Computing Teitelbaum's $\mathcal{L}$-invariant

Peter Mathias Gräf – peter.graef@iwr.uni-heidelberg.de

Interdisziplinäres Zentrum für Wissenschaftliches Rechnen – Ruprecht-Karls-Universität Heidelberg

Introduction

Let $f \in \mathcal{S}_{k+2}(\Gamma_0(pN))^{\text{new}}$ be a newform, where $k \geq 0$ is even, p is prime and N is an integer not divisible by p. Let χ be a Dirichlet character of conductor prime to pN such that $U_p f = \chi(p)p^{\frac{k}{2}} f$. By work of Mazur, Tate and Teitelbaum the order of vanishing of the p-adic L-function attached to f at the central point $s = \frac{k+2}{2}$ is one higher than that of the classical L-function attached to it. Moreover, they formulated the following **exceptional zero conjecture**: There exists an invariant $\mathcal{L}_p(f) \in \mathbb{C}_p$, depending only on the local Galois representation $\sigma_p(f)$ attached to f, such that

$$L'_p(f, \chi, \tfrac{k+2}{2}) = \mathcal{L}_p(f) \cdot L^{\text{alg}}(f, \chi, \tfrac{k+2}{2}).$$

By work of Greenberg-Stevens and many others, the qualitative behaviour of $\mathcal{L}_p(f)$ is well known, on the contrary, not a lot of quantitative data on $\mathcal{L}_p(f)$ for arbitrary even weight is available. For this purpose, our aim is to compute the $\mathcal{L}$-invariant defined by Teitelbaum in [5].

Objectives

- Find an **efficient method** for computing $\mathcal{L}$-invariants for modular forms of arbitrary even weight.
- Analyze the **distribution and behaviour** of the $\mathcal{L}$-invariants for growing weight.

Computing Teitelbaum's $\mathcal{L}$-operator

Teitelbaum's invariants $\mathcal{L}_p(f)$ are realized as the eigenvalues of an operator, called the **$\mathcal{L}$-operator,** defined on the finite-dimensional $\mathbb{C}_p$-vector space of harmonic cocycles on the **Bruhat-Tits tree $\mathcal{T}$**. There are three main difficulties in designing an efficient method to compute this operator (up to a prescribed precision) as a matrix with p-adic entries:

- The $\mathcal{L}$-operator in [5] is defined over $\mathbb{C}_p$. In order to keep the running time of our computations as low as possible, we show that the $\mathcal{L}$-operator can in fact be **defined over $\mathbb{Q}_p$**.
- In order to describe Γ-invariant harmonic cocycles on $\mathcal{T}$ by a finite amount of data, it is necessary to compute a **fundamental domain for the action of Γ on $\mathcal{T}$**, see [1].
- Coleman integrals enter into the definition of the $\mathcal{L}$-operator following [5] and a priori efficient computation of these seems to be completely out of reach. Teitelbaum proved in [5] that one can replace these Coleman integrals by p-adic integrals coming from harmonic cocycles. However, computing the integrals directly in terms of this new definition is much too slow to compute the $\mathcal{L}$-operator efficiently. An alternate approach was presented by Greenberg in [3] building on the **overconvergent methods** developed by Darmon, Pollack and Stevens.

To make the overconvergent method applicable in our setting, we prove a **control theorem for p-adic automorphic forms** of arbitrary even weight generalizing [3, Corollary 2].

Quotients of the Bruhat-Tits-tree

- Let B be a definite rational quaternion algebra of discriminant N, that is split at p.
- Let R be a maximal order in B.
- Let $\Gamma = R[\frac{1}{p}]_1^\times$ denote the units of reduced norm 1.
- Fix a splitting $\iota\colon B \otimes_\mathbb{Q} \mathbb{Q}_p \to \mathrm{M}_2(\mathbb{Q}_p)$ such that $\iota(R \otimes_\mathbb{Z} \mathbb{Z}_p) = \mathrm{M}_2(\mathbb{Z}_p)$ and regard Γ as a (discrete and cocompact) subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ via the splitting.

The **Bruhat-Tits tree $\mathcal{T}$** for $\mathrm{GL}_2(\mathbb{Q}_p)$ is the graph whose vertices $\mathcal{T}_0$ are the homothety classes of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^2$. Two vertices v and w are joined by an edge in $\mathcal{T}_1$ if there exist representative lattices L and L' such that $pL \subsetneq L' \subsetneq L$. The graph $\mathcal{T}$ is a $p+1$-regular tree. Via the reduction map, it encodes the geometry of the p-adic upper half plane $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$. The group $\mathrm{GL}_2(\mathbb{Q}_p)$ acts transitively on $\mathcal{T}$. The quotient $\Gamma \backslash \mathcal{T}$ is a finite graph. A **fundamental domain** for the action of Γ on the tree $\mathcal{T}$ can be efficiently computed, see [1].

