Boundary distributions on the Drinfeld period domain for GL_3

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Structure

Motivation

Modular forms and modular symbols

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$, the (complex) *upper half plane*.

Association:

$$\big\{\text{modular forms on } \mathbb{H}\big\} \xrightarrow{\text{Period integrals}} \big\{\text{modular symbols}\big\}$$

(analytic objects) (combinatorial objects)

This association is Hecke-equivariant!

Modular forms and modular symbols

Upshot:

- Modular symbols "know" special L-values.
- Great tool for explicit computations (for example to compute Heegner points).
- Modular symbols (and their overconvergent variant) show up in the construction of eigenvarieties.

Aim: Discuss a non-archimedean analogue of this construction.

The central objects

Notation

- Let K be a non-archimedean local field:

$$\begin{cases} \text{char}(K) = 0: & K/\mathbb{Q}_p \text{ finite extension} \\ \text{char}(K) = p > 0: & K = \mathbb{F}_q((t)),\ q = p^e \end{cases}$$

- Let π denote a uniformizing parameter in K and $\nu(\cdot)$ the normalized valuation on K.
- Let $\mathcal{O}_K$ denote the ring of integers of K.
- Let $\mathbb{C}_K$ denote the completion of an algebraic closure of K.
- Let $n \geq 2$ and $G = \mathrm{GL}_n(K)$. (In the sequel, mostly $n \in \{2, 3\}$.) The diagonal torus in G is denoted by T. The Borel subgroup of upper triangular matrices in denoted by B.

The Drinfeld period domain

The *Drinfeld period domain* for G is the space

$$\mathcal{X} := \mathcal{X}^{(n)} := \mathbb{P}_K^{n-1} \setminus \bigcup_{H \in \mathcal{H}} H,$$

where $\mathcal{H}$ denotes the set of all K-rational hyperplanes in $\mathbb{P}_K^{n-1}$.

- $\mathcal{X}$ is a rigid space over K.
- $\mathcal{X}$ carries a natural action by G.

Remark: For $n = 2$, we have

$$\mathcal{X}(\mathbb{C}_K) = \mathbb{P}^1(\mathbb{C}_K) \setminus \mathbb{P}^1(K) = \mathbb{C}_K \setminus K.$$

$\rightsquigarrow$ $\mathcal{X}$ serves as an analogue of $\mathbb{H}$.

The Bruhat-Tits building

The *Bruhat-Tits building* $\mathcal{T} := \mathcal{T}^{(n)}$ of G is the simplicial complex given as follows.

- The vertices $\mathcal{T}_0$ consist of homothety classes $[\Lambda]$ of lattices $\Lambda \subset K^n$.
- The m-cells $\mathcal{T}_m$ for $m \in \{1, \ldots, n-1\}$ consist of sets $\{[\Lambda_0], \ldots, [\Lambda_m]\}$ where

$$\pi\Lambda_0 \subsetneq \Lambda_m \subsetneq \Lambda_{m-1} \subsetneq \cdots \subsetneq \Lambda_0.$$

A pointed m-cell is an m-cell with a distinguished vertex $[\Lambda_0]$. The set of pointed m-cells is denoted by $\widehat{\mathcal{T}}_m$.

- The group G acts transitively on $\mathcal{T}$.
- There is a G-equivariant *reduction map* $\mathrm{red}\colon \mathcal{X} \to \mathcal{T}$. The preimages of the $(n-1)$-cells are certain multi-annuli in $\mathcal{X}$. This map encodes the structure of $\mathcal{X}$ as a rigid space.
- The flag variety G/B can be viewed as the boundary of $\mathcal{T}$.

The Bruhat-Tits building

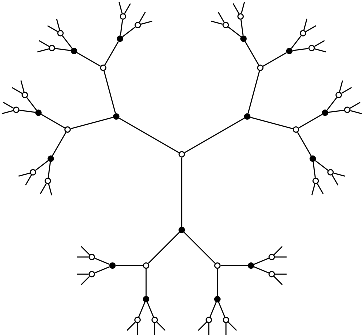

Figure The Bruhat-Tits building for $n = 2$ and $K = \mathbb{Q}_2$ or $K = \mathbb{F}_2((t))$.

