Prime numbers and cryptography

Proseminar/Seminar

Dr. B. S. Banwait, C. V. Sriram

Seminar Vorbesprechung
Wednesday, 16^{th} February 2022

Interdisciplinary Center for Scientific Computing

UNIVERSITÄT HEIDELBERG
ZUKUNFT SEIT 1386

Introduction

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Their study goes back to the very beginnings of mathematics.

Rhind Mathematical Papyrus, from ca. 1550 BC, contains *Egyptian fraction* expansions of some prime numbers.

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The first page of al-Kindi's manuscript *On Deciphering Cryptographic Messages*, ca 750 AD

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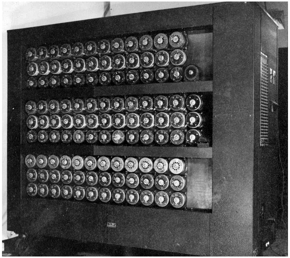

British *Bombe* machine for decrypting Enigma messages. Designed by Alan Turing.

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The most modern cryptosystems are based on elliptic curves.

Rubric of the Proseminar talks

Talk 1 - Prime numbers and complexity analysis

- Basic properties, including the Fundamental theorem of arithmetic, that any integer can be expressed *uniquely* as a product of prime numbers:

$$N = p_1^{e_1} \cdots p_r^{e_r}.$$

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- The Riemann zeta function
$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s},$$
and its Euler product decomposition
$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}}.$$

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These talks will give an overview of some of these fast methods.

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2. Find the smallest r such that $\mathrm{ord}_r(n) > (\log_2 n)^2$. (if r and n are not coprime, then skip this r)
3. For all $2 \le a \le \min(r, n-1)$, check that a does not divide n: If $a|n$ for some $2 \le a \le \min(r, n-1)$, output *composite*.
4. If $n \le r$, output *prime*.
5. For $a = 1$ to $\left\lfloor \sqrt{\varphi(r)} \log_2(n) \right\rfloor$ do

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These two talks will present the details and consequences.

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Sieving refers to progressively removing composite numbers up to a bound, leaving only the primes behind.

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and all of 2, 3, $(1 \pm \sqrt{-5})$ are all "prime" (actually, *irreducible*). This talk will give an overview of these key ideas.

Talk 10 - The number field sieve

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- Complexity
- Relation to the quadratic sieve

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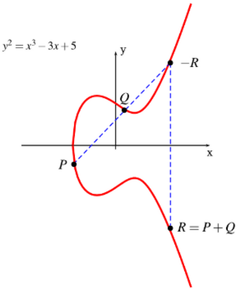

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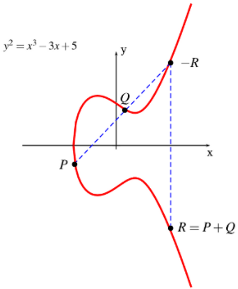
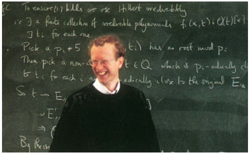


The identity of the group law is the point at infinity O_E, infinitely far up the y-axis.

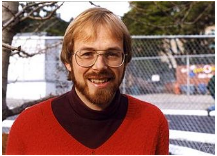
Modularity Theorem (Wiles, Taylor-Wiles, 1995)

Every elliptic curve $E/\mathbb{Q}$ arises from a (weight-2 cuspidal of level $\Gamma_0(N)$) modular form f_E such that the L-functions coincide:

$$L(E,s) = L(f_E,s).$$



Andrew J. Wiles



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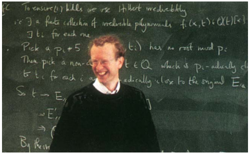
Corollary

Fermat's Last Theorem is true: for $n > 2$, the equation

$$x^n + y^n = z^n$$

admits only the trivial solutions (i.e. when $xyz = 0$).



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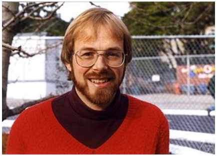

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They are also the subject of the Birch-Swinnerton-Dyer conjecture, one of the Clay Millenium Problems - solving it will earn you $1,000,000!