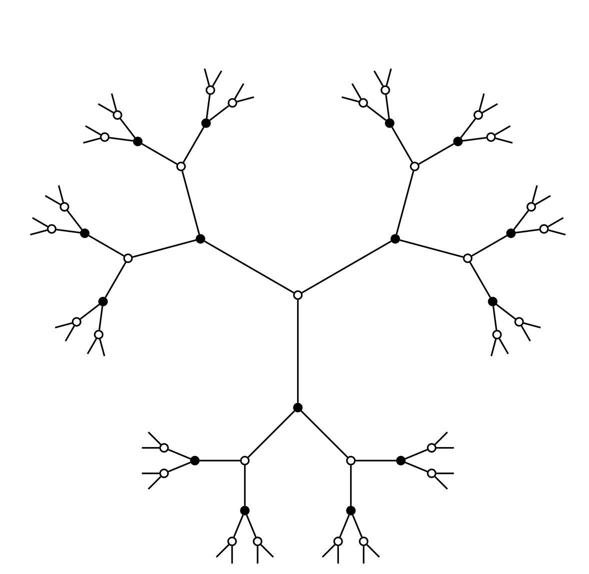

Figure 1: Bruhat-Tits tree for $p = 2$

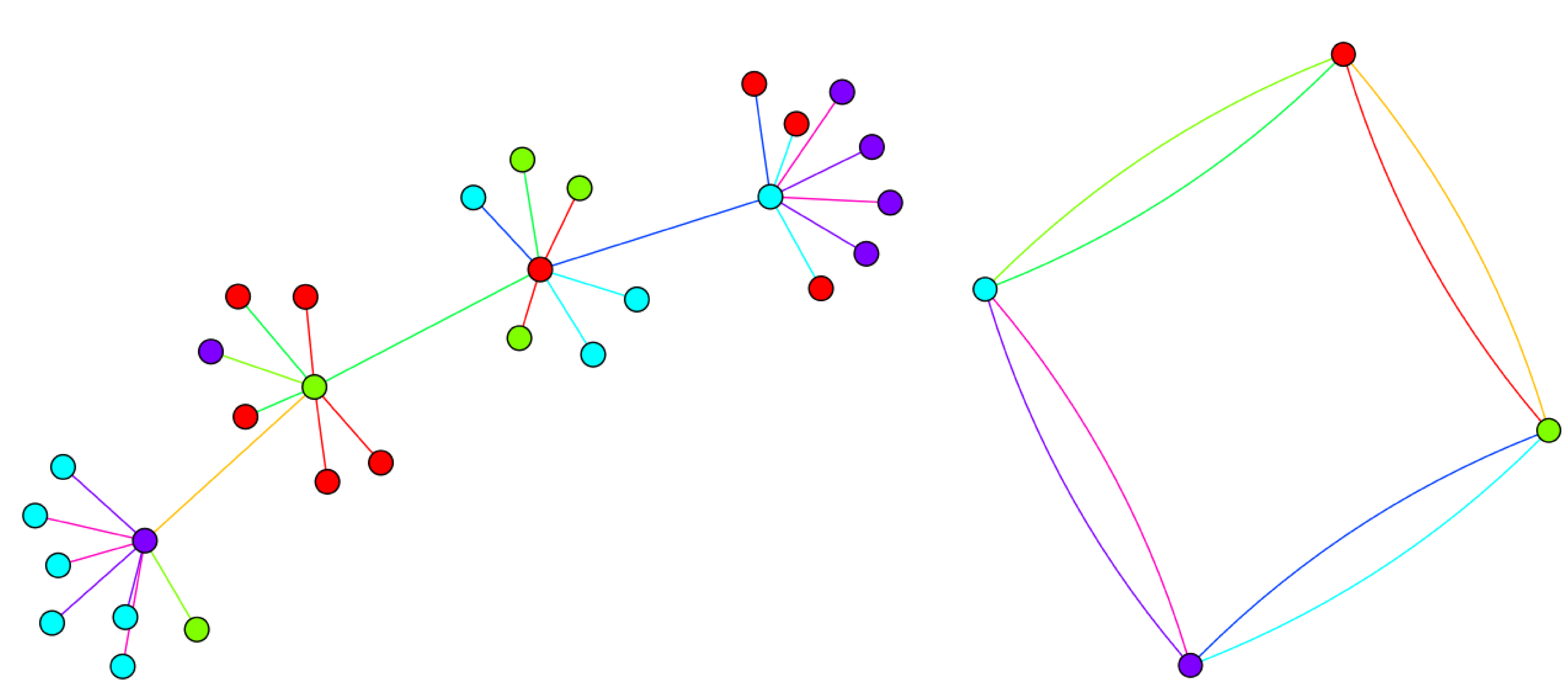

Figure 2: Fundamental domain and quotient graph for $p = 7$ and $N = 11$

The crucial step is to reduce the problem wether two edges (or vertices) are Γ-equivalent to a shortest vector search in a lattice by only using an approximation of ι up to some finite p-adic precision.

Teitelbaum's $\mathcal{L}$-operator

Let $P_k = \mathbb{Q}_p[x]_{\leq k}$ with the $\mathrm{GL}_2(\mathbb{Q}_p)$-action given by

$$(P \cdot g)(x) = \det(g)^{-\frac{k}{2}}(cx+d)^k P\left(\frac{ax+b}{cx+d}\right), \quad \text{for } P \in \mathcal{P}_k,\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}_p)$$

and denote by V_k its dual. The space of **Γ-invariant harmonic cocycles** $C_h(\Gamma, k)$ on $\mathcal{T}$ consists of maps $c\colon \mathcal{T}_1 \to V_k$ such that for all $v \in \mathcal{T}_0, e \in \mathcal{T}_1$ and $\gamma \in \Gamma$ one has

$$c(\overline{e}) = -c(e), \quad \textstyle\sum_{s(e)=v} c(e) = 0, \quad \gamma \cdot c(e) = c(\gamma e).$$

There is a Hecke-equivariant isomorphism $\mathcal{S}_{k+2}(\Gamma_0(pN), \mathbb{C}_p)^{\text{new}} \cong C_h(\Gamma, k) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$. Moreover, any $c \in C_h(\Gamma, k)$ gives rise to $\mathbb{Q}_p$-valued distribution μ_c on certain locally analytic functions on $\mathbb{P}^1(\mathbb{Q}_p)$. For $\tau \in \mathbb{Q}_{p^2} \setminus \mathbb{Q}_p$ define $\kappa_{\text{col}}^\tau(c)\colon \Gamma \to V_k$ by

