. \end{aligned}$$

Then $1 + I_n$ is a multiplicative group: $1 + x \in 1 + I_n$ has $\sum_{n \geq 0}(-x)^n$ as its inverse. $\square$

Lemma 4.37. *For $n \geq 1$, $\log$ is well-defined on $1 + I_n$ and gives an isomorphism of groups*

$$1 + I_n \xrightarrow{\sim} I_n.$$

Proof. $\log$ converges by [Lee09], corollary 3.30 and yields a map

$$\log\colon 1 + I_n \to \mathbb{Q}_p[\![U_n'/V_n']\!].$$

It remains to show that $x^k/k \in I_n$ for $x \in I_n, n \geq 1$, or equivalently $x^k \in p^{v_p(k)} I_n$ for all $k \geq 1$. To see this note that x^k is a $\mathbb{Z}_p[\![\langle \delta^{p^n} \rangle]\!]$-linear combination of elements of the form

$$\prod_{r=1}^{k} p^{v_p(i_r)} h_{n,i_r} \text{ with } i_1 \leq \ldots \leq i_k.$$

In the proof of 4.36 we saw that $p^{n+v_p(i_r)} \mid p^{v_p(i_r)+v_p(i_{r+1})} h_{n,i_r} h_{n,i_{r+1}}$. Consequently

$$\prod_{r=1}^{k} p^{v_p(i_r)} h_{n,i_r} = p^{n(k-1)+v_p(i_1)+\cdots v_p(i_{k-1})} \cdot y$$

with y either h_{n,i_k}, $h_{n,i_k+i_{k-1}}$ or $\sum_{v_p(l) \geq v_p(i_k)} h_{n,l}$ or a sum of these, according to the three cases in the proof of the lemma.

Accounting for the p-power coefficient of y in I_n we get $x^k \in p^{n(k-2)+v_p(i_1)+\cdots v_p(i_{k-1})} I_n \subseteq p^{v_p(k)} I_n$. Note that the induced homomorphism $\log_n \colon (1+I_n^i)/(1+I_n^{i+1}) \to I_n^i/I_n^{i+1}$ is an isomorphism, since

$$\log_n(1-x) = -x - x^2 \sum_{k \geq 2} \frac{x^{k-2}}{k} = -x.$$

Then $\log$ is an isomorphism by continuity, cf. Lemma 3.54 in [Lee09].

$\square$

Lemma 4.38. *There are the following commutative diagrams of $\mathcal{O}$-modules:*

$$\begin{tikzcd}
\mathcal{O}[\![\mathrm{Conj}(G)]\!] \arrow[r, "\varphi"] \arrow[d, "p \cdot \tau_{n-1}"] & \mathcal{O}[\![\mathrm{Conj}(G)]\!] \arrow[d, "\tau_n"] \\
\Lambda_{\mathcal{O}}(U'_{n-1}/V'_{n-1}) \arrow[r, "\varphi"] & \Lambda_{\mathcal{O}}(U'_n/V'_n),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}[\![\mathrm{Conj}(G)]\!]^{\wedge}_{S} \arrow[r, "\varphi"] \arrow[d, "p \cdot \tau_{n-1,S}"] & \mathcal{O}[\![\mathrm{Conj}(G)]\!]^{\wedge}_{S} \arrow[d, "\tau_{n,S}"] \\
\Lambda_{\mathcal{O}}(U'_{n-1}/V'_{n-1})^{\wedge}_{S} \arrow[r, "\varphi"] & \Lambda_{\mathcal{O}}(U'_n/V'_n)_S{}^{\wedge},
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{O}[\![U'_{m-1}/V'_{m-1}]\!] \arrow[r, "\varphi"] \arrow[d, "p \cdot \tau_{n-1}"] & \mathcal{O}[\![U'_m/V'_m]\!] \arrow[d, "\tau_n"] \\
\Lambda_{\mathcal{O}}(U'_{n-1}/V'_{n-1}) \arrow[r, "\varphi"] & \Lambda_{\mathcal{O}}(U'_n/V'_n),
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}[\![U'_{m-1}/V'_{m-1}]\!]^{\wedge}_{S} \arrow[r, "\varphi"] \arrow[d, "p \cdot \tau_{n-1,S}"] & \mathcal{O}[\![U'_m/V'_m]\!]^{\wedge}_{S} \arrow[d, "\tau_{n,S}"] \\
\Lambda_{\mathcal{O}}(U'_{n-1}/V'_{n-1})^{\wedge}_{S} \arrow[r, "\varphi"] & \Lambda_{\mathcal{O}}(U'_n/V'_n)^{\wedge}_{S}.
\end{tikzcd}$$

