Faculty of Mathematics and Computer Sciences

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Master thesis

in Mathematics

submitted by

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born in Mannheim

July 3, 2018

Algorithmic Aspects of Hecke-Eigensystems on Spaces of Harmonic Cocycles

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at the

Institute of Mathematics

under the supervision of

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It is possible to describe modular forms over function fields, also called Drinfeld modular forms, in a combinatorial setting. The realization is through harmonic cocycles on the Bruhat-Tits tree $\mathcal{T}$, which are further required to be equivariant under the action of some congruence subgroup Γ of $\mathrm{GL}_2(\mathbb{F}_q[T])$. Building on this, we can compute eigensystems for Hecke operators. For this purpose, we thoroughly study the Bruhat-Tits tree $\mathcal{T}$, its corresponding quotient graph $\Gamma\backslash\mathcal{T}$, and the values of harmonic cocycles on them. In particular, we construct a basis of the vector space of all Γ-equivariant harmonic cocycles with values in a vector space over a field of prime characteristic and determine transformation matrices of Hecke operators with regards to this choice of basis. Finally, we examine the corresponding eigenvalues.

Algorithmische Aspekte der Hecke-Eigensysteme auf Räumen harmonischer Kozykel

Modulformen über Funktionenkörper, auch Drinfeld'sche Modulformen genannt, können kombinatorisch beschrieben werden. Die Umsetzung erfolgt über harmonische Kozykel auf dem Bruhat-Tits Baum $\mathcal{T}$, die zusätzlich bezüglich der Operation einer Kongruenzuntergruppe Γ von $\mathrm{GL}_2(\mathbb{F}_q[T])$ eine Äquivarianz-Eigenschaft erfüllen. Dann ist es möglich, Eigensysteme für Hecke Operatoren zu berechnen. Dafür untersuchen wir ausgiebig den Bruhat-Tits Baum $\mathcal{T}$, den dazugehörenden Quotientengraphen $\Gamma\backslash\mathcal{T}$ und die Werte der harmonischen Kozykel auf diesen. Hierzu konstruieren wir eine Basis des Vektorraums aller Γ-äquivarianten harmonischen Kozykel mit Werten in einem Vektorraum über einem Körper mit Primzahl-Charakteristik und bestimmen bezüglich dieser Basiswahl die Darstellungsmatrizen der Hecke Operatoren. Abschließend untersuchen wir die zugehörigen Eigenwerte.

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1 Introduction

The theory of Hecke operators is central to modern number theory. While Hecke operators occur in a number of contexts, one usually describes them as operators acting on modular forms. This Master thesis is relying on a different perspective introduced in "The Poisson Kernel for Drinfeld Modular Curves" [Tei91] by J. Teitelbaum and concerns modular forms over function fields.

In his work, Teitelbaum interprets cusp forms for a congruence subgroup Γ of $\mathrm{GL}_2(\mathbb{F}_q[T])$ as harmonic cocycles which are defined on the directed edges of the Bruhat-Tits tree $\mathcal{T}$ and are Γ-equivariant under some $\mathrm{GL}_2(\mathbb{F}_q[T])$-action. He describes how to find a basis of the vector space $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ of all Γ-equivariant harmonic cocycles with values in a finite-dimensional vector space X over a field of prime characteristic.

Later on, building on the work of Teitelbaum, R. Butenuth introduces in [But07] Hecke operators on $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$. There, he revisits Serre's description of the Bruhat-Tits tree $\mathcal{T}$ in "Trees" [Ser80] and explicitly studies the quotient graphs $\Gamma\backslash\mathcal{T}$, where he chooses one of the following three congruence subgroups for a normalized polynomial N of $R := \mathbb{F}_q[T]$:

- $\Gamma(N) := \left\{ \gamma \in \mathrm{GL}_2(R) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$
- $\Gamma_1(N) := \left\{ \gamma \in \mathrm{GL}_2(R) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$
- $\Gamma_0(N) := \left\{ \gamma \in \mathrm{GL}_2(R) \,\middle|\, \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \pmod{N} \right\}$

Starting from there, he creates methods for implementing the quotient graphs in the computer algebra system `Magma` (see [BCP97, CBFe13]) and expands Teitelbaum's theory of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ specifically for the three congruence subgroups mentioned above.

The main goal of this thesis is to build on Butenuth's work and write programs for the implementation of the quotient graphs $\Gamma\backslash\mathcal{T}$, the evaluation of Γ-equivariant harmonic cocycles in $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ with values in a vector space X over a field of prime characteristic, and the calculation of transformation matrices for Hecke operators T_P with $\gcd(N, P) = 1$ and their eigenvalues in order to get a better understanding of the Hecke algebra generated by all Hecke operators.

Let k be a finite field of prime characteristic p with q elements, R the polynomial ring in one variable T with coefficients in k, and K the field of fractions of R. Furthermore, let v_∞ be the discrete valuation of K at infinity, K_∞ the completion of K with respect to v_∞, and $\mathcal{O}_\infty$ the valuation ring with uniformizer π_∞.

In Chapter 2, we recall some facts from graph theory and introduce the Bruhat-Tits tree $\mathcal{T}$, which has homothety classes of $\mathcal{O}_\infty$-lattices as vertices. Then, we describe a representation of its vertices and edges through matrices in $\mathrm{GL}_2(K_\infty)$. This representation allows an operation of $\mathrm{GL}_2(K_\infty)$ on the set of vertices and edges through regular matrix multiplication from left. Subsequently, we study the quotient graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$ and realize the quotient graph $\Gamma\backslash\mathcal{T}$ as a covering, where Γ is either one of the three congruence subgroups above or the congruence subgroup

$$\Gamma_0^1(N) := \left\{ \gamma \in \mathrm{GL}_2(R) \,\middle|\, \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \pmod{N},\ \det(\gamma) = 1 \right\}.$$

Finally, we explain how to implement the vertices and edges of the quotient graphs in `Magma`.

In Chapter 3, we introduce harmonic cocycles as functions on the directed edges of the Bruhat-Tits tree $\mathcal{T}$ and further study the Γ-equivariance property, which contributes to the fact that we only have to set the values of Γ-equivariant harmonic cocycles on the quotient graph $\Gamma\backslash\mathcal{T}$. The set of required edges will also be narrowed down by the introduction of so-called stable edges, which are defined by having trivial stabilizers in $\mathrm{GL}_2(R)$, allowing to explicitly construct a basis of the vector space $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$. Unfortunately, this description will only be useful if Γ is either $\Gamma(N)$ or $\Gamma_1(N)$. In the case of the congruence subgroups $\Gamma_0^1(N)$, $\Gamma_0(N)$, and $\mathrm{SL}_2(R)$, there are no stable edges and, additionally, the groups are not p'-torsion-free for primes $p' \neq p$. However, we will describe a way out and realize the vector spaces $\mathrm{C}_{\mathrm{har}}(\Gamma_0^1(N), X)$ and $\mathrm{C}_{\mathrm{har}}(\Gamma_0(N), X)$ as subspaces of $\mathrm{C}_{\mathrm{har}}(\Gamma_1(N), X)$ and $\mathrm{C}_{\mathrm{har}}(\mathrm{SL}_2(R), X)$ as a subspace of $\mathrm{C}_{\mathrm{har}}(\Gamma(N), X)$. The implementation in `Magma` for all cases is described at the end of the chapter.

The final Chapter 4 focuses on Hecke operators, which are defined as the composition of three separate maps. Their definition will allow an immediate implementation, after having realized harmonic cocycles in Chapter 3. Finally, we will explicitly calculate Hecke operators T_P on the vector space of Γ-equivariant harmonic cocycles with values in the r-th symmetric power of K^2, denoted by $\mathrm{Sym}^r(K^2)$, or its irreducible subrepresentations and examine their eigenvalues for polynomials P with increasing degree.

The implementation of all calculations has been carried out in `Magma`. The code is available upon request. For the visualization of all quotient graphs, I have been using the open source graph visualization software `Graphviz` (see [EGK+03]).

Acknowledgements

My adviser Prof. Dr. Gebhard Böckle introduced me to this topic and supported me whenever I needed further guidance, and I am grateful for his support. I also thank Dr. David Guiraud and Peter Gräf. Both were very welcoming and assisted me in countless, frequent meetings. Furthermore, I express my gratitude to Dr. Andreas Maurischat and the institute for providing access to the servers of the university and a license of the `Magma` software. Finally, I am grateful for the constant support of my family and friends.

2 Quotients of the Bruhat-Tits Tree

Throughout the whole thesis, let k be a finite field of prime characteristic p with q elements, R the polynomial ring in one variable T with coefficients in k, and K the field of fractions of R. Furthermore, let v_∞ be the discrete valuation of K at infinity, K_∞ the completion of K with respect to v_∞, and $\mathcal{O}_\infty$ the valuation ring with uniformizer π_∞.

The aim of this chapter is the computation of the quotient graph $\Gamma\backslash\mathcal{T}$ where Γ is a congruence subgroup of $\mathrm{GL}_2(R)$ and $\mathcal{T}$ the Bruhat-Tits tree. Therefore, we first study the quotient graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$ and show that for a subgroup Γ of $\mathrm{GL}_2(R)$ of finite index the quotient graph $\Gamma\backslash\mathcal{T}$ is a covering of $\mathrm{GL}_2(R)\backslash\mathcal{T}$, which only depends on a system of representatives of $\Gamma\backslash\mathrm{GL}_2(R)$. Since congruence subgroups are such subgroups, we eventually determine a system of representatives of $\Gamma\backslash\mathrm{GL}_2(R)$ to calculate the quotient graph $\Gamma\backslash\mathcal{T}$. We mainly follow [Ser80] and [But07].

2.1 Introduction to Graph Theory

Definition 2.1. A *graph* G consists of two sets V, E and two maps

$$E \longrightarrow V \times V, \quad e \longmapsto (o(e), t(e))$$

and

$$\bar{\cdot} : E \longrightarrow E, \quad e \longmapsto \bar{e}$$

which satisfy $\bar{\bar{e}} = e$, $\bar{e} \neq e$, and $o(e) = t(\bar{e})$ for every $e \in E$. In this context, the following notation is common:

- An element $v \in V$ is called a *vertex* of the graph G.
- An element $e \in E$ is called an *edge* of the graph G.
- Let $e \in E$ be an edge. Then, one refers to $o(e)$ as the *origin* and to $t(e)$ as the *target* of e.
- Let $v, w \in V$ be two vertices. If there is an edge $e \in E$ with $o(e) = v$ and $t(e) = w$, one says that v and w are *adjacent*.

An *orientation* of the graph G is a subset $O \subseteq E$ such that E is the disjoint union of O and $\overline{O} := \{\bar{e} \mid e \in O\}$.

Definition 2.2. Let G be a graph and v, w two vertices. A *path* of length n from v to w is an n-tuple $(e_1, \ldots, e_n)$ with edges e_i which satisfy $o(e_1) = v, t(e_i) = o(e_{i+1})$ for $i \in \{1, \ldots, n-1\}$, and $t(e_n) = w$. Furthermore, one declares:

- Let $(e_1, \ldots, e_n)$ be a path. A pair (e_i, e_{i+1}) with $e_i = \overline{e_{i+1}}$ is called *backtracking*.
- A path from v to itself without backtracking is called a *circuit*.
- If there exists a path from v to w for any two vertices $v, w \in V$, the graph G is called *connected*.

Definition 2.3. A *tree* $\mathcal{T}$ is a connected, non-empty graph without circuits.

Proposition 2.4. Let $\mathcal{T}$ be a tree and v, w two vertices. Then, there is exactly one path from v to w without backtracking.

Proof. There always exists a path from v to w, because a tree is a connected graph. If a path contains backtracking, one can omit the respective edges to get a path without backtracking. Uniqueness follows from the fact that two distinct paths without backtracking would give rise to a circuit of v or w. $\square$

Definition 2.5. A connected graph G is called *n-regular* if for every vertex v there are exactly n edges $e_1, \ldots, e_n$ with $o(e_i) = v$.

2.2 The Bruhat-Tits Tree

In this section, we introduce the central object we are interested in, namely the Bruhat-Tits tree. For this purpose, we first define the following equivalence relation on the set of $\mathcal{O}_\infty$-lattices in K_∞^2: Two $\mathcal{O}_\infty$-lattices L_1, L_2 are equivalent if and only if there exists an $x \in K_\infty^\times$ such that $L_2 = xL_1$.

Lemma 2.6. Let L be an $\mathcal{O}_\infty$-lattice in K_∞^2. Then, there exists a unique representative L' in every equivalence class such that $L' \subseteq L$ and $L' \nsubseteq \pi_\infty L$.

Proof. Let Λ' be an equivalence class and L' an arbitrary representative. According to the invariant factor theorem, there exists a basis $\{e_1, e_2\}$ of K_∞^2 and $a, b \in \mathbb{Z}$ such that $L = \langle e_1, e_2 \rangle_{\mathcal{O}_\infty}$ and $L' = \langle \pi_\infty^a e_1, \pi_\infty^b e_2 \rangle_{\mathcal{O}_\infty}$. The lattice L' is a subset of L if and only if $a, b \geq 0$. Therefore, consider

$$xL' = \langle \pi_\infty^{a+v_\infty(x)} e_1,\, \pi_\infty^{b+v_\infty(x)} e_2 \rangle_{\mathcal{O}_\infty}$$

for an $x \in K_\infty^\times$. By taking $x = \pi_\infty^{-\min(a,b)}$ and replacing L' by xL', one gets a representative L' of Λ' satisfying $L' \subseteq L$ and $L' \nsubseteq \pi_\infty L$. $\square$

Let L be an $\mathcal{O}_\infty$-lattice in K_∞^2 and L' a representative of an equivalence class Λ' constructed like in the proof above. Then, we have $L/L' \simeq \mathcal{O}_\infty/\pi_\infty^n\mathcal{O}_\infty$ where $n := |a-b|$. Note that n is neither dependent on the choice of the representative nor of the basis.

Definition 2.7. Let V be the set of all equivalence classes of $\mathcal{O}_\infty$-lattices in K_∞^2. The map

$$d : V \times V \longrightarrow \mathbb{N}_0, (\Lambda, \Lambda') \longmapsto |a-b|$$

is called the *distance* between Λ and Λ'.

We are now ready to define the Bruhat-Tits tree $\mathcal{T}$. Let V be the set of vertices consisting of all equivalence classes of $\mathcal{O}_\infty$-lattices in K_∞^2 and

$$E := \{(\Lambda_1, \Lambda_2) \in V \times V \mid d(\Lambda_1, \Lambda_2) = 1\}$$

the set of edges. Note that for two adjacent vertices Λ, Λ' there exist representatives L, L' with

$$\pi_\infty L = \langle \pi_\infty e_1, \pi_\infty e_2 \rangle_{\mathcal{O}_\infty} \subsetneq L' = \langle e_1, \pi_\infty e_2 \rangle_{\mathcal{O}_\infty} \subsetneq L = \langle e_1, e_2 \rangle_{\mathcal{O}_\infty} ,$$

which is equivalent to

$$\{0\} \subsetneq L/L' \subsetneq L/\pi_\infty L \simeq k^2 .$$

As a result, we obtain $l(L/L') = 1$.

To proof that this graph defines a tree, which also turns out to be $q+1$-regular, we need one final lemma.

Definition 2.8. Let M be a module. A *composition series* of M is a series of submodules

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

where M_i is a maximal proper submodule of M_{i+1} for each i.

Lemma 2.9. If a module $M \neq \{0\}$ is Artinian and Noetherian, it has a composition series.

Proof. See [Jac95, Theorem 3.5]. □

Theorem 2.10. The graph $\mathcal{T}$ is a $q+1$-regular tree and is called the *Bruhat-Tits tree.*

Proof. Let Λ, Λ' be two vertices and L, L' representatives such that $L' \subset L$. According to Lemma 2.9, there exists a composition series of L/L', which gives rise to a sequence

$$L' = L_n \subset L_{n-1} \subset \cdots \subset L_0 = L$$

with lengths $l(L_{i-1}/L_i) = 1$ for $i \in \{1, 2, \dots, n\}$. This determines a path from Λ to Λ' proving the graph is connected.

Let $\Lambda_1, \Lambda_2, \dots, \Lambda_n$ be a path without backtracking. Since Λ_k and Λ_{k+1} are adjacent, there exist representatives L_k and L_{k+1}, respectively, with $L_{k+1} \subseteq L_k$ and $l(L_k/L_{k+1}) = 1$. In particular, we have $l(L_0/L_n) = n$. In order to prove that the given path is not a circuit, it is sufficient to show $L_n \nsubseteq \pi_\infty L_0$. In this case, the equivalence class $\Lambda_n = [L_n]$ has to differ from $\Lambda_0 = [L_0]$. Otherwise, it would follow from Lemma 2.6 that $L_n = L_0$ which would contradict $l(L_0/L_n) = n$.

By construction, we get $L_1 \nsubseteq \pi_\infty L_0$ and, since the path is assumed to be without backtracking, also $\pi_\infty L_{n-2} \neq L_0$. Consequently, we have $L_{n-1} = L_n + \pi_\infty L_{n-2}$ or $L_{n-1} \equiv L_n \pmod{\pi_\infty L_0}$, which is non-trivial by induction hypothesis. As a result, the path is not a circuit and the graph $\mathcal{T}$ thereby a tree.

Now, let Λ be a vertex and Λ' an arbitrary neighbor. As in previous discussions, there are representatives L, L' such that $L/\pi_\infty L \simeq k^2$ is a free k-module of rank 2 with a submodule $L'/\pi_\infty L$ of rank 1. Thus, there is a one-to-one correspondence between the edges e with $o(e) = \Lambda$ and submodules of k^2 of rank 1, of which there are $\#\mathbb{P}^1(k) = q + 1$ many. $\square$

Remark 2.11. The operation of $\mathrm{GL}_2(K_\infty)$ on the set of vertices V and edges E of the Bruhat-Tits tree $\mathcal{T}$ is declared via regular matrix multiplication from left.

2.3 The Quotient Graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$

Now that we have introduced the Bruhat-Tits $\mathcal{T}$ tree, we are able to describe its quotient by the general linear group of $R = \mathbb{F}_q[T]$. To achieve this, though, we have to understand how $\mathrm{GL}_2(R)$ acts on the tree.

Proposition 2.12. There exists a bijection

$$V \simeq \mathrm{GL}_2(K_\infty)/K_\infty^\times \,\mathrm{GL}_2(\mathcal{O}_\infty)$$

where V is the set of vertices of $\mathcal{T}$.