H.P.F. Swinnerton-Dyer with B. Birch

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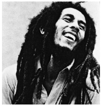

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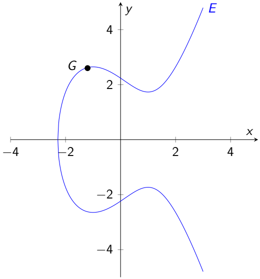

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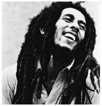

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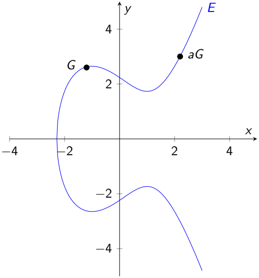

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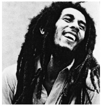

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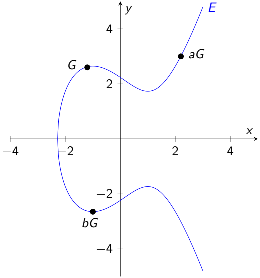

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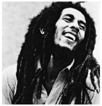

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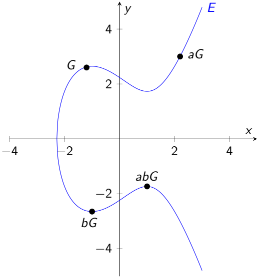

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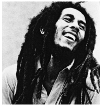

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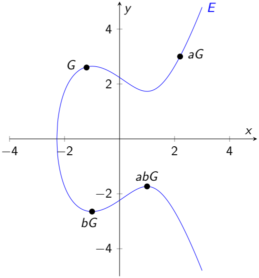
Private key: $b \in \mathbb{Z}$
Public key: $b \cdot G$

Shared Secret $= abG = baG$

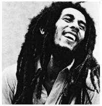
Talk 12 - Public-key cryptography

Elliptic curves over finite fields are also used extensively in public-key cryptography, via the Elliptic Curve Diffie-Hellman (ECDH) protocol:

Alice



Bob



Private key: $a \in \mathbb{Z}$
Public key: $a \cdot G$

Domain parameters:
$E, G \in E$

Private key: $b \in \mathbb{Z}$
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Shared Secret $= abG = baG$

This works because computing a from aG and G is a computationally infeasible problem!

Talk 13 - Factoring numbers with elliptic curves

In 1987 Hendrik W. Lenstra discovered a surprising method of factoring numbers using elliptic curves.

Hendrik W. Lenstra

Talk 14 - Post-quantum cryptogprahy, and SIDH

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IBM Q System One

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The United States' National Institute of Standards and Technology currently has an ongoing competition to select primitives that would be recommended for widespread use.

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Talk 14 - Post-quantum cryptogprahy, and SIDH

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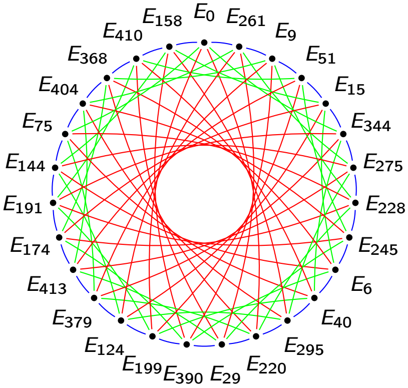
One primitive which has reached Round 3 of the competition - Supersingular Isogeny Key Encapsulation - is based on the theory of isogenies of elliptic curves.

Isogeny-based cryptography

The domain parameter consists of a supersingular isogeny graph of elliptic curves over a finite field:



Nodes: Supersingular elliptic curves $E_A : y^2 = x^3 + Ax^2 + x$ over $\mathbb{F}_{419}$.
Edges: 3-, 5-, and 7-isogenies (more details to come.)

