Approximate Solution of Eckhaus Equation Using Elzaki Decomposition Method

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Abstract

This article presents the application of Elzaki decomposition method to obtained approximate solution of Eckhaus equation. The method in comparison with other methods worked perfectly without any requirement for discretization or linearization. Solution obtained corresponds with exact solutions in literature

Keywords: Elzaki Transform, Adomain Decomposition Method, Eckhaus Equation.

1 Introduction

Linear and non-linear differential equations are often employed to describe physical phenomena that occurs in many branches of science and technology such as Computational chemistry, Quantum mechanics, Control theory, Optics and Plasma physics. Since the importance of linear and non-linear differential equation can not be underestimate in mathematical modeling, many researchers have examined and developed various methods to solve differential equations more accurately and efficiently. To mention but a few Adomian [1], developed the celebrated method of the 80th, the Adomian Decomposition Method(ADM). He [2-5] developed Homotopy Perturbation Method(HPM) for solutions of pysical problems.

This article uses Elzaki Decomposition Method(EDM) to solve Eckhaus equation which is a generalization of the non-linear Schrodinger equation with quintic non-linear term, it deffers to non-linear Schrodinger equation with respect to Hamiltonian structure and field commutation relation.

Elzaki Decomposition Method(EDM) is the combination of Elzaki transform recently introduced by Tarig Elzaki in 2010 with Adomian Decomposition Method(ADM) [6-8]. The transform can not solve or simplify the non-linear term, therefore ADM is used to decompose the non-linear term. The solution is provided in a series form.

2 Elzaki Transform

The above mentioned transform of function belongs to a class A,

$$A = \{u(t) : \exists m, k_1, k_2 > 0 : |u(t)| < me^{\frac{|t|}{k_j}}, t \in (-1)^j \times [0, \infty)\}$$

$u(t)$ is defined by $E[u(t)] = U(v)$ described as:

$$E[u(t)] = v\int_0^{\infty} u(t)e^{-\frac{t}{v}}dt = U(v), v \in (k_1, k_2) \tag{1}$$

The properties of Elzaki transform are:

1. $E\{t^n\} = n!v^{n+2}, n \geq 0$
2. $E\{e^{-at}\} = \dfrac{v^2}{1+av}$,
3. $E\sin at = \dfrac{av^3}{1+a^2v^2}$,
4. $E\{u(t)\} = \dfrac{U(v)}{v} - vu(0)$,
5. $E\{U^n(t)\} = \dfrac{U(v)}{v^n} - \sum_{k=0}^{n-1} v^{2-n+k}u^k(0)$

3. Basic idea of Elzaki Decomposition Method

A generalised second order non-linear partial differential equation of the form:

$$\begin{aligned} &Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \\ &u(x,0) = h(x), u_t(x,t) = f(x) \end{aligned} \tag{2}$$

is considered. D is the linear differential operator, R is the lower differential operator of order less than D, N represents the generalised non-linear differential operator and g(x,t) is the source term.

Applying Elzaki transform to both sides of Equ.(2) gives:

$$E[Du(x,t)]+E[Ru(x,t)]+E[Nu(x,t)]=E[g(x,t)] \ , \tag{3}$$

With the differentiation property of Elzaki transform and applying the initial conditions to Equ.(3) gives:

$$E[u(x,t)]=v^2E[g(x,t)]+v^2h(x)+v^3h-v^2E[Ru(x,t)+Nu(x,t)] \ , \tag{4}$$

Now we apply the Adomian Decompostion method to decompose the non-linear term by replacing the unknown function by an inifinite series:

$$u=\sum_{n=0}^{\infty}u_n(x,t) \, , \tag{5}$$

And the non-linear term by:

$$Nu=\sum_{n=0}^{\infty}A_n(u_{0,}u_{1,}u_2,...) \ , \tag{6}$$

Where $A_n(u_{0,}u_{1,}u_2,...)$ are the Adomian polynomials [5-6], to be determined recursively by the Adomian algorithm given as:

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}[N(\sum_{i=0}^{n}\lambda^i u_i)]_{\lambda=0} \qquad , \text{n=0,1,2,...} \tag{7}$$

substituting Equ.(5)-Equ.(7) into Equ.(4) gives:

$$E[\sum_{n=0}^{\infty}u_n(x,t)]=v^2E[g(x,t)]+v^2h(x)+v^3f(x)-v^2E[\sum_{n=0}^{\infty}u_{nxx}+\sum_{n=0}^{\infty}A_n], \tag{8}$$

$$\sum_{n=0}^{\infty}E[u_n(x,t)]=v^2E[g(x,t)]+v^2h(x)+v^3f(x)-v^2E[\sum_{n=0}^{\infty}u_{nxx}+\sum_{n=0}^{\infty}A_n] \tag{9}$$

Taking the Elzaki transform inverse on both sides of Equ.(9) gives:

$$u(x,t)=G(x,t)-E^{-1}[v^2E[\sum_{n=0}^{\infty}u_{nxx}+\sum_{n=0}^{\infty}A_n]], \tag{10}$$

where $G(x,t)$ arise from the source term and the prescribed initial conditions. The approximate expressions for Equ.(2) is obtained recursively as:

$$u_0(x,t) = G(x,t),$$

$$u_1(x,t) = E^{-1}[v^2[\sum_{n=0}^{\infty} u_{0xx} + \sum_{n=0}^{\infty} A_0]],$$

$$u_2(x,t) = E^{-1}[v^2[\sum_{n=0}^{\infty} u_{1xx} + \sum_{n=0}^{\infty} A_1]],$$

$$\vdots$$

and so on. With the help of Mathematica software one can obtain more terms.