Proof. Let Λ be a vertex, $L = \langle x_1, x_2 \rangle \in \Lambda$ a representative, and $\{e_1, e_2\}$ the standard basis of K_∞^2. Then, there exists a $\lambda \in \mathrm{GL}_2(K_\infty)$ such that $(x_1, x_2) = \lambda(e_1, e_2)$. Since $L = \langle x_1, x_2 \rangle_{\mathcal{O}_\infty}$ is an $\mathcal{O}_\infty$-lattice, multiplication with an element from $\mathrm{GL}_2(\mathcal{O}_\infty)$ does not change it, and the same is also true for its equivalence class with respect to multiplication with elements from $K_\infty^\times$. Thus, the map

$$V \longrightarrow \mathrm{GL}_2(K_\infty)/K_\infty^\times \,\mathrm{GL}_2(\mathcal{O}_\infty), \quad \Lambda \longmapsto \lambda$$

is a bijection. $\square$

Proposition 2.13. There exists a bijection

$$E \simeq \mathrm{GL}_2(K_\infty)/K_\infty^\times I$$

where E is the set of edges of $\mathcal{T}$ and

$$I := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) \,\middle|\, v_\infty(c) > 0 \right\} .$$

Proof. Let $e = (\Lambda_1, \Lambda_2)$ be an edge. Since Λ_1 and Λ_2 are adjacent, there exists a $\lambda \in \mathrm{GL}_2(K_\infty)$ such that

$$\begin{aligned} \Lambda_1 &= [L_1] = [\lambda(\mathcal{O}_\infty \oplus \mathcal{O}_\infty)] \, , \\ \Lambda_2 &= [L_2] = [\lambda(\mathcal{O}_\infty \oplus \pi_\infty \mathcal{O}_\infty)] \, . \end{aligned}$$

A quick calculation shows that multiplication with an element from $I \cap \mathrm{GL}_2(\mathcal{O}_\infty) = I$ does not change the lattices, and their classes are invariant under multiplication with elements from $K_\infty^\times$. Thus, the map

$$E \longrightarrow \mathrm{GL}_2(K_\infty)/K_\infty^\times I, \quad (\Lambda_1, \Lambda_2) \longmapsto \lambda$$

is a bijection. $\square$

Before we calculate the quotient graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$, we further improve our description of V through matrices. The proof of the following lemma is not necessarily insightful and can also be found in [But07, Lemma 1.18]. Since it will be part of the implementation in `Magma`, we depict it in a form useful for our purpose.

Lemma 2.14. The map

$$\left\{ \begin{pmatrix} \pi_\infty^n & y \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z}, \, y \bmod \pi_\infty^n \right\} \longrightarrow V, \quad A \longmapsto [A(\mathcal{O}_\infty \oplus \mathcal{O}_\infty)]$$

is bijective.

Proof. Let Λ be a vertex and

$$[\lambda] = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] \in \mathrm{GL}_2(K_\infty)/K_\infty^\times \, \mathrm{GL}_2(\mathcal{O}_\infty)$$

its equivalence class with representative λ according to Proposition 2.12. If $v_\infty(c) < v_\infty(d)$, exchange the columns, which corresponds to a multiplication from right with

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in K_\infty^\times \, \mathrm{GL}_2(\mathcal{O}_\infty) \, ,$$

and one still remains in the equivalence class of λ. That is why we can assume $v_\infty(c) \geq v_\infty(d)$ and get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{c}{d} & 1 \end{pmatrix} = \begin{pmatrix} a - \frac{bc}{d} & b \\ 0 & d \end{pmatrix} \xrightarrow{\cdot d^{-1}} \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} .$$

Since $x \in K_\infty^\times$, there exists an $\varepsilon \in \mathcal{O}_\infty^\times$ and $n \in \mathbb{Z}$ such that $x = \varepsilon \pi_\infty^n$. Thus, we finally receive a representative

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_\infty^n & y \\ 0 & 1 \end{pmatrix} .$$

It is still to be clarified under which condition this choice is unique. If we assume there are two in the equivalence class of λ, we have

$$\begin{pmatrix} \pi_\infty^n & y_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_\infty^m & y_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \pi_\infty^m \alpha + y_2\gamma & \pi_\infty^m \beta + y_2 \delta \\ \gamma & \delta \end{pmatrix} .$$

So, we get $\gamma = 0$ and $\delta = 1$, from which it follows $\alpha = 1$ and $m = n$. Thus, the entry y is unique up to an element from $\pi_\infty^n \mathcal{O}_\infty$. □

Definition 2.15. Let L_1, L_2 be two lattices. We define the integer

$$\chi(L_1, L_2) := l(L_1/L) - l(L_2/L)$$

where $L \subseteq L_1 \cap L_2$ is an arbitrary lattice.

Remark 2.16. The value $\chi(L_1, L_2)$ does not depend on the choice of the lattice $L \subseteq L_1 \cap L_2$.

Proposition 2.17. Let L be a lattice and $\sigma \in \mathrm{GL}_2(K_\infty)$. Then, we have

$$\chi(L, \sigma L) = v_\infty(\det(\sigma)) .$$

Proof. The invariant factor theorem guarantees the existence of a basis $\{e_1, e_2\}$ of L and integers a, b such that $\{\pi_\infty^a e_1, \pi_\infty^b e_2\}$ is a basis of σL. With regards to this basis, the matrix of σ is given by

$$\sigma = \begin{pmatrix} \pi_\infty^a & 0 \\ 0 & \pi_\infty^b \end{pmatrix} \sigma_0$$

where $\sigma_0 \in \mathrm{GL}_2(\mathcal{O}_\infty)$. Thus, the valuation of the determinant of σ is

$$v_\infty(\det(\sigma)) = v_\infty(\pi_\infty^{a+b}) + v_\infty(\det(\sigma_0)) = a + b .$$

Now, we consider the following cases:

- $a, b \geq 0$: Note that $L \cap \sigma L = \sigma L$.
- $a, b < 0$: Note that $L \cap \sigma L = L$.
- $a \geq 0, b < 0$: Note that $L \cap \sigma L = \langle \pi_\infty^a e_1, e_2 \rangle$.
- $a < 0, b \geq 0$: Note that $L \cap \sigma L = \langle e_1, \pi_\infty^b e_2 \rangle$.

Hence, $\chi(L, \sigma L) = a + b$ in all cases. □

Lemma 2.18. Let v be a vertex and $\sigma \in \mathrm{GL}_2(K_\infty)$. Then, we have

$$d(v, \sigma v) \equiv v_\infty(\det(\sigma)) \pmod 2 .$$

Proof. According to the proof of Proposition 2.17, we have

$$v_\infty(\det(\sigma)) = a + b .$$

With $d(v, \sigma v) = |a - b| \equiv a + b \pmod 2$, we get the desired equation. □

Remark 2.19. Lemma 2.18 will contribute to the fact that the quotient "graph" $\mathrm{GL}_2(R) \backslash \mathcal{T}$ is indeed a graph.

Lemma 2.20. Let $L_n := \mathcal{O}_\infty \oplus \pi_\infty^n \mathcal{O}_\infty$ and $\Lambda_n := [L_n]$ for $n \in \mathbb{N}_0$. The equivalence classes Λ_n are pairwise $\mathrm{GL}_2(R)$-inequivalent.

Proof. Let us assume that Λ_m and Λ_n are $\mathrm{GL}_2(R)$-equivalent for some $m, n \in \mathbb{N}$ with $m \neq n$ and let

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(R)$$

such that $\sigma \Lambda_m = \Lambda_n$. Without limitation, let $n = m + r$ for $r \in \mathbb{N}$. Since $\sigma L_m \in \Lambda_n$, we have also $\sigma L_m = \pi_\infty^i L_n$ for some $i \in \mathbb{Z}$. By definition and Proposition 2.17, we have

$$-(r + 2i) = \chi(L_m, \pi_\infty^i L_{m+r}) = \chi(L_m, \sigma L_m) = v_\infty(\det(\sigma)) = 0 ,$$

that is $r = -2i$. Thus, $\sigma L_m = \pi_\infty^i L_{m-2i}$ gives rise to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\mathcal{O}_\infty \oplus \pi_\infty^m \mathcal{O}_\infty) = \pi_\infty^i \mathcal{O}_\infty \oplus \pi_\infty^{m-i} \mathcal{O}_\infty .$$

By choosing specific vectors, this equation translates into

$$\begin{aligned} 0 \leq \deg(a) \leq i , \quad 0 \leq \deg(b) \leq -(m - i) , \\ 0 \leq \deg(c) \leq m - i , \quad 0 \leq \deg(d) \leq -i . \end{aligned}$$

Therefore, i has to be 0, which means $m = n$, contradicting the assumption. □

The main problem of the proof of the following theorem is to find suitable matrices such that we receive one of the lattices defined in Lemma 2.20. Fortunately, this has already been achieved in [But07, Satz 1.19], but once again, we revisit the proof to have a better understanding as to how it could be implemented in Magma later on.

Theorem 2.21. The quotient graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$ is given by $\Lambda_0 \to \Lambda_1 \to \Lambda_2 \to \dots$.

Proof. According to Lemma 2.14, it suffices to consider vertices which are represented by matrices of the form

$$\begin{pmatrix} \pi_\infty^n & y \\ 0 & 1 \end{pmatrix}$$

where n is an integer and $y \in K_\infty$. It is possible to assume that $0 < v_\infty(y) < n$, because y can be written as $y = f + g$ for $f \in R$ and $g \in K_\infty$ with $0 < v_\infty(g) < n$ such that

$$\begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_\infty^n & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_\infty^n & g \\ 0 & 1 \end{pmatrix}.$$

If $n \leq 0$, we have $y = 0$ and get

$$\begin{pmatrix} \pi_\infty^n & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} \pi_\infty^n & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_\infty^{-n} & 0 \\ 0 & \pi_\infty^{-n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pi_\infty^{-n} \end{pmatrix}.$$

In the case that $n > 0$, consider

$$\begin{aligned}
\begin{pmatrix} \pi_\infty^n & y \\ 0 & 1 \end{pmatrix} &\sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \pi_\infty^n & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \pi_\infty^n & y \end{pmatrix} \\
&\sim \begin{pmatrix} 0 & 1 \\ \pi_\infty^n & y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\pi_\infty^n y^{-1} & 1 \end{pmatrix} = \begin{pmatrix} -\pi_\infty^n y^{-1} & 1 \\ 0 & y \end{pmatrix} \\
&\sim \begin{pmatrix} -\pi_\infty^n y^{-1} & 1 \\ 0 & y \end{pmatrix} \begin{pmatrix} y^{-1} & 0 \\ 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} -\pi_\infty^n y^{-2} & y^{-1} \\ 0 & 1 \end{pmatrix} \\
&\sim \begin{pmatrix} \pi_\infty^{n-2v_\infty(y)} & y^{-1} \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

If $n - 2v_\infty(y) \leq 0$, then it is the same situation as in the case $n \leq 0$. Otherwise, replace n by $n - 2v_\infty(y)$ and repeat the calculation for $n > 0$. Note that y has also been replaced, namely by y^{-1}, which requires to be fragmented into its polynomial and non-polynomial parts, similar to y at the beginning.

In any case, one receives a representative of the equivalence class Λ_n. According to Lemma 2.20, all Λ_n are pairwise $\mathrm{GL}_2(R)$-inequivalent, and Λ_n and Λ_{n+1} are adjacent, because $d(\Lambda_n, \Lambda_{n+1}) = 1$. We conclude that the quotient graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$ is represented by $\Lambda_0 \to \Lambda_1 \to \Lambda_2 \to \dots$. □

Definition 2.22. Let $\pi\colon \mathcal{T} \to \mathrm{GL}_2(R)\backslash\mathcal{T}$ be the projection map. If $v \in V$ and $\pi(v) = \Lambda_i$, we call i the *stage* of v.

2.4 The Quotient Graph $\Gamma\backslash\mathcal{T}$

Having calculated the quotient graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$, we are now ready to set up the quotient graph $\Gamma\backslash\mathcal{T}$ for an arbitrary subgroup of $\mathrm{GL}_2(R)$ of finite index. For this purpose, we only need one final lemma, which describes the stabilizer subgroups of the standard lattices in $\mathrm{GL}_2(R)$.

Lemma 2.23. The stabilizer subgroup of the equivalence class Λ_n in $\mathrm{GL}_2(R)$ is equal to

$$G_n := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(R) \,\middle|\, a, d \in k^\times,\, b \in R,\, \deg(b) \leq n \right\}$$

for $n \geq 1$ and $G_0 := \mathrm{GL}_2(k)$ for $n = 0$.

Proof. First, consider the case $n = 0$. A matrix σ in $\mathrm{GL}_2(R)$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Lambda_0 = \Lambda_0$$

satisfies the following condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This means $a, d \in k$ and, since $\det(\sigma) \in k^\times$, also $b, c \in k$. Therefore, we get $\sigma \in G_0$. The other inclusion follows from the fact that equivalence classes are determined up to $x \in K_\infty^\times$. For $n \geq 1$, a matrix σ in $\mathrm{GL}_2(R)$ with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \Lambda_n = \Lambda_n$$

has to satisfy the following condition:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi_\infty^n \end{pmatrix} = \begin{pmatrix} a & b\pi_\infty^n \\ c & d\pi_\infty^n \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & \pi_\infty^n \end{pmatrix}.$$

From the first row, it follows that $\deg(a) = 0, \deg(b) \leq n$, and the second row sets $\deg(c) \leq -n, \deg(d) = 0$. Thus, one gets $\sigma \in G_n$. The other inclusion is obvious once one takes c to be 0 above. $\square$

Theorem 2.24. Let Γ be a subgroup of $\mathrm{GL}_2(R)$ of finite index and $\{s_1, \ldots, s_m\}$ be a system of representatives of $\Gamma\backslash\mathrm{GL}_2(R)$. Consider the disjoint union

$$U := \bigsqcup_{i=1}^{m} s_i(\mathrm{GL}_2(R)\backslash\mathcal{T})$$

and on it the relation defined through the following rules:

- Only vertices and edges of the same stage are to be identified.

- Two vertices $s_i(\Lambda_n), s_j(\Lambda_n)$ are to be identified if there exists a $g \in G_n$ such that $s_i g s_j^{-1} \in \Gamma$.

- Two edges $s_i((\Lambda_0, \Lambda_1)), s_j((\Lambda_0, \Lambda_1))$ are to be identified if there exists a $g \in G_0 \cap G_1$ such that $s_i g s_j^{-1} \in \Gamma$.

- Two edges $s_i((\Lambda_n, \Lambda_{n+1})), s_j((\Lambda_n, \Lambda_{n+1}))$ for $n \geq 1$ are to be identified if there exists a $g \in G_n$ such that $s_i g s_j^{-1} \in \Gamma$.

Then, we have $\Gamma \backslash \mathcal{T} = U/\sim$.

Proof. Since U contains at least one representative of every Γ-orbit of the set of vertices or edges of $\mathcal{T}$, we have $\Gamma\backslash\mathcal{T} \subseteq U$. Now, if any two vertices $s_i(\Lambda_k), s_j(\Lambda_l)$ were equivalent for $k \neq l$, that is of different stages, we would have $\Lambda_l = s_j^{-1}\gamma s_i \Lambda_k$, which contradicts Lemma 2.20. Thus, only vertices and edges of the same stage can possibly be identified. For instance, two vertices $s_i(\Lambda_n), s_j(\Lambda_n)$ are equivalent if and only if there exists a $\gamma \in \Gamma$ such that $\gamma s_j(\Lambda_n) = s_i(\Lambda_n)$ which, according to Lemma 2.23, translates into $s_i^{-1}\gamma s_j \in G_n$. The third and forth rule are proven similarly. $\square$

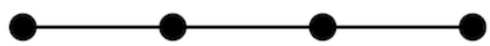

Figure 2.1: The quotient graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$

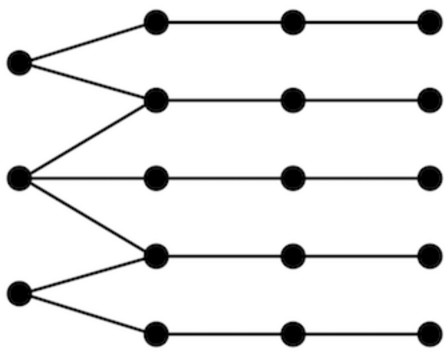

Figure 2.2: The quotient graph $\Gamma\backslash\mathcal{T}$ for $k = \mathbb{F}_2, \Gamma = \Gamma_1(T^2)$

Corollary 2.25. If $\Gamma = \mathrm{SL}_2(R)$, we have $\Gamma\backslash\mathcal{T} = \mathrm{GL}_2(R)\backslash\mathcal{T}$.

Proof. As the kernel of the determinant homomorphism, $\mathrm{SL}_2(R)$ is normal in $\mathrm{GL}_2(R)$, and we have

$$\mathrm{GL}_2(R)/\,\mathrm{SL}_2(R) \simeq k^\times\,.$$

By taking the system of representatives given by

$$\mathrm{GL}_2(R)/\,\mathrm{SL}_2(R) = \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix}\,,$$

we see that every representative is an element of the stabilizer group G_n of the vertex Λ_n for all $n \geq 0$. □

2.5 A System of Representatives of $\Gamma\backslash\,\mathrm{GL}_2(R)$

In this section, we introduce the congruence subgroups of $\mathrm{GL}_2(R)$ and notice that they are subgroups of finite indices. Subsequently, we determine a system of representatives of $\Gamma\backslash\,\mathrm{GL}_2(R)$ for four specific congruence subgroups.

Definition 2.26. Let N be a normalized polynomial of R. Define

$$\Gamma(N) \coloneqq \left\{ \gamma \in \mathrm{GL}_2(R) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}\,.$$

A *congruence subgroup* Γ is a subgroup of $\mathrm{GL}_2(R)$ with $\Gamma(N) \subseteq \Gamma$. The following three congruence subgroups will be of special interest:

- $\Gamma_1(N) \coloneqq \left\{ \gamma \in \mathrm{GL}_2(R) \,\middle|\, \gamma \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}$
- $\Gamma_0^1(N) \coloneqq \left\{ \gamma \in \mathrm{GL}_2(R) \,\middle|\, \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \pmod{N},\ \det(\gamma) = 1 \right\}$
- $\Gamma_0(N) \coloneqq \left\{ \gamma \in \mathrm{GL}_2(R) \,\middle|\, \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \pmod{N} \right\}$

Proposition 2.27. Let N be a normalized polynomial of R. Then, we have

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0^1(N) \subseteq \Gamma_0(N)\,.$$

Proof. The first and third inclusions are immediately clear. Hence, it remains to be seen that $\Gamma_1(N) \subset \Gamma_0^1(N)$. Let γ be in $\Gamma_1(N)$. Then, we have

$$\det(\gamma) = 1 \cdot 1 - b \cdot 0 \equiv 1 \pmod{N}\,.$$

So, there exists a polynomial x such that $\det(\gamma) = 1 + xN$, which has to be in $k^\times$, because γ is out of $\Gamma_1(N)$. In conclusion, either both x and N have to be constant or $x = 0$. Thus, $\det(\gamma) = 1$ and $\gamma \in \Gamma_0^1(N)$. □

2.5.1 A System of Representatives of $\mathrm{GL}_2(R)/\Gamma(N)$

The following lemma will be useful, when we calculate the quotient $\mathrm{GL}_2(R)/\Gamma(N)$, and its proof also provides a method to lift a matrix in $\mathrm{SL}_2(R/N)$ to a matrix in $\mathrm{SL}_2(R)$, as discussed in [But07, Satz 1.33]. We will need it, when we implement a system of representatives of $\mathrm{GL}_2(R)/\Gamma(N)$ in `Magma`.

Lemma 2.28. Let N be a normalized polynomial of R. Then, the map

$$\rho : \mathrm{SL}_2(R) \longrightarrow \mathrm{SL}_2(R/N), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a \bmod N & b \bmod N \\ c \bmod N & d \bmod N \end{pmatrix}.$$

is surjective.

Proof. Let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R/N).$$

Since $ad - bc \equiv 1 \pmod N$, we have $\gcd(c, d, N) = 1$. To find $s, t \in R$ such that $\gcd(c + sN, d + tN) = 1$, consider the following two cases: If $c \neq 0$, let $s = 0$ and choose $t \in R$ such that

$$\begin{aligned} t &\equiv 1 \pmod p, \; p \mid \gcd(c, d), \\ t &\equiv 0 \pmod p, \; p \nmid \gcd(c, d), \; p \mid c. \end{aligned}$$

The existence of such a t is guaranteed by the Chinese remainder theorem. If $c = 0$, we have $d \neq 0$ because of $\gcd(c, d, N) = 1$. So, one repeats the steps above with the roles of c and d interchanged.