Proof. Note that $v_p(p \cdot (d^{p^{n-1}-1})) = n+1$, hence $[\varepsilon^p, \delta^{p^{n-1}}] = \varepsilon^{p*(d^{p^{n-1}-1})} \in V'_n = \langle \varepsilon^{p^{n+1}} \rangle$ and $\delta^{p^{n-1}} g^p \delta^{-p^{n-1}} \overset{(*)}{=} g^p \in \Lambda_{\mathcal{O}}(U'_n/V'_n)$ for $g \in U'_n$.

For the upper left square consider $g \in U'_{n-1}$. Using $(*)$ we have

$$\begin{aligned}\tau_n \circ \varphi(\mathrm{class}_G(g)) &= \sum_{i=0}^{p^n-1} \mathrm{class}_G(\delta^i g^p \delta^{-1}) = p \cdot \sum_{i=0}^{p^{n-1}-1} \mathrm{class}_G(\delta^i g^p \delta^{-1}) \\ &= p \cdot \varphi\left(\sum_{i=0}^{p^{n-1}-1} \mathrm{class}_G(\delta^i g \delta^{-1})\right) = p \cdot \varphi \circ \tau_{n-1}(\mathrm{class}_G(g)).\end{aligned}$$

If $g \notin U'_{n-1}$ we have $\varphi \notin U'_n$ and so both ways around the diagram are the zero map.

Since φ and the $\tau_{*,S}$ are $\Lambda_{\mathcal{O}}(Z(G))_S$-linear we can also derive commutativity of the upper right square from this.

For the two lower squares let $g \in U'_{n-1}$. Then $\mathrm{Tr}_{m,n} \circ \varphi(g) = \sum_{i=0}^{p^{n-m}-1} \delta^{p^m \cdot i} g^p \delta^{-p^m \cdot i} = \varphi\left(\sum_{i=0}^{p^{n-m}-1} \delta^{p^{m-1} \cdot i} g \delta^{-p^{m-1} \cdot i}\right) = \varphi \circ \mathrm{Tr}_{m-1,n-1}(g)$ and analogous for the right one. $\square$

Lemma 4.39. *For $n \geq 1$, we have*

$$\tau_n(\mathcal{L}_G(x)) = \log(\theta_n(x)\varphi(\theta_{n-1}(x))^{-1}) \text{ for all } x \in K_1(\Lambda_{\mathcal{O}}(G))$$
$$\text{and } \tau_{n,S}(\mathcal{L}_{G,S}(x)) = \log(\theta_{n,S}(x)\varphi(\theta_{n-1,S}(x))^{-1}) \text{ for all } x \in K_1(\Lambda_{\mathcal{O}}(G)_S^{\wedge}).$$

Proof. The proof is lemma 3.50 in [Lee09] when one uses that $\Lambda_{\mathcal{O}}(G)_{S(G)} = \Lambda_{\mathcal{O}}(G)_{S(U'_n)} = \Lambda_{\mathcal{O}}(G)_{S(Z(U'_n))}$. $\square$

Proposition 4.40. *We have $\theta(K_1(\Lambda_{\mathcal{O}}(G))) \subseteq \Psi$ and $\theta_S(K_1(\Lambda_{\mathcal{O}}(G)_S)) \subseteq \Psi_S$.*

