Hyperderivatives on Drinfeld cusp forms via representation theory

(joint with G. Böckle and R. Perkins)

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Structure

1. Drinfeld modular forms
2. Hyperderivatives and the Frobenius
3. The link to representation theory
4. Numerical examples

Notation

Let p be prime and $q = p^e$.

$A = \mathbb{F}_q[t]$	$(\mathbb{Z})$
$F = \mathbb{F}_q(t)$	$(\mathbb{Q})$
$F_\infty = \mathbb{F}_q(\!(1/t)\!)$	$(\mathbb{R})$
$\mathbb{C}_\infty = \hat{\overline{F}}_\infty$	$(\mathbb{C})$

Let $\Omega = \mathbb{P}^1_{F_\infty} \setminus \mathbb{P}^1(F_\infty)$ the *Drinfeld period domain* viewed as a rigid space over F_∞. It carries a natural action of $\mathrm{GL}_2(F_\infty)$.

Drinfeld modular forms

Definition:

Let $k \in \mathbb{Z}$, $\ell \in \mathbb{Z}/(q-1)\mathbb{Z}$. A *Drinfeld modular (cusp) form* of weight k and type ℓ for $\Gamma = \mathrm{GL}_2(A)$ is a rigid analytic function $f\colon \Omega \to \mathbb{C}_\infty$ such that:

(i) $f(\gamma z) = \det(\gamma)^{-\ell}(cz+d)^k f(z)$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

(ii) f is holomorphic at ∞ (vanishes at ∞).

We denote by $M_{k,\ell}(\Gamma)$ and $S_{k,\ell}(\Gamma)$ the $\mathbb{C}_\infty$-vector spaces of Drinfeld modular forms and Drinfeld cusp forms.

For $\mathfrak{p} \subset A$ prime there is an attached *Hecke operator* $T_\mathfrak{p} \in \mathrm{End}(M_{k,\ell}(\Gamma))$, which stabilizes $S_{k,\ell}(\Gamma)$. The Hecke operators form a commutative (Hecke) algebra $\mathbb{T}$.

Drinfeld modular forms

Question: Are there *interesting* maps between the spaces $S_{k,\ell}(\Gamma)$ for varying k and ℓ that behave well with respect to the Hecke operators (defined over F)?

In the classical situation, one expects no such maps to exist: The *Maeda conjecture* predicts that $S_k^{\mathrm{cl}}(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q})$ is irreducible as a Hecke module.

For Drinfeld cups forms the behaviour is very different.

Hyperderivatives: (Bosser-Pellarin) Fix $k \geq 2$ and suppose that $s \geq 1$ satisfies $\binom{k+s-1}{i} \equiv 0 \pmod{p}$ for $i = 1, \ldots, s$. Then there exists a linear *hyperderivative map*

$$D_s\colon S_{k,\ell}(\Gamma) \to S_{k+2s,\ell+s}(\Gamma)$$

such that $D_s(T_\mathfrak{p} f) = T_\mathfrak{p}(D_s f)$. In general, D_s is neither trivial nor surjective.

(Note that Bosser-Pellarin use a different normalization for the Hecke operators, namely they have $D_s(T_\mathfrak{p} f) = \mathfrak{p}^{-s} T_\mathfrak{p}(D_s f)$.)

Frobenius: The map

$$\tau_p\colon S_{k,\ell}(\Gamma) \to S_{pk,p\ell}(\Gamma), \quad f \mapsto f^p$$

satisfies $\tau_p(T_\mathfrak{p} f) = T_\mathfrak{p} \tau_p(f)$.

The Steinberg module

The projective line $\mathbb{P}^1(F)$ carries a natural left action by $\mathrm{GL}_2(F)$. Let

$$\begin{aligned} \deg\colon \mathbb{Z}[\mathbb{P}^1(F) \times \mathrm{GL}_2(F)/\Gamma] &\to \mathbb{Z}[\mathrm{GL}_2(F)/\Gamma], \\ \textstyle\sum_i n_i(P_i, g_i\Gamma) &\mapsto \textstyle\sum_i n_i(g_i\Gamma), \end{aligned}$$

a map of $\mathbb{Z}[\mathrm{GL}_2(F)]$-modules.

Definition:

The *Steinberg module* St_Γ for Γ is the kernel in the short exact sequence

$$0 \to \mathrm{St}_\Gamma \to \mathbb{Z}[\mathbb{P}^1(F) \times \mathrm{GL}_2(F)/\Gamma] \xrightarrow{\deg} \mathbb{Z}[\mathrm{GL}_2(F)/\Gamma] \to 0.$$

It carries an action by the Hecke algebra $\mathbb{T}$.

Teitelbaum's isomorphism

Let $\Delta_k = \mathrm{Sym}^k((F^2)^*) \cong F[X, Y]_{\deg = k}$ and

$$V_{k,\ell} = \mathrm{Hom}_F(\Delta_{k-2} \otimes_F \det^{\ell-1}, F).$$

Theorem: (Teitelbaum)

There is a Hecke-equivariant isomorphism

$$S_{k,\ell}(\Gamma) \to (\mathrm{St}_\Gamma \otimes_{\mathrm{GL}_2(F)} V_{k,\ell}) \otimes_F \mathbb{C}_\infty.$$

The functor $\mathrm{St}_\Gamma \otimes_{\mathrm{GL}_2(F)} (\cdot)$

Theorem: (B-G-P)

The functor

$$\begin{aligned} \mathrm{Rep}^f_F(\mathrm{GL}_2(F)) &\to \mathrm{Mod}^f_{F[\mathbb{T}]} \\ M &\mapsto \mathrm{St}_\Gamma \otimes_{\mathrm{GL}_2(F)} M \end{aligned}$$

is exact.