Next, let r be a polynomial such that $ad - bc = 1 + rN$ and $f, g \in R$ with $1 = f(c+sN) + g(d+tN)$. Then, for $k \coloneqq -(r + at - bs)g$ and $l \coloneqq (r + at - bs)f$ a simple calculation shows

$$(a + kN)(d + tN) - (b + lN)(c + sN) = 1,$$

which gives rise to a lift of γ in $\mathrm{SL}_2(R)$. $\square$

Theorem 2.29. Let N be a normalized polynomial of R. Then,

$$\mathrm{GL}_2(R)/\Gamma(N) \simeq \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(R/N).$$

Proof. Consider the projection map

$$\rho : \mathrm{GL}_2(R) \longrightarrow \mathrm{GL}_2(R/N).$$

According to Lemma 2.28, it is a surjective homomorphism with kernel $\Gamma(N)$. Thus, we have an isomorphism

$$\mathrm{GL}_2(R)/\Gamma(N) \simeq \mathrm{GL}_2(R/N)\,.$$

Furthermore, since

$$\mathrm{GL}_2(R/N) = \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(R/N)\,,$$

the theorem is proven. □

Corollary 2.30. Let Γ be a congruence subgroup. Then, the index $[\mathrm{GL}_2(R) : \Gamma]$ is finite.

2.5.2 A System of Representatives of $\Gamma_1(N)\backslash \mathrm{GL}_2(R)$

Theorem 2.31. Let N be a normalized polynomial of R. Then, we have

$$\Gamma_1(N)\backslash \mathrm{GL}_2(R) \simeq \{(x,y) \in (R/N)^2 \mid \gcd(x,y) = 1\} \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix}\,.$$

Proof. Let $(c,d) \in (R/N)^2$ be a pair with $\gcd(c,d) = 1$. Then, there exist r, s in R/N such that $rc + sd = 1$, and we get a matrix

$$\begin{pmatrix} s & -r \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R/N)\,.$$

Since $\Gamma_1(N) \subseteq \mathrm{SL}_2(R)$, it is sufficient to consider the following action

$$\begin{aligned} \mathrm{SL}_2(R) \times \{(x,y) \in (R/N)^2 \mid \gcd(x,y) = 1\} &\longrightarrow \{(x,y) \in (R/N)^2 \mid \gcd(x,y) = 1\}, \\ \left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, (c,d)\right) &\longmapsto (\gamma s + \delta c, -\gamma r + \delta d)\,, \end{aligned}$$

which is motivated and well-defined by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} s & -r \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha s + \beta c & -\alpha r + \beta d \\ \gamma s + \delta c & -\gamma r + \delta d \end{pmatrix}\,.$$

The stabilizer of $(1,1)$ is determined by $\delta \equiv 1 \pmod N$ and $\gamma + \delta \equiv 1 \pmod N$, that is $\gamma \equiv 0 \pmod N$. Additionally, notice that $\alpha\delta - \beta\gamma \equiv 1 \pmod N$. In conclusion, we get $\alpha \equiv 1 \pmod N, \gamma \equiv 0 \pmod N$ and $\delta \equiv 1 \pmod N$. That is why the stabilizer of $(1,1)$ under the action is $\Gamma_1(N)$. □

2.5.3 A System of Representatives of $\Gamma_0^1(N)\backslash \operatorname{GL}_2(R)$

Lemma 2.32. Let N be a normalized polynomial of R. Then, $\sigma, \tau \in \operatorname{SL}_2(R)$ are in the same $\Gamma_0^1(N)$-right coset if and only if there exists an $x \in R$ with $\gcd(x, N) = 1$ such that $(\tau_{21}, \tau_{22}) \equiv x(\sigma_{21}, \sigma_{22}) \pmod N$.

Proof. In [But07, Lemma 1.22], this is proven for the case of $\Gamma_0(N)$, which is almost identical to the statement of this lemma. Namely, one only has to set $x = 1$ and $y = ab' - a'b$ in the proof there. Since it is rather technical, but neither insightful nor necessary with regards to the implementation, we will leave it at the reference. □

Theorem 2.33. Let N be a normalized polynomial of R. Then,

$$\Gamma_0^1(N)\backslash \operatorname{SL}_2(R) \simeq \mathbb{P}^1(R/N)\,.$$

Proof. By Lemma 2.32, there exists an inclusion

$$\Gamma_0^1(N)\backslash \operatorname{SL}_2(R) \hookrightarrow \mathbb{P}^1(R/N)\,,$$

so it remains to be seen that the other inclusion is true as well. For this purpose, let (x, y) be in

$$\mathbb{P}^1(R/N) = \{[(x, y)] \mid \ \gcd(x, y) = 1\}\,,$$

where the equivalence relation is defined up to a unit in R/N. Since $\gcd(x, y) = 1$, the pair (x, y) can be expanded to a matrix in $\operatorname{SL}_2(R/N)$, which then is lifted to a matrix in $\operatorname{SL}_2(R)$ with Lemma 2.28. Thus, we have a map

$$\mathbb{P}^1(R/N) \longrightarrow \Gamma_0^1(N)\backslash \operatorname{SL}_2(R)\,,$$

which is by Lemma 2.32 well-defined and injective. □

Corollary 2.34. Let N be a normalized polynomial of R. Then, we obtain

$$\Gamma_0^1(N)\backslash \operatorname{GL}_2(R) \simeq \mathbb{P}^1(R/N) \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix}.$$

Proof. Applying Theorem 2.33 to

$$\operatorname{GL}_2(R) = \operatorname{SL}_2(R) \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix} \simeq \Gamma_0^1(N)\mathbb{P}^1(R/N) \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix}.$$

yields the statement of the corollary.

□

2.5.4 A System of Representatives of $\Gamma_0(N)\backslash \mathrm{GL}_2(R)$

Lemma 2.35. Let N be a normalized polynomial of R. Then, $\sigma, \tau \in \mathrm{GL}_2(R)$ are in the same $\Gamma_0(N)$-right coset if and only if there exists an $x \in R$ with $\gcd(x, N) = 1$ such that $(\tau_{21}, \tau_{22}) \equiv x(\sigma_{21}, \sigma_{22}) \pmod N$.

Proof. See [But07, Lemma 1.22]. □

Theorem 2.36. Let N be a normalized polynomial of R. Then, we have

$$\Gamma_0(N)\backslash \mathrm{GL}_2(R) \simeq \mathbb{P}^1(R/N)\,.$$

Proof. This is proven similarly to Theorem 2.33 by using Lemma 2.35 instead of Lemma 2.32. □

2.6 Implementation in Magma

Now that we have discussed the theory, it is time to explain how to implement the quotient of the Bruhat-Tits tree by a congruence subgroup on a computer. According to Theorem 2.24, the quotient graph $\Gamma\backslash\mathcal{T}$ essentially consists of $m := [\mathrm{GL}_2(R) : \Gamma]$ copies of the quotient graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$ where several vertices and edges are identified. Since $\mathrm{GL}_2(R)\backslash\mathcal{T}$ has infinitely many vertices and edges, we need the identification process to stop at some point, in order to implement the quotient graph $\Gamma\backslash\mathcal{T}$ in `Magma`. Fortunately, the following theorem guarantees this.

Theorem 2.37. Let N be a normed polynomial in R with $n := \deg(N)$ and Γ be a congruence subgroup. If there is a $m_1 \geq n$ and $\gamma_1 \in G_{m_1}$ such that $s_i^{-1}\gamma_1 s_j \in \Gamma$ for $s_i, s_j \in \Gamma\backslash \mathrm{GL}_2(R)$ with $s_i \neq s_j$, there exists a $m_2 < n \leq m_1$ and $\gamma_2 \in G_{m_2}$ such that $s_i^{-1}\gamma_2 s_j \in \Gamma$.

Proof. Suppose there is a $m \in \mathbb{N}$ with $m \geq n$ such that $s_i(\Lambda_m), s_j(\Lambda_m)$ are to be identified. Thus, there exists

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G_m$$

with $s_i^{-1}\gamma s_j \in \Gamma$, $a, d \in k^\times$, and $b \in R$ with $\deg(b) \leq m$. Further, let $p, q \in R$ such that $b = pN + q$ and $\deg(q) < n$. Then, we obtain

$$s_i^{-1}\gamma s_j = s_i^{-1}\begin{pmatrix} a & pN+q \\ 0 & d \end{pmatrix} s_j \equiv s_i^{-1}\begin{pmatrix} a & q \\ 0 & d \end{pmatrix} s_j \pmod N\,.$$

Therefore, all identification must have been taken place until stage $\deg(q) < n$. □

2.6.1 A System of Representatives of $\Gamma\backslash \operatorname{GL}_2(R)$

Now that we know that the identification terminates after finitely many steps, we can examine which vertices and edges overlap. For this purpose, we need a system of representatives of $\Gamma\backslash \operatorname{GL}_2(R)$. The approach to the implementation of these systems requires different methods for different Γ, but we have already encountered all of them, when we studied $\Gamma\backslash \operatorname{GL}_2(R)$.

Starting with $\Gamma = \Gamma(N)$, we have

$$\operatorname{GL}_2(R)/\Gamma(N) \simeq \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix} \operatorname{SL}_2(R/N)\,,$$

according to Theorem 2.29. Thus, we only have to write a function, which can lift a matrix with entries in $\operatorname{SL}_2(R/N)$ to a matrix in $\operatorname{SL}_2(R)$. This is easily achieved following the proof of Lemma 2.28.

If $\Gamma = \Gamma_1(N)$, we proved in Theorem 2.31 that a system of representative is given by

$$\Gamma_1(N)\backslash \operatorname{GL}_2(R) \simeq \{(x,y) \in (R/N)^2 \mid \gcd(x,y) = 1\} \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix}\,.$$

Let $(x,y) \in (R/N)^2$ with $\gcd(x,y) = 1$. If the greatest common divisor is 1 after directly lifting the pair to R, we have already arrived at a matrix in $\operatorname{SL}_2(R)$. Otherwise, search for $s,t \in R$ such that $\gcd(x+sN, y+tN) = 1$, which is described in the proof of Lemma 2.28. The only inconvenience we face during this process is that, unfortunately, `Magma` does not provide a gcd-function for the quotient ring R/N. That is why we consider $\gcd(x,y) \bmod N$ in R, and check whether it results in a polynomial coinciding with a unit in R/N. This is equivalent to $\gcd(\gcd(x,y) \bmod N, N) = 1$ in R.

Finally, the implementation of

$$\Gamma_0^1(N)\backslash \operatorname{GL}_2(R) \simeq \mathbb{P}^1(R/N) \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix}\,,$$

according to Corollary 2.34, and

$$\Gamma_0(N)\backslash \operatorname{GL}_2(R) \simeq \mathbb{P}^1(R/N)\,,$$

by Theorem 2.36, work similarly.

2.6.2 The Quotient Graph $\Gamma\backslash\mathcal{T}$

For the implementation of the quotient graph it is beneficial to use the `MultiDigraph` object of `Magma`. It saves directed edges and also allows the existence of multiple edges from one vertex to an other. Moreover, Magma itself offers a wide range of functions in regard to graphs. Some important ones with their description in [CBFe13] are:

- `VertexSet(`G`)`, `EdgeSet(`G`)`
- `AssignLabel(`$\sim G, v, l$`)`: It assigns the label l to the vertex v in the graph G. We will label all vertices with a number, so we know afterwards, where they are in the graph and which representative $s_i \in \Gamma\backslash \operatorname{GL}_2(R)$ they have been assigned to.
- `IncidentEdges(`v`)`: It returns all the edges incident to and from the vertex v as a subset of `EdgeSet(`G`)`.

However, there are a few drawbacks. It is not possible to identify two edges directly, so one has to identify the corresponding vertices instead. For this purpose, there is the `Contract(`u, v`)` function, which identifies two vertices and assigns all neighbours of the vanishing vertex to the vertex with which it was identified. Unfortunately, once `Contract(`u, v`)` erases the second vertex, it also renumbers all following vertices. Hence, it becomes rather tiresome to keep track of numeration. In order to circumvent these problems, we use a different method to identify all vertices and edges. Once the process is finished, we save the data in the `MultiDigraph` object to benefit from `Magma`'s further features.

Let n be the degree of the monic polynomial N and m the cardinality of the representative system $\Gamma\backslash \operatorname{GL}_2(R)$, where Γ is one of the groups $\Gamma(N), \Gamma_0(N), \Gamma_0^1(N)$ and $\Gamma_1(N)$. We construct a quadratic matrix A with $m(n+1)$ rows and columns corresponding to all possible vertices. Keep in mind that there are $n+1$ stages, because stage 0 counts too. The entries of the matrix will be integers and the entry in the i-th row and j-th column states whether the i-th vertex is connected with the j-th vertex and how often. A negative entry means that the edges are incoming and not outgoing. Initially, we have m copies of $\operatorname{GL}_2(R)\backslash\mathcal{T}$ and the numeration of the vertices are as follows:

$$\begin{aligned}
1 &\longrightarrow 1+m \longrightarrow \cdots \longrightarrow 1+nm \\
2 &\longrightarrow 2+m \longrightarrow \cdots \longrightarrow 2+nm \\
&\vdots \\
m &\longrightarrow m+m \longrightarrow \cdots \longrightarrow m+nm
\end{aligned}$$

and the corresponding matrix will be

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

for example with $m = 3$ and $n = 2$. The identification process takes place in three steps and according to the rules of Theorem 2.24:

- Identify the edges between stage 0 and 1.

Algorithm 1 Identification of Edges of Stage 0

---

for $s_i, s_j \in \Gamma\backslash \mathrm{GL}_2(R), s_i \neq s_j, i \leq j$ **do**
  for $g \in G_0 \cap G_1$ **do**
    if $s_j g s_i^{-1} \in \Gamma$ **then**
      $A_{j+lm,k} := 0$ for $k \in \{1, \ldots, m(n+1)\}, l \in \{0, \ldots, (n-1)\}$;
    end if
  end for
end for

- Identify vertices of stage 0. During this process, we have to assign the values of the vanishing vertex to the vertex it is getting identified with. The half-line of both are otherwise unaffected.

- Identify vertices of stages ≥ 1 (and the corresponding edges). This is similar to the first step, where not only the vertex and edge are disappearing, but also the corresponding half-line.

Once the identification process is finished, remove all rows and columns which represent nonexisting vertices and transfer the data from the matrix to the object class `MultiDigraph`. At the same time, we assign all remaining vertices their number in the initial numeration above as a label. Then, the number mod m corresponds to its representative in $\Gamma\backslash \mathrm{GL}_2(R)$ and the number divided by m refers to its stage.

These assignments also indicate the representatives of all edges of stages $i \geq 1$, because they are identified if and only if their origins are identified. In the case of edges between vertices of stage 0 and 1, we have to be more careful and consider all possibilities, which are listed below:

- If two distinct edges e_i and e_j are identified, there are no complications, since one only has to omit the edge e_j represented by s_j.

- If only the origins of two edges e_i and e_j are identified, their representatives still correspond to the numeration of their targets $t(e_i)$ and $t(e_j)$. The edge e_j represented by s_j exists, but its origin is different.

- If only the targets of two edges e_i and e_j are identified, their representatives still correspond to the numeration of their origins $o(e_i)$ and $o(e_j)$. The edge e_j represented by s_j exists, but its target is different.

- If the origin as well as the target of two edges e_i and e_j are identified, we will have two edges from $o(e_i)$ to $t(e_i)$. The first one is represented by s_i and the second by s_j.

To keep track of the representatives of all edges, we store the information in `list`. Then, the assignment of the labels for vertices and edges can be implemented in `Magma` as explained in Algorithm 2.

Algorithm 2 Assignment of Vertices and Edges

```
x := 0;
for 1 ≤ i ≤ m(n + 1) do
    if not relevant[i] then
        RemoveRow(A, i − x);
        RemoveColumn(A, i − x);
        x := x + 1;
    else
        Include(~ActualNumber, i);
    end if
end for
Graph := MultiDigraph< NumberOfRows(A)| >; counter := 0;
for 1 ≤ i ≤ NumberOfRows(A) do
    AssignLabel(~Graph, v_i, Sprintf("%o", ActualNumber[i]));
    for 1 ≤ j ≤ NumberOfRows(A) do
        while A_ij > 0 do
            for 1 ≤ k ≤ m do
                if list[k] = (ActualNumber[i], ActualNumber[j]) then
                    AddEdge(~Graph, v_i, v_j, Sprintf("%o", k));
                    counter := 1; list[k] := < 0, 0 >; break k;
                end if
            end for
            if counter ≠ 1 then
                AddEdge(~Graph, v_i, v_j, Sprintf("%o", ActualNumber[i]));
            end if
            A_ij := A_ij − 1; counter := 0;
        end while
    end for
end for
```

3 Harmonic Cocycles

In this chapter, we introduce special functions called harmonic cocycles, which are defined on the edges of the Bruhat-Tits tree and will further satisfy equivariance under some action of a congruence subgroup. We will immediately notice that these functions constitute a finite-dimensional vector space if they map into a finite-dimensional vector space. Furthermore, it will suffice to specify the values of harmonic cocycles only on the stable edges since these will determine the values of the harmonic cocycles everywhere else. In the end, we determine a basis of the vector space of harmonic cocycles and discuss how to implement harmonic cocycles in `Magma`. Most of these results are from [Ser80] and [Tei91] except some technical proofs in Section 3.2.1, which rely on the work of [But07].

3.1 The Definition of Harmonic Cocycles

Definition 3.1. Let X be a vector space with an action of $\mathrm{GL}_2(R)$, Γ a congruence subgroup, and $\mathcal{T}$ the Bruhat-Tits tree. A map $c\colon E(\mathcal{T}) \longrightarrow X$ which satisfies

1. $\sum_{t(v)=e} c(e) = 0$ for every $v \in V(\mathcal{T})$,
2. $c(\overline{e}) = -c(e)$ for every $e \in E(\mathcal{T})$,

is called a *harmonic cocycle* with values in X. Furthermore, we define:

- A harmonic cocycle c is called Γ-*equivariant* if it satisfies $c(\gamma \cdot e) = \gamma \cdot c(e)$ for every $e \in E(\mathcal{T}), \gamma \in \Gamma$.
- A harmonic cocycle c is called *cuspidal* if there is a finite subgraph $S \subset \Gamma \backslash \mathcal{T}$ such that $c(e) \neq 0$ only for $e \in E(\pi^{-1}(S))$, where $\pi\colon \mathcal{T} \longrightarrow \Gamma\backslash\mathcal{T}$ is the projection.

This definition guarantees that if X is a finite-dimensional vector space over a field F with an action of $\mathrm{GL}_2(R)$ and Γ a congruence subgroup, the set $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ of all Γ-equivariant harmonic cocycles is not only a vector space over F but also finite-dimensional. In order to realize this, though, we need to prove the following property of the Bruhat-Tits tree $\mathcal{T}$.

Proposition 3.2. Let $v \in V(\mathcal{T})$ be a vertex with normal form

$$v = \begin{pmatrix} \pi_\infty^n & y \\ 0 & 1 \end{pmatrix} .$$

Then, the normal forms of all adjacent vertices are given by

$$\begin{pmatrix} \pi_\infty^{n+1} & y + x\pi_\infty^n \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} \pi_\infty^{n-1} & \overline{y} \\ 0 & 1 \end{pmatrix} ,$$

where $x \in k$ and $\overline{y}$ is the class of y mod $\pi_\infty^{n-1}\mathcal{O}_\infty$. If v is of stage 0, all adjacent vertices are of stage 1, and if v is of stage $i \geq 1$, exactly one adjacent vertex is of stage $i+1$ and the other q edges are of stage $i-1$.