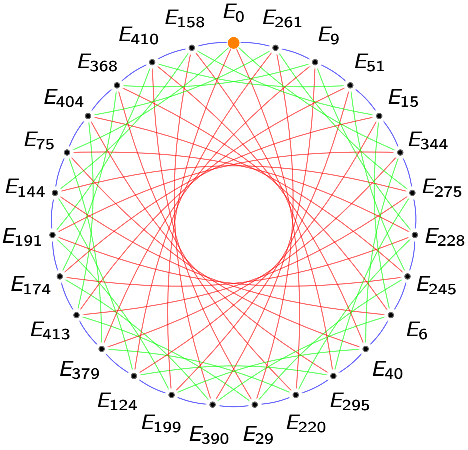
Diffie-Hellman on supersingular isogeny graphs

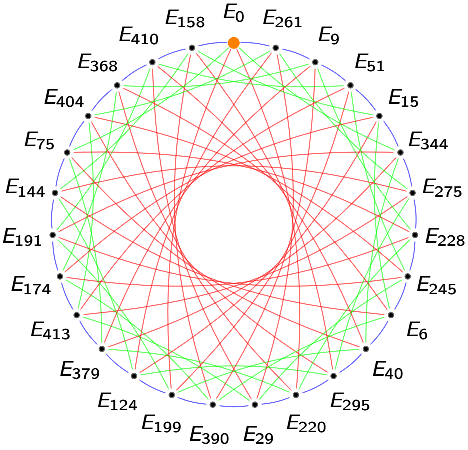
Alice

$a = [+, -, +, -]$

Bob

$b = [+, +, -, +]$



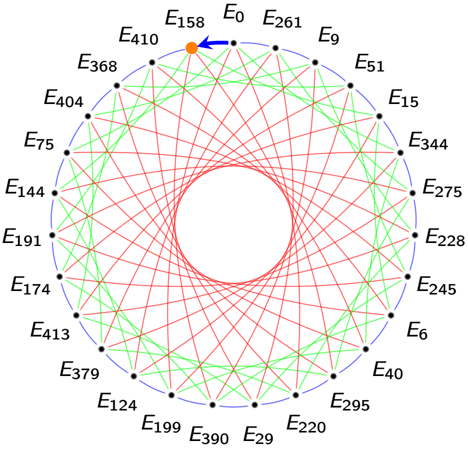


Diffie-Hellman on supersingular isogeny graphs

Alice

$a = [{\color{blue}+}, {\color{green}-}, {\color{blue}+}, {\color{red}-}]$

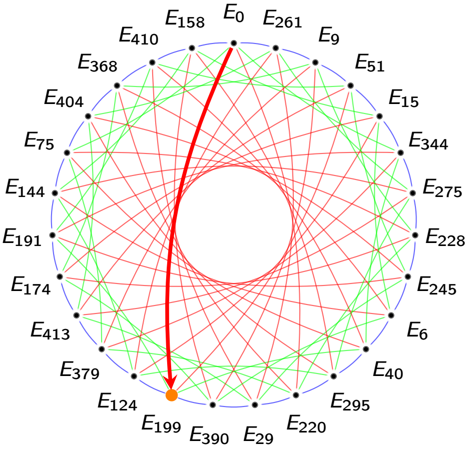
$\uparrow$



Bob

$b = [{\color{red}+}, {\color{green}+}, {\color{blue}-}, {\color{red}+}]$

$\uparrow$



Diffie-Hellman on supersingular isogeny graphs

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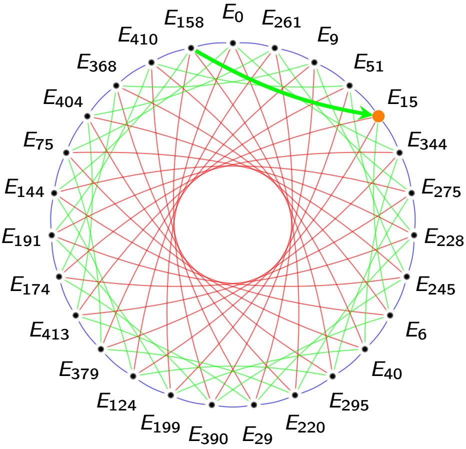
$\uparrow$



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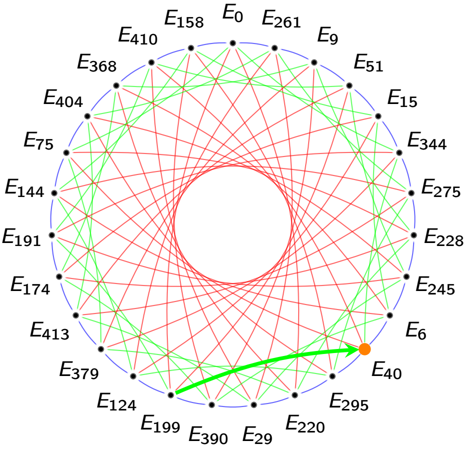