Proof. We proof the first assertion, the second follows analogously; compare proposition 3.57 in [Lee09]. Let $x \in K_1(\Lambda(G))$ and (x_n) be the image of x under θ. Denote by r the index of U'_n in G, then an application of K_1 to the commutative diagram

$$\begin{array}{ccc} \Lambda(G) & \longrightarrow & M_r(\Lambda(U)'_n) \\ \downarrow & & \downarrow \\ \Lambda(G/V_m) & \longrightarrow & M_r(\Lambda(U'_n/V_m)) \end{array}$$

yields

$$\begin{tikzcd}
K_1(\Lambda(G)) \arrow[r, "N_n"] \arrow[d, "p_m"'] & K_1(\Lambda(U)'_n) \arrow[d, "p_m"] \\
K_1(\Lambda(G/V'_n)) \arrow[r, "N_n"] & K_1(\Lambda(U'_n/V'_n)),
\end{tikzcd}$$

or $p_m \circ N_n = N_n \circ p_m$ for short. Consequently $p_m(x_n) = p_m \circ \theta_n(x) = p_m \circ p_n \circ N_n(x) = p_m \circ N_n(x) = N_n \circ p_m(x) = N_n \circ N_m \circ p_m(x) = N_n \circ p_m \circ N_m(x) = N_n \circ \theta_m(x) = N_n(x_m)$. This is exactly condition (i) in the definition of Ψ. A similar argument shows $\theta_S(x) \in \Psi_S$ for $x \in K_1(\Lambda(G)_S)$.

For $x \in K_1(\Lambda(G))$ let $q_n(x) := \theta_n(x)\varphi(\theta_{n-1}(x))^{-1} \in \Lambda(U'_n/V'_n)^\times$. Condition (ii) for Ψ is then rephrased as $q_n(x) \in 1 + I_n, 1 \leq n$.

By the previous lemma 4.39 we conclude $\mathrm{im}(\log \circ q_n) \subseteq I_n$. If $\log$ were to be injective on $\mathrm{im}(q_n)$ then $q_n(x)$ would be the unique preimage of $\tau_n \circ \mathcal{L}_G(x) \in I_n$ under $\log$, which by lemma 4.37 is in $1 + I_n$.

Hence let $0 = \log(q_n(x)) = \tau_n \circ \mathcal{L}_G(x)$. Here is an error in [Lee09]: τ_n is by no means injective, though it is the component of an injective map, τ. A correct argument can be found in proposition 5.1 in [Har10] which in our situation gives $\theta_n(x)\varphi(\theta_{n-1}(x))^{-1} \in 1+p\Lambda(U'_n/V'_n)$. Since $\log\colon 1+p\Lambda(U'_n/V'_n) \to p\Lambda(U'_n/V'_n)$ is an isomorphism, we can proceed as described above.

$\square$

Theorem 4.41. *The map θ is an isomorphism of abelian groups $\theta\colon K_1(\Lambda_{\mathcal{O}}(G)) \xrightarrow{\sim} \Psi$.*

Proof. Define

$$\widetilde{\mathcal{L}}\colon \Psi \to \Omega, \quad (x_n)_n \mapsto (y_n)n,$$

with $y_0 = \mathcal{L}_{U'_0/V'_0}(x_0)$ and $y_n = \log(x_n\varphi(x_{n-1})^{-1})$, for $n \geq 1$.

This is well-defined, since $y_n \in I_n$ by the definition of Ψ and Lemma 4.37. For $1 \le m \le n \le c$ the y_n satisfy

$$\begin{aligned}
\mathrm{Tr}_{m,n}(y_m) &= \mathrm{Tr}_{m,n}(\log(x_n\varphi(x_{n-1})^{-1})) \\
&= \mathrm{Tr}_{m,n}(\log(x_m)) - \mathrm{Tr}_{m,n} \circ \varphi \circ \log(x_{n-1}) \\
&= \log(N_{m,n}(x_m)) - \varphi \circ \mathrm{Tr}_{m-1,n-1} \circ \log(x_{m-1}) \\
&= \log(N_{m,n}(x_m)) - \varphi \circ \log \circ N_{m-1,n-1}(x_{m-1}) \\
&= \log(p_{n,m}(x_n)) - \varphi \circ \log \circ p_{n-1,m-1}(x_{n-1}) \\
&= p_{m,n}(\log(x_n)) - \varphi \circ p_{n-1,m-1} \circ \log(x_{n-1}) \\
&= p_{m,n} \circ \log(x_n\varphi(x_{n-1})^{-1}) = p_{n,m}(y_n).
\end{aligned}$$