Proof. The first part follows immediately from

$$\begin{pmatrix} \pi_\infty^{n+1} & y + x\pi_\infty^n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_\infty^n & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_\infty & x \\ 0 & 1 \end{pmatrix} ,$$
$$\begin{pmatrix} \pi_\infty^n & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi_\infty^{n-1} & \overline{y} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi_\infty & (y-\overline{y})\pi_\infty^{-(n-1)} \\ 0 & 1 \end{pmatrix} .$$

The assertion regarding the stages are easily proven by following the steps of the algorithm described in the proof of Theorem 2.21. □

Proposition 3.3. Let X be a finite-dimensional vector space over a field F with an action of $\mathrm{GL}_2(R)$ and let Γ be a congruence subgroup. Then, the set $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ of all Γ-equivariant harmonic cocycles is a finite-dimensional vector space over F.

Proof. It is immediately clear that $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ is a vector space. That is why we only have to prove it is of finite dimension. Therefore, let c be in $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ and $e = \sigma(\Lambda_i, \Lambda_{i+1}) \in E(\mathcal{T})$ with $\sigma \in \mathrm{GL}_2(R)$ be an edge of stage $i \geq 1$. Furthermore, let $v_1, \ldots, v_q$ be the adjacent vertices of stage $i-1$ of the vertex $\sigma\Lambda_i$. By definition, we have

$$c(e) = -c(\overline{e}) = \sum_{i=1}^{q} c(v_i, \sigma\Lambda_i) .$$

Thus, the values of c are completely determined by its values on edges of stage 0. So, let $e = \gamma s_k(\Lambda_0, \Lambda_1)$ be an edge of stage 0 such that $\gamma \in \Gamma$ and $s_k \in \Gamma \backslash \mathrm{GL}_2(R)$. Since c is Γ-equivariant, it follows

$$c(e) = \gamma \cdot c(s_k(\Lambda_0, \Lambda_1)) .$$

Consequently, c is fully declared by its values on $s_k(\Lambda_0, \Lambda_1)$ with $s_k \in \Gamma \backslash \mathrm{GL}_2(R)$. These essentially are the edges of stage 0 of the quotient graph $\Gamma\backslash\mathcal{T}$, of which there are only finitely many. Thus, if $\{x_1, \ldots, x_n\}$ is a basis of X, the set

$$\{e_{ij} \colon s_i(\Lambda_0, \Lambda_1) \longmapsto x_j \mid s_i(\Lambda_0, \Lambda_1) \in \Gamma\backslash\mathcal{T}\}$$

generates the vector space $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$. □

Corollary 3.4. Let X be a finite-dimensional vector space over a field F with an action of $\mathrm{GL}_2(R)$ and let Γ be a congruence subgroup. Then, if $m := \#\Gamma\backslash \mathrm{GL}_2(R)$ and $n := \dim(X)$, we have

$$\dim(\mathrm{C}_{\mathrm{har}}(\Gamma, X)) \leq m \cdot n\,.$$

Now, we have to further study the quotient graph $\Gamma\backslash\mathcal{T}$ and observe how many and what kind of edges of stage 0 occur. For this purpose, we introduce the notion of stable vertices and edges.

Definition 3.5. Let Γ be a congruence subgroup and $\mathcal{T}$ the Bruhat-Tits tree. A vertex $v \in V(\mathcal{T})$ or an edge $e \in E(\mathcal{T})$ is called Γ-*stable* if $\mathrm{Stab}_\Gamma(v) = \{1\}$ or $\mathrm{Stab}_\Gamma(e) = \{1\}$, respectively.

Example 3.6. If $\Gamma = \Gamma_0(N)$ and $q > 2$, there are no Γ-stable vertices or edges, because for all $v \in V(\mathcal{T})$, we have

$$\begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} \in \mathrm{Stab}_\Gamma(v)$$

for $1 \neq \varphi \in k^\times$ and similarly

$$\begin{pmatrix} \varphi & 0 \\ 0 & \varphi \end{pmatrix} \in \mathrm{Stab}_\Gamma(e) = \mathrm{Stab}_\Gamma(o(e)) \cap \mathrm{Stab}_\Gamma(t(e))$$

for all edges $e \in E(\mathcal{T})$. The same is also true for $\Gamma = \Gamma_0^1(N)$ and $q \neq 2^r$, because in this case, $\#k^\times = q - 1$ is even such that there always exists an element $\varphi \in k^\times$ of order 2.

The construction of a basis of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ will heavily rely on the existence of stable vertices and edges. That is why the following considerations only concern the congruence subgroups $\Gamma(N)$ and $\Gamma_1(N)$. In the case of $\Gamma_0^1(N)$ and $\Gamma_0(N)$, we will describe a different method once we have established a basis for the former congruence subgroups.

3.2 A Basis of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ if $\Gamma = \Gamma(N)$ or $\Gamma_1(N)$

Proposition 3.7. Let v be a Γ-stable vertex. Then, all $q + 1$ edges containing v as a vertex are also Γ-stable.

Proof. Let e be an arbitrary edge containing v. Then,

$$\mathrm{Stab}_\Gamma(e) = \mathrm{Stab}_\Gamma(o(e)) \cap \mathrm{Stab}_\Gamma(t(e)) = \{1\}$$

since either $o(e)$ or $t(e)$ is v, which is Γ-stable. $\square$

Theorem 3.8. Let Γ be a congruence subgroup and $v \in V(\mathcal{T})$ be a Γ-stable vertex. Then, there are exactly $q+1$ edges containing $\pi(v) \in V(\Gamma\backslash\mathcal{T})$ as a vertex.

Proof. Let $e_i := (v, v_i)$ be all $q+1$ edges with v as a vertex. If two of the edges e_m and e_n were equivalent in $\Gamma\backslash\mathcal{T}$, there would exist $1 \neq \gamma \in \Gamma$ such that $\gamma[e_m] = [e_n]$. If γv were equal to v_n, then

$$1 = d(v, v_n) = d(v, \gamma v) \equiv 0 \pmod{2},$$

by Lemma 2.18. Thus, it would have to be $\gamma v = v$. But this contradicts the assumption that v is Γ-stable. □

Remark 3.9. In general, the converse of this theorem is not true. For example, consider $k = \mathbb{F}_3, N = T^3 + T, \Gamma = \Gamma_0(N)$. The quotient graph $\Gamma\backslash\mathcal{T}$ has a vertex with four adjacent edges, but the Bruhat-Tits tree itself has no stable vertices, as discussed in Example 3.6.

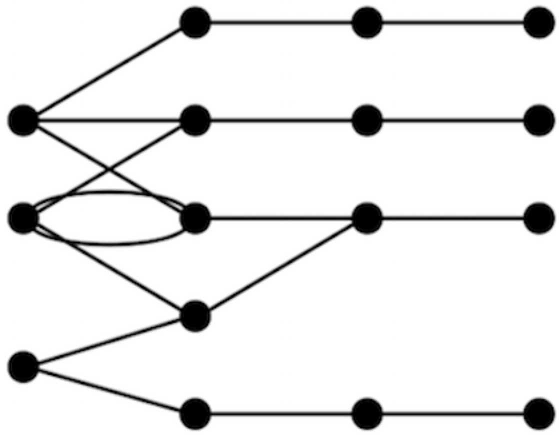

Figure 3.1: The quotient graph $\Gamma\backslash\mathcal{T}$ for $k = \mathbb{F}_3, \Gamma = \Gamma_0(T^3 + T)$

According to Theorem 3.8, if we have a stable vertex v of stage 0, all edges containing v will correspond to distinct edges in the quotient graph. Thus, there will also be $q+1$ edges in the quotient graph such that we can omit exactly one of them when choosing a basis, because the value of a Γ-equivariant harmonic cocycle on the omitted edge can be calculated by its values on the other q edges.

Therefore, it remains to be seen how many edges of stage 0 contribute to the basis of the vector space $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ if the initial vertex is unstable. This will require a lot more work than before and will be dealt with in the following section.

3.2.1 The Situation at an Unstable Vertex

Lemma 3.10. If $\deg(N) \geq 1$ and $l \neq p$ is prime, the congruence subgroups $\Gamma(N)$ and $\Gamma_1(N)$ are l-torsion-free.

Proof. Since $\Gamma(N) \subseteq \Gamma(P)$ for every prime divisor P of N, we assume that N is prime. Let $\gamma \in \Gamma(N)$ with $\gamma^l = 1$ for an arbitrary prime $l \neq p$. It suffices to show that $\gamma \equiv 1 \bmod (N)^m$ for all $m \in \mathbb{N}$, because $\bigcap_{m\in\mathbb{N}}(N)^m = \{0\}$. We prove it by induction, beginning with $m = 1$ which is by definition true. Suppose that $\gamma \equiv 1 \bmod (N)^m$ for a $m \geq 1$. This means there exists a matrix $\sigma \in \mathrm{GL}_2(R)$ such that $\gamma = 1 + N^m\sigma$. It follows that

$$1 = \gamma^l = (1 + N^m\sigma)^l = \sum_{i=0}^{l} \binom{l}{i} (N^m\sigma)^i \equiv 1 + lN^m\sigma \bmod (N)^{m+1}\,.$$

Since l and p are coprime, we get $\sigma \equiv 0 \bmod (N)$ and thus $\gamma \equiv 1 \bmod (N)^{m+1}$.

Now, let $\gamma \in \Gamma_1(N)$ with $\gamma^l = 1$ for an arbitrary prime $l \neq p$. Note that $\Gamma(N)$ is a normal subgroup of $\Gamma_1(N)$, because it is normal in $\mathrm{GL}_2(R)$ as the kernel of the projection

$$\mathrm{GL}_2(R) \longrightarrow \mathrm{GL}_2(R/N)\,.$$

Consequently, $\Gamma_1(N)/\Gamma(N) \simeq R/N$ is a p-group with $[\Gamma_1(N) : \Gamma(N)] = q^{\deg(N)}$ many elements. For the image $\overline{\gamma}$ of γ under the canonical projection we have

$$\overline{\gamma}^l = \overline{\gamma^l} = \overline{1} = 1\,.$$

Since l and p are coprime, it follows $l = 1$. $\square$

Example 3.11. If $q \neq 2$ and l is a prime divisor of $q - 1$, the congruence subgroups $\Gamma_0^1(N)$ and $\Gamma_0(N)$ are not l-torsion-free. In order to prove this, consider the matrix

$$\gamma = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in \Gamma_0^1(N) \subseteq \Gamma_0(N)$$

for $a, d \in k^\times$ and $ad = 1$. Then, $\gamma^{q-1} = 1$ and thus $\left(\gamma^{\frac{q-1}{l}}\right)^l = 1$.

Lemma 3.12. Let G be a subgroup of $\mathrm{GL}_2(R)$ which is l-torsion-free for all prime $l \neq p$. Then, $(q-1)(q^2-1)$ is a divisor of the index $[\mathrm{GL}_2(R) : G]$.

Proof. Consider the action

$$\mathrm{GL}_2(k) \times \mathrm{GL}_2(R)/G \longrightarrow \mathrm{GL}_2(R)/G\,, \quad (\sigma, gG) \longmapsto \sigma gG\,.$$

The stabilizer subgroup of an arbitrary element $gG \in \mathrm{GL}_2(R)/G$ under this action is

$$\mathrm{Stab}_{\mathrm{GL}_2(k)}(gG) = \{\sigma \in \mathrm{GL}_2(k) \mid \sigma g \in gG\} = \{\sigma \in \mathrm{GL}_2(k) \mid \sigma \in gGg^{-1}\}\,.$$

Let σ be an element of $\mathrm{Stab}_{\mathrm{GL}_2(k)}(gG)$. Then, there exists an $h \in G$ such that

$$\sigma^l = (ghg^{-1})^l = gh^l g^{-1} \neq 1$$

for all prime $l \neq p$. That is why the stabilizer subgroup $\mathrm{Stab}_{\mathrm{GL}_2(k)}(gG)$ of an arbitrary element gG is also l-torsion-free. Since $\#\,\mathrm{GL}_2(k) = q(q-1)(q^2-1)$ and $\mathrm{Stab}_{\mathrm{GL}_2(k)}(gG) \subseteq \mathrm{GL}_2(k)$, $\mathrm{Stab}_{\mathrm{GL}_2(k)}(gG)$ is even a p-group.

Now, let $g_1, \ldots, g_m$ be a system of representatives of the orbits under the action above. According to the orbit-stabilizer theorem, it follows for $q = p^r$ that

$$[\mathrm{GL}_2(R) : G] = \sum_{i=1}^{m} \frac{\#\,\mathrm{GL}_2(k)}{\#\,\mathrm{Stab}_{\mathrm{GL}_2(k)}(g_i G)} = (q-1)(q^2-1) \sum_{i=1}^{m} p^{r-r_i}\,.$$

Thus, $(q-1)(q^2-1)$ is a divisor of the index $[\mathrm{GL}_2(R) : G]$ □

Lemma 3.13. Let G be a non-trivial p-subgroup of $\mathrm{GL}_2(K_\infty^2)$. Then, there is exactly one element $x \in \mathbb{P}^1(K_\infty)$ with $G \cdot x = x$.

Proof. First, we prove the existence of an element $x \in \mathbb{P}^1(K_\infty)$ with the required property. Since G is a non-trivial p-group, there exists a matrix $\sigma \neq 1$ in the center of G of order p^m for some $m \in \mathbb{N}$. The matrix is unipotent because

$$0 = \sigma^{p^m} - 1 = (\sigma - 1)^{p^m}\,.$$

That is why it is conjugated to a matrix

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Thus, the set $V^{\langle\sigma\rangle}$ consisting of all elements in K_∞^2 fixed by σ is a one-dimensional subspace of K_∞^2. Furthermore, since σ is an element of the center of G, we have

$$\sigma(g(v)) = g(\sigma(v)) = g(v)$$

for all $g \in G$ and $v \in V^{\langle\sigma\rangle}$. This induces an action

$$G/\langle\sigma\rangle \times V^{\langle\sigma\rangle} \longrightarrow V^{\langle\sigma\rangle}\,,$$

which has to be trivial because there are no endomorphisms of K_∞ of order p. Thus, $x := V^{\langle\sigma\rangle}$ defines an element of $\mathbb{P}^1(K_\infty)$ which is invariant under G.

Now, suppose there are two distinct elements $[x], [y] \in \mathbb{P}^1(K_\infty)$ with this property. Let σ be an arbitrary element of G. With regards to the basis $\{x, y\}$, σ must be of form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

when we consider $\sigma[x] = [x]$, and of form

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix},$$

because of $\sigma[y] = [y]$. Subsequently, σ is a diagonal matrix. Since G is a p-group, there exists a $m \in \mathbb{N}$ such that

$$1 = \sigma^{p^m} = \begin{pmatrix} a^{p^m} & 0 \\ 0 & d^{p^m} \end{pmatrix}$$

and, because $0 = \deg(a^{p^m}) = p^m \deg(a)$, we have $a \in k^\times$. As a result, we get that the order of a divides $q - 1$ and p^m. This is possible if and only if the order of a is 1 and thus $a = 1$. Similarly, we get $d = 1$. This means $\sigma = 1$, which contradicts the fact that G is non-trivial. □

Definition 3.14. If two paths of infinite length in $\mathcal{T}$ differ at only finitely many edges, they are called equivalent. Their corresponding classes are *ends* of $\mathcal{T}$. The ends of the quotient graph $\Gamma\backslash\mathcal{T}$ are called *cusps*.

Corollary 3.15. Let Γ be a subgroup of $\mathrm{GL}_2(R)$ which is l-torsion-free for all prime $l \neq p$ and let v be an Γ-unstable vertex of the Bruhat-Tits $\mathcal{T}$. Then, there exists a unique end $b_v \colon v \to v_1 \to v_2 \to \dots$ with initial vertex v and $\mathrm{Stab}_\Gamma(v) \subseteq \mathrm{Stab}_\Gamma(b_v)$. The end consists exclusively of Γ-unstable vertices.

Proof. The stabilizer group $\mathrm{Stab}_\Gamma(v)$ is non-trivial, since v is unstable; finite; and a p-group, because Γ is l-torsion-free for every prime $l \neq p$. According to Lemma 3.13, there exists an $x \in \mathbb{P}^1(K_\infty)$ which is invariant under the action of $\mathrm{Stab}_\Gamma(v)$. Since there is a one-to-one correspondence between $\mathbb{P}^1(K_\infty)$ and the ends of $\mathcal{T}$, $\mathrm{Stab}_\Gamma(v)$ leaves exactly one end $b_v \colon v \to v_1 \to v_2 \to \dots$ invariant. Subsequently, we have $\mathrm{Stab}_\Gamma(v) \subseteq \mathrm{Stab}_\Gamma(v_i)$. Thus, b_v consists only of unstable edges. □

The $\Gamma(N)$-Stable Vertices and Edges

Now, we can explicitly calculate the stabilizer of vertices and edges. The case $\Gamma = \Gamma(N)$ is the simplest, because $\Gamma(N)$ is a normal subgroup of $\mathrm{GL}_2(R)$. That is why we can directly calculate the stabilizer groups of vertices.

Proposition 3.16. Let $v \in V(\mathcal{T})$ be a vertex of stage $i \leq 1$. Then, v is $\Gamma(N)$-stable if and only if $i < \deg(N)$.

Proof. Since $v \in V(\mathcal{T})$ is a vertex of stage $i \leq 1$, there exists a $\sigma \in \mathrm{GL}_2(R)$ such that $v = \sigma\Lambda_i$. Thus, the stabilizer group is given by

$$\mathrm{Stab}_{\Gamma(N)}(v) = \Gamma(N) \cap \mathrm{Stab}_{\mathrm{GL}_2(R)}(v) = \Gamma(N) \cap \sigma G_i \sigma^{-1},$$

where $\mathrm{Stab}_{\mathrm{GL}_2(R)}(\Lambda_i) = G_i$ according to Lemma 2.23. Since $\Gamma(N)$ is a normal subgroup of $\mathrm{GL}_2(R)$, we obtain

$$\begin{aligned}\sigma^{-1}\,\mathrm{Stab}_{\Gamma(N)}(v)\sigma &= \Gamma(N) \cap G_i \\ &= \begin{cases} \{1\} & \text{if } i < \deg(N)\,, \\ \left\{ \begin{pmatrix} 1 & xN \\ 0 & 1 \end{pmatrix} \,\middle|\, \deg(x) + \deg(N) \leq i \right\} & \text{if } i \geq \deg(N)\,. \end{cases}\end{aligned}$$

As a result, $v \in V(\mathcal{T})$ is $\Gamma(N)$-stable if and only if $i < \deg(N)$. □

Proposition 3.17. Every vertex $v \in V(\mathcal{T})$ of stage 0 is $\Gamma(N)$-stable.

Proof. According to Proposition 3.16, if $\deg(N) \geq 2$, all vertices of stage 1 are stable. If v was unstable, there would exist an end of unstable vertices by Corollary 3.15, including an unstable vertex of stage 1.

Now, let $\deg(N) = 1$. Then, there is only one vertex of stage 0 in $\Gamma(N)\backslash\mathcal{T}$, because

$$\mathrm{GL}_2(R)/\Gamma(N) \simeq \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(R/N) \simeq \mathrm{GL}_2(k)\,.$$

So, all vertices of stage 0 are identified with Λ_0, whose stabilizer group is given by

$$\mathrm{Stab}_{\Gamma(N)}(\Lambda_0) = \Gamma(N) \cap G_0 = \{1\}\,.$$

Consequently, every vertex $v \in V(\mathcal{T})$ of stage 0 is $\Gamma(N)$-stable. □

Corollary 3.18. Let $e \in E(\mathcal{T})$ be an edge between vertices of stage i and $i + 1$. Then, e is $\Gamma(N)$-stable if and only if $i < \deg(N)$.

