Numerical relativity
LN-1&2/Bas_FS25

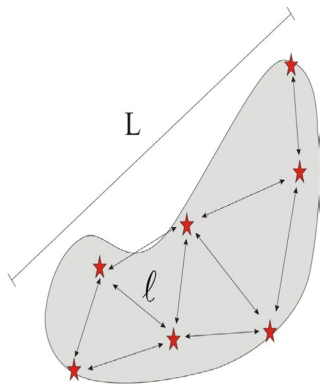
The fluid and gas phenomena

In nature there are at least four states of matter known:

Solid, Fluid, Gas, Plasma.

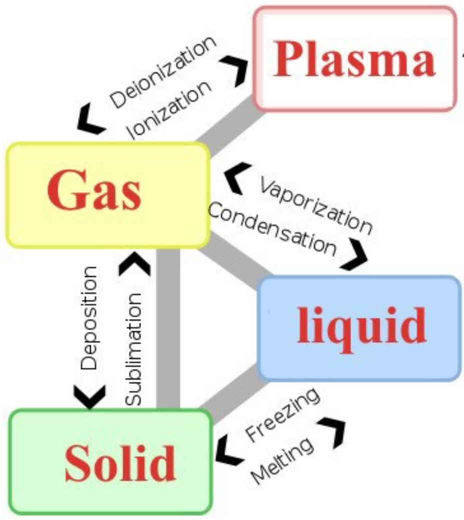
But what determines the state of matter and how to decide whether a sample of particles is a continuum or not?

Hypothesis: *the sample of particles is said to be a continuum, if the macroscopic properties, such as temperature, velocity etc., can still be associated with any arbitrary chosen small volume of it.*



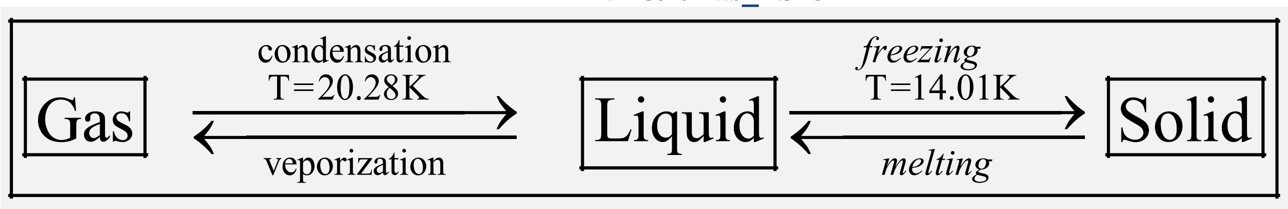
Crucial here is the length scale, ℓ, or equivalently the so-called **mean free path**: the mean distance particles may move before colliding with their neighbours.
Let "L" be the length characterizing the global size of the sample, then, we must first require that $\ell << L$ in order to ensure that there is a sufficiently large number of particles.
Based on terrestrial observations, ℓ appears to correlate with the temperature, i.e., $\ell \propto T$ and that for $T > T_c$, the matter may undergo phase transitions from one state into another.
Therefore, we may argue that if the resistive forces exceed the attractive inter-particle forces, a phase transition may, though not necessary, occur, i.e.

$$\nabla P > f_{\text{binding interatomic force}} \Rightarrow \textit{Phase transition.}$$



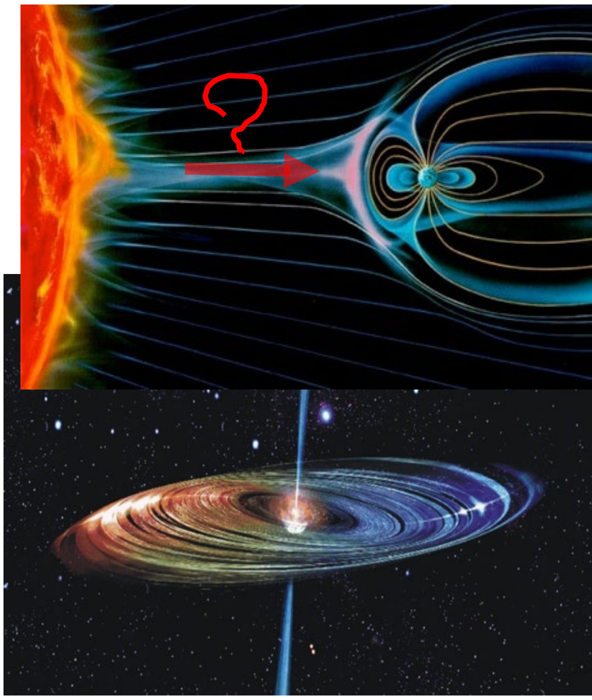
In general, ∇P and **f** operate on different length scales, hence decoupling is possible. However, gravity could still oppose the pressure and lead to condensation.