Question: Can one describe Hyperderivatives and Frobenius in terms of representation theory, i.e. via the functor above?

Hyperderivatives

Fix $k \geq 2$ and suppose that $s \geq 1$ satisfies $\binom{k+s-1}{i} \equiv 0 \pmod{p}$ for $i = 1, \ldots, s$. Then the map

$$\begin{aligned} \mathcal{D}_s \colon \Delta_{k-2+2s} \otimes_F \det^s &\to \Delta_{k-2} \\ X^i Y^j &\mapsto (-1)^s \binom{i}{s} X^{i-s} Y^{j-s} \end{aligned}$$

is $\mathrm{GL}_2(F)$-equivariant.

Via Teitelbaum's isomorphism (and after dualizing) we have

$$D_s = \mathcal{D}_s^*,$$

i.e. these are the hyperderivatives of Bosser-Pellarin.

The Frobenius

Denote by σ the Frobenius of (the perfect field) $\mathbb{C}_\infty$. Let $k \geq 2$ and $m \in \mathbb{Z}$. The map

$$\mathcal{C}_p \colon \sigma_*((\Delta_{pk-2} \otimes \det^{pm-1}) \otimes_F \mathbb{C}_\infty) \to (\Delta_{k-2} \otimes \det^{m-1}) \otimes_F \mathbb{C}_\infty$$

given by

$$\mathcal{C}_p(aX^{i-1}Y^{j-1}) = \begin{cases} a^{\frac{1}{p}} X^{\frac{i}{p}-1} Y^{\frac{j}{p}-1}, & \text{if } p \mid i, \\ 0, & \text{otherwise,} \end{cases}$$

is $\mathrm{GL}_2(F)$-equivariant and surjective. Via Teitelbaum's isomorphism (and after dualizing) it becomes the Frobenius on Drinfeld cusp forms.

Remark: This map can be interpreted in terms of the *Cartier operator* on $\mathbb{A}^2_{\mathbb{C}_\infty}$.

Remarks

- **Upshot:** All of the constructions presented in this talk work over any base A and for general congruence subgroups $\Gamma \subset \mathrm{GL}_2(F)$. In particular, the representation theoretic construction of the hyperderivative maps provides a natural generalization of the maps of Bosser-Pellarin to this general setting.
- It is a natural question to ask wether one can find other interesting maps via representation theory; questions along these lines will be discussed in the next talk by R. Perkins.

Example: Let $q = 3$. We have chains of hyperderivatives

$$S_{62,0}(\Gamma) \xrightarrow{D_2} S_{66,0}(\Gamma) \xrightarrow{D_{16}} S_{98,0}(\Gamma) \xrightarrow{D_2} S_{102,0}(\Gamma)$$

and a direct map

$$S_{62,0}(\Gamma) \xrightarrow{D_{20}} S_{102,0}(\Gamma)$$

but *no* direct maps

$$S_{62,0}(\Gamma) \xrightarrow{D_{18}} S_{98,0}(\Gamma) \quad \text{and} \quad S_{66,0}(\Gamma) \xrightarrow{D_{18}} S_{102,0}(\Gamma).$$

One has $D_2 D_{16} = D_{16} D_2 = 0$. There are also Frobenius maps

$$S_{22,0}(\Gamma) \xrightarrow{\tau_3} S_{66,0}(\Gamma) \quad \text{and} \quad S_{34,0}(\Gamma) \xrightarrow{\tau_3} S_{102,0}(\Gamma).$$

We have computed the action of the Hecke operator T_t on the image of all of these maps using Sage (with the normalization as in Bosser-Pellarin).

$$S_{22,0}(\Gamma) \xrightarrow{\tau_3} S_{66,0}(\Gamma), \quad S_{62,0}(\Gamma) \xrightarrow{D_2} S_{66,0}(\Gamma), \quad S_{66,0}(\Gamma) \xrightarrow{D_{16}} S_{98,0}(\Gamma), \quad S_{98,0}(\Gamma) \xrightarrow{D_2} S_{102,0}(\Gamma), \quad S_{62,0}(\Gamma) \xrightarrow{D_{20}} S_{102,0}(\Gamma), \quad S_{34,0}(\Gamma) \xrightarrow{\tau_3} S_{102,0}(\Gamma)$$

$S_{22,0}(\Gamma)$

$(X - t^4) \cdot (X + t^{10} + t^2)$

$S_{66,0}(\Gamma)$

$(X - t^4) \cdot (X - t^{12}) \cdot (X + t^{30} + t^6)$
$(X^2 + t^{28}X + t^{38} + t^{30} - t^{12})$
(irreducible of degree 3)

$S_{62,0}(\Gamma)$

$(X - t^2) \cdot (X - t^8)$
$(X - t^{10}) \cdot (X + t^{28} + t^4)$
(irreducible of degree 3)

$S_{98,0}(\Gamma)$

$(X - t^2) \cdot (X - t^8) \cdot (X - t^{20})$
$(X - t^{28}) \cdot (X + t^{46} + t^{22})$
$(X^2 + t^4X - t^{92} + t^{86} + t^{14})$
$(X^2 + t^{44}X + t^{70} + t^{62} - t^{44})$
(irreducible of degree 3)

$S_{34,0}(\Gamma)$

$(X - t^{10}) \cdot (X + t^{16} + t^8)$
$(X^2 + t^2X - t^{32} + t^{30} + t^6)$

$S_{102,0}(\Gamma)$

$(X - t^4) \cdot (X - t^{22}) \cdot (X - t^{28})$
$(X - t^{30}) \cdot (X + t^{48} + t^{24})$
$(X^2 + t^6X - t^{96} + t^{90} + t^{18})$
$(X^2 + t^{10}X + t^{92} - t^{84} + t^{12})$
(irreducible of degree 3)