Now let $0 = m < n$. We have $p_{n,0}(\log(x_n)) = \log(p_{n,0}(x_n)) = \log(N_{0,n}(x_0)) = \mathrm{Tr}_{0,n}(\log(x_0))$. Using this we get

$$\begin{aligned}
\mathrm{Tr}_{0,n}\left(\frac{1}{p}\log \circ \varphi(x_0)\right) &= \varphi\Big(\mathrm{Tr}_{0,n-1}(\log(x_0))\Big) \\
&= \varphi\Big(p_{n-1,0}(\log(x_{n-1}))\Big) \\
&= p_{n,0}\Big(\log \circ \varphi(x_{n-1})\Big).
\end{aligned}$$

Applying this to the definition of the integral logarithm we have

$$\begin{aligned}
\mathrm{Tr}_{0,n}(y_0) &= \mathrm{Tr}_{0,n}\left(\log(x_0) - \frac{1}{p}\log(\varphi(x_0))\right) \\
&= p_{n,0}\Big(\log(x_n) - \log(\varphi(x_{n-1}))\Big) = p_{n,0}(y_n).
\end{aligned}$$

Define the continuous group homomorphisms

$$\begin{aligned}
&\tilde{\omega}\colon \Omega \to U_0'/V_0', \quad (x_n)_n \mapsto x_0 \text{ if } x_0 \in U_0'/V_0' \text{ and} \\
&\tilde{\theta}\colon \mu_{\mathcal{O}} \times G^{ab} \to \Psi, \quad (\zeta, g) \mapsto (\zeta g^{p^n})_n.
\end{aligned}$$

Lemma 4.34 shows that $\tilde{\theta}$ is well-defined.

We will show that the following diagram is commutative with exact rows:

$$\begin{tikzcd}
1 \arrow[r] & \mu_{\mathcal{O}} \times G^{ab} \arrow[r] \arrow[d, "id"] & K_1(\Lambda_{\mathcal{O}}(G)) \arrow[r, "\mathcal{L}_G"] \arrow[d, "\theta"] & \mathcal{O}[\![\mathrm{Conj}(G)]\!] \arrow[r, "\omega"] \arrow[d, "\tau", "\wr"'] & G^{ab} \arrow[r] \arrow[d, "id"] & 1 \\
1 \arrow[r] & \mu_{\mathcal{O}} \times G^{ab} \arrow[r, "\tilde{\theta}"] & \Psi \arrow[r, "\widetilde{\mathcal{L}}"] & \Omega \arrow[r, "\tilde{\omega}"] & U_0'/V_0' \arrow[r] & 1.
\end{tikzcd} \tag{4.9}$$

The left square commutes by Lemma 4.34. The middle square commutes by diagram (4.8) in lemma 4.35. The right square commutes by definition of U_0'. The upper row is exact by lemma 4.29.

It remains to proof exactness of the lower row.

Injectivity of $\tilde{\theta}$ is clear from looking at the $n = 0$ component of the image.

Now we show exactness at Ψ. Commutativity of diagram 4.9 and exactness of the upper row imply $\mathrm{im}(\tilde{\theta})$. Now let $x \in \ker(\tilde{\mathcal{L}})$. By lemma 4.37 $\log\colon 1 + I_n \to I_n$ is injective for all $n \geq 1$ and we can write $x = (\phi^n(x_0))_n$ for some $x_0 \in \Lambda(U_0'/V_0')^\times$. Choose $y \in K_1(\Lambda(G))$ with $p_*(y) = x_0$. Using condition (i) of Ψ for the last step we have $\theta_n(y) = p_* \circ \mathrm{N}(y) = \mathrm{N} \circ p_*(y) = \mathrm{N}(x_0) = \varphi^n(x) \in \Lambda(U_n'/V_0')^\times$, i.e. $\theta(y) = x$. Since τ is an isomorphism, y must be in $\ker(\mathcal{L}_G)$ and it gives rise to an inverse image of x under $\tilde{\theta}$.