Most abundant chemical element in the universe is hydrogen.
Hydrogen makes **75%** of its mass and 90% of the number of its atoms.
The different phases of hydrogen can be summarized as follows:

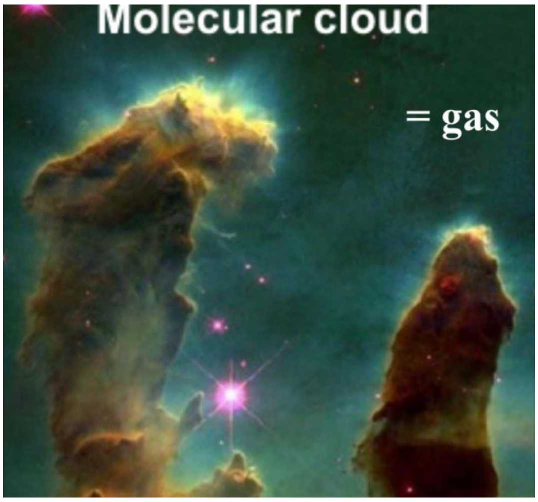


Q1: Consider Helium 4 (^{4}He): what are $T_{condens.}$ and T_{freaze}? Does ^{4}He undergo deposition?

In the case that gas is heated beyond a critical temperature, $T > T_c$, the electrons in the outer shell become electric-unbound to the nucleus and the gas undergoes a phase transition into plasma: it becomes electrical and magnetic conducting medium. The solar wind is a good example hereof. Also, highly collimated and relativistic jets that have been observed to emanate from the centres of galaxies are considered to be made of virially hot electric charged electron-proton or electron-positron particles, hence of plasmas.



Molecular clouds, where the embryos of stars are created are made of almost pure gas.



Conditions for treating a sample of moving particles as a fluid flow (?)

A sample of particles can be treated as a continuum, if the mean collision rate of two arbitrary neighbouring particles occurs at much shorter time scale than the time needed for acoustic waves to across the entire volume or the particles to move from one boundary to another.

In the case that the internal collisional velocity of the particles is comparable to the bulk velocity of the sample, then the sample can be treated as continuum, if:

$$\mathbf{a} \ll \boldsymbol{\ell} \ll \mathbf{L},$$

where: $\boldsymbol{\ell}$ is the collisional mean free path, i.e., the average distance travelled by a particle between two successive collisions. **L** is the length scale characterizing the global volume of the sample and **a** is the average radius of a particle.

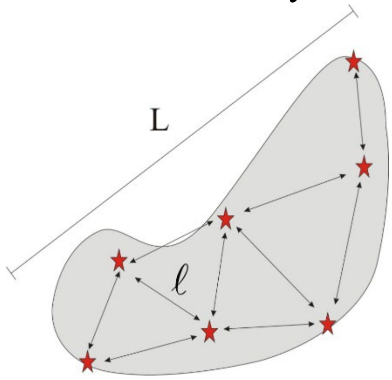


Fig. 1.1: A sample of interacting particles in a volume V. L is the macroscopic length scale and ℓ is the mean free path between two neighbouring particles.

Example 1:

Consider a box of 1 cm^3 of air under standard temperature and pressure (STP).
Can this sample be treated as continuum?