Proof. According to Proposition 3.7, every edge which contains a vertex of stage $i < \deg(N)$ is stable. These are the only stable edges, because if an edge between vertices of stage i and $i + 1$ was stable for $i \geq \deg(N)$, the corresponding vertex of stage i would also have to be stable, which contradicts the previous lemma. □

Thus, if $\Gamma = \Gamma(N)$, there are no unstable vertices of stage 0 and the quotient graph exclusively has stable vertices of stage 0 with $q + 1$ stable edges, of which one is to be omitted to construct a basis of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$.

The $\Gamma_1(N)$-Stable Vertices and Edges

Next, since $\Gamma_1(N)$ is not normal in $\mathrm{GL}_2(R)$, the expression

$$\mathrm{Stab}_{\Gamma_1(N)}(v) = \Gamma_1(N) \cap \mathrm{Stab}_{\mathrm{GL}_2(R)}(v) = \Gamma_1(N) \cap \sigma G_i \sigma^{-1}\,,$$

does not simplify. Thus, the calculations are more difficult in this case.

Lemma 3.19. Let $\gamma \in \mathrm{GL}_2(R/N)$, $\pi : \mathrm{GL}_2(R) \to \mathrm{GL}_2(R/N)$ be the projection map, and $\overline{\Gamma} \coloneqq \pi(\Gamma_1(N))$.

1. Let $i \geq 1$ and $\overline{G_i} \coloneqq \pi(G_i)$. Then, we have
$$\gamma^{-1}\overline{\Gamma}\gamma \cap \overline{G_i} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| \deg(x) \leq i, \frac{N}{\gcd(N, \gamma_{21})} \middle| x \right\} .$$

2. Let $\overline{G_0} \coloneqq \pi(G_0 \cap G_1)$. Then, we have
$$\gamma^{-1}\overline{\Gamma}\gamma \cap \overline{G_0} = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in k \right\} .$$

Proof. See [But07, Lemma 2.32]. □

Proposition 3.20. Let $v \in V(\mathcal{T})$ be a $\Gamma_1(N)$-unstable vertex of stage $i \geq 1$ and $b_v \colon v = v_1 \to v_2 \to \dots$ its unique end from Corollary 3.15. Then, we have $\mathrm{Stab}_{\Gamma_1(N)}(v_i) \subseteq \mathrm{Stab}_{\Gamma_1(N)}(v_{i+1})$ and $\#\,\mathrm{Stab}_{\Gamma_1(N)}(v_{i+1})/\,\mathrm{Stab}_{\Gamma_1(N)}(v_i) = q$.

Proof. See [But07, Satz 2.36]. □

Proposition 3.21. Let $e = (v_1, v_2) \in E(\mathcal{T})$ be a $\Gamma_1(N)$-unstable edge between vertices of stage 0 and 1 and $b_v \colon v = v_1 \to v_2 \to \dots$ its unique end from Corollary 3.15. Then, we have $\mathrm{Stab}_{\Gamma_1(N)}(e) \subseteq \mathrm{Stab}_{\Gamma_1(N)}(v_2)$, $\#\,\mathrm{Stab}_{\Gamma_1(N)}(v_2)/\,\mathrm{Stab}_{\Gamma_1(N)}(e) = q$, $\mathrm{Stab}_{\Gamma_1(N)}(v_i) \subseteq \mathrm{Stab}_{\Gamma_1(N)}(v_{i+1})$, and $\#\,\mathrm{Stab}_{\Gamma_1(N)}(v_{i+1})/\,\mathrm{Stab}_{\Gamma_1(N)}(v_i) = q$ for $i \geq 2$.

Proof. See [But07, Satz 2.37]. □

Corollary 3.22. Let $e = \gamma(\Lambda_i, \Lambda_{i+1}) \in E(\mathcal{T})$ be an edge between vertices of stage i and $i+1$ and let $M(\gamma^{-1}) \coloneqq \frac{N}{\gcd(N,c)}$ where $c \coloneqq (\gamma^{-1})_{2,1}$. If $\deg(M(\gamma^{-1})) > i$, the edge e is $\Gamma_1(N)$-stable. Otherwise, the stabilizer group of e is given by
$$\mathrm{Stab}_{\Gamma_1(N)}(e) = \gamma \left\{ \begin{pmatrix} 1 & M(\gamma^{-1})x \\ 0 & 1 \end{pmatrix} \middle| x \in R, \deg(x) + \deg(M(\gamma^{-1})) \leq i \right\} \gamma^{-1} .$$

Corollary 3.23. Let $v \in V(\mathcal{T})$ be a $\Gamma_1(N)$-unstable vertex of stage 0 and the path $b_v \colon v = v_1 \to v_2 \to \dots$ its unique end from Corollary 3.15. Then, we have $\mathrm{Stab}_{\Gamma_1(N)}(v) = \mathrm{Stab}_{\Gamma_1(N)}(v, v_2)$. Furthermore, if $e_i = (v, w_i)$ with $i = 1, \dots, q$ are the other edges containing v as initial vertex, all e_i are $\Gamma_1(N)$-stable and $\Gamma_1(N)$-equivalent.

Proof. According to Corollary 3.15, the group $\mathrm{Stab}_{\Gamma_1(N)}(v)$ stabilizes the whole end $b_v \colon v = v_1 \to v_2 \to \dots$, so we have $\mathrm{Stab}_{\Gamma_1(N)}(v) \subseteq \mathrm{Stab}_{\Gamma_1(N)}(v_i)$ for $i \geq 1$. Thus, it follows
$$\mathrm{Stab}_{\Gamma_1(N)}(v, v_2) = \mathrm{Stab}_{\Gamma_1(N)}(v) \cap \mathrm{Stab}_{\Gamma_1(N)}(v_2) = \mathrm{Stab}_{\Gamma_1(N)}(v) .$$

Now, let $e = (v, v_2)$ be the $\Gamma_1(N)$-unstable edge of the unique end and $e_i = (v, w_i)$ with $i = 1, \ldots, q$ the remaining q edges at the vertex v. Consider the action of $\operatorname{Stab}_{\Gamma_1(N)}(e)$ on these edges. By definition, the group stabilizes their initial vertex $o(e_i) = v$, and has to act transitively on the set of terminal vertices $t(e_i) = w_i$. Otherwise, there would exist a non-trivial element stabilizing a w_i, which would also stabilize the edge e_i. Consequently, $\operatorname{Stab}_{\Gamma_1(N)}(e_i)$ would be non-trivial and, according to Corollary 3.22, would have cardinality q. Since $\#\operatorname{Stab}_{\Gamma_1(N)}(v) = q$, we would get

$$\operatorname{Stab}_{\Gamma_1(N)}(e_i) = \operatorname{Stab}_{\Gamma_1(N)}(v) \cap \operatorname{Stab}_{\Gamma_1(N)}(w_i) = \operatorname{Stab}_{\Gamma_1(N)}(v)\,.$$

Then, $\operatorname{Stab}_{\Gamma_1(N)}(v)$ would stabilize a whole new end introduced by v and w_i, which would contradict Corollary 3.15. Thus, all e_i are stable edges and result in the same edge in the quotient graph $\Gamma\backslash\mathcal{T}$. $\square$

According to Corollary 3.23, the vertex v only has two adjacent edges in the quotient graph, with exactly one being stable and the other one being unstable. As a result, if $c\colon E(\mathcal{T}) \longrightarrow X$ is a $\Gamma_1(N)$-equivariant harmonic cocycle, its value on the edge $e = (v, v_2)$ is given by

$$c(e) = -c(\overline{e}) = \sum_{i=1}^{q} c(\overline{e_i}) = -\sum_{i=1}^{q} c(e_i) = -\sum_{\sigma\in U} c(\sigma \cdot e') = -\sum_{\sigma\in U} \sigma \cdot c(e')\,,$$

where $U \coloneqq \operatorname{Stab}_{\Gamma_1(N)}(e)$ and e' is an arbitrary $e_i = (v, w_i)$. Thus, if we have an unstable vertex of stage 0, we only have to set the value on one edge instead of q or $q+1$ edges.

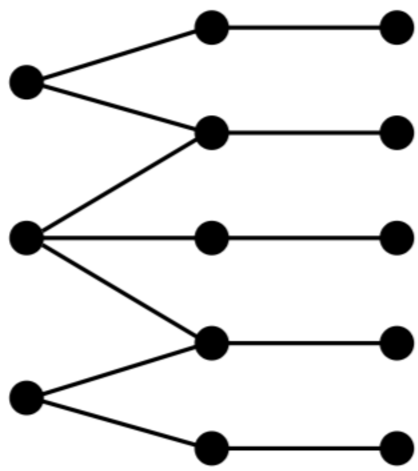

Figure 3.2: The quotient graph $\Gamma\backslash\mathcal{T}$ for $k = \mathbb{F}_2, \Gamma = \Gamma_1(T^2)$

3.2.2 Result

Corollary 3.24. Let X be a vector space with dimension d, m_v the number of stable vertices of stage 0 in $\Gamma\backslash\mathcal{T}$, and m_e the number of stable edges between stage 0 and 1 in $\Gamma\backslash\mathcal{T}$. Then, we have

$$\dim(\mathrm{C}_{\mathrm{har}}(\Gamma, X)) = d(m_e - m_v)\,.$$

If $\{e_1, e_2 \ldots, e_{m_e-m_v}\}$ is a subset of stable edges as required and $\{v_1, v_2 \ldots, v_d\}$ a basis of X,

$$\{c_{i,j}\colon e_i \longmapsto v_j \,|\, i \in \{1, 2, \ldots m_e - m_v\},\, j \in \{1, 2, \ldots, d\}\}$$

is a basis of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$.

3.3 A Basis of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ if $\Gamma = \Gamma_0^1(N)$, $\Gamma_0(N)$, or $\mathrm{SL}_2(R)$

Now, as discussed in the previous section, while these results are always true if Γ is $\Gamma(N)$ or $\Gamma_1(N)$, there are quite a few restrictions if Γ is $\Gamma_0^1(N)$, $\Gamma_0(N)$, or $\mathrm{SL}_2(R)$. This is because of the fact that there are no stable edges depending on $\#k = q$ and there exist isolated vertices depending on the choice of $N \in R$. In the following page, we discuss a resort such that we can fully describe the vector space of Γ-equivariant harmonic cocycles in all cases.

Proposition 3.25. Let X be a vector space with dimension d and N a normalized polynomial of R. Then, we have

$$\mathrm{C}_{\mathrm{har}}(\Gamma_0(N), X) \subseteq \mathrm{C}_{\mathrm{har}}(\Gamma_0^1(N), X) \subseteq \mathrm{C}_{\mathrm{har}}(\Gamma_1(N), X) \subseteq \mathrm{C}_{\mathrm{har}}(\Gamma(N), X)\,.$$

Proof. In Proposition 2.27, we proved

$$\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0^1(N) \subseteq \Gamma_0(N)\,.$$

Thereby, the first assertion follows immediately. □

Definition 3.26. Let G be a group and H a subgroup of G. Then, the set

$${}^{G}\,\mathrm{C}_{\mathrm{har}}(H, X) \coloneqq \{c \in \mathrm{C}_{\mathrm{har}}(H, X) \,|\, c(g \cdot e) = g \cdot c(e)\, \forall g \in G\}$$

defines the subspace of all G-equivariant elements of $\mathrm{C}_{\mathrm{har}}(H, X)$.

Proposition 3.27. Let X be a finite-dimensional vector space, N a normalized polynomial of R, and Γ either Γ_0^1 or Γ_0. Then, we have

$$\mathrm{C}_{\mathrm{har}}(\Gamma, X) = {}^{\Gamma/\Gamma_1}\,\mathrm{C}_{\mathrm{har}}(\Gamma_1, X)\,.$$

Proof. Let $c \in \mathrm{C}_{\mathrm{har}}(\Gamma, X)$, that is a Γ-equivariant harmonic cocycle. By Proposition 3.25, c is also an element of $\mathrm{C}_{\mathrm{har}}(\Gamma_1, X)$, thus $c \in {}^{\Gamma/\Gamma_1}\,\mathrm{C}_{\mathrm{har}}(\Gamma_1, X)$.

Conversely, let $c \in {}^{\Gamma/\Gamma_1}\,\mathrm{C}_{\mathrm{har}}(\Gamma_1, X)$. It is a harmonic cocycle, so we only have to prove Γ-equivariance. For this purpose, let $\gamma \in \Gamma$. Then, there exist $\gamma_1 \in \Gamma/\Gamma_1$ and $\gamma_2 \in \Gamma_1$ such that $\gamma = \gamma_1\gamma_2$. It follows

$$c(\gamma \cdot e) = \gamma_1 \cdot c(\gamma_2 \cdot e) = (\gamma_1\gamma_2) \cdot c(e) = \gamma \cdot c(e)\,,$$

where the first equation is because of Γ/Γ_1-equivariance and the second equation is due to Γ_1-equivariance. $\square$

Remark 3.28. With Proposition 3.27, it is now possible to compute $\mathrm{C}_{\mathrm{har}}(\Gamma_0^1, X)$ and $\mathrm{C}_{\mathrm{har}}(\Gamma_0, X)$ as subspaces of $\mathrm{C}_{\mathrm{har}}(\Gamma_1, X)$. For this purpose, we calculate the kernel of the map

$$\mathrm{C}_{\mathrm{har}}(\Gamma_1, X) \longrightarrow \mathrm{C}_{\mathrm{har}}(\Gamma_1, X), \quad c \longmapsto [e \mapsto g^{-1} \cdot c(g \cdot e)]$$

for every $g \in \Gamma/\Gamma_1$. The intersection delivers the desired subspace. It is essential that Γ_1 is a normal subgroup of Γ_0^1 and Γ_0, because otherwise the map would not be well-defined. If we consider a congruence subgroup Γ such that Γ_1 is not a normal subgroup of Γ, for example $\Gamma = \mathrm{SL}_2(R)$, we can compute $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ as a subspace of $\mathrm{C}_{\mathrm{har}}(\Gamma(N), X)$, since $\Gamma(N)$ is a normal subgroup of every congruence subgroup.

Further, especially with regards to complexity, it may be wiser to implement $\mathrm{C}_{\mathrm{har}}(\Gamma(N) \cap \Gamma_1(P))$ where $P \in R$ is a polynomial with $\gcd(N, P) = 1$. By choosing P appropriately, that is of lower degree, the computation gets less expensive.

3.4 Improvements in the Evaluation of a Harmonic Cocycle

With what we have described in the previous sections, we are able to evaluate a Γ-equivariant harmonic cocycle on an arbitrary edge of the Bruhat-Tits tree $\mathcal{T}$. Nevertheless, there is still room for improvement, especially when we consider that right now, if we want to evaluate on an edge of stage i sufficiently large, we will have to evaluate on several edges of prior stages.

3.4.1 Cuspidality

In a first improvement, we show that if the vector space X is defined over a field of prime characteristic p, all Γ-equivariant harmonic cocycles are cuspidal, that is after a specific stage i, the values will always be 0. For this purpose, we need the following lemma.

Lemma 3.29. Let Γ be a congruence subgroup and $U_n \coloneqq \mathrm{Stab}_\Gamma(\Lambda_n)$. Then, for a sufficiently large n, we have $\#U_n/U_{n-1} = q$. In this case, the edge $(\Lambda_n, \Lambda_{n+1})$ is invariant under the action of U_n/U_{n-1}. On all other edges e with origin Λ_n, the action of U_n/U_{n-1} is transitive.

Proof. By definition, there exists an $N \in R$ such that $\Gamma(N)$ is a subgroup of Γ. We define $U'_n \coloneqq \mathrm{Stab}_{\Gamma(N)}(\Lambda_n)$. If $n < \deg(N)$, $U'_n = \{1\}$. Otherwise, we have

$$\#U'_n/U'_{n-1} = q = \#G_n/G_{n-1}\,.$$

Since

$$U'_n/U'_{n-1} \subseteq U_n/U_{n-1} = U_n/U_n \cap G_{n-1} = G_{n-1}U_n/G_{n-1} \subseteq G_n/G_{n-1}\,,$$

we finally conclude $U_n/U_{n-1} = U'_n/U'_{n-1}$ for $n > \deg(N)+1$. Furthermore, it follows directly from the definition of U_n that U_n/U_{n-1} leaves the edge $(\Lambda_n, \Lambda_{n+1})$ fixed. As to its action on the other edges, we notice

$$\#U_n/U_{n-1}(\Lambda_{n-1}, \Lambda_n) = \#G_n/G_{n-1}(\Lambda_{n-1}, \Lambda_n) = q\,,$$

because $U'_n/U'_{n-1} = G_n/G_{n-1}$ acts transitively on the edges $e \neq (\Lambda_n, \Lambda_{n+1})$ which contain Λ_n as a vertex. $\square$

Theorem 3.30. Let X be a finite-dimensional vector space over a field of prime characteristic p with an action of $\mathrm{GL}_2(R)$ and let Γ be a congruence subgroup. Then, every Γ-equivariant harmonic cocycle is cuspidal.

Proof. Let c be a Γ-equivariant harmonic cocycle. In Chapter 2, we proved that $\Gamma\backslash\mathcal{T}$ consists of a finite subgraph and finitely many cusps. Therefore, it suffices to consider the values of c on sequences in $\mathcal{T}$ corresponding to cusps in $\Gamma\backslash\mathcal{T}$.

First, let $(e_n)_{n\in\mathbb{N}_0}$ be the sequence with edges $e_n \coloneqq (\Lambda_n, \Lambda_{n+1})$. We define the subspaces

$$X_n \coloneqq \{x \in X \mid \sigma \cdot x = x \,\forall \sigma \in U_n\}$$

of X, where U_n denotes the stabilizer subgroup of Λ_n in Γ. Since $U_n \subseteq U_{n+1}$ for $n \in \mathbb{N}$ and $c(e_n) \in X_n$, $(X_n)_{n\in\mathbb{N}}$ is by definition a descending sequence of subspaces of the finite-dimensional vector space X. That is why there exists a $m_1 \in \mathbb{N}$ such that

$$X_{m_1} = X_{m_1+1} = X_{m_1+2} = \dots\,.$$

Simultaneously, we know that $\#U_n/U_{n-1} = q$ for $n \geq m_2$ with a sufficiently large $m_2 \in \mathbb{N}$, which exists according to Lemma 3.29. Further, U_n/U_{n-1} acts transitively on the q edges containing Λ_n unequal to $e_n = (\Lambda_n, \Lambda_{n+1})$. Thus, for $n \geq \max(m_1, m_2)$, we have

$$c(e_{n+1}) = \sum_{i=1}^{q} c(\gamma_i \cdot e_n) = \sum_{i=1}^{q} \gamma_i \cdot c(e_n) = \sum_{i=1}^{q} c(e_n) = qc(e_n) = 0$$

where $\{\gamma_1, \dots, \gamma_q\}$ is a system of representatives for U_n/U_{n-1}. The third equation is true because of $\gamma_i \cdot c(e_n) = c(\gamma_i \cdot e_n) = c(e_n) \in X_{n-1} = X_n$.