The central triangle

Let $k \geq 0$ with $n \mid k$. We want to find maps and relations between the following three objects:

$$\mathcal{O}_{\mathcal{X}}(k+n) \quad \text{———} \quad C_{\mathrm{har}}(\mathcal{T}, V_k)$$

$$\mathrm{St}_n^{\mathrm{an}}(k)'$$

The holomorphic discrete series representation

Definition:

The *holomorphic discrete series representation* $\mathcal{O}_{\mathcal{X}}(k+n)$ of weight $k+n$ is the space $\mathcal{O}_{\mathcal{X}}$ of global rigid analytic functions on $\mathcal{X}$ endowed with a weight-$(k+n)$ action by G:

$$g_* f(\omega) = \det(g)^{(k+n)/n} j(g,\omega)^{-(k+n)} f(g\omega), \quad f \in \mathcal{O}_{\mathcal{X}}, \omega \in \mathcal{X}, g \in G.$$

Example: $k=0$: $\mathcal{O}_{\mathcal{X}}(n) \cong \Omega_{\mathcal{X}}^{n-1}$.

Remark: Invariants under arithmetic group $\rightsquigarrow$ analogue of modular forms.

The central triangle

Let $k \geq 0$ with $n \mid k$. We want to find maps and relations between the following three objects:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}}(k+n) & \text{———————} & C_{\mathrm{har}}(\mathcal{T}, V_k) \\ & \diagdown \qquad \diagup & \\ & \mathrm{St}_n^{\mathrm{an}}(k)' & \end{array}$$

Harmonic Cocycles

Let $V_k := (\mathrm{Sym}^k((\mathbb{C}_K^n)^*) \otimes_{\mathbb{C}_K} \det^{-k/n})^*$.

Definition:

A map $c\colon \widehat{\mathcal{T}}_{n-1} \to V_k$ is called a *harmonic cocycle* if:

(i) Let $\sigma \in \widehat{\mathcal{T}}_{n-1}$ and let ρ_σ be a generator of the group fixes σ modulo the group that fixes σ pointwise. Then

$$c(\rho_\sigma \sigma) = (-1)^{n-1} c(\sigma).$$

(ii) Let $\tau \in \mathcal{T}_{n-2}$. Then

$$\sum_{\sigma \mapsto \tau} c(\sigma) = 0,$$

where the sum is over all pointed $(n-1)$-cells $\sigma \in \widehat{\mathcal{T}}_{n-1}$ sharing the face τ, each with distinguished vertex opposite to τ.

The G-module of harmonic cocycles is denoted by $C_{\mathrm{har}}(\mathcal{T}, V_k)$.

The central triangle

Let $k \geq 0$ with $n \mid k$. We want to find maps and relations between the following three objects:

$$\mathcal{O}_{\mathcal{X}}(k+n) \quad \text{———} \quad C_{\mathrm{har}}(\mathcal{T}, V_k)$$

$$\mathrm{St}_n^{\mathrm{an}}(k)'$$

The locally analytic Steinberg representation

Let $\chi_k\colon T \to K^\times$ be the character given by $t \mapsto \det(t)^{-k/n} t_{nn}^k$. Let

$\mathcal{A}_k := \mathrm{Ind}_B^G(\chi_k) = \{f \in C^{\mathrm{an}}(G, \mathbb{C}_K) \mid f(gb) = \chi_k(b^{-1})f(g),\ g \in G, b \in B\}$

the locally analytic induction from B to G of χ_k. This is a locally analytic G-representation in the sense of Schneider-Teitelbaum.

Let $B \subsetneq P \subset G$ be a parabolic subgroup. Then

$$\mathcal{A}_{P,k} := \mathrm{Ind}_P^G(\mathrm{Ind}_B^{P,\mathrm{alg}}(\chi_k) \otimes_K \mathbb{C}_K)$$

is naturally a G-submodule of $\mathcal{A}_k$.

Definition:

The *locally analytic Steinberg representation of G of weight k* is the G-module

$$\mathrm{St}_n^{\mathrm{an}}(k) := \mathcal{A}_k / \sum_{B \subsetneq P \subset G} \mathcal{A}_{P,k}.$$

The GL_2-case (after Schneider-Teitelbaum)

The residue map

Note that $V_k \cong \mathcal{P}_k^*$ with $\mathcal{P}_k := \mathbb{C}_K[x]_{\deg \leq k}$.