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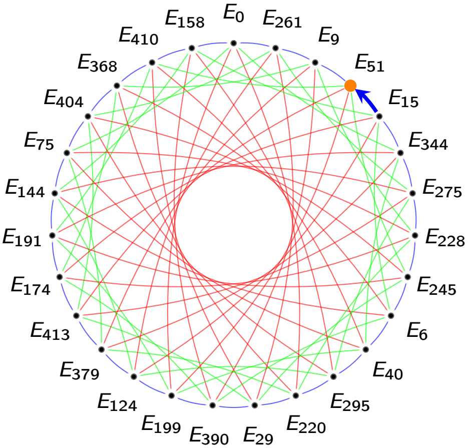

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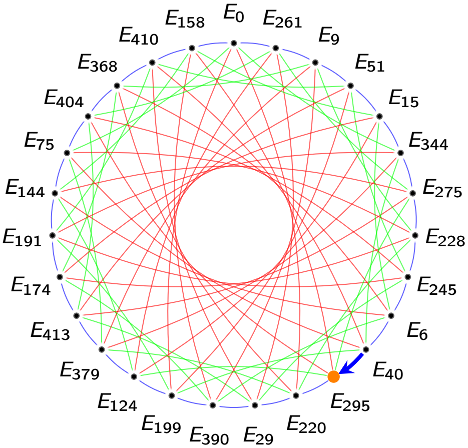

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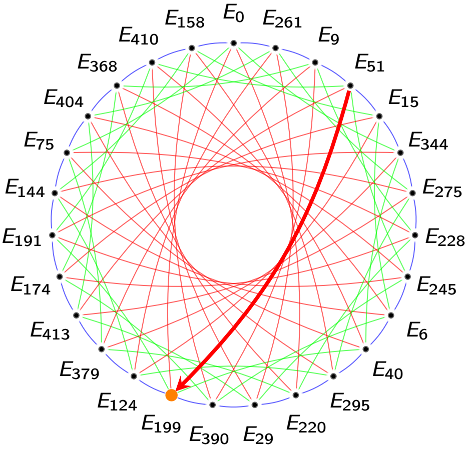

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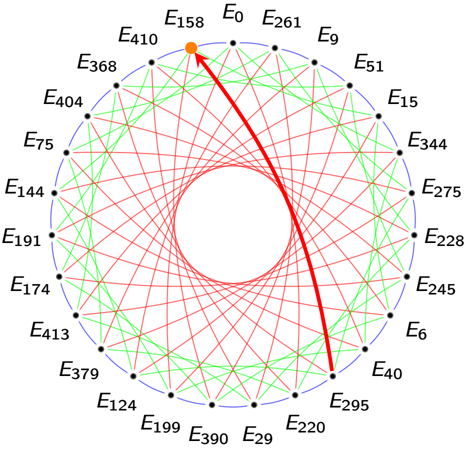

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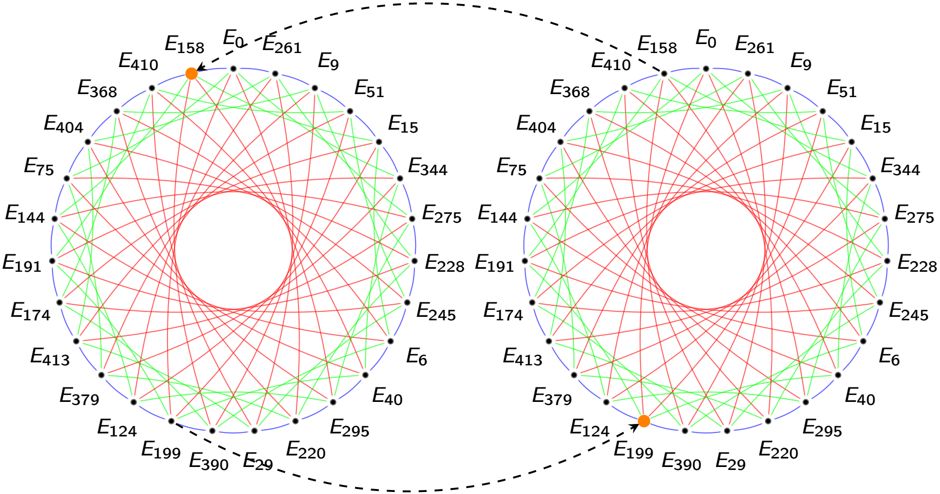