Now, let $(e_n)_{n\in\mathbb{N}_0}$ be an arbitrary sequence of edges in $\mathcal{T}$ corresponding to a cusp in $\Gamma\backslash\mathcal{T}$. Then, it is $\mathrm{GL}_2(R)$-equivalent to the standard cusp from the beginning, so there exists a $\sigma \in \mathrm{GL}_2(R)$ such that $e_n = \sigma(\Lambda_n, \Lambda_{n+1})$ for all $n \geq m$ for some $m \in \mathbb{N}$. Since the map

$$\bar{c} : E(\mathcal{T}) \longrightarrow X, e \longmapsto \sigma^{-1} \cdot c(\sigma \cdot e) .$$

defines a $\sigma^{-1}\Gamma\sigma$-equivariant harmonic cocycle, it disappears on the standard cusp. Consequently, c itself vanishes on $(e_n)_{n\in\mathbb{N}_0}$. □

Corollary 3.31. Let X be a vector space with dimension d over a field of prime characteristic p and let Γ be either $\Gamma(N)$ or $\Gamma_1(N)$. If $c\colon E(\mathcal{T}) \to X$ is Γ-equivariant harmonic cocycle, we have $c(e) = 0$ for all edges $e \in E(\mathcal{T})$ between vertices of stage i and $i+1$ with $i \geq n+d+1$.

Proof. In the proof of Theorem 3.30, we have noticed that a Γ-equivariant harmonic cocycle vanishes on edges between vertices of stage i and $i+1$, when the chain of subspaces X_i terminates and we have $\#\,\mathrm{Stab}_\Gamma(e_i)/\,\mathrm{Stab}_\Gamma(e_{i-1}) = q$. The latter requirement is always fulfilled for unstable edges, which at the latest appear after stage n, according to Theorem 3.8. The subspaces will terminate at maximum after d steps, because X is d-dimensional. □

3.4.2 The Source of Unstable Edges

Next, we study the concept of the source of unstable edges. This will allow us to calculate the value of a harmonic cocycle on an unstable edge by its value on a stable edge, which introduces an end through the unstable edge. For the whole section, let Γ be either $\Gamma(N)$ or $\Gamma_1(N)$.

Definition 3.32. Let $e \in E(\mathcal{T})$ be Γ-unstable. We define the *source* of e by

$$\mathrm{Src}_\Gamma(e) := \{e' \in E(\mathcal{T}) \mid e' \text{ stable; } \exists e'' \text{ unstable: } o(e'') = t(e'), e \in b_{o(e'')}\} .$$

For a Γ-stable edge $e \in E(\mathcal{T})$, we set $\mathrm{Src}_\Gamma(e) = \{e\}$.

Proposition 3.33. Let $e = (v_1, v_2)$ be an unstable edge such that the stage of v_1 is smaller than the stage of v_2. If $e_1, \dots, e_q$ are the remaining edges which contain v_1, then we have

$$\mathrm{Src}(e) = \bigsqcup_{i=1}^{q} \mathrm{Src}(e_i) .$$

Proof. Since e is unstable and the stabilizer groups grow with a factor of q, there exists a $m \in \mathbb{N}$ such that $\#\,\mathrm{Stab}_\Gamma(e) = q^m$. If $m = 1$, then all e_i are stable, which means that $\operatorname{Src}(e_i) = \{e_i\}$. Furthermore, we have $t(e_i) = o(e)$, so $e_i \in \operatorname{Src}(e)$. Now, let $e' \in \operatorname{Src}(e)$. By definition, e is the first unstable edge with $t(e') = o(e)$ and $e \in b_{t(e')}$, so $e' = e_i$ for some i. If $m > 1$, all e_i are unstable. We have $\operatorname{Src}(e_i) \subseteq \operatorname{Src}(e)$, because $b_{o(e)} \subset b_{o(e_i)}$. On the other hand, if e' is an element from $\operatorname{Src}(e)$, its path to infinity must include some e_i, proving the other inclusion. □

Corollary 3.34. Let $e = (v_1, v_2)$ be an edge. Then, $\# \operatorname{Src}(e) = \# \operatorname{Stab}_\Gamma(e) < \infty$.

Proof. The source of a stable edge only has one element. If e is an unstable edge, according to Proposition 3.33, its source is the union of the sources of finitely many adjacent edges of a prior stage. This terminates in finitely many steps with the source of stable edges, which only have one element themselves. By induction, we get

$$\# \operatorname{Src}(e) = \sum_{i=1}^{q} \# \operatorname{Src}(e_i) = \sum_{i=1}^{q} q^{m-1} = q^m ,$$

which is equal to $\# \operatorname{Stab}_\Gamma(e)$. □

Proposition 3.35. Let e be an unstable edge and e' an element of its source. Then, $\operatorname{Src}(e) = \operatorname{Stab}_\Gamma(e) \cdot e'$.

Proof. Since $e' \in \operatorname{Src}(e)$, there is an end $b_{t(e')} \colon t(e') \to e'' \to \dots$ such that $e \in b_{t(e')}$. Then, for $\gamma \in \operatorname{Stab}_\Gamma(e)$ the end $\gamma e' \to \gamma e'' \to \dots$ consists of unstable edges with the exception of $\gamma e'$, and contains $\gamma e = e$. Thus, $\gamma e' \in \operatorname{Src}(e)$ and

$$\operatorname{Stab}_\Gamma(e) e' \subseteq \operatorname{Src}(e) .$$

According to Corollary 3.34, both sets have the same number of elements. Hence, they are equal. □

Corollary 3.36. For all $e \in E(\mathcal{T}), \gamma \in \Gamma$, we get $\operatorname{Src}(\gamma e) = \gamma \operatorname{Src}(e)$.

Proof. According to Proposition 3.35, we have

$$\operatorname{Src}(\gamma e) = \operatorname{Stab}_\Gamma(\gamma e)\gamma e' = \gamma \ \operatorname{Stab}_\Gamma(e)\gamma^{-1}\gamma e' = \gamma \operatorname{Src}(e) .$$

□

Theorem 3.37. Let $c \colon E(\mathcal{T}) \longrightarrow X$ be a Γ-equivariant harmonic cocycle and let $e \in E(\mathcal{T})$ be unstable. Then

$$c(e) = \sum_{e' \in \operatorname{Src}(e)} c(e') .$$

Similarly, if $\overline{c}$ is a Γ-equivariant, harmonic map with values in X defined on all stable edges, then

$$c(e) := \sum_{e' \in \operatorname{Src}(e)} \overline{c}(e')$$

defines a Γ-equivariant harmonic cocycle $c \colon E(\mathcal{T}) \longrightarrow X$.

Proof. First, let $c\colon E(\mathcal{T}) \longrightarrow X$ be a Γ-equivariant harmonic cocycle, $e \in E(\mathcal{T})$ an unstable edge, and $e_1, \dots, e_q$ the other q adjacent edges. We prove the theorem by induction for $m \in \mathbb{N}, \# \operatorname{Stab}_\Gamma(e) = q^m$. Let $m = 1$. Then, by definition

$$\sum_{i=1}^{q} c(e_i) + c(\overline{e}) = 0\,.$$

Now, let $m > 1$. Applying Proposition 3.33 and by induction hypothesis for $\# \operatorname{Stab}_\Gamma(e_i)$, we get

$$c(e) = \sum_{i=1}^{q} c(e_i) = \sum_{i=1}^{q} \sum_{e' \in \operatorname{Src}(e_i)} c(e') = \sum_{e' \in \operatorname{Src}(e)} c(e')\,.$$

Finally, if $\overline{c}$ is a Γ-equivariant, harmonic map with values in X defined on all stable edges, then

$$c(e) \coloneqq \sum_{e' \in \operatorname{Src}(e)} \overline{c}(e')$$

is, according to Corollary 3.34, a finite sum and, according to Corollary 3.36, a Γ-equivariant map. Furthermore, if e is an unstable edge and $e_1, \dots, e_q$ as before,

$$c(e) = \sum_{e' \in \operatorname{Src}(e)} \overline{c}(e') = \sum_{i=1}^{q} \sum_{e' \in \operatorname{Src}(e_i)} \overline{c}(e') = \sum_{i=1}^{q} c(e_i)\,.$$

Therefore, c defines a harmonic cocycle. $\square$

Subsequently, if we want to evaluate a Γ-equivariant harmonic cocycle c on an unstable edge e, we can search for a stable edge e' of a prior stage introducing a path to infinity through e, evaluate c on e', and then calculate $c(e)$ by operating on $c(e')$ with $\operatorname{Stab}_\Gamma(e)$.

3.5 Implementation in `Magma`

Let X be a vector space over a field F of prime characteristic p with a $\operatorname{GL}_2(R)$-action. In order to evaluate a Γ-equivariant harmonic cocycle $c : E(\mathcal{T}) \longrightarrow X$ on an arbitrary edge $e \in E(\mathcal{T})$, there are essentially three major steps.

1. Let $A \in \operatorname{GL}_2(K_\infty)$ be a matrix which represents the edge e. Find elements $\gamma \in \Gamma$ and $\sigma \in \Gamma \backslash \operatorname{GL}_2(R)$ such that $A = \gamma\sigma(\Lambda_n, \Lambda_{n+1})$ for some $n \in \mathbb{N}$.
2. Evaluate c on the edge $\sigma(\Lambda_n, \Lambda_{n+1})$, that is on the corresponding edge $[e]$ of the quotient graph $\Gamma \backslash \mathcal{T}$.
3. Since $c(e) = c(\gamma \cdot [e]) = \gamma \cdot c([e])$, apply the $\operatorname{GL}_2(R)$-action on X.

First, let us discuss how to find the required matrices. Notice that while it is possible to consider matrices in $\mathrm{GL}_2(K_\infty)$ in `Magma`, there are quite a few complications regarding the use. For example, one can implement K_∞ as a Laurent series ring in π_∞ with coefficients from k, and an arbitrary element x of K_∞ is then given by

$$x = \sum_{i=m}^{\infty} x_i \pi_\infty^i = x_m \pi_\infty^m + x_{m+1}\pi_\infty^{m+1} + \dots$$

for $m \in \mathbb{Z}$, which is an infinite series. `Magma` can only save it up to a precision, which is 20 if not specified and not sufficient from a theoretical point of view. Even if we just use elements in the field of fractions K of R but make use of operations in K_∞, there is still the possibility that inverting an element will turn it into an infinite series. For example, consider

$$\frac{T+1}{T^2+1} \longmapsto \pi_\infty^{-1} + \pi_\infty^2 + 2\pi_\infty^4 + \pi_\infty^6 + 2\pi_\infty^8 + \pi_\infty^{10} + 2\pi_\infty^{12} + \pi_\infty^{14} + 2\pi_\infty^{16} + \pi_\infty^{18} + O(\pi_\infty^{19}) ,$$

where we chose $k = \mathbb{F}_3$ and embedded K into K_∞. However, according to all our previous considerations, we only need to operate in K, where we can invert these elements precisely. The only disadvantage from not using K_∞ is that we have to pass on preassigned functions such as `Valuation` and `Coefficient` for elements in K_∞, which can be easily implemented separately, though.

3.5.1 Representatives of Edges

Now, let $e = (v, w)$ be an edge. Our goal is to find $\gamma \in \Gamma$, $s_k \in \Gamma\backslash \mathrm{GL}_2(R)$ such that

$$e = (v, w) = \gamma s_k(\Lambda_i, \Lambda_j) ,$$

where $|i - j| = 1$. Therefore, we first need two functions which assign to the two vertices v, w their normal forms

$$v = \begin{pmatrix} \pi_\infty^{n_v} & y_v \\ 0 & 1 \end{pmatrix} , \; w = \begin{pmatrix} \pi_\infty^{n_w} & y_w \\ 0 & 1 \end{pmatrix}$$

and then their representatives on the quotient graph $\mathrm{GL}_2(R)\backslash\mathcal{T}$

$$v = \sigma_v \Lambda_i \, , \; w = \sigma_w \Lambda_j \, .$$

Both algorithms are thoroughly discussed in the proof of Lemma 2.14 and Theorem 2.21, respectively. At first, σ_v is not necessarily equal to σ_w, but Proposition 2.13 provides the existence of a $\sigma \in \mathrm{GL}_2(R)$ such that

$$e = (v, w) = (\sigma_v \Lambda_i, \sigma_w \Lambda_j) = \sigma(\Lambda_i, \Lambda_j) \, .$$

Therefore, we have

$$\sigma_v \alpha_v = \sigma = \sigma_w \alpha_w \, ,$$

where $\alpha_v \in G_i, \alpha_w \in G_j$. Now, consider the following four cases:

- If $1 \leq i < j$, we have $G_i \subseteq G_j$. As a result, we get

$$e = (\sigma_v \Lambda_i, \sigma_w \Lambda_j) = (\sigma_v \Lambda_i, \sigma_v \alpha_v \alpha_w^{-1} \Lambda_j) = \sigma_v (\Lambda_i, \Lambda_j) \,.$$

- If $1 \leq j < i$, we have $G_j \subseteq G_i$. Similarly, it follows

$$e = (\sigma_v \Lambda_i, \sigma_w \Lambda_j) = (\sigma_w \alpha_w \alpha_v^{-1} \Lambda_i, \sigma_w \Lambda_j) = \sigma_w (\Lambda_i, \Lambda_j) = \sigma_w \overline{(\Lambda_j, \Lambda_i)} \,.$$

- If $i = 0$ and $j = 1$, we still obtain $\sigma_v \alpha_v = \sigma = \sigma_w \alpha_w$ for $\alpha_v \in G_0, \alpha_w \in G_1$, but since $G_0 = \mathrm{GL}_2(k) \nsubseteq G_1$, we have

$$e = (\sigma_v \Lambda_0, \sigma_w \Lambda_1) = (\sigma \alpha_v^{-1} \Lambda_0, \sigma \alpha_w^{-1} \Lambda_1) = \sigma (\Lambda_0, \Lambda_1) \,.$$

  In order to find σ, we run through G_0 and G_1 and search for $\alpha_v \in G_0, \alpha_w \in G_1$ satisfying $\sigma_v \alpha_v = \sigma_w \alpha_w$.

- If $i = 1$ and $j = 0$, it similarly follows $e = \sigma(\Lambda_1, \Lambda_0) = \sigma \overline{(\Lambda_0, \Lambda_1)}$.

In all cases, once we have established $e = \sigma(\Lambda_i, \Lambda_j)$ for $\sigma \in \mathrm{GL}_2(R)$, all that remains is to find the corresponding edge on the quotient graph $\Gamma \backslash \mathcal{T}$. Algorithm 3 does exactly that.

Algorithm 3 Find a representative on the quotient graph $\Gamma \backslash \mathcal{T}$

```
σ, Λ_i given
Γ := Γ(N), Γ_1(N), Γ_0^1(N), or Γ_0(N)
for g ∈ Γ\ GL_2(R) do
    if σg^{-1} ∈ Γ then
        return σ * g^{-1}, g, Λ_i, Λ_j;
    end if
end for
```

Thus, we get $\gamma_1 \in \Gamma$ and $\gamma_{1,r} \in \Gamma \backslash \mathrm{GL}_2(R)$ such that

$$e = \gamma_1 \gamma_{1,r} (\Lambda_i, \Lambda_j) \,.$$

Theoretically, we have found the edge $[e]$ in the quotient graph $\Gamma \backslash \mathcal{T}$, which corresponds to our initial edge e in the Bruhat-Tits tree. However, it may be that $[e]$ no longer exists in the quotient graph in `Magma` if it has been identified with an other edge. Fortunately, we collected the information on edge equivalency when we implemented the quotient graph. Thus, we just have to go through that list and get matrices $\gamma_2 \in \Gamma, \gamma_{2,r} \in \Gamma \backslash \mathrm{GL}_2(R), g \in G_{\min(i,j)}$ or possibly $g \in G_0 \cap G_1$ such that

$$\gamma_{1,r} = \gamma_2 \gamma_{2,r} g \,,$$

where $\gamma_{2,r} \in \Gamma\backslash \mathrm{GL}_2(R)$ is an existing edge in the quotient graph which is equivalent to $\gamma_{1,r}$. Consequently, we have

$$e = \sigma(\Lambda_i, \Lambda_j) = \gamma_1\gamma_{1,r}(\Lambda_i, \Lambda_j) = \gamma_1\gamma_2\gamma_{2,r}(\Lambda_i, \Lambda_j)\,,$$

where $\sigma \in \mathrm{GL}_2(R), \gamma_1, \gamma_2 \in \Gamma, \gamma_{1,r}\gamma_{2,r} \in \Gamma\backslash \mathrm{GL}_2(R)$.

3.5.2 Evaluating a Harmonic Cocycle on an Edge

Now, let c be a Γ-equivariant harmonic cocycle given by its coordinates with regards to the basis in Corollary 3.24 and $e = (v, w) = (\sigma_v\Lambda_i, \sigma_w\Lambda_j)$ an edge. We assume $i < j$, otherwise return $-c(\overline{e})$. If $i > \deg(N) + \dim(X)$, then, according to Corollary 3.31, we have $c(e) = 0$.

If e is unstable, search for an edge e' in the source of e. To go through all edges of prior stages, starting with stage $i-1$ and terminal vertex v, use Proposition 3.2. Once this is achieved, return

$$c(e) = \sum_{\sigma\in\mathrm{Stab}_\Gamma(e)} \sigma \cdot c(e')\,.$$

Thus, we can restrict ourselves to the case that e is stable, and then, all edges of prior stages will also be stable. If e is of stage $i \geq 1$ and $e_1, \ldots, e_q$ are all edges with $t(e_i) = o(e)$, evaluate

$$c(e) = \sum_{i=1}^{q} c(e_i)\,.$$

In finitely many steps, we will be at edges of stage 0, where the values are either directly given by the coordinates of c and the action of Γ, or one lands on one of the omitted stable edges or an unstable edge. If we have arrived on one of the omitted stable edges, let $e_1, \ldots, e_q$ be the other adjacent edges of stage 0. Then, we have

$$c(e) = -\sum_{i=1}^{q} c(e_i)\,.$$

If we have landed on an unstable edge, let e' be one of the other q adjacent edges of stage 0, which are all stable, and we finally get

$$c(e) = -\sum_{\sigma\in\mathrm{Stab}_\Gamma(e)} \sigma \cdot c(e')\,.$$

The stabilizer subgroups of edges are easily implemented with the knowledge of Proposition 3.16 and Corollary 3.22. If the vector space X is compatible with homomorphisms in `Magma`, the $\mathrm{GL}_2(R)$-action in `Magma` is given by

emb := hom< X -> X | $\sigma \cdot x_1$, …, $\sigma \cdot x_d$>;

where $\{x_1, \ldots, x_d\}$ is a basis of X and $\sigma \cdot x_i$ is the image of the i-th vector under the $\mathrm{GL}_2(R)$-action. Otherwise, we have to declare X as a formal d-dimensional vector space and define a separate function, which implements the operation through regular matrix multiplication.

3.5.3 A Basis of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ if $\Gamma = \Gamma_0^1(N)$, $\Gamma_0(N)$, or $\mathrm{SL}_2(R)$

Now that we have discussed how to evaluate a Γ-equivariant harmonic cocycle on an arbitrary edge of the Bruhat-Tits tree $\mathcal{T}$, where Γ is one of the two congruence subgroups $\Gamma(N), \Gamma_1(N)$ or in restricted cases even $\Gamma_0^1(N), \Gamma_0(N)$, we want to compute the vector space $\mathrm{C}_{\mathrm{har}}(\Gamma_0^1(N), X)$, $\mathrm{C}_{\mathrm{har}}(\Gamma_0(N), X)$, and $\mathrm{C}_{\mathrm{har}}(\mathrm{SL}_2(R), X)$. In the following lines, we describe the method for the former two vector spaces, but if we replace Γ_1 by $\Gamma(N)$, we immediately get the case for $\mathrm{C}_{\mathrm{har}}(\mathrm{SL}_2(R), X)$.