Theorem: (Schneider (1984))

There is a G-equivariant residue map $\mathrm{Res}_k \colon \mathcal{O}_{\mathcal{X}}(k+2) \to C_{\mathrm{har}}(\mathcal{T}, V_k)$ given by

$$\mathrm{Res}_k(f)(\sigma)(x^i) = \mathrm{res}_\sigma(\omega^i f(\omega)\mathrm{d}\omega) \quad \text{for } f \in \mathcal{O}_{\mathcal{X}}, \sigma \in \widehat{\mathcal{T}}_1,$$

where $\mathrm{res}_\sigma(\cdot)$ is the residue in the series expansion on the oriented annulus given by the preimage of (the interior of) σ under the reduction map.

The Poisson kernel

Note that we have $G/B \cong \mathbb{P}^1$ and pullback under $K \hookrightarrow \mathbb{P}^1(K)$ gives an isomorphism

$$\mathrm{St}_2^{\mathrm{an}}(k) \cong C^{\mathrm{an}}(\mathbb{P}^1, k)/\mathcal{P}_k.$$

Here, $C^{\mathrm{an}}(\mathbb{P}^1, k)$ is the space of locally analytic functions on $\mathbb{P}^1(K)$ except for a possible pole of order $\leq k$ at ∞.

Theorem: (Teitelbaum (1990))

There is a G-equivariant Poisson kernel $I_k \colon \mathrm{St}_2^{\mathrm{an}}(k)' \to \mathcal{O}_{\mathcal{X}}(k+2)$ given by

$$I_k(\lambda)(\omega) = \lambda(x \mapsto \theta(x, \omega)) \quad \text{for } \lambda \in \mathrm{St}_2^{\mathrm{an}}(k)', \omega \in \mathcal{X},$$

where $\theta(x, \omega) = 1/(\omega - x)$, the *kernel function*.

Extending distributions

Let $C^b_{\mathrm{har}}(\mathcal{T}, V_k) \subset C_{\mathrm{har}}(\mathcal{T}, V_k)$ be the subspace of *bounded* harmonic cocycles.

Theorem: (Amice-Vélu (1975), Vishik (1976), Schneider (1984), Teitelbaum (1990))

There is a G-equivariant injective map

$$L_k \colon C^b_{\mathrm{har}}(\mathcal{T}, V_k) \to \mathrm{St}_2^{\mathrm{an}}(k)'.$$

Idea: $c \in C_{\mathrm{har}}(\mathcal{T}, V_k)$ defines a distribution on locally polynomial functions (of degree $\leq k$). If this distribution is bounded, it can be uniquely extended to allow integration of locally analytic functions.

Remark: The case $k = 0$ is particularly simple: Approximate continuous (or locally analytic) function on $\mathbb{P}^1(K)$ by locally constant ones.

The central triangle

If we transfer the notion of boundedness to the other spaces, we obtain the triangle:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}}(k+2)^b & \xrightarrow{\mathrm{Res}_k} & C^b_{\mathrm{har}}(\mathcal{T}, V_k) \\ & {\scriptstyle I_k}\nwarrow \qquad \swarrow{\scriptstyle L_k} & \\ & \mathrm{St}_2^{\mathrm{an}}(k)^{\prime,b} & \end{array}$$

Theorem: (Teitelbaum (1990))

We have $\mathrm{Res}_k \circ I_k \circ L_k = \mathrm{id}$.

Applications

All maps are G-equivariant $\rightsquigarrow$ Can take invariants under arithmetic groups.

- $\mathrm{char}(K) = 0$: Construction of p-adic $\mathcal{L}$-invariants: Teitelbaum (1990), Iovita-Spieß (2003), Chida-Mok-Park (2015)
- $\mathrm{char}(K) = p > 0$: Eichler-Shimura isomorphism for Drinfeld modular forms: Böckle (2012), Hecke-module structures of spaces of Drinfeld modular forms: Böckle-G.-Perkins (2019)
- Explicit computations: Böckle-Butenuth (2012), G. (2019)

The GL_3-case

Going beyond GL_2

- Schneider-Teitelbaum (1997): Any $n \geq 2$, $\mathrm{char}(K) = 0$ and $k = 0$.
- All other cases for $n \geq 3$ have been open.
- We consider $n = 3$, K of arbitrary characteristic and any $k \geq 0$. ($\rightsquigarrow$ restrict n, allow any K and k)

The residue map

Schneider-Teitelbaum: Construct $\mathrm{Res}_0 \colon \mathcal{O}_{\mathcal{X}}(3) \to C_{\mathrm{har}}(\mathcal{T}, \mathbb{C}_K)$ in analogy with the GL_2-case.