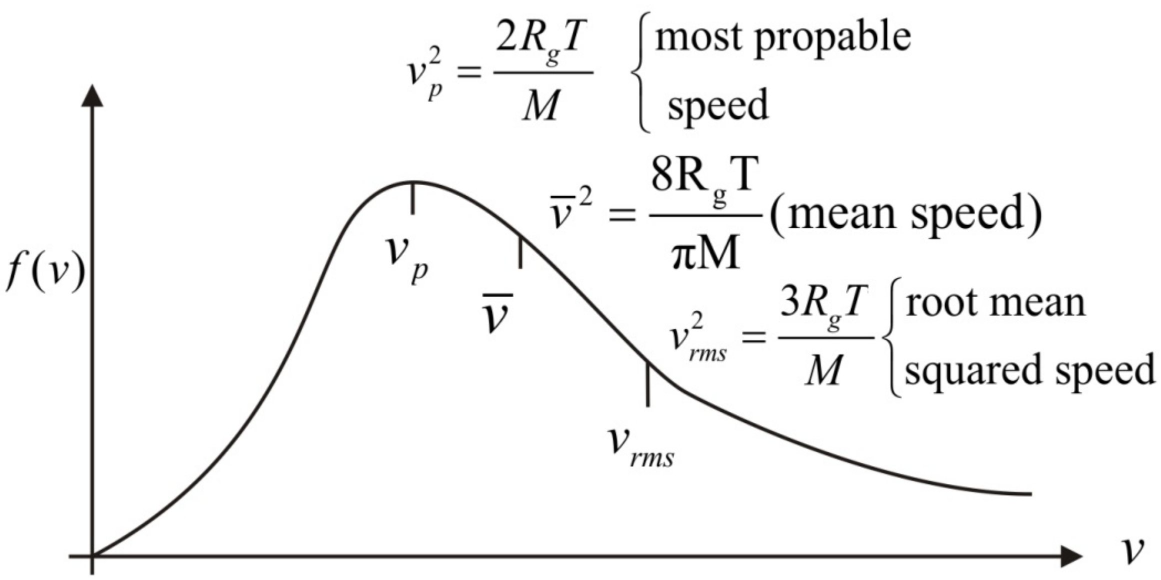
Answer:
The number of particles in this box is: $\mathbf{n \approx 10^{-3}\, N_{AV} \approx 6 \times 10^{20}}$**, where the Avogadro's number** $\mathbf{N_{AV} = 6.022 \times 10^{23}}$ **and** $\mathbf{\langle a \rangle}$ **(** $\mathbf{\approx 10^{-8}}$ **cm**) is the typical radius of an air particle.

Mean free path?

The distribution of velocities of air particles, assuming the air as ideal gas in thermal equilibrium, is given by the Maxwellian distribution:

$$f(v) = 4\pi\,[\frac{M}{2\pi R_g T}]^2\; v^2\; e^{-[\frac{Mv^2}{2R_g T}]},$$

where M = molar mass (*M* = *N*N_Av *m0)* and R_g = gas constant.



Where
$$\frac{\partial f}{\partial v} = 0 \Rightarrow v_p = \sqrt{\frac{2R_gT}{M}}. \qquad \boxed{v_p \overset{air,\,300K}{=} 422\ \text{m/s}}$$

$$\bar{v} = \int_0^\infty v f(v)\,dv = \sqrt{\frac{8R_gT}{\pi M}} = \frac{2}{\sqrt{\pi}} v_p$$

$$v_{rms} = (\int_0^\infty v^2 f(v)\,dv)^{1/2} = \sqrt{\frac{3R_gT}{M}} = \sqrt{\frac{3}{2}} v_p$$

Noting that the sound speed of a gas in thermal equilibrium reads:

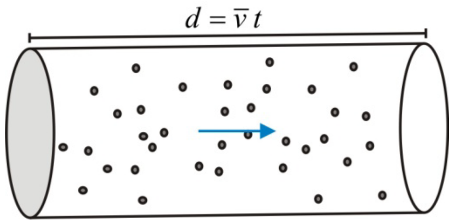
$$v_s = \sqrt{\gamma \frac{P}{\rho}}, \qquad \gamma_{air} = \frac{c_p}{c_V} = 1.4 \quad \Rightarrow \boxed{v_p > v_s}$$

Q: Does this imply that air particles are moving with supersonic velocity?

But what is the average distance a particle may move freely before colliding with its neighbours?

Concept: mean free path:

Assume we have a cylinder, through which the air particles are flowing.



The average distance travelled by a particle is: $d = \bar{v}\, t$.

The volume of the cylinder is V= A d, where A(= [π a^2]×n_v) is the area occupied by each particle times the number of particles = cross section of the cylinder.