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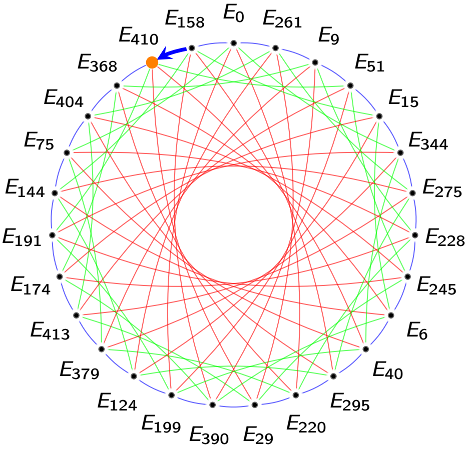

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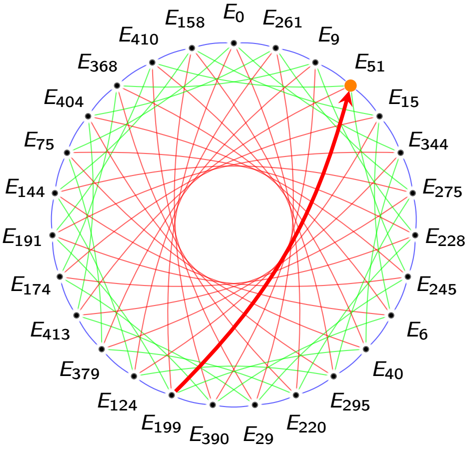

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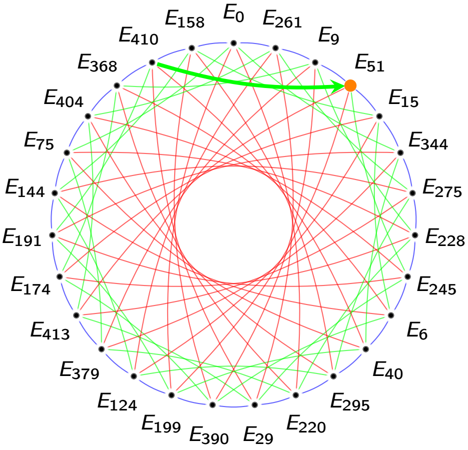

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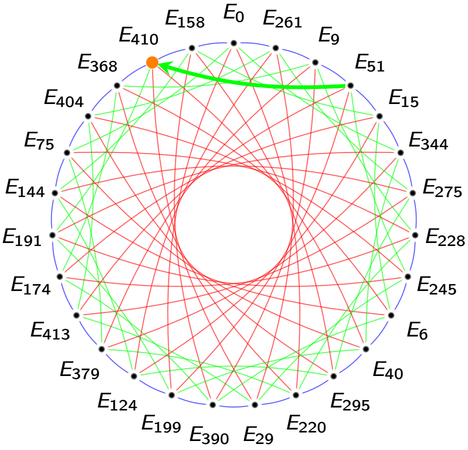

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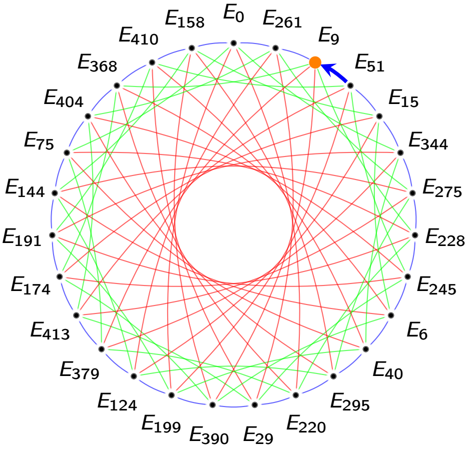