Idea: Construct a *translation map* $t_k \colon \mathcal{O}_{\mathcal{X}}(k+3) \hookrightarrow \mathcal{O}_{\mathcal{X}}(3) \otimes_{\mathbb{C}_K} V_k$ and consider

$$\mathcal{O}_{\mathcal{X}}(k+3) \xrightarrow{t_k} \mathcal{O}_{\mathcal{X}}(3) \otimes_{\mathbb{C}_K} V_k \xrightarrow{\mathrm{Res}_0 \otimes \mathrm{id}} C_{\mathrm{har}}(\mathcal{T}, \mathbb{C}_K) \otimes_{\mathbb{C}_K} V_k \to C_{\mathrm{har}}(\mathcal{T}, V_k).$$

(In the GL_2-case: Schneider-Stuhler (1991))

Theorem: ($k = 0$: Schneider-Teitelbaum (1997), $k > 0$: G. (2020))

There is a G-equivariant residue map $\mathrm{Res}_k \colon \mathcal{O}_{\mathcal{X}}(k+3) \to C_{\mathrm{har}}(\mathcal{T}, V_k)$ given by the analogous formula as in the GL_2-case.

The Poisson kernel

Fix the *Plücker-embedding* $pl\colon G/B \to \mathbb{P}^2(K) \times \mathbb{P}^2(K)$ given by

$$g \mapsto ([\alpha_1(g) : \alpha_2(g) : \alpha_3(g)], [\beta_1(g) : \beta_2(g) : \beta_3(g)]),$$

where the column vector $(\alpha_1(g), \alpha_2(g), \alpha_3(g))$ is the first column of g and $\beta_i(g)$ is the determinant of the 2×2 submatrix of g consisting of the first two columns and row $4 - i$ removed. It is a closed immersion.

The kernel function: (Schneider-Teitelbaum (1997)) Define $\theta\colon G/B \times \mathcal{X} \to \mathbb{C}_K$ by

$$\theta(g, \omega) = \frac{\alpha_1(g)}{\alpha_1(g)\omega_1 + \alpha_2(g)\omega_2 + \alpha_3(g)} \cdot \frac{\beta_1(g)}{\beta_1(g)\omega_2 + \beta_2(g)}.$$

The Poisson kernel

Problem: The function $\theta(g, \omega)$ is *not* locally analytic everywhere.

- This did not occur in the GL_2-case!
- Major obstacle for integration in the case $k > 0$.

Proposition: (G. (2020))

There exists an explicit locally analytic representative $\hat{\theta}(g, \omega)$ for the class of $\theta(g, \omega)$ in $\mathrm{St}_3^{\mathrm{con}} := C(G/B, \mathbb{C}_K)/\sum_{B \subsetneq P \subset G} C(G/P, \mathbb{C}_K)$.

Idea: Analyse the locus where $\theta(g, \omega)$ is not locally analytic. It turns out that this is an explicit $\mathbb{P}^1 \subset G/B$ that is linked to a parabolic subgroup $P \subset G$. We modify the kernel function in a natural way on an open neighborhood of this locus.

The Poisson kernel

We can prove the following theorem.

Theorem: (G. (2020))

There is a G-equivariant Poisson kernel $I_k \colon \mathrm{St}_3^{\mathrm{an}}(k)' \to \mathcal{O}_{\mathcal{X}}(k+3)$ given by

$$I_k(\lambda)(\omega) = \lambda(g \mapsto \det(g)^{-2k/3}\beta_1(g)^k\hat{\theta}(g,\omega)) \quad \text{for } \lambda \in \mathrm{St}_3^{\mathrm{an}}(k)', \omega \in \mathcal{X}.$$

Idea: We first prove the theorem for $k = 0$. Then we consider the diagram:

$$\begin{array}{ccc} \mathrm{St}_3^{\mathrm{an}}(k)' & \longrightarrow & \mathrm{St}_3^{\mathrm{an}}(0)' \otimes_{\mathbb{C}_K} V_k \\ {\scriptstyle I_k}\Big\downarrow & & \Big\downarrow{\scriptstyle I_0 \otimes \mathrm{id}} \\ \mathcal{O}_{\mathcal{X}}(k+3) & \xrightarrow{t_k} & \mathcal{O}_{\mathcal{X}}(3) \otimes_{\mathbb{C}_K} V_k \end{array}$$

Extending distributions

The final step is to develop an analogue of the theorem of Amice-Vélu and Vishik.To develop as systematic approach for this, we introduce the following space of *(generalized) automorphic forms*:

$$\mathbb{A}(V_k) := \left\{\varphi\colon G \to V_k \mid \varphi(xgh) = \varphi(g)\cdot h \text{ for } x \in K^\times, h \in \mathcal{I}\right\},$$

where $\mathcal{I} \subset \mathrm{GL}_3(\mathcal{O}_K)$ denotes the Iwahori subgroup of matrices that are upper triangular mod π.