Total mass
$\doteq V \cdot n \equiv \text{const.}$

Thus, the mean free path reads:

$$< \ell >= \frac{\int d\ dv}{\int dv} = \frac{1}{V_{tot}} \int d\ dv = \frac{1}{V_{tot}} \int \bar{v}t \frac{1}{n^2} dn = \frac{\bar{v}t}{V_{tot}\, n}$$

$$= \frac{\text{distance travelled}}{\text{volume} \times \text{number of partices}} = \frac{\bar{v}t}{\pi a^2 \bar{v} t n_v} = \frac{1}{\pi a^2 n_v} = \frac{1}{\sqrt{2}\sigma n_v}$$

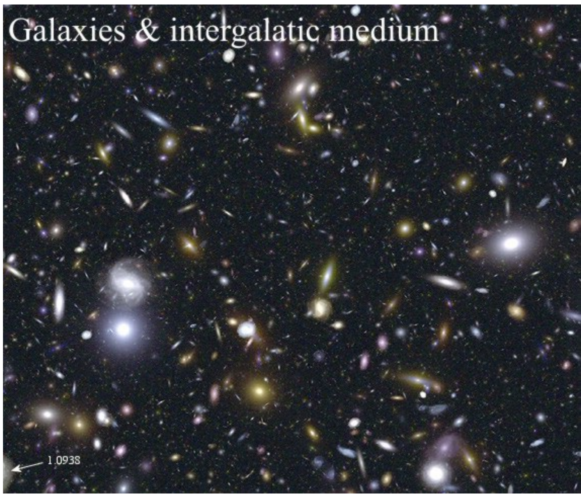
$$\Leftrightarrow \quad < \ell > \approx \frac{1}{10^{-15} 10^{20}} = 10^{-5}\, cm$$

$$\Leftrightarrow \quad a << \ell << L \Leftrightarrow Continuum$$

Q: A piece of pure iron has the volume of one liter. How many particles are contained in this liter under standard conditions?

Example 2:

- In intergalactic medium (IGM), the number density $n \sim 10^{-3}\ cm^{-3}$



$$\Rightarrow \ \ell = \frac{1}{\sigma n_v} \approx \frac{1}{10^{-18}}\ \text{cm}$$

$$= 10^{18}\ \text{cm} \approx 1\ \text{ly}$$

$$L_{\text{cloud}} \approx \text{kpc}\ (10^{21} - 10^{23}\ \text{cm})$$

$$\Rightarrow a << \ell << L \ \Leftrightarrow \ \text{Continuum}$$

- In interstellar medium (ISM), the number density $n \sim 1\ cm^{-3}$.

$$\Rightarrow \ \ell = \frac{1}{\sigma n_v} = \frac{1}{10^{-15}}\ \text{cm} = 10^{15}\ \text{cm}$$

$$L_{\text{cloud}} \approx 10^{19} - 10^{20}\ \text{cm}$$

$$\Rightarrow a << \ell << L \ \Leftrightarrow \ \text{Continuum}$$

Example 3:

By stellar clusters, we mean samples of millions of stars held together about their own center of mass.

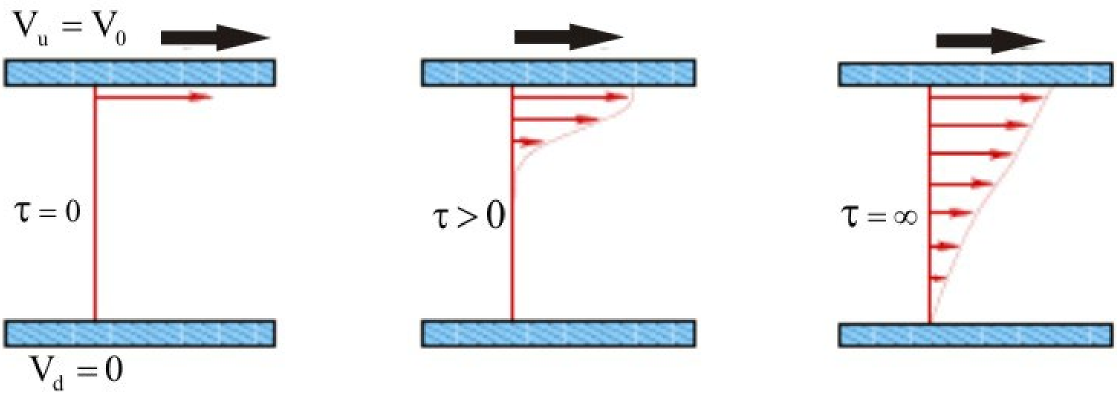
In stellar clusters **$\ell << L$. $<v>_{st} \sim <v>_{cluster}$.**