For this purpose, let $\{e_1, \ldots, e_m\}$ be a set of Γ_1-stable edges of the Bruhat-Tits tree $\mathcal{T}$ such that $\mathrm{C}_{\mathrm{har}}(\Gamma_1, X)$ can be fully declared, and let $\{x_1, \ldots, x_n\}$ be a basis of X. Further, let $\{b_1, \ldots, b_{m \cdot n}\}$ be a basis of $\mathrm{C}_{\mathrm{har}}(\Gamma_1, X)$ like in Corollary 3.24 such that

$$b_{(j-1)\cdot m+k}(e_j) = x_k \,.$$

Let Γ be either Γ_0^1 or Γ_0 and compute a system of representatives of

$$\begin{aligned}
\Gamma_0^1/\Gamma_1 &= \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \,\middle|\, x \in (R/N)^\times \right\} , \\
\Gamma_0/\Gamma_1 &= \begin{pmatrix} k^\times & 0 \\ 0 & 1 \end{pmatrix} \left\{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \,\middle|\, x \in (R/N)^\times \right\} ,
\end{aligned}$$

or if we consider $\mathrm{SL}_2(R)$, the group

$$\mathrm{SL}_2(R)/\Gamma(N) = \mathrm{SL}_2(R/N)\,.$$

These matrices have to be lifted again, which was previously described in Lemma 2.28. Alternatively, since $\Gamma(N)$ is normal in every congruence subgroup Γ and $\Gamma_1(N)$ is normal in $\Gamma_0^1(N)$ and $\Gamma_0(N)$, we know that $\Gamma/\Gamma(N)$, $\Gamma_0^1(N)/\Gamma_1(N)$, and $\Gamma_0(N)/\Gamma_1(N)$ are groups generated by finitely many elements. Then, it is sufficient and more efficient to only require equivariance with regards to a set of generators.

Next, compute for all $i \in \{1, \ldots, m \cdot n\}$ the expression

$$g^{-1} \cdot b_i(g \cdot e_j) = \sum_{k=1}^{n} f_{jk}^{(i)} x_k \quad \forall g \in \Gamma/\Gamma_1,\, j \in \{1, \ldots, m\}\,,$$

with the help of the cocycle function for Γ_1, which was previously implemented.

These computations give rise to new matrices

$$F^i = \begin{pmatrix} f_{11}^{(i)} & \dots & f_{1n}^{(i)} \\ \vdots & \ddots & \vdots \\ f_{m1}^{(i)} & \dots & f_{mn}^{(i)} \end{pmatrix}$$

for $i \in \{1, \dots, m \cdot n\}$. Finally, create the matrix

$$F_g = \begin{pmatrix} F_{11}^1 & \dots & F_{1n}^1 & \dots & F_{m1}^1 & \dots & F_{mn}^1 \\ \vdots & \ddots & & & & & \vdots \\ \vdots & & \ddots & & & & \vdots \\ \vdots & & & \ddots & & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ \vdots & & & & & \ddots & \vdots \\ F_{11}^{m\cdot n} & \dots & F_{1n}^{m\cdot n} & \dots & F_{m1}^{m\cdot n} & \dots & F_{mn}^{m\cdot n} \end{pmatrix}$$

and calculate the kernel of $\mathbb{E} - F_g$ for all $g \in \Gamma/\Gamma_1$. As a result, the intersection over all $g \in \Gamma/\Gamma_1$ induces the vector space $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$.

4 Hecke Operators

Throughout the chapter, let N be a monic polynomial in R and Γ one of the five congruence subgroups $\Gamma(N)$, $\Gamma_1(N)$, $\Gamma_0^1(N)$, $\Gamma_0(N)$, or $\mathrm{SL}_2(R)$ of $\mathrm{GL}_2(R)$. In this chapter, we will study linear operators T_P called Hecke operators for an irreducible polynomial P in R with $\gcd(N,P)=1$ on $\mathrm{C}_{\mathrm{har}}(\Gamma,X)$, the vector space of Γ-equivariant harmonic cocycles with values in a vector space X over a field of prime characteristic p. In Section 4.1, we mainly follow [But07].

4.1 The Definition of the Hecke Operator

In order to introduce the Hecke operator, we first have to consider several lemmas. We define

$$\Gamma^0(P) \coloneqq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, b \equiv 0 \pmod{P} \right\} .$$

Lemma 4.1. The map

$$\phi_P : \Gamma \cap \Gamma_0(P) \longrightarrow \Gamma \cap \Gamma^0(P), \quad \gamma \longmapsto \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

is bijective.

Proof. First, we clarify that the map ϕ_P is well-defined. For this purpose, let

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \cap \Gamma_0(P) .$$

Then, we have

$$\phi_P(\gamma) = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & bP \\ cP^{-1} & d \end{pmatrix} .$$

Since $\gamma \in \Gamma_0(P)$ and $\gcd(N,P)=1$, P divides c and N still divides cP^{-1}. Thus, $\phi_P(\gamma) \in \Gamma \cap \Gamma^0(P)$. To prove that the map ϕ_P is bijective, we define the map

$$\psi_P : \Gamma \cap \Gamma^0(P) \longrightarrow \Gamma \cap \Gamma_0(P), \quad \gamma \longmapsto \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} \gamma \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} ,$$

which is similarly well-defined and obviously the inverse map to ϕ_P. $\square$

Lemma 4.2. Let

$$\iota : \mathrm{C}_{\mathrm{har}}(\Gamma, X) \longrightarrow \mathrm{C}_{\mathrm{har}}(\Gamma \cap \Gamma^0(P), X)$$

be the inclusion map. We define

$$\phi_P^* : \mathrm{C}_{\mathrm{har}}(\Gamma \cap \Gamma^0(P), X) \longrightarrow \mathrm{C}_{\mathrm{har}}(\Gamma \cap \Gamma_0(P), X)$$

through

$$\phi_P^*(c)(e) \coloneqq \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot c\left(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} e\right)$$

and the map

$$\mathrm{tr} : \mathrm{C}_{\mathrm{har}}(\Gamma \cap \Gamma_0(P), X) \longrightarrow \mathrm{C}_{\mathrm{har}}(\Gamma, X)$$

by setting

$$\mathrm{tr}(c)(e) \coloneqq \sum_{\sigma \in (\Gamma \cap \Gamma_0(P)) \backslash \Gamma} \sigma^{-1} \cdot c(\sigma e)$$

for an edge $e \in E(\mathcal{T})$. Then, all three maps are well-defined and linear.

Proof. The linearity of all three maps is clear, so it remains to be seen that they are well-defined. Let us begin with the inclusion map ι. Since a Γ-equivariant harmonic cocycle is also $\Gamma \cap \Gamma^0(P)$-equivariant, it is immediately clear that the map ι is well-defined.

Now, let c be in $\mathrm{C}_{\mathrm{har}}(\Gamma \cap \Gamma^0(P), X)$. Then

$$\sum_{t(e)=v} \phi_P^*(c)(e) = \sum_{t(e)=v} \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot c\left(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} e\right) = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot \sum_{t(e)=w} c(e) = 0\,,$$

where

$$w = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} v\,.$$

Furthermore, for $\gamma \in \Gamma \cap \Gamma_0(P)$, we have

$$\begin{aligned} \phi_P^*(\gamma e) &= \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot c\left(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \gamma e\right) \\ &= \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} c\left(\phi_P(\gamma) \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} e\right) = \gamma \phi_P^*(e)\,, \end{aligned}$$

where we used Lemma 4.1 in the last equation. Finally, consider the map tr. First, notice that the definition is not dependent on the choice of a system of representatives. To see this, let $\{\sigma_1, \dots, \sigma_m\}$ and $\{\tau_1, \dots, \tau_m\}$ be two systems of representatives. Then, there are $\gamma_i \in \Gamma \cap \Gamma_0(P)$ such that $\tau_i = \gamma_i \sigma_i$. Consequently, we have

$$\sum_{i=1}^{m} \tau_i^{-1} \cdot c(\tau_i e) = \sum_{i=1}^{m} \sigma_i^{-1} \gamma_i^{-1} \cdot c(\gamma_i \sigma_i e) = \sum_{i=1}^{m} \sigma_i^{-1} \cdot c(\sigma_i e)\,.$$

Let c be in $\mathrm{C}_{\mathrm{har}}(\Gamma \cap \Gamma_0(P), X)$. Then,

$$\sum_{t(e)=v} \mathrm{tr}(c)(e) = \sum_{t(e)=v} \sum_{\sigma \in (\Gamma \cap \Gamma_0(P)) \backslash \Gamma} \sigma^{-1} \cdot c(\sigma e) = \sum_{\sigma \in (\Gamma \cap \Gamma_0(P)) \backslash \Gamma} \sigma^{-1} \cdot \sum_{t(e)=\sigma v} c(e) = 0\,.$$

Additionally, for $\gamma \in \Gamma$, we have

$$\mathrm{tr}(c)(\gamma e) = \sum_{\sigma \in (\Gamma \cap \Gamma_0(P)) \backslash \Gamma} \sigma^{-1} \cdot c(\sigma \gamma e) = \sum_{\sigma \gamma^{-1} \in (\Gamma \cap \Gamma_0(P)) \backslash \Gamma} \gamma \sigma^{-1} \cdot c(\sigma e) = \gamma \cdot \mathrm{tr}(c)(e)\,.$$

Thus, $\mathrm{tr}(c)$ is indeed a Γ-equivariant harmonic cocycle. □

Definition 4.3. Let P be an irreducible polynomial in R with $\gcd(N, P) = 1$. The map

$$\mathrm{T}_P : \mathrm{C}_{\mathrm{har}}(\Gamma, X) \longrightarrow \mathrm{C}_{\mathrm{har}}(\Gamma, X), \quad c \longmapsto (\mathrm{tr} \circ \phi_P^* \circ \iota)(c)$$

is called *Hecke operator*.

Remark 4.4. According to Lemma 4.2, the map T_P is well-defined and linear. Thus, it is an operator.

4.2 Implementation in `Magma`

Let P be an irreducible polynomial in R with $\gcd(N, P) = 1$. In order to implement a Hecke operator in `Magma`, we write out its definition

$$\mathrm{T}_P : \mathrm{C}_{\mathrm{har}}(\Gamma, X) \xrightarrow{\iota} \mathrm{C}_{\mathrm{har}}(\Gamma \cap \Gamma^0(P), X) \xrightarrow{\phi_P^*} \mathrm{C}_{\mathrm{har}}(\Gamma \cap \Gamma_0(P), X) \xrightarrow{\mathrm{tr}} \mathrm{C}_{\mathrm{har}}(\Gamma, X)\,,$$

or element-wise

$$\begin{aligned} \mathrm{T}_P(c)(e) = \mathrm{tr}(\phi_P^*(\iota(c)))(e) = \mathrm{tr}(\phi_P^*(c))(e) &= \sum_{\sigma \in (\Gamma \cap \Gamma_0(P)) \backslash \Gamma} \sigma^{-1} \cdot \phi_P^*(c)(\sigma e) \\ &= \sum_{\sigma \in (\Gamma \cap \Gamma_0(P)) \backslash \Gamma} \sigma^{-1} \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} \cdot c\left(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \sigma e\right) \end{aligned}$$

for $c \in \mathrm{C}_{\mathrm{har}}(\Gamma, X)$ and $e \in E(\mathcal{T})$. Therefore, we will obtain its matrix with regards to the basis $\{e_1, \dots, e_n\}$ of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ as discussed in Corollary 3.24 once we have a system of representatives of $(\Gamma \cap \Gamma_0(P)) \backslash \Gamma$ and computed the values of the basis elements e_i on the edges

$$\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \sigma e\,,$$

where e runs through all edges between vertices of stage 0 and 1 which contribute to the basis.

4.2.1 A System of Representatives of $(\Gamma \cap \Gamma_0(P))\backslash\Gamma$

Since N and P are coprime, it follows from the Chinese remainder theorem

$$\begin{aligned}(\Gamma \cap \Gamma_0(P))\backslash\Gamma \times \Gamma\backslash \mathrm{GL}_2(R) &= (\Gamma \cap \Gamma_0(P))\backslash \mathrm{GL}_2(R) \\ &\simeq \Gamma\backslash \mathrm{GL}_2(R) \times \Gamma_0(P)\backslash \mathrm{GL}_2(R)\,.\end{aligned}$$

Thus, to implement a system of representatives of $(\Gamma \cap \Gamma_0(P))\backslash\Gamma$, we start with a matrix $\gamma_0 \in \Gamma_0(P)\backslash \mathrm{GL}_2(R)$ and lift it to a matrix γ_1 such that

$$\begin{aligned}\gamma_1 &\equiv \mathbb{E} \bmod N\,, \\ \gamma_1 &\equiv \gamma_0 \bmod P\,.\end{aligned}$$

Now, γ_1 is not necessarily in $\mathrm{GL}_2(R)$. That is why we lift it, following the proof of Lemma 2.28, to a matrix $\gamma_2 \in \mathrm{GL}_2(R)$ such that $\gamma_2 \equiv \gamma_1 \bmod NP$.

4.2.2 Evaluation on the Basis Edges

Let σ be an element of $(\Gamma \cap \Gamma_0(P))\backslash\Gamma$ and $e = \tau(\Lambda_0, \Lambda_1)$ with $\tau \in \Gamma\backslash \mathrm{GL}_2(R)$ an edge between vertices of stage 0 and 1 contributing to the basis. Consider the edge

$$e' = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \sigma e = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \sigma\tau(\Lambda_0, \Lambda_1)\,.$$

With the help of Section 3.5.2, a harmonic cocycle is easily evaluated on the edge e'. To further decrease complexity, it is advised to buffer the values on the edges of the quotient graph, because e' will often be of the same stage. Then, we can avoid computing the harmonic cocycle on the same edge repeatedly and only have to apply the corresponding Γ-action on the previously calculated values.

4.2.3 Implementation for $\Gamma(N)$ or $\Gamma_1(N)$

Let Γ be either $\Gamma(N)$ or $\Gamma_1(N)$. Furthermore, let $\{e_1, \dots, e_m\}$ be all stable edges of stage 0 in $\Gamma\backslash\mathcal{T}$ contributing to a basis of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ and $\{x_1, \dots, x_n\}$ a basis of the vector space X. Then, we choose the following basis of the vector space $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ of all Γ-equivariant harmonic cocycles:

$$b_{i,j} : \Gamma\backslash E(\mathcal{T}) \longrightarrow X, \quad e_i \longmapsto x_j\,.$$

We arrange the basis according to $b_{1,1}, b_{1,2}, \dots, b_{1,n}, b_{2,1}, \dots, b_{m,n}$. The remaining implementation is straightforward and detailed in Algorithm 4.

Algorithm 4 Calculating the Hecke Operator T_P

Require: P with $\gcd(N, P) = 1$;
$T_P :=$ ZeroMatrix(K, $m \cdot n$, $m \cdot n$);
for $1 \le i \le m$ **do**
Let $e_i = \gamma_i(\Lambda_0, \Lambda_1)$.
for $1 \le j \le n$ **do**
value := Vec ! 0;
for $\sigma \in (\Gamma \cap \Gamma_0(P))\backslash\Gamma$ **do**
Find $\gamma \in \Gamma$ and $\gamma' \in \Gamma\backslash \operatorname{GL}_2(R)$ such that

$$\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \sigma\gamma_i(\Lambda_0, \Lambda_1) = \gamma\gamma'(\Lambda_i, \Lambda_j)$$

with $|i - j| = 1$. Evaluate

$$\text{add} := \sigma^{-1} \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} \gamma \cdot b_{i,j}(\gamma'(\Lambda_i, \Lambda_j))\,.$$

value := value + add;
end for
list := Coordinates(Vec, value);
for $1 \le k \le n$ **do**
$(T_P)_{i,k+j\cdot n}$:= list[k];
end for
end for
end for

4.2.4 Implementation for $\Gamma_0^1(N)$, $\Gamma_0(N)$, or $\operatorname{SL}_2(R)$

In Section 3.5.3, we computed a basis for $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ if Γ is $\Gamma_0^1(N)$, $\Gamma_0(N)$, or $\operatorname{SL}_2(R)$. Therefore, we only have to take those basis vectors and simply follow the steps in Section 4.2.3. At first, one gets a $d_1 \times d_2$ matrix T'_P with d_1 being the dimension of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$ and d_2 the dimension of $\mathrm{C}_{\mathrm{har}}(\Gamma_1(N), X)$ or $\mathrm{C}_{\mathrm{har}}(\Gamma(N), X)$. Since we are dealing with an operator on $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$, it is possible to rewrite the columns of T'_P in the basis of $\mathrm{C}_{\mathrm{har}}(\Gamma, X)$.

If $\Gamma = \operatorname{SL}_2(R)$, we can arbitrarily choose a polynomial N, as long as N and P are coprime. To reduce complexity, it is recommended to choose a polynomial N with $\deg(N)$ as small as possible such that $\gcd(N, P) = 1$ is still satisfied. If we compute T_P with two distinct N, the resulting transformation matrices may be different. This is to be expected because the vector space $\mathrm{C}_{\mathrm{har}}(\Gamma(N), X)$ is dependent on the choice of N. Thus, the matrices will be equal up to a change of coordinates.

4.3 Values in the r-th Symmetric Power of K^2

Let $n \geq 0$ be an integer and $V(n)$ the vector space of all homogeneous polynomials in two variables X, Y of degree $n-1$ over the field K_∞. Then, we define

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K_\infty) : \sigma \cdot \begin{pmatrix} X \\ Y \end{pmatrix} \coloneqq \det(\sigma)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} .$$

If we choose its dual space $V(-n) = \mathrm{Hom}(V(n), K_\infty)$, we consider the action

$$(\sigma \cdot \varphi)(v) \coloneqq \varphi(\sigma^{-1} \cdot v)$$

for $\varphi \in V(-n), v \in V(n)$.

According to [Tei91, Theorem 16], if $n \geq 2$ is an integer, there exists an isomorphism

$$S_n(\Gamma) \longrightarrow \mathrm{C}_{\mathrm{har}}(\Gamma, n)$$

where $S_n(\Gamma)$ is the vector space of *cusp forms of weight n for* Γ and $\mathrm{C}_{\mathrm{har}}(\Gamma, n)$ is the vector space of all Γ-equivariant harmonic cocycles with values in $V(1-n) \otimes \det^{-1}$. The elements of $\mathrm{C}_{\mathrm{har}}(\Gamma, n)$ are also called *harmonic cocycles of weight n for* Γ.

That is why, in this section, we specifically choose the vector space X to be the r-th symmetric power of $V \coloneqq K^2$ or an irreducible subrepresentation of $\mathrm{Sym}^r(V)$. To introduce a $\mathrm{GL}_2(K)$-action on $\mathrm{Sym}^r(V)$, we identify it with the vector space of homogeneous polynomials in two variables X, Y of degree r and take the action above, for X and its dual space $X^* = \mathrm{Hom}(\mathrm{Sym}^r(V), K)$, respectively.

According to the work of [Ure13, 3.4 Fazit], irreducible subrepresentations of $\mathrm{Sym}^r(V)$ are given by the dominant weights $(m, n) \in \mathbb{N}^2$ such that $m \geq n, m+n = r$, and $m - n = \sum_{i=0}^{l} k_i p^i$. Then, we have

$$L(m, n) \simeq L(m-n, 0) \otimes \wedge^2(V)^{\otimes n} \simeq \bigotimes_{i=0}^{l} (\mathrm{Sym}^{k_i}(V))^{[i]} \otimes \wedge^2(V)^{\otimes n} ,$$

where $(\mathrm{Sym}^{k_i}(V))^{[i]}$ denotes the vector space $\mathrm{Sym}^{k_i}(V)$ with the following action

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K) : \sigma \cdot_{[i]} \begin{pmatrix} X \\ Y \end{pmatrix} \coloneqq \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}^i \cdot \begin{pmatrix} X \\ Y \end{pmatrix} .$$

Alternatively, according to [Ure13, Theorem 4.9], there is an embedding of $L(r, 0)$ into $\mathrm{Sym}^r(V)$ through

$$L(r, 0) = \begin{cases} \mathrm{Sym}^r(V) & \text{, if } 0 \leq r \leq p-1, \\ \left\langle X^j Y^{r-j} \,\middle|\, p \nmid \binom{r}{j} \right\rangle & \text{, else.} \end{cases}$$

In this case, the action on $L(r, 0)$ is given by the regular action on $\mathrm{Sym}^r(V)$.