There are four natural Hecke operators acting on $\mathbb{A}(V_k)$: Two U-operators $(U_{\pi,i})_{i\in\{1,2\}}$ and two Atkin-Lehner operators $(W_{\pi,i})_{i\in\{1,2\}}$.

We obtain a G-equivariant embedding $C_{\mathrm{har}}(\mathcal{T}, V_k) \hookrightarrow \mathbb{A}(V_k)$ whose image $\mathbb{A}(V_k)^{\mathrm{new}}$ consists of eigenforms for all four operators with explicit eigenvalues: $\rightsquigarrow$ Can also transfer the notion of boundedness and consider a space $\mathbb{A}(V_k)_b^{\mathrm{new}}$.

Extending distributions

Definition:

An eigenform $\varphi \in \mathbb{A}(V_k)_b$ is called *non-critical* if it lifts uniquely to an eigenform in spaces of automorphic forms with overconvergent and partially overconvergent coefficients (i.e. the analogue of Coleman classicality holds).

Idea: In this non-critical case, we can use the values of the lifts as local building blocks for the desired extension of the distribution. This is inspired by the GL_2-case, where similar automorphic forms have been used to explicitly compute these distributions, but not to construct them.

Theorem: (G. (2020))

Assume that every automorphic form in $\mathbb{A}(V_k)_b^{\mathrm{new}}$ is non-critical. Then we obtain the desired G-equivariant map

$$L_k\colon C_{\mathrm{har}}^b(\mathcal{T}, V_k) \to \mathrm{St}_3^{\mathrm{an}}(k)'.$$

A control theorem

Let $\alpha_1, \alpha_2 \in \mathcal{O}_K \setminus \{0\}$. We say that the pair (α_1, α_2) *has small slope* if

$$\nu(\alpha_i) \leq \nu_i^{\mathrm{crit}} \quad \text{where} \quad \nu_i^{\mathrm{crit}} = \begin{cases} k, & i = 1, \\ 0, & i = 2, \end{cases}$$

for $i \in \{1, 2\}$.

Theorem: (G. (2020))

Let $\alpha_1, \alpha_2 \in \mathcal{O}_K \setminus \{0\}$ be such that the pair (α_1, α_2) has small slope. Then each form in $\mathbb{A}(V_k)_b^{(U_{\pi,i}=\alpha_i)_{i\in\{1,2\}}}$ is non-critical.

Non-critical forms

The bounds in the previous theorem are consistent with the standard literature, e.g. Williams (2018), Bellaïche-Chenevier (2019).

But: Since the forms in $\mathbb{A}(V_k)_b^{\mathrm{new}}$ have slopes $(2k/3, k/3)$, this only gives the existence of L_k for $k = 0$.

Conjecture: (G. (2020))

Every automorphic form in $\mathbb{A}(V_k)_b^{\mathrm{new}}$ is non-critical.

Hope: The forms in $\mathbb{A}(V_k)_b^{\mathrm{new}}$ are very special: They are not just eigenforms for the two U-operators, but also for the Atkin-Lehner operators. We want to exploit these additional symmetries.

Remark: While we did not obtain the existence of L_k in general, the transfer to a lifting question makes the problem a lot more accessible. Systematically, this seems to be the correct point of view.

The central triangle

Finally we obtain the triangle:

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}}(k+3)^b & \xrightarrow{\mathrm{Res}_k} & C^b_{\mathrm{har}}(\mathcal{T}, V_k) \\ & {\scriptstyle I_k} \nwarrow \quad \swarrow {\scriptstyle L_k} & \\ & \mathrm{St}_3^{\mathrm{an}}(k)^{\prime,b} & \end{array}$$

Theorem: (G. (2020)

Assume that our conjecture holds. Then we have $\mathrm{Res}_k \circ I_k \circ L_k = \mathrm{id}$.