$$\kappa_{\text{col}}^\tau(c)(\gamma)(P) = \frac{1}{2}\mathrm{Tr}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}\left(\int_{\mathbb{P}^1(\mathbb{Q}_p)} P(x) \log_p\left(\frac{x - \gamma\tau}{x - \tau}\right) \mathrm{d}\mu_c(x)\right) \in \mathbb{Q}_p, \quad \text{for } \gamma \in \Gamma,\ P \in \mathcal{P}_k.$$

This induces the Coleman integration map $\kappa_{\text{col}}\colon C_h(\Gamma, k) \to H^1(\Gamma, V_k)$. There is also a combinatorial Hecke-equivariant isomorphism $\kappa_{\text{sch}}\colon C_h(\Gamma, k) \to H^1(\Gamma, V_k)$ due to Schneider. The composition

$$\mathcal{L} = \kappa_{\text{col}} \circ (\kappa_{\text{sch}})^{-1}\colon H^1(\Gamma, V_k) \to H^1(\Gamma, V_k)$$

is **Teitelbaum's $\mathcal{L}$-operator**, whose eigenvalues are the $\mathcal{L}$-invariants of the associated newforms.

A control theorem for p-adic automorphic forms

To compute the Coleman integrals, we first construct a covering of $\mathbb{P}^1(\mathbb{Q}_p)$ given by edges in $\mathcal{T}$ such that the integrands have a nice expression. Then the remaining step is to compute the moments $m(\mu_c, g, i) = \mu_c(g\mathbb{Z}_p)(x^i \cdot g^{-1})$ for all $g \in \mathrm{GL}_2(\mathbb{Q}_p)$, $i \geq 0$. These moments are encoded in values of rigid analytic automorphic forms for Γ. Let $\mathcal{A}_k(\Gamma)$ denote the vector space of these forms and $\mathbb{A}_k(\Gamma)$ the space of **p-adic automorphic forms**. We have $C_h(\Gamma, k) \cong (\mathbb{A}_k(\Gamma)^{p-\text{new}})^{U_p = p^{k/2}}$. By methods along the lines of [4], we obtain the following **control theorem**:

- The restriction of the specialization map $\rho\colon \mathcal{A}_k(\Gamma)^{U_p = p^{k/2}} \to \mathbb{A}_k(\Gamma)^{U_p = p^{k/2}}$ is an isomorphism.
- Let φ_c the p-adic automorphic form corresponding to c and denote the above lift by Φ_c. Then
$$\Phi_c(g)(x^i) = m(\mu_c, g, i).$$
- For any lift $\Phi_0 \in \mathcal{A}_k(\Gamma)$ of φ_c defined over $\mathbb{Z}_p$, $n \geq 1$ and $i \in \{1, \ldots, n\}$, we have
$$((p^{-k/2}U_p)^n \Phi_0)(g)(x^{k+i}) = m(\mu_c, g, k+i) \pmod{p^{n-i+1}}.$$

Computational Results for $p = 2$

Let $d_k(p, N) = \dim_\mathbb{C} \mathcal{S}_k(\Gamma_0(pN))^{\text{new}}$ and denote by $\alpha_\mathcal{L}(k, p, N)$ the slopes of the $\mathcal{L}$-operator.
Let $\alpha_\mathcal{L}^\pm(k, p, N)$ be the slopes with respect to the Atkin-Lehner involution at N.

k	$d_k(2,3)$	$\alpha_\mathcal{L}(k,2,3)$	
		$\alpha_\mathcal{L}^+(k,2,3)$	$\alpha_\mathcal{L}^-(k,2,3)$
6	1	0_1	
8	1		-1_1
10	1		0_1
12	3	-1_1	-4_2
14	1	-1_1	
16	3	-4_2	-2_1
18	3	-4_2	-1_1
20	3	-2_1	-6_2
22	3	-2_1	-4_2
24	5	-6_2	$-2_1, -7_2$
26	3	-6_2	-2_1
28	5	$-2_1, -7_2$	-5_2
30	5	$-2_1, -7_2$	-6_2
32	5	-5_2	$-3_1, -7_2$

Table 1: $p = 2$, $N = 3$

k	$d_k(2,5)$	$\alpha_\mathcal{L}(k,2,5)$	
		$\alpha_\mathcal{L}^+(k,2,5)$	$\alpha_\mathcal{L}^-(k,2,5)$
6	3	-2_2	0_1
8	1		-1_1
10	3	-5_2	0_1
12	5	-2_2	$-1_1, -4_2$
14	3	-3_2	-1_1
16	5	-5_2	$-2_1, -4_2$
18	7	-5_4	$-1_1, -4_2$
20	5	-3_2	$-2_1, -6_2$
22	7	$-1_2, -8_2$	$-2_1, -4_2$
24	9	-5_4	$-2_1, -6_2, -7_2$
26	7	$-\frac{11}{2}_4$	$-2_1, -6_2$
28	9	$-1_2, -8_2$	$-2_1, -5_2, -7_2$
30	11	$-4_2, -7_4$	$-2_1, -6_2, -7_2$
32	9	$-\frac{11}{2}_4$	$-3_1, -5_2, -7_2$

Table 2: $p = 2$, $N = 5$

Conjectures

- For $k \in 4\mathbb{Z}$, $k \geq 8$, we have
$$\alpha_\mathcal{L}^-(k,2,3) = \alpha_\mathcal{L}^+(k+4,2,3) = \alpha_\mathcal{L}^+(k+6,2,3) = \alpha_\mathcal{L}^-(k+10,2,3)$$
and consequently $\alpha_\mathcal{L}(k,2,3) = \alpha_\mathcal{L}(k+6,2,3)$.
- For $k \geq 6$ we have $\alpha_\mathcal{L}^-(k,2,5) = \alpha_\mathcal{L}(k,2,3)$. Moreover, for $k \in 2 + 4\mathbb{Z}$, $k \geq 6$, we have
$$\alpha_\mathcal{L}^+(k,2,5) = \alpha_\mathcal{L}^+(k+6,2,5).$$

Future Directions

- The above conjectures show only a small part of the visible patterns and observations. We are gathering more data for other primes, and are computing $\mathcal{L}$-invariants of classical and p-adic modular forms building on work of Lauder.
- Our observations are related to recent conjectures of Bergdall and Pollack on the size of the Coleman family through f and its relation with the $\mathcal{L}$-invariant.

References

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