For the exactness at Ω, observe that by lemma 4.29 applied to $G = U_0'/V_0'$ we have $\mathrm{im}(\widetilde{\mathcal{L}}) \subseteq \ker(\widetilde{\omega})$. On the other hand, since τ is an isomorphism,

$$\ker(\widetilde{\omega}) = \tau(\ker(\omega)) = \tau(\mathrm{im}(\mathcal{L}_G)) = \mathrm{im}(\widetilde{\mathcal{L}} \circ \theta) \subseteq \mathrm{im}(\widetilde{\mathcal{L}}).$$

So we get $\mathrm{im}(\widetilde{\mathcal{L}}) = \ker(\widetilde{\omega})$. The five lemma finishes the proof. $\square$

Now let G be a group satisfying assumptions a) and b) in the beginning of this chapter, i.e. $G = \Delta \times P$, where P is the maximal pro-p subgroup of a one-dimensional quotient of the false Tate group and Δ is a cyclic subgroup of μ_{p-1}. Recall that $c := c_G \in \mathbb{N}$ such that $\varepsilon^{p^c} = 1$. Following Kakde (cf. prop. 86 in [Kak10]), we now bring the strategy from section 2.5 to work: Every character $\chi \in \widehat{\Delta}$ takes its values in an unramified extension $\mathcal{O}_\chi$ of $\mathbb{Z}_p$. For a profinite group W denote the Iwasawa algebra $\Lambda_{\mathcal{O}_\chi}(W)$ just by $\Lambda_\chi(W)$. For $n \leq c$ let $\zeta_n \in \Lambda(\Delta \times U_n'/V_n')_S^\times$ be the abelian p-adic zeta function for the admissible extension $F_\infty^{V_n'}/F_\infty^{\Delta \times U_n'}$.

Proposition 4.42. *The NMC is true for F_∞/F with $\mathrm{Gal}(F_\infty/F) = G$ if and only if $(\chi(\zeta_n)) \in \Psi_S$ for all characters χ of Δ.*

Proof. Fix $\chi \in \widehat{\Delta}$. Denote by C_χ the class $[\Lambda_{\mathcal{O}_\chi} \otimes_{\Lambda(G)} C(F_\infty/F)] \in K_0(\Lambda_{\mathcal{O}_\chi}(P), \Lambda_{\mathcal{O}_\chi}(P)_S)$. Chose an element $f_\chi \in K_1(\Lambda_{\mathcal{O}_\chi}(P)_S)$ with $\partial(f_\chi) = -[C_\chi]$ and denote the image $\theta_S(f) \in \prod_{n \leq c} \Lambda_{\mathcal{O}_\chi}(U_n'/V_n')_S^\times$ by (f_n). It is $(f_n) \in \Psi_S$ by proposition 4.40. By exactness of the localisation sequence for $\Lambda_\chi(P) \to \Lambda_\chi(P)_S$ we get $u_n := \chi(\zeta_n) f_n^{-1} \in \Lambda_\chi(P)^\times$. Assuming $(\chi(\zeta_n)) \in \Psi_S$ we have

$(u_n) \in \Psi_S \cap \prod_n \Lambda_\chi(U'_n/V'_n)^\times = \Psi$. The last equality holds since the norms maps in the definition of Ψ_S restrict to the ones in the definition of Ψ. Denote the unique preimage of (u_n) under θ by $u \in K_1(\Lambda_\chi(P))$ and let $L_\chi := uf$.

Denote by e_χ the idempotents corresponding to the decomposition $\Lambda(\Delta \times P) = \sum_\chi \Lambda_\chi(P)$. We claim that $\zeta = \sum_\chi e_\chi L_\chi$ is the p-adic zeta function for F_∞/F. By defintion ζ is a characteristic element, i.e. $\partial(\zeta) = -[C(F_\infty/F)]$.

It remains to proof the interpolation property: Let ρ be an irreducible Artin representation of P. Then there is an $n \leq c$ and a one-dimensional Artin representation ρ_n of U'_n, such that $\rho = \chi \operatorname{ind}_{U'_n}^P \rho_n$. Consequently, for $\chi \in \widehat{\Delta}$ and $r \in \mathbb{N}$ divisible by $p-1$ it is

$$\zeta(\chi\rho\kappa_F^r) = \zeta_n(\chi\rho_n\kappa^r_{F_\infty^{U'_n}}) = L(\chi\rho, 1-r).$$

$\square$

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