Stars may migrate from the cluster without direct interaction with other stars. ⇒ **normal fluid description here is not appropriate** (the collision is not diffusive)

Basic properties of fluid flows:

- **Mass density** $\rho = \frac{mass}{volume}$ **= the amount of matter contained in a unit of volume. The units of** $[\rho] = [M/L^3] = g/cm^3, kg/m^3$.

- **Viscosity: is a physical property of the fluid that offers resistance to shear forces. The particles re-arrange their internal motion and interaction in such a manner to yield a force that appose their global motion. It is also a measure for the interaction between particles that gives rise to resistance and deformation of particle motions, i.e., internal friction.**



It appears that each fluid has its own resistivity and it is defined as follows:
Let τ be a measure for the force per unit area, i.e.,

$$\tau = \left[\frac{\text{Force}}{Area}\right] = \left[\frac{m\,a}{\text{L}^2}\right] = \left[\frac{\text{MLT}^{-2}}{\text{L}^2}\right] = \left[\text{ML}^{-1}\text{T}^{-2}\right]$$

$$\text{But as } \left[\frac{\text{dV}}{\text{dy}}\right] = \frac{\text{LT}^{-1}}{\text{L}} = \text{T}^{-1}$$

$$\text{and as } \mu = \frac{\tau}{\text{dV}/\text{dy}} \Rightarrow [\mu] = \frac{[\tau]}{[\text{dV}/\text{dy}]} = \frac{\text{ML}^{-1}\text{T}^{-2}}{\text{T}^{-1}} = \text{ML}^{-1}\text{T}^{-1}$$

$$\boxed{\tau_{ij} = -\mu\left(\frac{dV_j}{dx_i} + \frac{dV_i}{dx_j}\right) + \frac{2}{3}\mu\delta_{ij}\frac{dV_k}{dx_k}} \Leftrightarrow \tau = \text{coefficient} \times \begin{Bmatrix} \text{the rate of deformation} \\ \text{of V in } x_i\text{-directions} \end{Bmatrix}$$

where μ is called the dynamical viscosity.

A simple example: $\tau = \mu\frac{dV}{dy}, \quad \Leftrightarrow \quad$ (The Newton's law of viscosity)

For τ=const. $\Rightarrow \mu \propto 1/\left[\frac{dV}{dy}\right]$

⇔ largest deformations occur in low viscous flows, whereas high viscous flows are resistant against deformations.

$$\boxed{\begin{aligned} \mu_{air} &= 1.78\times10^{-4} \text{ g/cm/s} \\ \mu_{water} &= 1.14\times10^{-2} \text{ g/cm/s} \\ \mu_{oil} &= 19 \text{ g/cm/s} \end{aligned}}$$

Example: From the viscosity per unit mass, we obtain the coefficient, called **Kinematic viscosity:** $\nu = \frac{\mu}{\rho}$.

Note that not all viscous fluid flows obey the Newtonian's law of viscosity. The viscosity law in ketchup, polymers, blood and many other types of fluids may obey other laws, such as : $\boldsymbol{\tau \propto (dV/dy)^n}$: there are classified as **Non-Newtonian fluids**.

- Pressure: $P = \frac{\text{Force}}{\text{Area}} = \frac{F}{A} \Rightarrow [P] = NL^{-2}$

  Thus $P = \frac{dF}{dA} \Rightarrow F = \int_A \vec{P}\cdot dA.$

- **Boyle's law: PV = Const.**

The equations of fluid dynamics:

$$\iint_{\partial S} F\cdot\vec{n}\,ds = \iiint_V \nabla\cdot F\,dV$$

$$\frac{\partial}{\partial t}\left[\iiint_V \rho\,dV\right]+\iiint_V \nabla\cdot F\,dV=0,\quad F=\rho\vec{V}$$

$$\iiint_V\left[\frac{\partial\rho}{\partial t}+\nabla\cdot F\right]=0$$

$$\Rightarrow\frac{\partial\rho}{\partial t}+\nabla\cdot\rho\vec{V}=0$$

Let **V** be a volume filled with a plasma, whose surface is ∂S.