The above can be written in a more compact form:

$$u_0(x,t) = G(x,t)$$
$$u_{n+1}(x,t) = E^{-1}[vE[u_{nxx} + A_n]], n \geq 0$$

4 Application of the method

In this section, we consider the Eckhaus equation in order to demonstrate the effectiveness of the method.

$$iu_t + u_{xx} + 2|u|^2 u + |u|^4 u - 2\beta_i(|u|^2)u = 0 \tag{11}$$

With $u(x,0) = e^{ix}$

Neglecting the last term, we have:

$$iu_t + u_{xx} + 2|u^2|u + |u|^4 u = 0 \tag{12}$$

Applying Elzaki transform to Equ.(12) we have:

$$E[u(x,t)] = v^2 u(x,0) + ivE[u_{xx} + 2|u|^2 + |u|^4 u] \tag{13}$$

Considering the non-linear terms

$$E[u(x,t)] = v^2 u(x,0) + ivE[\sum_{n=0}^{\infty} u_{nxx}] + 2viE[\sum_{n=0}^{\infty} A_n] + ivE[\sum_{n=0}^{\infty} A_n^*] \tag{14}$$

Applying the initial condition and taking the inverse Elzaki transform, we have:

$$u(x,t) = e^{ix} + iE^{-1}[vE[u_{nxx}]] + iE^{-1}[2vE[A_n]] + iE^{-1}[vE[A_n^*]] \tag{15}$$

The approximate expression for Equ.(12) is obtained recursively and solution

takes series form:

$$u_0(x,t) = e^{ix} \tag{16}$$

$$u_{n+1}(x,t) = iE^{-1}[vE[u_{nxx}]] + iE^{-1}[2v[A_n]] + iE^{-1}[vE[A_n^*]], \quad n \geq 0 \tag{17}$$

The Adomian is determined using Equ.(7) as follows:

$$A_0 = u_0^2 \overline{u_0}$$

$$A_1 = 2u_0 \overline{u_0} u_1 + u_0^2 \overline{u_1} \tag{18}$$

$$A_2 = 2u_0 \overline{u_0} u_2 + u_0^2 \overline{u_2} + \overline{u_0} u_1^2 + 2u_0 \overline{u_1} u_1$$

$$A_3 = \overline{u_1} u_1^2 + 2\overline{u_1} u_1^2 + 2\overline{u_0} u_1 u_2 + 4u_0 \overline{u_1} u_2 + 2u_0 \overline{u_0} u_3 + u_0^2 u_3$$

$\vdots$

and so on.

For the second non-linear term:

$$A_0^* = u_0^3 \overline{u_0}^2$$

$$A_1^* = 3u_0^2 \overline{u_0}^2 u_1 + 2u_0^3 \overline{u_0} u_1 \tag{19}$$

$$A_2^* = 3u_0 \overline{u_0}^2 u_1^2 + 6u_0^2 \overline{u_0}^2 u_1^2 + u_0^3 \overline{u_1} u_1 + 3u_0^2 \overline{u_0}^2 u_2 + 2u_0^3 \overline{u_0} u_2$$

$\vdots$ and so on.

Using the above recursive relationship we generate the first few terms:

$$u_0(x,t) = e^{ix} \tag{20}$$

$$u_1(x,t) = iE^{-1}[vE[u_{0xx}]] + iE^{-1}[2vE[A_n]] + iE^{-1}[vE[A_n^*]] \tag{21}$$

$$= -2ite^{ix}$$

$$u_2(x,t) = iE^{-1}[vE[u_{1xx}]] + iE^{-1}[2vE[A_1]] + iE^{-1}[vE[A_1^*]] \tag{22}$$

$$= \frac{-4t^2 e^{ix}}{2!}(4 + e^{2ix})$$

$$u_3(x,t) = iE^{-1}[vE[u_{2xx}]] + iE^{-1}[2vE[A_n]] + iE^{-1}[vE[A_2^*]] \tag{23}$$

$$= \frac{8it^3 e^{ix}}{3!}(18.5 + 10e^{2ix} + e^{4ix})$$

$\vdots$

and so on.

With the help of Mathematica software one can obtain more terms.

Assume $(18.5 + 10e^{2ix} + e^{4ix}) = \phi_2$ and $(4 + e^{2ix}) = \phi_1$, the sum of the terms is given as:

$$u(x,t) = (1 - (2it) + \frac{(2it)^2}{2!}\phi_1 - \frac{(2it)^3}{3!}\phi_2 + \cdots)e^{ix} \tag{24}$$

Neglecting the $\phi^{'s}$ we have:

$$u_n(x,t) = (\sum_{n=0}^{\infty}(-1)^n \frac{(2it)^n}{n!})e^{ix} \tag{25}$$

in series form and it corresponds with the exact solution in literature.

$$u(x,t) = e^{i(x-2t)} \tag{26}$$

5 Conclusion

In this article, the Elzaki Decomposition Method(EDM) has been successfully used to solve Eckhaus equation with initial conditions. The method is easy, reliable and accurate compare to other methods.

The solution reflect that the method is a powerful and efficient technique in finding approximate solutions of non-linear differential equations. A new efficient recurrent relation was obtained which implies that the method is highly accurate numerical solution for non-linear problems in computation chemistry, physic and quantum mechanics.

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