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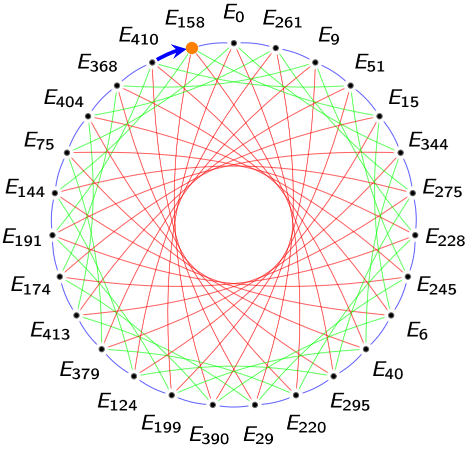

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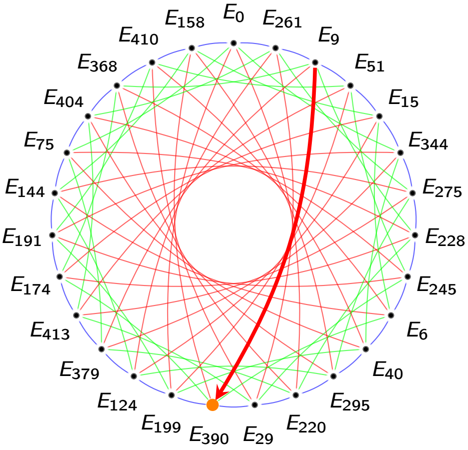

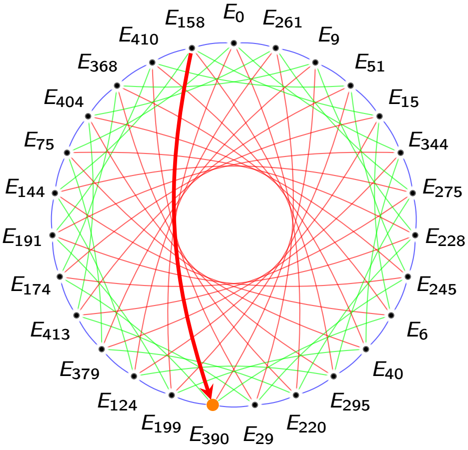

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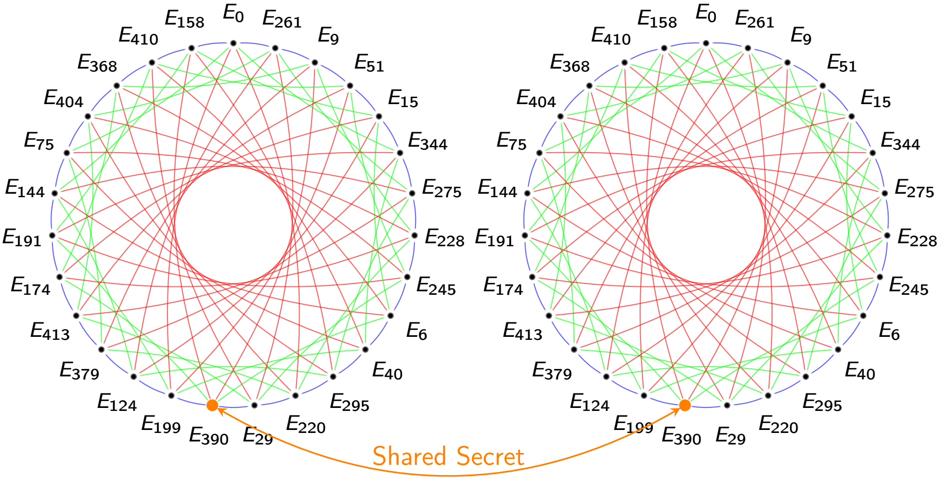

Practicalities

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- Students can ask questions to the instructors before March 28th also – Proseminar talks to C. V. Sriram, and (Pro)Seminar talks to B. S. Banwait.

Questions?

Fragen?

Intentional bilingual pause slide for questions

Absichtliche zweisprachige Pausenfolie für Fragen