Assume that $f_{\text{source}} = f_{\text{sink}} = 0$, ↔ no generation and destruction of matter.

Then the rate of change of the total mass inside V is uniquely determined through the net flux across the surface ∂S.

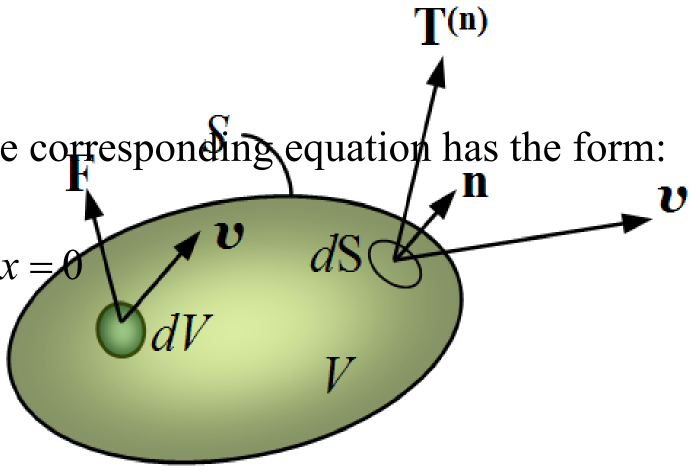
Mass conservation

$$\frac{\partial M}{\partial t} = -\iint_{\partial S} F\cdot\vec{n}\,ds$$

According to the divergence theorem:

A quantity is said to be conservative, if the corresponding equation has the form:

$$\frac{\partial[q]}{\partial t}+div\cdot Flux=0$$



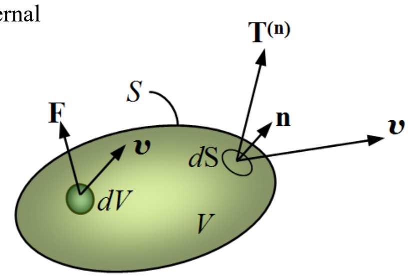
Momentum in fluids:

The momentum in fluids is a conserved vector quantity, which describes the motion of a collection of particle subject to external and internal forces:

$$\frac{\partial}{\partial t}[\text{Momentum}]+\text{div}\cdot[\text{Flux of momentum}]= \text{f}_{\text{internal}}+\text{f}_{\text{external}}$$

$$\frac{\partial}{\partial t}\rho V_j+\frac{\partial}{\partial x_k}(\rho V_j V_k)=-\frac{\partial}{\partial x_k}P\delta_{jk}+\frac{\partial}{\partial x_k}\sigma_{jk}+\rho f_j$$

where $\text{V}=(\text{V}_1,\text{V}_2,V_3)$, $\text{P}=\text{pressure}$, $\sigma_{jk}=nonisotropic$ (traceless) component of internal interaction (friction).



Energy equation:

Assume we have the control volume **dV** filled with a hot, radiative and magnetized plasma moving with velocity $\vec{\text{V}}$ under the influence of a gravitational field, then the total energy of the plasma in **V** at a certain time "t" :

$$\text{E}_{\text{total}}=\text{E}_{\text{kinetic}}+\text{E}_{\text{internal}}+\text{E}_{\text{magnetic}}+\text{E}_{\text{radiative}}+\text{E}_{\text{potential}}+\text{E}_{\text{extra}}$$

$$\frac{\partial}{\partial t}E_{tot}+div\cdot[Flux]=\Gamma_{gain}-\Lambda_{loss}$$

Assume that:

$$\text{E}_{\text{kinetic}}=\text{E}_{\text{magnetic}}=\text{E}_{\text{radiative}}=\text{E}_{\text{potential}}=\text{E}_{\text{extra}}=\text{constant},$$

Then we are left with an equation that describes the time-evolution of the internal energy:

$$\frac{\partial}{\partial t}\rho E_{\text{int}} + div\cdot[\rho E_{\text{int}}\vec{V}] = -\underbrace{P\nabla\cdot\vec{V}}_{>0,<0?} + \Gamma_{gain} - \Lambda_{loss}$$