As for the implementation in `Magma`, since there is no special object for $\mathrm{Sym}^r(V)$ in the library, we realize it or its dual space as a formal $r+1$-dimensional vector space over K using `VectorSpace(K,r+1)`. The corresponding action of a $\sigma \in \mathrm{GL}_2(K)$ on a vector v is given by regular vector-matrix multiplication $v^T A$, where

$$A = \begin{pmatrix} \sigma \cdot e_1 \\ \vdots \\ \sigma \cdot e_{r+1} \end{pmatrix}$$

with a basis $\{e_1, \ldots, e_{r+1}\}$ of either $\mathrm{Sym}^r(V)$ or its dual space. If we choose

$$\{X^i Y^{r-i} \,|\, i = 0, 1, \ldots, r\}$$

as a basis for $\mathrm{Sym}^r(V)$, we get

$$\begin{aligned}
\sigma \cdot (X^i Y^{r-i}) &= \det(\sigma)^{-i} (dX - bY)^i \det(\sigma)^{-(r-i)} (-cX + aY)^{r-i} \\
&= \det(\sigma)^{-r} \left(\sum_{k=0}^{i} \binom{i}{k} (dX)^k (-bY)^{i-k} \right) \left(\sum_{l=0}^{r-i} \binom{r-i}{l} (-cX)^l (aY)^{r-i-l} \right) \\
&= \det(\sigma)^{-r} \sum_{k=0}^{i} \sum_{l=0}^{r-i} (-1)^{i-k+l} \binom{i}{k} \binom{r-i}{l} a^{r-i-l} b^{i-k} c^l d^k X^{k+l} Y^{r-(k+l)} \,.
\end{aligned}$$

Introducing $j = k + l$ and

$$s_{i,j} \coloneqq \sum_{0 \le k \le i,\, 0 \le l \le r-i,\, k+l=j} (-1)^{i-k+l} \binom{i}{k} \binom{r-i}{l} a^{r-i-l} b^{i-k} c^l d^k$$

give rise to

$$\sigma \cdot (X^i Y^{r-i}) = \det(\sigma)^{-r} \sum_{j=0}^{r} s_{i,j} X^j Y^{r-j} \,.$$

Similarly, let

$$v = \sum_{j=0}^{r} v_j X^j Y^{r-j} \in \mathrm{Sym}^r(V)$$

and consider the dual basis $\{(X^i Y^{r-i})^* \,|\, i = 0, 1, \ldots, r\}$. Then, we have

$$\begin{aligned}
(\sigma \cdot (X^i Y^{r-i})^*)(v) &= (X^i Y^{r-i})^* (\sigma^{-1} \cdot v) \\
&= (X^i Y^{r-i})^* \left(\sum_{j=0}^{r} v_j (aX + bY)^j (cX + dY)^{r-j} \right) \\
&= \sum_{j=0}^{r} v_j \sum_{k=0}^{j} \sum_{l=0}^{r-j} \binom{j}{k} \binom{r-j}{l} a^k b^{j-k} c^l d^{r-j-l} \delta_{i,k+l}
\end{aligned}$$

with the Kronecker delta $\delta_{i,j}$. Thus, if

$$s_{i,j} := \sum_{0\le k\le j,\, 0\le l\le r-j,\, k+l=i} \binom{j}{k}\binom{r-j}{l} a^k b^{j-k} c^l d^{r-j-l}\,,$$

we finally get

$$\sigma \cdot (X^i Y^{r-i})^* = \sum_{j=0}^{r} s_{i,j} (X^j Y^{r-j})^*\,.$$

Also, if there is a twist, that is a contribution by $\wedge^2(V)^{\otimes n}$, one further has to consider an additional factor $\det(\sigma)^n$.

4.4 Examples

Before we list several examples of how the characteristic polynomials of Hecke operators T_P with increasing $\deg(P)$ change, let us consider some concrete Hecke operators and their transformation matrices. For this purpose, let $X(n) := V(1-n) \otimes \det^{2-n}$.

First, we start with Hecke operators on $\mathrm{C}_{\mathrm{har}}(\Gamma_1(T), X(5))$ over $\mathbb{F}_3$, that means the cocycles have values in $V(-4) \otimes \det^{-3}$. For the irreducible polynomials $P = T+1$ and $Q = T^2+1$, we get

$$T_P = \begin{pmatrix} 1 & 0 & 2T & 0 \\ 0 & T+1 & 0 & 0 \\ 0 & 0 & T+1 & 0 \\ 0 & 2T^3 & 0 & 1 \end{pmatrix},$$

$$T_Q = \begin{pmatrix} 1 & 0 & 2T^2 & 0 \\ 0 & T^2+1 & 0 & 0 \\ 0 & 0 & T^2+1 & 0 \\ 0 & 2T^4 & 0 & 1 \end{pmatrix}.$$

Calculating $T_P T_Q - T_Q T_P = 0$ shows that both matrices commutate, which is to be expected of Hecke operators. Now, if we consider both on $\mathrm{C}_{\mathrm{har}}(\Gamma_0(T), X(5))$, we get the notification "`Matrix with 0 rows and 0 columns`", that is the dimension of $\mathrm{C}_{\mathrm{har}}(\Gamma_0(T), X(5))$ is zero. This result is in compliance with the following theorem.

Theorem 4.5. If $q \neq 2$, N a prime polynomial, and $q-1 \nmid n$, we have

$$\dim(S_n(\Gamma_0(N))) = 0\,.$$

Proof. [Gek86, Section VII.6] ☐

Thus, let us consider Hecke operators on $\mathrm{C}_{\mathrm{har}}(\Gamma_1(T), X(6))$ over $\mathbb{F}_3$, that means the cocycles have values in $V(-5) \otimes \det^{-4}$. For the irreducible polynomials $P = T+1$ and $Q = T^2 + 1$, we get

$$T_P = \begin{pmatrix} 1 & 0 & 2T^2 + T & 0 & 0 \\ 0 & 1 & 0 & 2T & 0 \\ 0 & 0 & T^2 + 2T + 1 & 0 & 0 \\ 0 & 2T^3 & 0 & 1 & 0 \\ 0 & 0 & 2T^4 + T^3 & 0 & 1 \end{pmatrix},$$
$$T_Q = \begin{pmatrix} 1 & 0 & 2T^4 + T^2 & 0 & 0 \\ 0 & T^4 + 1 & 0 & 0 & 0 \\ 0 & 0 & T^4 + 2T^2 + 1 & 0 & 0 \\ 0 & 0 & 0 & T^4 + 1 & 0 \\ 0 & 0 & 2T^6 + T^4 & 0 & 1 \end{pmatrix}.$$

Again, the operators commutate. Moving over to $\mathrm{C}_{\mathrm{har}}(\Gamma_0(T), X(6))$, we have

$$T_P = \begin{pmatrix} 1 & 2T^2 + T & 0 \\ 0 & T^2 + 2T + 1 & 0 \\ 0 & 2T^4 + T^3 & 1 \end{pmatrix},$$
$$T_Q = \begin{pmatrix} 1 & 2T^4 + T^2 & 0 \\ 0 & T^4 + 2T^2 + 1 & 0 \\ 0 & 2T^6 + T^4 & 1 \end{pmatrix}.$$

This suggests that the first, third, and last basis vector of $\mathrm{C}_{\mathrm{har}}(\Gamma_1(T), X(6))$ generate $\mathrm{C}_{\mathrm{har}}(\Gamma_0(T), X(6))$, which can easily be verified by our designated function.

Finally, let us consider Hecke operators on $\mathrm{C}_{\mathrm{har}}(\Gamma_1(T), X)$ over $\mathbb{F}_3$ but with values in the irreducible subrepresentation $L(4,1)$ of $\mathrm{Sym}^5(V)$. According to previous discussions, we have

$$L(4,1) = L(3,0) \otimes \wedge^2(V)$$

and

$$L(3,0) = \left\langle X^j Y^{3-j} \,\middle|\, p \nmid \binom{3}{j} \right\rangle = \langle X^0 Y^3, X^3 Y^0 \rangle \subset \mathrm{Sym}^3(V)\,.$$

Then, the Hecke operator for $P = T+1$ operating on cocycles with values in $L(4,1)$,

$\mathrm{Sym}^3(V) \otimes \det$, and $\mathrm{Sym}^5(V)$ are

$$T_P = \begin{pmatrix} \frac{1}{T+1} & 0 \\ 0 & \frac{1}{T+1} \end{pmatrix},$$

$$T_P = \begin{pmatrix} \frac{1}{T+1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{T+1} \end{pmatrix},$$

$$T_P = \begin{pmatrix} 1 & 0 & 2T^5 & 0 & T^5 & 0 \\ 0 & 2T+1 & 0 & 2T^3 & 0 & 0 \\ 0 & 0 & 1 & 0 & T^3 & 0 \\ 0 & T & 0 & 1 & 0 & 0 \\ 0 & 0 & 2T & 0 & 2T+1 & 0 \\ 0 & T & 0 & 2T^3 & 0 & 1 \end{pmatrix},$$

respectively. The first matrix has the eigenvalue P^{-1} twice, the second has 1 and P^{-1} each twice, and the final matrix has 1 twice and otherwise only eigenvalues, which are not in K.

4.4.1 Characteristic Polynomials of Hecke Operators

Finally, we compute several Hecke operators T_P and their characteristic polynomial for P with increasing degree. Then, we examine the change in the characteristic polynomial. For this purpose, we vary the field $k = \mathbb{F}_q$, the congruence subgroups Γ, and the dimension of X. The result is presented in tables on the following pages.

n	Characteristic Polynomial of T_P with $P = T + 1$
2	$(X+1)^2$
3	$(X+1)^4$
4	$(X+1)^4 \cdot (X+T+1)^2$
5	$(X+1)^4 \cdot (X^2+T^3+1)^2$
6	$(X+1)^4 \cdot (X+T+1)^4 \cdot (X+T^2+1)^2$
n	Characteristic Polynomial of T_P with $P = T^2 + T + 1$
2	$(X+1)^2$
3	$(X+1)^4$
4	$(X+1)^4 \cdot (X+T^2+T+1)^2$
5	$(X+1)^4 \cdot (X^2+T^6+T^3+1)^2$
6	$(X+1)^4 \cdot (X+T^2+T+1)^4 \cdot (X^2+T^4+T^2+1)^2$
n	Characteristic Polynomial of T_P with $P = T^3 + T + 1$
2	$(X+1)^2$
3	$(X+1)^4$
4	$(X+1)^4 \cdot (X+T^3+T+1)^2$
5	$(X+1)^4 \cdot (X^2+T^9+T^3+1)^2$
6	$(X+1)^4 \cdot (X+T^3+T+1)^4 \cdot (X^2+T^6+T^2+1)^2$

Table 4.1: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\Gamma(T), X(n))$, $\mathbb{F}_2$

n	Characteristic Polynomial of T_P with $P = T + 1$
2	$(X+1)^{16}$
3	$(X+1)^{32}$
4	$(X+1)^{32} \cdot (X+T+1)^{16}$
5	$(X+1)^{32} \cdot (X^2+T^3+1)^{16}$
6	$(X+1)^{32} \cdot (X+T+1)^{32} \cdot (X+T^2+1)^{16}$
n	Characteristic Polynomial of T_P with $P = T^2 + T + 1$
2	$(X+1)^{16}$
3	$(X+1)^{32}$
4	$(X+1)^{32} \cdot (X+T^2+T+1)^{16}$
5	$(X+1)^{32} \cdot (X^2+T^6+T^3+1)^{16}$
6	$(X+1)^{32} \cdot (X+T^2+T+1)^{32} \cdot (X+T^4+T^2+1)^{16}$

Table 4.2: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\Gamma(T^2), X(n))$, $\mathbb{F}_2$

n	Characteristic Polynomial of T_P with $P = T + 1$
2	$(X + 1)$
3	$(X + 1)^2$
4	$(X + 1)^2 \cdot (X + T + 1)$
5	$(X + 1)^2 \cdot (X^2 + T^3 + 1)$
6	$(X + 1)^2 \cdot (X + T + 1)^2 \cdot (X + T^2 + 1)$
n	Characteristic Polynomial of T_P with $P = T^2 + T + 1$
2	$(X + 1)$
3	$(X + 1)^2$
4	$(X + 1)^2 \cdot (X + T^2 + T + 1)$
5	$(X + 1)^2 \cdot (X^2 + T^6 + T^3 + 1)$
6	$(X + 1)^2 \cdot (X + T^2 + T + 1)^2 \cdot (X + T^4 + T^2 + 1)$

Table 4.3: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\Gamma_1(T), X(n))$, $\mathbb{F}_2$

n	Characteristic Polynomial of T_P with $P = T + 1$
2	$(X + 1)^4$
3	$(X + 1)^8$
4	$(X + 1)^8 \cdot (X + T + 1)^4$
5	$(X + 1)^8 \cdot (X^2 + T^3 + 1)^4$
6	$(X + 1)^8 \cdot (X + T + 1)^8 \cdot (X + T^2 + 1)^4$
n	Characteristic Polynomial of T_P with $P = T^2 + T + 1$
2	$(X + 1)^4$
3	$(X + 1)^8$
4	$(X + 1)^8 \cdot (X + T^2 + T + 1)^4$
5	$(X + 1)^8 \cdot (X^2 + T^6 + T^3 + 1)^4$
6	$(X + 1)^8 \cdot (X + T^2 + T + 1)^8 \cdot (X + T^4 + T^2 + 1)^4$

Table 4.4: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\Gamma_1(T^2), X(n))$, $\mathbb{F}_2$

n	Characteristic Polynomial of T_P with $P = T + 1$
2	$(X+1)^{16}$
3	$(X+1)^{32}$
4	$(X+1)^{32} \cdot (X+T+1)^{16}$
5	$(X+1)^{32} \cdot (X^2+T^3+1)^{16}$
6	$(X+1)^{32} \cdot (X+T+1)^{32} \cdot (X+T^2+1)^{16}$
n	Characteristic Polynomial of T_P with $P = T^2 + T + 1$
2	$(X+1)^{16}$
3	$(X+1)^{32}$
4	$(X+1)^{32} \cdot (X+T^2+T+1)^{16}$
5	$(X+1)^{32} \cdot (X^2+T^6+T^3+1)^{16}$
6	$(X+1)^{32} \cdot (X+T^2+T+1)^{32} \cdot (X+T^4+T^2+1)^{16}$

Table 4.5: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\Gamma_1(T^3), X(n))$, $\mathbb{F}_2$

n	Characteristic Polynomial of T_P with $P = T + 1$
2	$(X+2)$
3	$(X+2)^2$
4	$(X+2)^2 \cdot (X+T+2)$
5	$(X+2)^2 \cdot (X+2T+2)^2$
6	$(X+2)^2 \cdot (X+T^2+2) \cdot (X+2T^2+2) \cdot (X+2T^2+T+2)$
n	Characteristic Polynomial of T_P with $P = T^2 + 1$
2	$(X+2)$
3	$(X+2)^2$
4	$(X+2)^2 \cdot (X+2T^2+2)$
5	$(X+2)^2 \cdot (X+2T^2+2)^2$
6	$(X+2)^2 \cdot (X+2T^4+2)^2 \cdot (X+2T^4+T^2+2)$

Table 4.6: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\Gamma_1(T), X(n))$, $\mathbb{F}_3$

n	Characteristic Polynomial of T_P with $P = T + 1$
2	$(X+2)^9$
3	$(X+2)^{18}$
4	$(X+2)^{18} \cdot (X+T+2)^9$
5	$(X+2)^{18} \cdot (X+2T+2)^{18}$
6	$(X+2)^{18} \cdot (X+T^2+2)^9 \cdot (X+2T^2+2)^9 + (X+2T^2+T+2)^9$
n	Characteristic Polynomial of T_P with $P = T^2 + 1$
2	$(X+2)^9$
3	$(X+2)^{18}$
4	$(X+2)^{18} \cdot (X+2T^2+2)^9$
5	$(X+2)^{18} \cdot (X+2T^2+2)^{18}$
6	$(X+2)^{18} \cdot (X+2T^4+2)^{18} \cdot (X+2T^4+T^2+2)^9$

Table 4.7: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\Gamma_1(T^2), X(n))$, $\mathbb{F}_3$

n	Characteristic Polynomial of T_P with $P = T + 1$
2	$(X+2)$
3	
4	$(X+2)^2 \cdot (X+T+2)$
5	
6	$(X+2)^2 \cdot (X+T^2+2) \cdot (X+2T^2+2) \cdot (X+2T^2+T+2)$
n	Characteristic Polynomial of T_P with $P = T^2 + 1$
2	$(X+2)$
3	
4	$(X+2)^2 \cdot (X+2T^2+2)$
5	
6	$(X+2)^2 \cdot (X+2T^4+2)^2 \cdot (X+2T^4+T^2+2)$

Table 4.8: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\Gamma_0^1(T), X(n))$, $\mathbb{F}_3$

n	Characteristic Polynomial of T_P with $P = T + 1$
2	$(X + 2)$
3	
4	$(X + 2)^2$
5	
6	$(X + 2)^2 \cdot (X + 2T^2 + T + 2)$
n	Characteristic Polynomial of T_P with $P = T^2 + 1$
2	$(X + 2)$
3	
4	$(X + 2)^2$
5	
6	$(X + 2)^2 \cdot (X + 2T^4 + T^2 + 2)$

Table 4.9: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\Gamma_0(T), X(n))$, $\mathbb{F}_3$

n	Characteristic Polynomial of T_P with $P = T + 1$
2	
$3 \dots 5$	$(X + 1)$
6	$(X + 1) \cdot (X + T + 1)$
n	Characteristic Polynomial of T_P with $P = T^2 + T + 1$
2	
$3 \dots 5$	$(X + 1)$
6	$(X + 1) \cdot (X + T^2 + T + 1)$

Table 4.10: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\mathrm{SL}_2(R), X(n))$, $\mathbb{F}_2$

n	Characteristic Polynomial of T_P with $P = T + 1$
$2, 3$	
4	$(X + 2)$
5	
6	$(X + 2)$
n	Characteristic Polynomial of T_P with $P = T^2 + 1$
$2, 3$	
4	$(X + 2)$
5	
6	$(X + 2)$
n	Characteristic Polynomial of T_P with $P = T^3 + T^2 + 2T + 1$
$2, 3$	
4	$(X + 2)$
5	
6	$(X + 2)$

Table 4.11: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\mathrm{SL}_2(R), X(n))$, $\mathbb{F}_3$

n	Characteristic Polynomial of T_P with $P = T + 1$
$2 \dots 5$	
6	$(X + 4)$
n	Characteristic Polynomial of T_P with $P = T^2 + T + 1$
$2 \dots 5$	
6	$(X + 4)$
n	Characteristic Polynomial of T_P with $P = T^3 + T^2 + 3T + 1$
$2 \dots 5$	
6	$(X + 4)$
$7 \dots 9$	
10	$(X + 4)$

Table 4.12: Hecke Operator T_P on $\mathrm{C}_{\mathrm{har}}(\mathrm{SL}_2(R), X(n))$, $\mathbb{F}_5$

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 3. Juli 2018 ....